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RATIONAL CURVES ON HYPERSURFACES
[after A. Givental]

by **Rahul PANDHARIPANDE**

0. INTRODUCTION

We describe here a remarkable relationship studied by Givental between hypergeometric series and the quantum cohomology of hypersurfaces in projective space [G1]. As the quantum product involves genus 0 Gromov-Witten invariants, a connection between hypergeometric series and the geometry of rational curves on the hypersurfaces is made. While the most general context for such relationships has not yet been understood, analogous results for complete intersections in smooth toric varieties and flag varieties have been pursued by Givental [G2] and Kim [Ki].

The first discovery in this subject was the startling prediction from Mirror symmetry by Candelas, de la Ossa, Green, and Parkes [COGP] of the numbers of rational curves on quintic 3-folds in $\mathbf{P}^4$. We recount an equivalent form of their original prediction. Let $I_i(t)$ be defined by:

$$\sum_{i=0}^{3} I_i H^i = \sum_{d=0}^{\infty} e^{(H+d)t} \frac{\prod_{r=1}^{5d}(5H+r)}{\prod_{r=1}^{d}(H+r)^5} \mod H^4.$$

The functions $I_i(t)$ are a basis of solutions of the Picard-Fuchs differential equation

$$\Big(\frac{d}{dt}\Big)^4 I = 5e^t\Big(5\frac{d}{dt}+1\Big)\Big(5\frac{d}{dt}+2\Big)\Big(5\frac{d}{dt}+3\Big)\Big(5\frac{d}{dt}+4\Big)I$$

arising in the B-model from the variation of Hodge structures of a specific family of Calabi-Yau 3-folds. Let n_d be the virtual number of degree d rational curves on a general quintic 3-fold in $\mathbf{P}^4$. Let the change of variables $T(t) = I_1/I_0\ (t)$ define a new coordinate T. The functions $J_i = I_i/I_0\ (T)$ in the new variable were predicted to equal an A-model series:

$$\sum_{i=0}^{3} J_i H^i = e^{HT} + \frac{H^2}{5}\sum_{d=1}^{\infty} n_d d^3 \sum_{k=1}^{\infty} \frac{e^{(H+kd)T}}{(H+kd)^2} \mod H^4 \tag{1}$$

and satisfy the differential equation:

$$\frac{d^2}{dT^2}\frac{1}{K(e^T)}\frac{d^2}{dT^2}\, J_i = 0, \quad \text{where} \quad K(e^T) = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{e^{dT}}{1-e^{dT}}. \tag{2}$$

As the enumerative geometry of quintic 3-folds was not known to have any structure at all, these formulas were completely unexpected. There is a large literature in both physics and mathematics on Mirror symmetry for Calabi-Yau 3-folds. The A-model / B-model framework is described in [W1]. A mathematical perspective can be found in [Mo], [CK]. Higher dimensional Calabi-Yau manifolds are considered in [GMP].

The numbers n_d in (1) and (2) have the following interpretation: if the rational curves on the general quintic 3-fold Q were nonsingular, isolated with balanced normal bundle, and disjoint, then n_d would simply be the number of degree d rational curves on Q. However, this strong assumption is false [V] – there exist nodal degree 5 rational curves on Q. The nonexistence of families of rational curves in Q is still open (Clemens' conjecture). A mathematically precise statement of conjecture (1) requires a substantial program to define n_d: moduli spaces of maps, their virtual classes, and Gromov-Witten invariants. This program has been completed in both symplectic and algebraic geometry through the recent work of many mathematicians (see the foundational papers [RT], [KM]). The virtually enumerative numbers n_d are defined via the genus 0 Gromov-Witten invariants N_d of the quintic by a formula accounting for multiple cover contributions [AM], [M], [Vo]:

$$\sum_{d=1}^{\infty} N_d q^d = \sum_{d=1}^{\infty}\sum_{k=1}^{\infty} n_d k^{-3} q^{kd}. \tag{3}$$

The numbers n_d are enumerative at least for $d \leq 9$ [K], [KJ1]. An outlook in higher degrees may be found in [KJ2].

The central relationship in Givental's work may be explained as follows. Let X be a hypersurface in $\mathbf{P}^m$ of degree $l \leq m+1$. A correlator S_X is defined via the quantum product and related quantum differential equations associated to X (see Section 2). S_X is closely related to the hypergeometric series:

$$S_X^* = \sum_{d=0}^{\infty} e^{(H+d)t} \frac{\Pi_{r=1}^{ld}(lH+r)}{\Pi_{r=1}^{d}(H+r)^{m+1}} \bmod H^m. \tag{4}$$

The precise relationship is divided in 3 cases.

(i) If $l < m$, then $S_X^* = S_X$.
(ii) If $l = m$, then $e^{-m!e^t} S_X^* = S_X$.
(iii) If $l = m+1$, then S_X^* and S_X are related by an explicit transformation (see Section 4).

In the case of the quintic 3-fold, S_X is exactly the right side of (1) (see Section 4.5). The transformation (iii) then specializes to the Mirror symmetry prediction proving (1). Equation (2) is a consequence of the quantum differential equation. The results in cases (i)

and (ii) have direct applications to the quantum cohomology ring of the corresponding hypersurfaces (see Section 3). Givental has suggested that cases (i) and (ii) correspond to non-compact Mirrors.

The plan of the paper is as follows. Section 1 contains a rapid review of Gromov-Witten invariants, descendents, quantum products, and quantum differential equations. In Section 2, a new quantum product $*_X$ is defined on the cohomology of the ambient space $\mathbf{P}^m$. The $*_X$-product greatly clarifies the relationship between quantum structures on X and $\mathbf{P}^m$. Givental's correlator S_X is naturally defined via differential equations arising from the $*_X$-product. This product appears in [G1], [Ki] and was explained to the author by T. Graber. Section 3 covers cases (i) and (ii) where $l \leq m$. These are much easier than the Calabi-Yau case which is established in Section 4. The treatment in Sections 3 and 4 follows [G1] with some augmentation and modification.

The main technical tool needed in Givental's approach is an explicit localization formula in equivariant cohomology for the natural torus action on the moduli space of maps $\overline{M}_{0,n}(\mathbf{P}^m, d)$. As this moduli space is a nonsingular stack, the Bott residue formula holds. The fixed point loci of the torus action as well as the precise equivariant normal bundle determinations have been explained in detail in [Ko]. For Givental's arguments in the smooth toric case [G2], a virtual localization formula [GP] is necessary as the moduli space of maps may be quite ill-behaved.

The number $n_1 = 2875$ of lines on a general quintic 3-fold was obtained in the 19^{th} century by Schubert (via intersection calculations in the Grassmannian $\mathbf{G}(\mathbf{P}^1, \mathbf{P}^4)$). The numbers $n_2 = 609250$ and $n_3 = 317206375$ of conics and twisted cubics were computed by S. Katz [K] and Ellingsrud and Strømme [ES1] respectively. Localization was first applied to the enumerative geometry of quintics in [ES2]. The method of torus localization on $\overline{M}_{0,n}(\mathbf{P}^m, d)$ was developed by Kontsevich in [Ko] precisely to attack the Mirror prediction. The resulting formulas determined all the numbers n_d by a complex sum over graphs. This summation yielded the first mathematical computation of $n_4 = 242467530000$ [Ko]. An important aspect of the argument in Sections 3 and 4 is an organization of graph sums.

A complete proof of the Mirror prediction for quintics by Lian, Liu, and Yau using localization formulas has appeared recently in [LLY]. The argument announced by Givental in [G1] yields a complete proof of (i)-(iii). It is the latter proof that is explained here (see [LLY], [G3] for a comparison of viewpoints). Givental's work is also discussed in [CK] and [BDPP]. So far, mathematical approaches to the A-model series do not involve the B-model at all. While these results verify the predictions of [COGP], the full correspondence of Mirror symmetry remains to be mathematically explained.

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1. THE QUANTUM DIFFERENTIAL EQUATION

1.1. Descendents

A nonsingular algebraic variety X is *convex* if

$$H^1(\mathbf{P}^1, \mu^* T_X) = 0$$

for all maps $\mu : \mathbf{P}^1 \to X$. The main examples of compact convex varieties are $X = \mathbf{G}/\mathbf{P}$ where $\mathbf{G}$ is a linear algebraic group and $\mathbf{P}$ is a parabolic subgroup. The case of most interest here is $X = \mathbf{P}^m$. The space $\overline{M}_{0,n}(X, \beta)$ of n-pointed genus 0 stable maps representing the class $\beta \in H_2(X, \mathbb{Z})$ is a coarse moduli space with quotient singularities (or a nonsingular Deligne-Mumford stack) of pure dimension

$$\dim(X) + \int_\beta c_1(T_X) + n - 3$$

in case X is convex [Ko], [FP]. Let $e_i : \overline{M}_{0,n}(X, \beta) \to X$ be the i^{th} evaluation map. Let ψ_i be the first Chern class of the i^{th} cotangent line bundle L_i on $\overline{M}_{0,n}(X, \beta)$. The fiber of L_i over the moduli point

$$[\mu : (C, p_1, \ldots, p_n) \to X]$$

is the cotangent space of C at p_i. The Chern classes ψ_i are elements in $H^2(\overline{M}_{0,n}(X, \beta), \mathbb{Q})$. In [W2], invariants of X are defined by integrals over the moduli space of maps. The genus 0 *gravitational descendents* are the invariants:

$$\langle\, \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \,\rangle_\beta^X = \int_{\overline{M}_{0,n}(X,\beta)} e_1^*(\gamma_1) \cup \psi_1^{a_1} \cup \cdots \cup e_n^*(\gamma_n) \cup \psi_n^{a_n} \tag{5}$$

where $\gamma_i \in H^*(X, \mathbb{Q})$ and the a_i are nonnegative integers. As usual, the invariants are defined to vanish unless the dimension of the integrand is correct. When the a_i are all 0, the gravitational descendents specialize to the *Gromov-Witten invariants* of X. For simplicity, $\tau_0(\gamma)$ will often be denoted by γ in (5).

In this preliminary section, three topics are covered. Basic properties of the descendent integrals are treated first. Next, a fundamental solution of the quantum differential equation obtained from the flat connection in the Dubrovin formalism is derived. Finally, an explicit form of this solution in case $X = \mathbf{P}^m$ is given. The main sources in the mathematics literature for this material are [D], [G1].

The formulas of fundamental class and divisor for Gromov-Witten invariants (see [RT], [KM]) take a slightly different form for gravitational descendents. These formulas are given in Section 1.2 and are closely related to Witten's equations in [W2]. Together with the topological recursion relations, these formulas are sufficient to reconstruct the genus

0 descendents from the Gromov-Witten invariants and to derive a fundamental solution to the quantum differential equation.

X will be assumed to be convex. This hypothesis leads to great simplification in the genus 0 case: no virtual fundamental class considerations are needed at this point. The results, however, are valid for general nonsingular projective X.

1.2. **The string, dilaton, and divisor equations**

Let the map

$$\nu : \overline{M}_{0,n+1}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$$

be the natural contraction morphism forgetting the last point. Contraction is possible only when $\beta = 0$, $n \geq 3$ or $\beta \neq 0$, $n \geq 0$. Three basic equations hold for descendent invariants: the string, dilaton, and divisor equations. They apply when contraction is possible and the class assigned to the last marking is of total codimension 0 or 1.

I. The string equation. Let $T_0 \in H^*(X,\mathbb{Q})$ be the unit:

$$\langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\ T_0\ \rangle_\beta =$$

$$\sum_{i=1}^{n}\langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_{i-1}}(\gamma_{i-1})\ \tau_{a_i-1}(\gamma_i)\ \tau_{a_{i+1}}(\gamma_{i+1})\cdots\tau_{a_n}(\gamma_n)\ \rangle_\beta.$$

II. The dilaton equation:

$$\langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\ \tau_1(T_0)\ \rangle_\beta = (-2+n)\cdot\langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\ \rangle_\beta.$$

III. The divisor equation. Let $\gamma \in H^2(X,\mathbb{Q})$:

$$\langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\ \gamma\ \rangle_\beta = (\int_\beta \gamma)\ \cdot\ \langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\ \rangle_\beta$$

$$+\sum_{i=1}^{n}\langle\ \tau_{a_1}(\gamma_1)\cdots\tau_{a_{i-1}}(\gamma_{i-1})\ \tau_{a_i-1}(\gamma_i\cup\gamma)\ \tau_{a_{i+1}}(\gamma_{i+1})\cdots\tau_{a_n}(\gamma_n)\ \rangle_\beta.$$

In these formulas, any term with a negative exponent on a cotangent line class is defined to be 0.

The proofs of these equations in the convex genus 0 case rely on a comparison result for cotangent lines: $\psi_i = \nu^*(\psi_i) + [D_{i,n+1}]$. Here, $D_{i,n+1}$ denotes the boundary divisor in $\overline{M}_{0,n+1}(X,\beta)$ determined by the curve and point partition (see [FP]):

$$(\beta_1 = 0, \{i, n+1\} \mid \beta_2 = \beta, \{1,\ldots,\hat{i},\ldots,n\}).$$

Equations I-III follow easily from this comparison result. In the nonconvex or higher genus cases, the string and divisor equation hold exactly in the form above. The dilaton equation is true in genus g with the factor $(-2+n)$ replaced by $(2g-2+n)$. The proofs in this greater generality require properties of the virtual fundamental class.

In genus 0, the descendent integrals actually carry no more information than the Gromov-Witten invariants (see [Du]):

PROPOSITION 1. — *The genus* 0 *descendents of* X *can be uniquely reconstructed from the genus* 0 *Gromov-Witten invariants.*

The proof is via the topological recursion relations. For $n \geq 3$, consider the map

$$\nu : \overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,3}$$

forgetting all data except the first 3 markings. Again, a comparison result for cotangent lines is needed: $\psi_1 - \nu^*(\psi_1)$ is seen to equal a linear combination of boundary divisors of $\overline{M}_{0,n}(X, \beta)$. Since ψ_1 is 0 in $H^2(\overline{M}_{0,3}, \mathbb{Q})$, ψ_1 is a boundary class on $\overline{M}_{0,n}(X, \beta)$. The divisors which occur in $\psi_1 - \nu^*(\psi_1)$ are those with point splitting $A \cup B$ where $1 \in A$ and $\{2, 3\} \subset B$. No multiplicities occur (this may be seen, for example, by intersections with curves).

LEMMA 1. — *Let* $n \geq 3$. *The following boundary expression for* ψ *holds:* $\psi_1 = \sum_\Gamma D$, *where the sum is over all boundary divisors with point splitting separating* 1 *from* $\{2, 3\}$.

Using Lemma 1 together with the recursive structure of the boundary (see [Ko], [BM], [FP]), a *topological recursion relation* among genus 0 descendent integrals is obtained. First, let $T_0, \ldots, T_m$ be a basis of $H^*(X, \mathbb{Q})$ (we assume here the cohomology is all even dimensional to avoid signs). Let $g_{ef} = \int_X T_e \cup T_f$ be the intersection pairing, and let g^{ef} be the inverse matrix. The recursion relation is:

$$\langle\, \tau_{a_1}(\gamma_1)\, \tau_{a_2}(\gamma_2)\, \tau_{a_3}(\gamma_3) \prod_{i \in S} \tau_{d_i}(\delta_i)\, \rangle_\beta = \tag{6}$$

$$\sum \langle\, \tau_{a_1 - 1}(\gamma_1) \prod_{i \in S_1} \tau_{d_i}(\delta_i)\, T_e\, \rangle_{\beta_1} g^{ef} \langle\, T_f\, \tau_{a_2}(\gamma_2)\, \tau_{a_3}(\gamma_3) \prod_{i \in S_2} \tau_{d_i}(\delta_i)\, \rangle_{\beta_2}.$$

The sum is over all stable splittings $\beta_1 + \beta_2 = \beta$, $S_1 \cup S_2 = S$, and over the diagonal splitting indices e, f: the class $\sum g^{ef} T_e \otimes T_f$ is the Poincaré dual of the diagonal $\Delta \subset X \times X$.

The proof of the Reconstruction Theorem follows easily from (6). The argument is via induction on the number of cotangent line classes. A descendent with no cotangent line classes is a Gromov-Witten invariant by definition. All $\beta = 0$ invariants are determined by the classical cohomology of X together with well-known formulas for cotangent line class integrals on $\overline{M}_{0,n}$ [W2]. The topological recursion relations reduce descendents with at least 3 markings to integrals with fewer cotangent line classes. Let $\langle I \rangle_{\beta \neq 0}$ be a descendent integral with only 2 markings. Let H be an ample divisor on X. Add an extra marking subject to the divisor H condition: $\langle I \cdot H \rangle_\beta$. The divisor equation then relates $\langle I \rangle_\beta$ and $\langle I \cdot H \rangle_\beta$ modulo descendents with fewer cotangent lines. Since $\langle I \cdot H \rangle_\beta$ has 3 markings, equation (6) equates $\langle I \cdot H \rangle_\beta$ with an expression involving descendents with fewer cotangent lines. Similarly, if $\langle I \rangle_{\beta \neq 0}$ is an integral with only 1 marking, then $\langle I \cdot H \cdot H \rangle_\beta$ is considered. This completes the proof of Proposition 1.

1.3. **The fundamental solution**

Gravitational descendents arise in fundamental solutions of the quantum differential equation of X. Givental derives a solution naturally via a related torus action and equivariant considerations. The solution is rederived here using the topological recursion relations (as suggested to the author by S. Katz). It is also possible to derive the solution from the WDVV-equations for descendents. This solution was found by Witten and Dijkgraaf; it also appears in [Du].

As before, let $T_0, \ldots, T_m$ be a homogeneous basis of $V = H^*(X, \mathbb{Q})$ such that T_0 is the ring identity and T_m is its Poincaré dual. The tangent space of V at every point is canonically identified with V. Let $\partial_0, \ldots, \partial_m$ be the corresponding tangent fields. Let $\gamma = \sum t_i T_i$ be coordinates on V defined by the basis. Let $F = \sum f^i \partial_i$ be a vector field. Let Φ be the quantum potential defined by the Gromov-Witten invariants:

$$\Phi(t_0, \ldots, t_m) = \sum_{n \geq 3} \sum_{\beta} \frac{1}{n!} \langle \gamma^n \rangle_\beta.$$

In the basic case $X = \mathbf{G}/\mathbf{P}$, the potential is a formal series in $\mathbb{Q}[[t_i]]$ (in general additional variables are needed). The quantum product is defined by:

$$\partial_i * \partial_j = \Phi_{ijr} g^{rs} \partial_s. \tag{7}$$

Define a (formal) connection $\nabla_\hbar$ on the tangent bundle of V by:

$$\nabla_{\hbar,i}(F) = \hbar \frac{\partial F}{\partial t_i} - \partial_i * F = \sum \left(\hbar \frac{\partial f^s}{\partial t_i} - \Phi_{ijr} g^{rs} f^j \right) \partial_s.$$

The WDVV-equations imply that $\nabla_\hbar$ is flat (see [D], [KM], [G1]). Therefore, flat vector fields F exist formally. The equations for flat solutions F are: $\hbar \partial F / \partial t_i = \partial_i * F$. This is the quantum differential equation.

Following [G1], define a matrix of formal functions in $\mathbb{Q}[[\hbar^{-1}, t_i]]$:

$$\Psi_{ab} = g_{ab} + \sum_{n \geq 0,\ \beta,\ (n,\beta) \neq (0,0)} \frac{1}{n!} \langle T_a \cdot \frac{T_b}{\hbar - \psi} \cdot \gamma^n \rangle_\beta$$

where $0 \leq a, b \leq m$. The matrix may be written more explicitly as:

$$\Psi_{ab} = g_{ab} + \sum \sum_{k \geq 0} \frac{\hbar^{-k-1}}{n!} \langle T_a \cdot \tau_k(T_b) \cdot \gamma^n \rangle_\beta$$

with the same summation conventions on n and β.

PROPOSITION 2. — *Ψ yields a fundamental solution of the quantum differential equation:*

$$\nabla_{\hbar,i} \sum \Psi_{ab} g^{as} \partial_s = 0. \tag{8}$$

The constant term of the solution $\Psi_{ab} g^{as}$ is the identity matrix.

Proof. By the definitions, it follows:

$$\hbar \frac{\partial \Psi_{ab}}{\partial t_i} g^{as} = \sum_{n \geq 0,\ \beta} \sum_{k \geq 0} \frac{\hbar^{-k}}{n!} \langle T_a \cdot \tau_k(T_b) \cdot T_i \cdot \gamma^n \rangle_\beta \ g^{as}. \tag{9}$$

The coefficient of ∂_s in the second term $\partial_i * \Psi_{ab} g^{aj} \partial_j$ in the covariant derivative is:

$$\Phi_{ijr} g^{rs} \Psi_{ab} g^{aj} = \Phi_{ibr} g^{rs} + \tag{10}$$

$$\sum \sum_{k \geq 0} \frac{\hbar^{-k-1}}{n_1! n_2!} \langle T_a \cdot \tau_k(T_b) \cdot \gamma^{n_1} \rangle_{\beta_1} \ g^{aj} \langle T_j \cdot T_i \cdot T_r \cdot \gamma^{n_2} \rangle_{\beta_2} \ g^{rs}$$

where the first sum is over stable splittings $n_1 + n_2 = n$, $\beta_1 + \beta_2 = \beta$, (and, of course, the repeated indices).

The $k = 0$ terms of (9) sum to exactly the first term $\Phi_{ibr} g^{rs}$ of (10). The $k \geq 1$ terms of (9) may be replaced via the topological recursion relations (6) relative to the first 3 markings to obtain precisely the second term in (10). □

The fundamental solution takes a simpler form when restricted to the space $H^2(X, \mathbb{Q})$ – that is, after passing to the small quantum cohomology. The *small quantum product* is obtained by setting all variables t_i to 0 in the formula (7) which do not correspond to cohomology basis elements in $H^2(X, \mathbb{Q})$. Let $T_1, \ldots, T_k$ span $H^2(X, \mathbb{Q})$. Let T denote the vector of cohomology classes $(T_1, \ldots, T_k)$. Let t denote the vector of variables $(t_1, \ldots, t_k)$. For $\beta \in H_2(X, \mathbb{Z})$, let v_β denote the vector of constants $(\int_\beta T_1, \ldots, \int_\beta T_k)$. We assume the classes $T_1, \ldots, T_k$ pair *non-negatively* with all effective curve classes in X (this is possible for $X = \mathbf{G}/\mathbf{P}$, but required only for simplicity). By the divisor formula, the small product may be written as:

$$\partial_i * \partial_j = \sum_\beta e^{v_\beta \cdot t} \ \langle \ T_i \cdot T_j \cdot T_r \ \rangle_\beta \ g^{rs} \ \partial_s. \tag{11}$$

The matrix Ψ can be written after restriction to $H^2(X, \mathbb{Q})$ as:

$$\Psi_{ab} = \sum_\beta e^{v_\beta \cdot t} \langle T_a \cdot \frac{e^{T \cdot t/\hbar} T_b}{\hbar - \psi} \rangle_\beta. \tag{12}$$

Again, the divisor equation is used. The formula (12) is a sum of descendents with 2 markings. The function of cohomology and cotangent classes at the second point is expanded to define the invariant. A convention is made in (12) regarding the $\beta = 0$ case (as 2 point degree 0 invariants are not defined):

$$\langle T_a \cdot \frac{e^{T \cdot t/\hbar} T_b}{\hbar - \psi} \rangle_0 = \langle T_a \cdot e^{T \cdot t/\hbar} T_b \cdot 1 \rangle_0.$$

The series (12) is viewed as a formal power series in the variables $\hbar^{-1}, t_i$, and e^{t_i}. More precisely, the series is an element of $\mathbb{Q}[\hbar^{-1}, t_i][[e^{t_i}]]$. Modulo the variables t_i and e^{t_i}, $\Psi_{ab} g^{as}$ is the constant identity matrix.

The small quantum differential equation is:

$$1 \leq i \leq k, \quad \hbar \frac{\partial F}{\partial t_i} = \partial_i * F$$

where F is a vector field function of only t, and the product is the small quantum product. For $1 \leq i \leq k$, the small analogue of (8) holds for (12):

$$\hbar \frac{\partial \Psi_{ab}}{\partial t_i} g^{as} \partial_s = \partial_i * \Psi_{ab} g^{as} \partial_s.$$

In fact, the restricted matrix $\Psi_{ab} g^{as}$ provides a fundamental solution to this small quantum differential equation. Only the small quantum objects will be considered in this paper.

1.4. **Projective space**

We now let $X = \mathbf{P}^m$. Let H denote the hyperplane class in $H^2(\mathbf{P}^m, \mathbb{Q})$. Let $T_i = H^i$ be the cohomology basis. Let $t = t_1$. The small quantum ring structure is:

$$QH_s^* \mathbf{P}^m = \mathbb{Q}[\partial_1, e^t]/(\partial_1^{m+1} - e^t)$$

(see [G1], [FP]). Let $\sum_0^m f^i \partial_i$ be a vector field where $f^i = f^i(t)$. The small quantum differential equation is then the following system:

$$i > 0, \quad \hbar \frac{\partial f^i}{\partial t} = f^{i-1}$$

$$\hbar \frac{\partial f^0}{\partial t} = e^t f^m.$$

The function f^m determines a vector solution if and only if it is annihilated by the operator $\mathcal{D} = (\hbar d/dt)^{m+1} - e^t$. A (formal) fundamental solution to the equation $\mathcal{D} f = 0$ is given by the following expression:

$$S = S_{\mathbf{P}^m} = \sum_{d \geq 0} \frac{e^{(H/\hbar + d)t}}{\prod_{r=1}^d (H + r\hbar)^{m+1}} \quad mod\ H^{m+1}. \tag{13}$$

S is expanded in powers of H (subject to $H^{m+1} = 0$) as:

$$S = \sum_{b=0}^m S_b H^{m-b}$$

where S_b is a formal series in $\mathbb{Q}[\hbar^{-1}, t][[e^t]]$. It is easily checked by formula (13) that $\mathcal{D}$ annihilates S_b. Define the matrix M of functions by

$$M_b^s = (\hbar \frac{d}{dt})^{m-s} S_b.$$

Modulo t and e^t, the only contribution to M_b^s occurs in the $d = 0$ summand in (13); it is the identity matrix. $M_b^s \partial_s$ defines a fundamental solution to the small quantum differential equation. By uniqueness,

$$\Psi_{ab} g^{as} = M_b^s. \tag{14}$$

The required uniqueness statement here depends on the equality modulo t, e^t and the fact that the solutions lie in $\mathbb{Q}[\hbar^{-1}, t][[e^t]]$.

Consider the hyperplane embedding $X = \mathbf{P}^m \subset \mathbf{P}^{m+1}$. The solution $S_{\mathbf{P}^m}(t, \hbar = 1)$ agrees with the hypergeometric series $S^*_{\mathbf{P}^m}$ defined in equation (4) for the hyperplane $X \subset \mathbf{P}^{m+1}$. This is the first example of Givental's correspondence (i) of Section 0.

Equations (13) and (14) together with the solution (12) compute all 2 point invariants of $\mathbf{P}^m$ with a cotangent line class on 1 point. For example, tracing through the equations yields:

$$\langle\ \tau_{dm+d-2}(T_m)\ \rangle_d = \int_{\overline{M}_{0,1}(\mathbf{P}^m,d)} e_1^*(H^m) \cup \psi^{dm+d-2} = \frac{1}{(d!)^{m+1}}.$$

The solution to the small quantum differential equation provides an elegant organization of these 2 point descendents.

For a general space X, define $S_b = \Psi_{0b}$. Let $\mathcal{D}(\hbar, \hbar\partial/\partial t_i, e^{t_i})$ be a differential operator which is a polynomial in the operators $\hbar, \hbar\partial/\partial t_i$, and e^{t_i}.

LEMMA 2. — *If $\mathcal{D}S_b = 0$ for all $0 \leq b \leq m$, then the equation $\mathcal{D}(0, \partial_i, e^{t_i}) = 0$ holds in $QH^*_s(X)$.*

Proof. Let $M = \Psi_{ab}g^{as}$ be the solution matrix. Let $p = \mathcal{D}(0, \partial_i, e^{t_i})$ in $QH^*_s(X)$. Let P be the matrix of quantum multiplication by p in the basis $\partial_0, \ldots, \partial_m$. By the quantum differential equation,

$$\mathcal{D}M = P \cdot M + \sum_{k \geq 1}^{K} \hbar^k C_k \cdot M \tag{15}$$

where C_k is a matrix with coefficients in $\mathbb{Q}[[e^{t_i}]]$ and $K < \infty$. M is certainly invertible in the coefficient ring $\mathbb{Q}[[\hbar^{-1}, t_i, e^{t_i}]]$. The vector $(S_0, \ldots, S_m)$ is the m^{th} row of the solution matrix M. Hence, after multiplying (15) by M^{-1} from the right and using the $\hbar$-grading, it follows that the m^{th} row of P vanishes. Poincaré duality on X and the definition of the quantum product then imply $p = 0$. □

2. THE $*_X$-PRODUCT INDUCED BY A HYPERSURFACE

2.1. The product construction

Let $Y = \mathbf{P}^m$ (or more generally a homogeneous variety $Y = \mathbf{G}/\mathbf{P}$). Let L be an ample line bundle on Y. Let $s \in H^0(Y, L)$ be a section with nonsingular zero locus X. Assume the dimension of X is at least 3. The isomorphisms

$$H_2(Y, \mathbb{Z}) \tilde{=} H_2(X, \mathbb{Z}), \quad H^2(Y, \mathbb{Q}) \tilde{=} H^2(X, \mathbb{Q}) \tag{16}$$

hold by the Lefschetz theorem.

Gromov-Witten invariants are defined on X via the virtual fundamental class since the moduli space of maps is in general ill-behaved. In our special situation, the virtual class on $\overline{M}_{0,0}(X,\beta)$ has a simple interpretation. Let $\pi : U \to \overline{M}_{0,0}(Y,\beta)$ be the universal curve. Let $\mu : U \to Y$ be the universal map. It is not hard to check the following facts. The sheaf $E_\beta = \pi_*\mu^*(L)$ is a vector bundle of rank $\int_\beta c_1(L) + 1$ equipped with a canonical section s_E obtained from s. The moduli space

$$\overline{M}_{0,0}(X,\beta) \subset \overline{M}_{0,0}(Y,\beta)$$

is the stack theoretic zero locus of s_E. The virtual fundamental class of $\overline{M}_{0,0}(X,\beta)$ is simply the refined top Chern class of E_β with respect to the section s_E: it is a Chow class on $\overline{M}_{0,0}(X,\beta)$ of expected dimension. Let $\nu : \overline{M}_{0,n}(Y,\beta) \to \overline{M}_{0,0}(Y,\beta)$ be the forgetful morphism. The virtual class of $\overline{M}_{0,n}(X,\beta)$ is the refined top Chern class of the canonical pull-backs of E_β and s_E via ν. For notational convenience, the ν pull-back of E_β will also be denoted E_β. The algebraic theory of the virtual fundamental class is developed in [LT], [B], [BF]. The above construction appears in [Ko].

Let $T_0, \ldots, T_m$ be a basis of $H^*(Y,\mathbb{Q})$ satisfying the conventions of Section 1.3. As before, we assume that $T_1, \ldots, T_k$ is a basis of $H^2(Y,\mathbb{Q})$ pairing *non-negatively* with all effective curve classes in Y. Let $t_1, \ldots, t_k$ be the corresponding divisor variables, let $q_i = e^{t_i}$, and let q denote $(q_1, \ldots, q_k)$. Let $QH^*_s(X) = H^*(X,\mathbb{Q}) \otimes \mathbb{Q}[[q]]$ with the canonical free $\mathbb{Q}[[q]]$-module structure. The small quantum product on $QH^*_s(X)$ is $\mathbb{Q}[[q]]$-linear and defined via the 3 point genus 0 invariants as in (11). Associativity is established via properties of the virtual fundamental class [LT], [B].

A new $\mathbb{Q}[[q]]$-linear small quantum product $*_X$ is now defined on the free module $H^*(Y,\mathbb{Q}) \otimes \mathbb{Q}[[q]]$. Let $i : X \to Y$ denote the inclusion. Let

$$i^* : H^*(Y,\mathbb{Q}) \otimes \mathbb{Q}[[q]] \to H^*(X,\mathbb{Q}) \otimes \mathbb{Q}[[q]] \tag{17}$$

denote the canonical $\mathbb{Q}[[q]]$-linear pull-back. The new product $*_X$ on the right side of (17) will satisfy a homomorphism property: for $a, b \in H^*(Y,\mathbb{Q})$,

$$i^*(a *_X b) = (i^*a) * (i^*b) \tag{18}$$

where the right side of (18) is the product in $QH^*_s(X)$.

The product $*_X$ is defined by new integrals over the moduli space of maps to Y. Consider again the vector bundle E_β on $\overline{M}_{0,n}(Y,\beta)$. For each marking j, there is a canonical bundle sequence obtained by evaluation:

$$0 \to E'_{\beta,j} \to E_\beta \to e_j^*(L) \to 0 \tag{19}$$

The bundle $E'_{\beta,j}$ is of rank $\int_\beta c_1(L)$. If $\beta = 0$, then $E'_{\beta,j} = 0$. Define new integrals: for cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(Y,\mathbb{Q})$,

$$\langle \gamma_1 \cdots \gamma_{n-1} \cdot \tilde{\gamma}_n \rangle_\beta^{Y/X} = \int_{\overline{M}_{0,n}(Y,\beta)} \prod_{i=1}^{n} e_i^*(\gamma_i) \cup c_{\mathrm{top}}(E'_{\beta,n}). \tag{20}$$

The tilde in the argument of (20) denotes the marking with respect to which the construction (19) is undertaken. Define the new product $a *_X b$ for $a, b \in H^*(Y, \mathbb{Q})$ by:

$$a *_X b = \sum_\beta q^{v_\beta} \langle a \cdot b \cdot \tilde{T}_e \rangle_\beta^{Y/X} g^{ef} T_f$$

where $q^{v_\beta} = q_1^{\int_\beta T_1} \cdots q_k^{\int_\beta T_k}$.

PROPOSITION 3. — *The product $*_X$ defines a commutative, associative, unital ring structure on $H^*(Y, \mathbb{Q}) \otimes \mathbb{Q}[[q]]$ (with unit T_0).*

Proof. The product is commutative by the symmetry of the integrals (20) in the first two factors. The unital property of T_0 follows for exactly the same reasons as in the usual quantum product: only degree 0 terms contribute to a product with T_0 and $\langle T_0 \cdot \gamma \cdot \tilde{T}_e \rangle_0^{Y/X} = \int_Y \gamma \cup T_e$.

In the usual quantum product, associativity is a consequence of the basic boundary linear equivalence on $\overline{M}_{0,4}$ pulled back to $\overline{M}_{0,4}(Y, \beta)$. A slight twist is needed here. Let $a, b, c, \gamma \in H^*(Y, \mathbb{Q})$. Let $D(1,2 \mid 3,4)$ and $D(1,4 \mid 2,3)$ be the divisors on $\overline{M}_{0,4}(Y, \beta)$ determined by the associated point splittings (see [FP]). Let

$$\omega = e_1^*(a) \cup e_2^*(b) \cup e_3^*(c) \cup e_4^*(\gamma) \cup c_{\text{top}}(E'_{\beta,4}).$$

There is an equality:

$$\int_{D(1,2 \mid 3,4)} \omega = \int_{D(1,4 \mid 2,3)} \omega.$$

The recursive structure of the boundary and the simple behavior of the restriction of $E'_{\beta,4}$ yields:

$$\int_{D(1,2 \mid 3,4)} \omega = \sum_{\beta_1 + \beta_2 = \beta} \langle a \cdot b \cdot \tilde{T}_e \rangle_{\beta_1}^{Y/X} g^{ef} \langle T_f \cdot c \cdot \tilde{\gamma} \rangle_{\beta_2}^{Y/X},$$

$$\int_{D(1,4 \mid 2,3)} \omega = \sum_{\beta_1 + \beta_2 = \beta} \langle a \cdot \tilde{\gamma} \cdot T_e \rangle_{\beta_1}^{Y/X} g^{ef} \langle \tilde{T}_f \cdot b \cdot c \rangle_{\beta_2}^{Y/X}.$$

Associativity now follows easily. □

The following Proposition is due to T. Graber.

PROPOSITION 4. — *The pull-back i^* is a ring homomorphism from*

$$QH_s^*(Y/X) = (H^*(Y, \mathbb{Q}) \otimes \mathbb{Q}[[q]], *_X)$$

to $QH_s^*(X)$.

Proof. Let $a, b \in H^*(Y, \mathbb{Q})$. Let $a *_X b = \sum_\beta q^{v_\beta} c_\beta$. Let the product of the pull-backs be: $i^*(a) * i^*(b) = \sum_\beta q^{v_\beta} c'_\beta$. The equality $i^*(c_\beta) = c'_\beta$ must be proven.

Consider the following fiber square:

$$\begin{array}{ccc} Z & \longrightarrow & \overline{M}_{0,3}(Y,\beta) \\ {\scriptstyle e_3}\downarrow & & \downarrow{\scriptstyle e_3} \\ X & \xrightarrow{i} & Y \end{array}$$

where Z is the zero locus of $e_3^*(s) \in H^0(e_3^*(L))$. Let $\gamma = e_1^*(a) \cup e_2^*(b)$ in $H^*(\overline{M}_{0,3}(Y,\beta), \mathbb{Q})$. Then,

$$c_\beta = e_{3*}(\gamma \cup c_{\text{top}}(E'_{\beta,3})).$$

By properties of the Gysin map [F],

$$i^*(c_\beta) = e_{3*}i^!(\gamma \cup c_{\text{top}}(E'_{\beta,3})). \tag{21}$$

Recall the embedding

$$\overline{M}_{0,3}(X,\beta) \subset \overline{M}_{0,3}(Y,\beta) \tag{22}$$

as the zero section of E_β. By the realization of the virtual fundamental class of maps to X,

$$c'_\beta = e_{3*}0^!_{E_\beta}(\gamma). \tag{23}$$

The inclusion (22) factors through Z. There is an equality of classes on Z:

$$0^!_{E_\beta}(\gamma) = 0^!_{e_3^*(L)}(\gamma \cup c_{\text{top}}(E'_{\beta,3})) = i^!(\gamma \cup c_{\text{top}}(E'_{\beta,3})).$$

The first equality follows from sequence (19); the second is by definition. Equation (21) and (23) yield the equality $i^*(c_\beta) = c'_\beta$, which concludes the proof. □

The small quantum differential equation for the product $*_X$ may also be considered:

$$\hbar \frac{\partial}{\partial t_i} F = \partial_i *_X F. \tag{24}$$

Again the coordinates $t = (t_1, \ldots, t_k)$ correspond to the cohomology basis of $H^2(Y, \mathbb{Z})$ and F is a vector field function of t. The fundamental solution takes a form similar to (12). Let

$$\Psi_{ab} = \sum_\beta e^{v_\beta \cdot t} \langle \tilde{T}_a \cdot \frac{e^{T \cdot t/\hbar} T_b}{\hbar - \psi} \rangle_\beta^{Y/X}. \tag{25}$$

Again, a convention is required for $\beta = 0$:

$$\langle \tilde{T}_a \cdot \frac{e^{T \cdot t/\hbar} T_b}{\hbar - \psi} \rangle_0^{Y/X} = \langle \tilde{T}_a \cdot e^{T \cdot t/\hbar} T_b \cdot 1 \rangle_0^{Y/X} = \langle T_a \cdot e^{T \cdot t/\hbar} T_b \cdot 1 \rangle_0^{Y}.$$

The fundamental solution of (24) is $\Psi_{ab} g^{as} \partial_s$. This may be proven as a specialization of the fundamental solution

$$\Psi_{ab} = g_{ab} + \sum_{n \geq 0,\ \beta,\ (n,\beta) \neq (0,0)} \frac{1}{n!} \langle \tilde{T}_a \cdot \frac{T_b}{\hbar - \psi} \cdot \gamma^n \rangle_\beta$$

of the differential equation obtained from large product $*_X$

$$\partial_i *_X \partial_j = \sum_{n\geq 0} \sum_{\beta} \frac{1}{n!} \langle\, T_i \cdot T_j \cdot \tilde{T}_r \cdot \gamma^n \,\rangle_{\beta}^{Y/X} g^{rs} \partial_s, \quad \gamma = \sum_{i=0}^{m} t_i T_i$$

via analogous topological recursion relations for the tilde integrals (following the proofs in Section 1). Alternatively, (25) may be proven to be a fundamental solution directly from the definition (24) together with the tilde topological recursion relations.

2.2. **Projective space**

Let $Y = \mathbf{P}^m$. Let X be a hypersurface of degree l. Consider the fundamental solution of (24) given by the matrix $\Psi_{ab} g^{as}$. Let $t = t_1$. Let $S_b = \Psi_{ab} g^{a,m-1}$ be the $(m-1)^{st}$ row of the solution matrix. A Lemma analogous to Lemma 2 holds. Let $\mathcal{D}(\hbar, \hbar\partial/\partial t, e^t)$ be a differential operator which is a polynomial in the operators $\hbar, \hbar\partial/\partial t$, and e^t.

LEMMA 3. — *If $\mathcal{D}S_b = 0$ for all $0 \leq b \leq m$, then the equation $\mathcal{D}(0, \partial_1, e^t) = 0$ holds in $QH_s^*(X)$.*

Proof. Let $p = \mathcal{D}(0, \partial_1, e^t)$ in $QH_s^*(Y/X)$. Let P be the matrix of quantum $*_X$-multiplication by p in the basis $\partial_0, \ldots, \partial_m$. By the argument used in the proof of Lemma 2, it follows that the $(m-1)^{st}$ row of P vanishes. The definition of the product $*_X$ together with classical intersection theory on Y then implies $p = f(e^t)\partial_m$. Hence, $i^*(p) = 0$. The Lemma then follows from Proposition 4. □

The functions S_b will be calculated for hypersurfaces of degree $l \leq m+1$ via torus localization. For notational simplicity, let $H^b = T_b$. We first organize the functions S_b in a more convenient form:

$$\tilde{S}(t, \hbar) = \sum_{b=0}^{m} S_b H^{m-b} \in H^*(Y, \mathbb{Q})[\hbar^{-1}, t][[e^t]].$$

The definitions imply:

$$\tilde{S}(t, \hbar) = \sum_{b=0}^{m} \sum_{d\geq 0} e^{dt} \langle \tilde{H} \cdot \frac{e^{Ht/\hbar} T_b}{\hbar - \psi} \rangle_d^{Y/X} H^{m-b} \tag{26}$$

$$= \sum_{d\geq 0} e^{(H/\hbar + d)t} e_{2*}\left(\frac{1}{l} \cdot \frac{c_{\mathrm{top}}(E_d)}{\hbar - \psi_2}\right). \tag{27}$$

In degree 0, the convention

$$e_{2*}\left(\frac{c_{\mathrm{top}}(E_d)}{\hbar - \psi_2}\right) = l \cdot H \in H^*(Y, \mathbb{Q}) \tag{28}$$

is taken.

LEMMA 4. — *$\tilde{S}(t, \hbar)$ is divisible by H.*

Proof. The degree 0 contribution is clearly divisible by H by (28). It suffices to show the integrals

$$\langle \tilde{H} \cdot \frac{T_m}{\hbar - \psi} \rangle_d^{Y/X} = \int_{[\overline{M}_{0,2}(X,d)]^{\mathrm{vir}}} \frac{1}{l} \cdot \frac{e_2^*(T_m)}{\hbar - \psi_2}$$

vanish for $d > 0$. In fact, these are zero for a trivial reason: the point class T_m does not intersect X in Y. □

The correlator $S(t, \hbar) = l \cdot \tilde{S}(t, \hbar)$ is more convenient for the calculations to come. The dependence on l will be made explicit as a third argument $S(t, \hbar, l)$. Givental's correlator S_X for a hypersurface $X \subset \mathbf{P}^m$ is defined by

$$S_X(t, \hbar) = \frac{1}{lH} S(t, \hbar, l). \tag{29}$$

The correlators S_X considered in the correspondences (i)-(iii) of Section 0 are evaluated at $\hbar = 1$. It is $S(t, \hbar, l)$ which is explicitly calculated in Sections 3 and 4 via fixed point localization for the natural torus action on the moduli space of maps.

3. HYPERSURFACES: CASES (i) AND (ii)

3.1. The T-action

We first set notation for the required torus action. Let λ denote the set $\{\lambda_0, \ldots, \lambda_m\}$. Let $\mathbf{R} = \mathbb{Q}[\lambda]$ be the standard presentation of the equivariant cohomology ring of the torus $\mathbf{T} = (\mathbb{C}^*)^{m+1}$: λ_i is the equivariant first Chern class of the dual of the standard representation of the i^{th} factor $\mathbb{C}^*$. Let $\mathbf{T}$ act on $W = \mathbb{C}^{m+1}$ via the standard diagonal representation. A $\mathbf{T}$-action on $\mathbf{P}(W) = \mathbf{P}^m$ is canonically obtained. The $\mathbf{T}$-action lifts canonically to $\mathcal{O}_{\mathbf{P}^m}(1)$ (and therefore to $\mathcal{O}_{\mathbf{P}^m}(k)$ for every k). Let $H \in H_{\mathbf{T}}^*(\mathbf{P}^m)$ be the equivariant first Chern class of $\mathcal{O}_{\mathbf{P}^m}(1)$. All equivariant cohomology groups will be taken with $\mathbb{Q}$-coefficients. The standard presentation of $H_{\mathbf{T}}^*(\mathbf{P}^m)$ is:

$$H_{\mathbf{T}}^*(\mathbf{P}^m) \cong \mathbb{Q}[H, \lambda]/(\Pi_{i=0}^m (H - \lambda_i)). \tag{30}$$

The fixed points $\{p_0, \ldots, p_m\}$ of the $\mathbf{T}$-action on $\mathbf{P}^m$ correspond to the basis vectors in $\mathbb{C}^{m+1}$. Let $\phi_i \in H_{\mathbf{T}}^m(\mathbf{P}^m)$ denote (the dual of) the equivariant fundamental class of the point p_i. There is a naturally graded equivariant push-forward:

$$\int_{\mathbf{P}^m} : H_{\mathbf{T}}^*(\mathbf{P}^m) \to \mathbf{R}.$$

For $x, y \in H_{\mathbf{T}}^*(\mathbf{P}^m)$, an inner product $\langle x, y \rangle = \int_{\mathbf{P}^m} x \cup y$ is defined. The elements ϕ_i satisfy the following basis property:

$$x = y \iff \forall i,\ \langle \phi_i, x \rangle = \langle \phi_i, y \rangle. \tag{31}$$

Property (31) follows easily from the presentation (30).

There is a canonically induced $\mathbf{T}$-action on the stack $\overline{M}_d = \overline{M}_{0,2}(\mathbf{P}^m, d)$. The torus acts on a stable map to $\mathbf{P}^m$ by translating the image. The $\mathbf{T}$-fixed locus has been determined in [Ko], and is (necessarily) a nonsingular substack. Each domain component of a $\mathbf{T}$-fixed map must have as image in $\mathbf{P}^m$ a $\mathbf{T}$-orbit of dimension 0 or 1. The 0-dimensional orbits are the $m+1$ fixed points, and the 1-dimensional orbits are the $\binom{m+1}{2}$ lines connecting these points. Moreover, for a map to be $\mathbf{T}$-fixed, all nodes, marking, and ramifications point must have fixed images in $\mathbf{P}^m$. As a result, the components of the $\mathbf{T}$-fixed locus of $\overline{M}_d$ are in bijective correspondence with graph types describing the configurations of the domain components and the markings of the map.

More precisely, the graphs arising in this correspondence are triples (Γ, μ, δ) where Γ is 2-pointed tree,

$$\mu : \mathrm{Vert}(\Gamma) \to \{p_0, \ldots, p_m\}, \quad \text{and} \quad \delta : \mathrm{Edge}(\Gamma) \to \mathbb{Z}^{>0}.$$

Let $\zeta = [\mu : (C, x_1, x_2) \to \mathbf{P}^m] \in \overline{M}_d^{\mathbf{T}}$. The graph (Γ, μ, δ) associated to the fixed component containing ζ is constructed as follows. The vertices $v \in \mathrm{Vert}(\Gamma)$ correspond to *connected components* $D(v)$ of $\mu^{-1}(\{p_0, \ldots, p_m\})$. $D(v)$ may be 0 or 1 dimensional. The map μ on $\mathrm{Vert}(\Gamma)$ is determined by: $\mu(v) = \mu(D(v))$. An edge e connecting vertices v, v' corresponds to a component $D(e) = \mathbf{P}^1 \subset C$ incident to $D(v), D(v')$ and lying over the $\mathbf{T}$-invariant line L connecting the fixed points $p_\mu(v), p_\mu(v')$. The degree of the map from $D(e)$ to L is $\delta(e)$ (this map is uniquely determined (up to isomorphism) by $\delta(e)$ since it is unramified over $L \setminus \{p_\mu(v), p_\mu(v')\}$). The markings on C determine vertex markings on Γ. We will let the symbol Γ denote the entire decorated graph structure. Let

$$\widetilde{M}_\Gamma = \Pi_{v \in \mathrm{Vert}(\Gamma)} \, \overline{M}_{0,\mathrm{val}(v)},$$

where $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ are taken to be points. The valence of a vertex includes the markings as well as the incident edges. The fixed component associated to a graph Γ is simply the stack quotient: $\overline{M}_\Gamma = \widetilde{M}_\Gamma / \mathbf{G}$, where $\mathbf{G}$ is an associated automorphism group (see [GP]). The order of $\mathbf{G}$ is equal to

$$\Pi_{e \in \mathrm{Edge}(\Gamma)} \delta(e) \cdot |\mathrm{Aut}(\Gamma)|,$$

where $\mathrm{Aut}(\Gamma)$ is the automorphism group of the decorated graph. An excellent description of these graphs may be found in [Ko].

Let $X \subset \mathbf{P}^m$ be a hypersurface of degree $l \leq m+1$. We follow the notation of Section 2 (with the l dependence explicit in the correlators). As all structures in the definition of $S(t, \hbar, l)$ are canonically $\mathbf{T}$-equivariant, an element $S_{\mathbf{T}}(t, \hbar, l) \in H^*_{\mathbf{T}}(\mathbf{P}^m, \mathbb{Q})[[\hbar^{-1}, t, e^t]]$ is defined by:

$$S_{\mathbf{T}}(t, \hbar, l) = \sum_{d \geq 0} e^{(H/\hbar + d)t} e_{2*}\left(\frac{c_{\mathrm{top}}(E_d)}{\hbar - \psi_2}\right). \tag{32}$$

In the following definitions and computations, all the geometric structures (schemes, sheaves, maps, push-forwards, cohomology groups) will be given their canonical $\mathbf{T}$-equivariant interpretations.

3.2. **Linear recursions in cases (i) and (ii)**

The calculation of $S_{\mathbf{T}}(t,\hbar,l)$ relies on linear recursions obtained from localization. The recursions involve a related set of equivariant correlators $Z_i \in \mathbf{R}[[\hbar^{-1},q]]$ defined for $0 \leq i \leq m$ by:

$$Z_i(q,\hbar,l) = 1 + \sum_{d>0} q^d \int_{\overline{M}_d} \frac{E'_{d,2}}{\hbar - \psi_2} e_2^*(\phi_i). \tag{33}$$

The integral on the right is the equivariant push-forward to a point. $E'_{d,2}$ in the integrand denotes the top Chern class of the bundle $E'_{d,2}$. The latter convention will be kept throughout this section: bundles appearing in integrands will always denote their top Chern classes. By the definitions of $E'_{d,2}$, $S_{\mathbf{T}}(t,\hbar,l)$, and the exact sequence (19), we see:

$$\langle \phi_i, S_{\mathbf{T}}(t,\hbar,l)\rangle = e^{\lambda_i t/\hbar} l \lambda_i Z_i(e^t,\hbar,l), \tag{34}$$

where the pairing $\langle , \rangle$ is taken to be linear in the auxiliary parameters $\hbar^{-1}, t, e^t$. The degree 0 term involves a matching of conventions. By equation (34) and property (31), the correlators Z_i determine $S_{\mathbf{T}}$.

The dimension of $\overline{M}_d$ is $(m+1)d+m-1$, the rank of $E'_{d,2}$ is ld, and the codimension of the class $e_2^*(\phi)$ is m. Therefore, for $l \leq m$, initial terms in the $(1/\hbar)$-expansion of (33) with ψ_2-degree less than $(m+1-l)d-1$ vanish by dimension considerations. Hence, we find:

$$Z_i(q,\hbar,l \leq m) = 1 + \sum_{d>0} \left(\frac{q}{\hbar^{m+1-l}}\right)^d \int_{\overline{M}_d} \frac{\psi_2^{(m+1-l)d-1}}{1-\psi_2/\hbar} E'_{d,2} e_2^*(\phi_i). \tag{35}$$

We rescale the q-dependence in these cases by the following definition: $z_i(Q,\hbar,l \leq m) = Z_i(Q\hbar^{m+1-l},\hbar,l)$. Such vanishing does not apply in the Calabi-Yau case.

Equivariant integrals over $\overline{M}_d$ may be computed via $\mathbf{T}$-equivariant localization. Let I be an equivariant class in $H^*_{\mathbf{T}}(\overline{M}_d)$. Let G_d denote the set of components of the $\mathbf{T}$-fixed stack $\overline{M}_d^{\mathbf{T}}$. The elements of G_d are labelled by decorated graphs Γ. The equivariant integral of I over $\overline{M}_d$ equals a sum of contributions over the set G_d:

$$\int_{\overline{M}_d} I = \sum_{\Gamma \in G_d} \int_{\overline{M}_\Gamma} \frac{I}{N_\Gamma}, \tag{36}$$

where N_Γ is the equivariant normal bundle to $\overline{M}_\Gamma \subset \overline{M}_d$. Equation (36) is the Bott residue formula in equivariant cohomology. Explicit formulas for the equivariant Euler class of N_Γ in terms of tautological classes in $H^*_{\mathbf{T}}(\overline{M}_\Gamma)$ have been obtained by Kontsevich in [Ko] (see also [GP], [CK]). Givental requires the full formulas four times. In each case, the formulas are used to verify an algebraic equality – the method is straightforward

algebraic manipulation. The computations in the Calabi-Yau case will be covered in some detail. The others will be omitted.

Let $d > 0$. Fix an index $0 \leq i \leq m$, and consider the integrals in (33) and (35). In order to analyze these integrals, it is necessary to study the graphs associated to the fixed loci. We partition the set G_d into 3 disjoint subsets:

$$G_d = G_d^{i*} \cup G_d^{i0} \cup G_d^{i1}.$$

The set G_d^{i*} consists of the fixed loci for which the 2^{nd} marked point on the domain curve is *not* mapped to $p_i \in \mathbf{P}^m$. The set G_d^{i0} consists of loci for which an *irreducible component* of the domain curve containing the 2^{nd} marked point is collapsed to p_i. Finally, G_d^{i1} consists of loci for which the 2^{nd} marking is mapped to p_i without lying on a collapsed component. Let $G_d^i = G_d^{i0} \cup G_d^{i1}$. Let G^{i0} and G^{i1} denote the unions $\bigcup_{d>0} G_d^{i0}$ and $\bigcup_{d>0} G_d^{i1}$ respectively. The three graph types have the following basic properties:

Type G_d^{i*}. Let $\Gamma \in G_d^{i*}$. As $e_2^*(\phi_i)$ vanishes when restricted to $\overline{M}_\Gamma$, the contribution of Γ to the integrals in (33) and (35) via (36) is 0.

Type G_d^{i0}. Let $\Gamma \in G_d^{i0}$. Let $v \in \mathrm{Vert}(\Gamma)$ be the vertex at which the 2^{nd} marking is incident. $D(v)$ is collapsed to p_i. The restriction of ψ_2 to $\overline{M}_\Gamma$ carries the trivial $\mathbf{T}$-action. Hence, a simple nilpotency result holds:

$$\psi_2^{\dim(v)+1} = 0 \in H_{\mathbf{T}}^*(\overline{M}_\Gamma) \tag{37}$$

where $\dim(v) = \mathrm{val}(v) - 3$ is the dimension of $\overline{M}_{0,\mathrm{val}(v)}$. The valence bound $\mathrm{val}(v) \leq d+2$ is easily deduced for $\Gamma \in G_d^{i0}$. It is achieved for graphs with d edges and both markings incident to v. The order of nilpotency of ψ_2 is thus bounded by d.

Type G_d^{i1}. Let $\Gamma \in G_d^{i1}$. Again let $v \in \mathrm{Vert}(\Gamma)$ be the vertex at which the 2^{nd} marking is incident. In this case, $D(v)$ is a point. The vertex v is incident to a unique edge e of Γ. Let e connect vertices v, v'. Let $j = \mu(v')$. If $\delta(e) < d$, let Γ_j be the 2-pointed graph obtained by contracting e: Γ_j is the complete subgraph of Γ not containing v with the 2^{nd} marking placed at v'. The graph Γ_j is an element of $G_{d-\delta(e)}^j$. Note also that $|\mathrm{Aut}(\Gamma)| = |\mathrm{Aut}(\Gamma_j)|$. It is this pruning of graphs which provides recursion relations for the correlators z_i from equivariant localization.

The first application of localization is the following basic observation:

LEMMA 5. — *The correlators $Z_i(q, \hbar, l \leq m+1)$ are naturally elements of the ring $\mathbb{Q}(\lambda, \hbar)[[q]]$:*

$$Z_i(q, \hbar, l) = 1 + \sum_d q^d \, \zeta_{id}(\lambda, \hbar).$$

Moreover, the rational functions $\zeta_{id}(\lambda, \hbar)$ are regular at all the values

$$\hbar = \frac{\lambda_i - \lambda_j}{n}$$

where $i \neq j$ and $n \geq 1$.

Proof. A priori, $\zeta_{id}(\lambda, \hbar) \in \mathbf{R}[[\hbar^{-1}]]$:

$$\zeta_{id}(\lambda, \hbar) = \sum_{k=0}^{\infty} \int_{\overline{M}_d} \frac{\psi_2^k E'_{d,2}}{\hbar^{k+1}} e_2^*(\phi_i). \tag{38}$$

Let $\Gamma \in G_d$. The contribution of Γ to ζ_{id} is :

$$\mathrm{Cont}_\Gamma(\zeta_{id}) = \sum_{k=0}^{\infty} \int_{\overline{M}_\Gamma} \frac{\psi_2^k E'_{d,2}}{\hbar^{k+1} N_\Gamma} e_2^*(\phi_i).$$

By the Type G_d^{i*} vanishing, we obtain:

$$\zeta_{id} = \sum_{\Gamma \in G_d^{i0}} \mathrm{Cont}_\Gamma(\zeta_{id}) + \sum_{\Gamma \in G_d^{i1}} \mathrm{Cont}_\Gamma(\zeta_{id}). \tag{39}$$

Let $\Gamma \in G_d^{i0}$. By the Type G_d^{i0} nilpotency condition, we see:

$$\mathrm{Cont}_\Gamma(\zeta_{id}) = \sum_{k=0}^{d-1} \frac{P_{\Gamma,k}(\lambda)}{\hbar^{k+1}} \tag{40}$$

where $P_{\Gamma,k}(\lambda) \in \mathbb{Q}(\lambda)$. Let $\Gamma \in G_d^{i1}$. The restriction of ψ_2 to $\overline{M}_\Gamma$ is topologically trivial with $(\lambda_j - \lambda_i)/\delta(e)$ as equivariant class (we adhere to the notation of Type G_d^{i1} above). Hence, the contributions of Γ to the terms $k \geq 0$ form a geometric series. The series sum is:

$$\mathrm{Cont}_\Gamma(\zeta_{id}) = \frac{P_\Gamma(\lambda)}{(\hbar + \frac{\lambda_i - \lambda_j}{\delta(e)})}, \tag{41}$$

where $P_\Gamma(\lambda) \in \mathbb{Q}(\lambda)$. By equations (39-41), $\zeta_{id} \in \mathbb{Q}(\lambda, \hbar)$. The explicit forms of (40) and (41) prove the regularity claim at $\hbar = (\lambda_i - \lambda_j)/n$. □

The contributions of G^{i0} and G^{i1} to the integrals in (35) yield linear recursion relations for the correlators $z_i(Q, \hbar, l \leq m)$. The contribution of G^{i0} will be the initial part of the relation. This contribution is analyzed first.

LEMMA 6. — *The contribution $C_i(Q, \hbar, l)$ of graph type G^{i0} to $z_i(Q, \hbar, l)$ is determined in cases* (i) *and* (ii) *by the following results*:

$$C_i(Q, \hbar, l < m) = 0 \tag{42}$$

$$C_i(Q, \hbar, m) = -1 + exp\big(-m!Q + \frac{(m\lambda_i)^m}{\Pi_{\alpha \neq i}(\lambda_i - \lambda_\alpha)} Q\big). \tag{43}$$

Proof. Let $\Gamma \in G_d^{i0}$. We follow the notation of Type G_d^{i0} above. For $d > 0$ and $l < m$, we see $(m + 1 - l)d - 1 \geq d$. Hence, the restriction of the integrand of (35) to $\overline{M}_\Gamma$ vanishes by (37) and the valence bound in these cases. Thus, $C_i(Q, \hbar, l < m) = 0$.

In case $l = m$, the $(1/\hbar)$-expansion of the integrand (35) contains only one possibly nonvanishing term after restriction to $\overline{M}_\Gamma$: $\psi_2^{d-1} E'_{d,2} e_2^*(\phi_i)$. This term also vanishes unless

the valence of v is $d+2$. As previously remarked, the graphs $\Gamma \in G_d$ with valence $d+2$ at a vertex v are particularly simple: they must have d edges (all of degree 1) and both markings incident at the vertex v. It is then straightforward to explicitly compute the contribution $C_i(Q,\hbar,m)$ from the combinatorics of these simple graphs via the localization formula. This is the first localization computation needed by Givental. □

The linear recursion relations for $z_i(Q,\hbar,l\leq m)$ are given by:

$$z_i(Q,\hbar,l\leq m) = 1 + C_i(Q,\hbar,l) + \sum_{j\neq i}\sum_{d>0} Q^d\, C_i^j(d,\hbar,l)\; z_j(Q,\frac{\lambda_j-\lambda_i}{d},l), \tag{44}$$

where the recursion coefficients are:

$$C_i^j(d,\hbar,l) = \frac{1}{\frac{\lambda_i-\lambda_j}{\hbar}+d}\; \frac{\prod_{r=1}^{ld}\frac{ld\lambda_i}{\lambda_j-\lambda_i}+r}{\prod_{\alpha=0}^{m}\prod_{r=1,(\alpha,r)\neq(j,d)}^{d}\frac{d(\lambda_i-\lambda_\alpha)}{\lambda_j-\lambda_i}+r}. \tag{45}$$

The initial term C_i is the contribution of G^{i0} to z_i. The double sum is the contribution of G^{i1}. A truly remarkable feature of (44) is the $\hbar=(\lambda_j-\lambda_i)/d$ substitution on the right. This substitution is well defined by Lemma 5. Its origin is a normal bundle factor in the localization formula. Let $\Gamma\in G_d^{i1}$. We follow the notation of Type G_d^{i1} above. If $\delta(e)=d$, then the contribution of Γ to z_i equals $Q^dC_i^j(d,\hbar,l)$. Assume $\delta(e)<d$. Let Γ_j be the contracted graph obtained from $\Gamma\in G_d^{i1}$ as described in Type G_d^{i1} above. The linear recursion is obtained by the following equation:

$$\mathrm{Cont}_\Gamma(z_i(Q,\hbar,l\leq m)) = Q^{\delta(e)}C_i^j(\delta(e),\hbar,l)\cdot \mathrm{Cont}_{\Gamma_j}(z_j(Q,\frac{\lambda_j-\lambda_i}{\delta(e)},l)), \tag{46}$$

where Cont_Γ denote the contribution of Γ to the argument. Equations (44 -45) are deduced directly from (46) by summing over all graphs $\Gamma\in G^{i1}$. The flag (v',e) in the graph Γ corresponds to a node in the domain curve. The normal bundle of $\overline{M}_\Gamma\subset\overline{M}_d$ has a line bundle quotient obtained from the deformation space of this node. This nodal deformation is absent in the normal bundle contributions for the graph Γ_j, but appears algebraically in the evaluation of the correlator z_j at $\hbar=(\lambda_j-\lambda_i)/d$. Once this graph pruning strategy is noticed and the explicit recursions given, it is nothing more than an algebraic check to prove equation (46) from the localization formulas. This is the second needed localization computation.

Define the correlators $Z_i^*\in\mathbf{R}[[\hbar^{-1},q]]$ by

$$Z_i^*(q,\hbar,l\leq m+1) = \sum_{d=0}^{\infty} q^d\; \frac{\prod_{r=1}^{ld}(l\lambda_i+r\hbar)}{\prod_{\alpha=0}^{m}\prod_{r=1}^{d}(\lambda_i-\lambda_\alpha+r\hbar)}. \tag{47}$$

For all $l\leq m+1$, $Z_i^*\in\mathbb{Q}(\lambda,\hbar)[[q]]$, and the correlators Z_i^* satisfy the regularity property of Lemma 5. Let $z_i^*(Q,\hbar,l\leq m)=Z_i^*(Q\hbar^{(m+1-l)},\hbar,l)$. A direct algebraic computation shows the correlators $z_i^*(Q,\hbar,l<m)$ satisfy the recursions (44-45). Hence, $z_i^*(Q,\hbar,l<m)=z_i(Q,\hbar,l<m)$ since the recursions clearly have a unique solution.

Define the correlators $S^*_{\mathbf{T}} \in H^*_{\mathbf{T}}(\mathbf{P}^m)[[\hbar^{-1}, t, e^t]]$ by:

$$S^*_{\mathbf{T}}(t, \hbar, l \leq m+1) = \sum_{d \geq 0} \frac{e^{(H/\hbar + d)t} \prod_{r=0}^{ld} (lH + r\hbar)}{\prod_{\alpha=0}^{m} \prod_{r=1}^{d} (H - \lambda_\alpha + r\hbar)}. \tag{48}$$

The following equality holds:

$$\langle \phi_i, S^*_{\mathbf{T}}(t, \hbar, l \leq m+1) \rangle = e^{\lambda_i t/\hbar} l \lambda_i Z^*_i(e^t, \hbar, l). \tag{49}$$

For $l < m$, we have in addition:

$$\begin{aligned} e^{\lambda_i t/\hbar} l \lambda_i Z^*_i(e^t, \hbar, l < m) &= e^{\lambda_i t/\hbar} l \lambda_i Z_i(e^t, \hbar, l) \\ &= \langle \phi_i, S_{\mathbf{T}}(t, \hbar, l) \rangle. \end{aligned}$$

Hence, by property (31),

$$S^*_{\mathbf{T}}(t, \hbar, l < m) = S_{\mathbf{T}}(t, \hbar, l). \tag{50}$$

The non-equivariant correlator S_X is defined by:

$$S_X = \frac{1}{lH} \, S_{\mathbf{T}}(t, \hbar, l \leq m+1)|_{\lambda_i = 0, \ \hbar = 1}$$

(see Section 2.2). Case (i) of the main result (Section 0) is then proven by (50) and a calculation of the non-equivariant restriction of $S^*_{\mathbf{T}}$.

For $l = m$, a direct calculation shows the slightly modified correlator $e^{-m!Q} z^*_i(Q, \hbar, m)$ satisfies (44-45). Therefore, $e^{-m!Q} z^*_i(Q, \hbar, m) = z_i(Q, \hbar, m)$ by uniqueness. The equality $e^{-m!e^t/\hbar} S^*_{\mathbf{T}}(t, \hbar, m) = S_{\mathbf{T}}(t, \hbar, m)$ then follows analogously. Givental's relationship (ii) is obtained.

By Lemma 3, relations in the quantum cohomology ring $QH^*_s(X)$ of the hypersurface X are obtained from differential operators annihilating $S(t, \hbar, l)$. The differential equation:

$$\left(\hbar \frac{d}{dt}\right)^m S(t, \hbar, l < m) = l e^t \prod_{r=1}^{l-1} \left(l\hbar \frac{d}{dt} + r\hbar\right) S(t, \hbar, l)$$

yields the relation: $H^m = l^l q H^{l-1}$ in $QH^*_s(X_{l<m})$. Restricted cases of this relation were proven previously by Beauville [Be]. Similarly, the equation:

$$\left(\hbar \frac{d}{dt} + m! e^t\right)^m S(t, \hbar, m) = m e^t \prod_{r=1}^{m-1} \left(m\hbar \frac{d}{dt} + m m! e^t + r\hbar\right) S(t, \hbar, m)$$

yields the relation: $(H + m!q)^m = m^m q (H + m!q)^{m-1}$ in $QH^*_s(X_m)$.

4. CASE (iii): CALABI-YAU

4.1. Linear recursions

For the Calabi-Yau case $l = m+1$, no initial terms vanish in (33) for dimensional reasons. It will be useful to write Z_i in a partially expanded form.

$$Z_i(q,\hbar,m+1) = 1+\sum_{d>0} q^d\Big(\sum_{k=0}^{d-1}\hbar^{-k-1}\int_{\overline{M}_d}\psi_2^k E'_{d,2}e_2^*(\phi_i)\Big) \tag{51}$$

$$+\sum_{d>0}\big(\frac{q}{\hbar}\big)^d\int_{\overline{M}_d}\frac{\psi_2^d}{\hbar-\psi_2}E'_{d,2}e_2^*(\phi_i). \tag{52}$$

In this case, the definition $z_i(Q,\hbar,m+1) = Z_i(Q\hbar,\hbar,m+1)$ is made. The treatment of the $\hbar$ factor differs in case (iii) from the treatment given in (i) and (ii).

The linear recursions for z_i in the Calabi-Yau case take the form:

$$z_i(Q,\hbar,m+1) = 1+\sum_{d>0}\frac{Q^d}{d!}R_{id} + \sum_{d>0}\sum_{j\neq i} Q^d\ C_i^j(d,\hbar,m+1)\ z_j(Q,\frac{\lambda_j-\lambda_i}{d},m+1),$$

where $R_{id} = \sum_{j=0}^d R_{id}^j\hbar^{d-j}$ is a polynomial in $\mathbf{R}[\hbar]$ of $\hbar$-degree at most d, and the recursion coefficient C_i^j is determined by:

$$C_i^j(d,\hbar,m+1) = \frac{1}{\lambda_i-\lambda_j+d\hbar}\ \frac{\prod_{r=1}^{(m+1)d}(m+1)\lambda_i+r\frac{\lambda_j-\lambda_i}{d}}{d!\prod_{\alpha\neq i}\prod_{r=1,\ (\alpha,r)\neq(j,d)}^{d}\lambda_i-\lambda_\alpha+r\frac{\lambda_j-\lambda_i}{d}}. \tag{53}$$

The proof of this recursion relation is similar (but not identical) to the proofs in cases (i) and (ii) of (44). Equation (44) was derived by separating the contribution of graph types G^{i0} and G^{i1}. Here, we instead separate the contributions of the terms (51) and (52) in the expansion of $Z_i(q,\hbar,m+1)$. The contribution of (51) is easily seen to be of the form $1+\sum_{d>0}Q^dR_{id}/d!$ given by the initial term in the recursion – the polynomials R_{id} are not specified. Next, the contribution of (52) is analyzed. The graph type G^{i0} contributions to (52) vanish by the argument in Section 3 since ψ_2-degree is too high. Hence, only graphs of type G^{i1} contribute to (52). The graph pruning strategy is now applied as before.

In this case, we include the details of the required localization calculation. Let $\Gamma\in G_d^{i1}$. Two equations are needed. Let Γ be the unique graph of type G_d^{i1} with a single edge e connecting fixed points i and $j\neq i$ and satisfying $\delta(e)=d$. The first equation is:

$$\mathrm{Cont}_\Gamma\Big(Q^d\int_{\overline{M}_d}\frac{\psi_2^d}{\hbar-\psi_2}E'_{d,2}e_2^*(\phi_i)\Big) = Q^dC_i^j(d,\hbar,m+1). \tag{54}$$

The proof is by a computation of the left contribution. The stack $\overline{M}_\Gamma$ is 0 dimensional with $|\mathbf{G}|=d$; the space $\widetilde{M}_\Gamma$ is a regular point. Let

$$\mu : (C,x_1,x_2)\to\mathbf{P}^m$$

be the fixed map corresponding to Γ. The restriction of the equivariant top Chern class (or Euler class) of $E'_{d,2}$ to $\overline{M}_\Gamma$ is:

$$c_{\text{top}}(E'_{d,2})|_{\overline{M}_\Gamma} = \Pi_{r=1}^{(m+1)d}(m+1)\lambda_i + r\frac{\lambda_j - \lambda_i}{d}.$$

The Euler class of N_Γ is obtained from $H^0(C, \mu^*(T_{\mathbf{P}^m}))$ (after subtracting the trivial weight obtained from the unique infinitesimal automorphism fixing x_1 and x_2). The weights of $H^0(C, \mu^*(T_{\mathbf{P}^m}))$ are determined by the μ pull-back of the Euler sequence:

$$\begin{aligned} c_{\text{top}}(N_\Gamma) &= \Pi_{\alpha=0}^{m}\Pi_{r=0,\ (\alpha,r)\neq(i,0),(j,d)}^{d}\ \lambda_i - \lambda_\alpha + r\frac{\lambda_j - \lambda_i}{d} \\ &= d!(\frac{\lambda_j - \lambda_i}{d})^d \cdot \Pi_{\alpha\neq i}(\lambda_i - \lambda_\alpha) \cdot \\ &\quad \Pi_{\alpha\neq i}\Pi_{r=1,\ (\alpha,r)\neq(j,d)}^{d}\ \lambda_i - \lambda_\alpha + r\frac{\lambda_j - \lambda_i}{d}. \end{aligned}$$

The classes ψ_2 and $e_2^*(\phi_i)$ restrict to $(\lambda_j - \lambda_i)/d$ and $\Pi_{\alpha\neq i}(\lambda_i - \lambda_\alpha)$ respectively. Since,

$$\text{Cont}_\Gamma\Big(Q^d \int_{\overline{M}_d} \frac{\psi_2^d}{\hbar - \psi_2} E'_{d,2} e_2^*(\phi_i)\Big) = \frac{Q^d}{|\mathbf{G}|} \int_{\widetilde{M}_\Gamma} \frac{\psi_2^d}{(\hbar - \psi_2)\cdot N_\Gamma} E'_{d,2} e_2^*(\phi_i),$$

equation (54) is an algebraic consequence of these weight calculations (pulled back to $H_{\mathbf{T}}^*(\widetilde{M}_\Gamma)$).

Next, let $\Gamma \in G_d^{i1}$ satisfy $\delta(e) < d$ (following the notation of Type G^{i1} in Section 3). The second equation is:

$$\text{Cont}_\Gamma\Big(Q^d \int_{\overline{M}_d} \frac{\psi_2^d}{\hbar - \psi_2} E'_{d,2} e_2^*(\phi_i)\Big) = \tag{55}$$

$$Q^{\delta(e)} C_i^j(\delta(e), \hbar, m+1) \cdot \text{Cont}_{\Gamma_j}\Big((\frac{\lambda_j - \lambda_i}{\delta(e)} Q)^{d-\delta(e)} \int_{\overline{M}_{d-\delta(e)}} \frac{E'_{d-\delta(e),2}}{\frac{\lambda_j - \lambda_i}{\delta(e)} - \psi_2} e_2^*(\phi_j)\Big).$$

The proof is by expanding the left contribution.

Standard weight calculations (via a natural restriction sequence of sections of $\mu^*(\mathcal{O}_{\mathbf{P}^m}(m+1))$ to the component $D(e)$) yield the following product formula of weights:

$$c_{\text{top}}(E'_{d,2})|_{\overline{M}_\Gamma} = c_{\text{top}}(E'_{d-\delta(e),2})|_{\overline{M}_{\Gamma_j}} \cdot \Pi_{r=1}^{(m+1)\delta(e)}(m+1)\lambda_i + r\frac{\lambda_j - \lambda_i}{\delta(e)}. \tag{56}$$

The normal bundle N_Γ is determined in K-theory as a sum of two pieces. Let $[\mu : (C, x_1, x_2) \to \mathbf{P}^m] \in \overline{M}_\Gamma$. The first piece is topologically trivial with weight obtained from the representation $H^0(C, \mu^*(T_{\mathbf{P}^m}))$ after removing the infinitesimal automorphisms. The second piece is a direct sum of line bundles obtained from the deformation spaces of the nodes of C forced by Γ. Recall, the second marking of Γ_j corresponds to the forced node of C lying on $D(e)$.

The normal bundle piece obtained from $H^0(C, \mu^*(T_{\mathbf{P}^m}))$ may be decomposed via restriction to $D(e)$ as in (56). Note that $\widetilde{M}_\Gamma$ and $\widetilde{M}_{\Gamma_j}$ are canonically isomorphic. Via this

isomorphism, we find

$$\frac{e_2^*(\phi_i)}{N_\Gamma} = \frac{1}{\frac{\lambda_j-\lambda_i}{\delta(e)} - \psi_2} \frac{e_2^*(\phi_j)}{N_{\Gamma_j}} \cdot \frac{1}{\delta(e)!(\frac{\lambda_j-\lambda_i}{\delta(e)})^{\delta(e)} \cdot \Pi_{\alpha\neq i} \Pi_{r=1,\ (\alpha,r)\neq(j,\delta(e))}^{\delta(e)} \lambda_i - \lambda_\alpha + r\frac{\lambda_j-\lambda_i}{\delta(e)}}, \tag{57}$$

where the left and right sides are naturally classes on $\widetilde{M}_\Gamma$ and $\widetilde{M}_{\Gamma_j}$ respectively. The first term on the right is the nodal deformation corresponding to the pruned node.

It is important to realize the treatment of the second marking differs for $\widetilde{M}_\Gamma$ and $\widetilde{M}_{\Gamma_j}$. The natural pull-back of ψ_2 to $\widetilde{M}_\Gamma$ is of pure weight $(\lambda_j - \lambda_i)/\delta(e)$.

Finally, we have $|\mathbf{G}_\Gamma| = \delta(e)|\mathbf{G}_{\Gamma_j}|$. As the Γ and Γ_j contributions in (55) may be integrated on the tilde space (with automorphism corrections), equation (55) now follows algebraically. The linear recursions are obtained from (54) and (55) by summing over graphs of type G^{i1}. This is the third use of the full localization formulas for the moduli space of maps.

4.2. **Polynomiality**

The Calabi-Yau case is difficult for several reasons. The recursion relations for z_i are not yet determined as the functions R_{id} are unknown. It is necessary to find additional conditions satisfied by the correlators z_i. Givental's idea here is to prove a polynomiality constraint satisfied by a related double correlator Φ. Define $\Phi(z,q) \in \mathbb{Q}(\lambda,\hbar)[[z,q]]$ by:

$$\Phi(z,q) = \sum_{i=0}^{m} \frac{(m+1)\lambda_i}{\Pi_{j\neq i}(\lambda_i - \lambda_j)}\, e^{\lambda_i z} Z_i(qe^{z\hbar}, \hbar, m+1) Z_i(q, -\hbar, m+1). \tag{58}$$

A constraint on $\Phi(z,q)$ may be interpreted as a further condition on the correlators z_i.

A geometric construction is needed for the polynomiality constraint. Consider a new 1-dimensional torus $\mathbb{C}^*$. Let $\mathbb{Q}[\hbar]$ be the standard presentation of the equivariant cohomology ring of $\mathbb{C}^*$ (again, $\hbar$ is the first Chern class of the dual of the standard representation of $\mathbb{C}^*$). Let $\mathbb{C}^*$ act on the vector space $V = \mathbb{C}^2$ via the exponential weights $(0,-1)$. Let y_1, y_2 be the respective fixed points for the induced action on $\mathbf{P}^1 = \mathbf{P}(V)$. The equivariant Chern classes of the tangent representations at the fixed points are $\hbar, -\hbar$ respectively. Recall from Section 3.1 the $\mathbf{T}$-action on W. There are naturally induced $(\mathbb{C}^* \times \mathbf{T})$-actions on $\mathbf{P}(V) \times \mathbf{P}(W)$ and $\overline{M}_{0,2}(\mathbf{P}(V) \times \mathbf{P}(W), (1,d))$. The space of interest to us will be:

$$L_d = e_1^{-1}\big(\{y_1\} \times \mathbf{P}(W)\big) \cap e_2^{-1}\big(\{y_2\} \times \mathbf{P}(W)\big) \subset \overline{M}_{0,2}(\mathbf{P}(V) \times \mathbf{P}(W), (1,d)).$$

L_d is easily seen to be a nonsingular, $(\mathbb{C}^* \times \mathbf{T})$-equivariant substack.

Let L'_d denote the polynomial space $\mathbf{P}(W \otimes Sym^d(V^*))$ with the canonical $(\mathbb{C}^* \times \mathbf{T})$-representation. A degree d algebraic map $\mathbf{P}(V) \to \mathbf{P}(W)$ canonically yields a point in L'_d. There is a natural $(\mathbb{C}^* \times \mathbf{T})$-equivariant morphism

$$\mu : M_{0,2}(\mathbf{P}(V) \times \mathbf{P}(W), (1,d)) \to L'_d$$

obtained by identifying an element of the left moduli space with the graph of a uniquely determined map $\mathbf{P}(V) \to \mathbf{P}(W)$. It may be shown that μ extends to a $\mathbb{C}^* \times \mathbf{T}$-equivariant morphism from the stack $\overline{M}_{0,2}(\mathbf{P}(V) \times \mathbf{P}(W), (1,d))$ [G1], [LLY]. Let $\mu : L_d \to L'_d$ be the induced map. Let $P \in H^*_{\mathbb{C}^* \times \mathbf{T}}(L'_d)$ be the first Chern class of $\mathcal{O}_{L'_d}(1)$. Let E_d be the equivariant bundle on L_d with fiber over a stable map $[(f_V \times f_W) : C \to P(V) \times P(W)]$ equal to $H^0(C, f_W^*(\mathcal{O}_{\mathbf{P}(W)}(m+1)))$.

LEMMA 7. — *There is an equality:*

$$\Phi(z,q) = \sum_{d \geq 0} q^d \int_{L_d} e^{\mu^*(P)\cdot z} E_d, \tag{59}$$

where the integral on the right is the $(\mathbb{C}^* \times \mathbf{T})$*-equivariant push forward to a point.*

Proof. This is the fourth (and last) localization calculation on the moduli space of maps needed by Givental. The remarkable feature of this equality is the following. On the left side of (59), $\hbar$ is a formal parameter. On the right side, it is an element of equivariant cohomology. As L_d is a nonsingular stack, the $\mathbb{C}^* \times \mathbf{T}$-localization formula yields an explicit graph summation answer for the integral on the right which is directly matched with (58).

The first step is identify the graph types of the fixed loci of L_d. Recall the definitions of G^{i0} and G^{i1} from Section 3. Let $G^i = G^{i0} \cup G^{i1} \cup \{\mathrm{Triv}(i)\}$ where $\mathrm{Triv}(i)$ is the edgeless two pointed graph with a single vertex v satisfying $\mu(v) = p_i$. Let $\deg(\mathrm{Triv(i)}) = 0$. The components of $L_d^{\mathbb{C}^* \times \mathbf{T}}$ are in bijective correspondence to triples (i, Γ_1, Γ_2) where $0 \leq i \leq m$ and $\Gamma_1, \Gamma_2 \in G^i$ satisfy $\deg(\Gamma_1) + \deg(\Gamma_2) = d$. The graphs Γ_1, Γ_2 describe the configurations lying over the points $y_1, y_2 \in \mathbf{P}(V)$ respectively. A fixed map

$$\mu : (C, x_1, x_2) \to \mathbf{P}(V) \times \mathbf{P}(W)$$

in the corresponding component satisfies the following properties. The domain is a union of three subcurves $C = C_1 \cup C_m \cup C_2$. The curve C_m is mapped isomorphically by μ to $\mathbf{P}(V) \times \{p_i\}$. C_1 and C_2 contain x_1 and x_2 and lie over y_1 and y_2 respectively. The Lemma will follow from the calculation of the contribution of (i, Γ_1, Γ_2) to the integral in (59).

Let $\Gamma = (i, \Gamma_1, \Gamma_2)$. Let d_1, d_2 equal $\deg(\Gamma_1), \deg(\Gamma_2)$ respectively. We treat the generic case: $d_1, d_2 > 0$. The degenerate cases in which either Γ_1 or Γ_2 equals $\mathrm{Triv}(i)$ are computed analogously. The contribution equation is:

$$\begin{aligned} \mathrm{Cont}_\Gamma(q^d \int_{L_d} e^{\mu^*(P)\cdot z} E_d) \;=\; & \frac{(m+1)\lambda_i}{\Pi_{\alpha \neq i} \lambda_i - \lambda_\alpha} e^{\lambda_i z} \cdot \\ & (qe^{z\hbar})^{d_1} \cdot \mathrm{Cont}_{\Gamma_1}\Big(\int_{\overline{M}_{d_1}} \frac{E'_{d_1,2}}{\hbar - \psi_2} e_2^*(\phi_i)\Big) \cdot \\ & q^{d_2} \cdot \mathrm{Cont}_{\Gamma_2}\Big(\int_{\overline{M}_{d_2}} \frac{E'_{d_2,2}}{-\hbar - \psi_2} e_2^*(\phi_i)\Big). \end{aligned}$$

The contribution equation in the degenerate cases is identical (with the convention $\mathrm{Cont}_{\mathrm{Triv(i)}} = 1$).

The equation is proven by expanding the localization formula for the left contribution. Note first that the fixed stack $\overline{M}_\Gamma \subset L_d$ is naturally isomorphic to $\overline{M}_{\Gamma_1} \times \overline{M}_{\Gamma_2}$. As

$$\mu(\overline{M}_\Gamma) = [\mathbb{C}_i \otimes [(y_1^*)^{d_2}(y_2^*)^{d_1}]],$$

the class $\mu^*(P)$ is pure weight equal to $\lambda_i + d_1\hbar$. The class $c_{\mathrm{top}}(E_d)|_{\overline{M}_\Gamma}$ is pure weight and factors as:

$$(m+1)\lambda_i \cdot c_{\mathrm{top}}(E'_{d_1,2})|_{\overline{M}_{\Gamma_1}} \cdot c_{\mathrm{top}}(E'_{d_2,2})|_{\overline{M}_{\Gamma_2}}$$

by the restriction sequence to C_m. Similarly,

$$\frac{\Pi_{\alpha \neq i}\lambda_i - \lambda_\alpha}{N_\Gamma}$$

is computed to equal the product of $e_2^*(\phi_i)/((\hbar - \psi_2)N_{\Gamma_1})$ from $\overline{M}_{\Gamma_1}$ with $e_2^*(\phi_i)/((-\hbar - \psi_2)N_{\Gamma_2})$ from $\overline{M}_{\Gamma_2}$. This normal bundle expression is obtained by the restriction sequence of tangent sections to C_m and an accounting of nodal deformations. As the N_Γ is the normal bundle in L_d, only tangent sections of $H^0(C, \mu^*(\mathbf{P}(V))$ vanishing at the markings x_1 and x_2 appear in the normal bundle expression. The contribution equation now follows directly.

Equation (59) is obtained from the contribution equation, the definition of $Z_i(q, \hbar, m+1)$, and a sum over graphs. □

By Lemma 7, $\Phi(z,q)$ may be rewritten as:

$$\Phi(z,q) = \sum_{d \geq 0} q^d \int_{L'_d} e^{Pz} \mu_*(c_{\mathrm{top}}(E_d)). \tag{60}$$

The group $\mathbb{C}^* \times \mathbf{T}$ acts with $(m+1)(d+1)$ isolated fixed points on L'_d. A weight calculation of the representation $W \otimes Sym^d(V^*)$ yields the standard presentation:

$$H^*_{\mathbb{C}^* \times \mathbf{T}}(L'_d) = \mathbb{Q}[P, \lambda, \hbar]/(\Pi_{\alpha=0}^m \Pi_{r=0}^d (P - \lambda_\alpha - r\hbar)).$$

As $\mu_*(c_{\mathrm{top}}(E_d)) \in H^{(m+1)d+1}_{\mathbb{C}^* \times \mathbf{T}}(L'_d)$, there is a unique polynomial

$$E_d^Z(P, \hbar, \lambda) \in \mathbb{Q}[P, \lambda, \hbar]$$

of homogeneous degree $(m+1)d+1$ satisfying $\mu_*(c_{\mathrm{top}}(E_d)) = E_d^Z(P, \lambda, \hbar)$ in $H^*_{\mathbb{C}^* \times \mathbf{T}}(L'_d)$. The Bott residue formula for the integral in (60) then yields:

$$\Phi(z,q) = \frac{1}{2\pi i} \oint e^{Pz} \sum_{d \geq 0} \frac{q^d E_d^Z(P, \lambda, \hbar)}{\Pi_{\alpha=0}^m \Pi_{r=0}^d (P - \lambda_\alpha - r\hbar)} dP. \tag{61}$$

Givental's polynomiality constraint is the following: $\Phi(z,q)$ is expressible as a residue integral of the form (61) where $E_d^Z(P, \lambda, \hbar) \in \mathbb{Q}[P, \lambda, \hbar]$ is of P-degree at most $(m+1)d+m$.

4.3. Correlators of class $\mathcal{P}$

Let $\{Y_i(q,\hbar)\}_{i=0}^m \subset \mathbf{R}[[\hbar^{-1},q]]$ be a set of functions (called correlators). Assume the correlators Y_i satisfy the rationality and regularity conditions of Lemma 5: $Y_i \in \mathbb{Q}(\lambda,\hbar)[[q]]$ with no poles at $\hbar = (\lambda_i - \lambda_j)/n$ (for all $j \neq i$ and $n \geq 1$). Let $y_i(Q,\hbar) = Y_i(Q\hbar,\hbar)$. Let y_i satisfy the following recursion relation:

$$y_i(Q,\hbar) = 1 + \sum_{d>0} \frac{Q^d}{d!} I_{id} + \sum_{d>0} \sum_{j\neq i} Q^d \, C_i^j(d,\hbar,m+1) \; y_j(Q, \frac{\lambda_j - \lambda_i}{d}), \tag{62}$$

where $I_{id} = \sum_{j=0}^d I_{id}^j \hbar^{d-j} \in \mathbb{Q}(\lambda)[\hbar]$ is an element of $\hbar$-degree at most d. The recursions (62) clearly determines y_i uniquely from the initial data I_{id}. A direct algebraic consequence of (62) is the existence of a unique expression:

$$y_i(Q,\hbar) = \sum_{d\geq 0} Q^d \frac{N_{id}}{d! \Pi_{j\neq i} \Pi_{r=1}^d (\lambda_i - \lambda_j + r\hbar)}, \tag{63}$$

where $N_{id} \in \mathbb{Q}(\lambda)[\hbar]$ is a polynomial of $\hbar$-degree at most $(m+1)d$, and $N_{i0} = 1$. We may also consider the double correlator $\Phi^Y \in \mathbb{Q}(\lambda,\hbar)[[z,q]]$:

$$\Phi^Y(z,q) = \sum_{i=0}^m \frac{(m+1)\lambda_i}{\Pi_{j\neq i}(\lambda_i - \lambda_j)} \; e^{\lambda_i z} Y_i(qe^{z\hbar},\hbar) Y_i(q,-\hbar). \tag{64}$$

After the substitution of (63) in (64), a straightforward algebraic computation shows:

$$\Phi^Y(z,q) = \frac{1}{2\pi i} \oint e^{Pz} \sum_{d\geq 0} \frac{q^d E_d^Y(P,\lambda,\hbar)}{\Pi_{\alpha=0}^m \Pi_{r=0}^d (P - \lambda_\alpha - r\hbar)} dP, \tag{65}$$

where $E_d^Y = \sum_{k=0}^{(m+1)d+m} f_k(\lambda,\hbar) P^k$ is the unique function of P-degree at most $(m+1)d+m$ determined by the values at the $(m+1)(d+1)$ evaluations $P = \lambda_i + r\hbar$ $(0 \leq i \leq m,$ $0 \leq r \leq d)$:

$$E_d^Y(\lambda_i + r\hbar) = (m+1)\lambda_i N_{ir}(\hbar) N_{i(d-r)}(-\hbar). \tag{66}$$

In general, the coefficients $f_k(\lambda,\hbar) \in \mathbb{Q}(\lambda,\hbar)$ will be rational functions. The correlators Y_i satisfy Givental's polynomiality condition if $E_d^Y \in \mathbb{Q}[P,\lambda,\hbar]$.

LEMMA 8. — *The correlators Y_i satisfy Givental's polynomiality condition if and only if $\Phi^Y(z,q) \in \mathbb{Q}[\lambda,\hbar][[z,q]]$.*

Proof. By the Bott residue formula, the integral

$$\frac{1}{2\pi i} \oint \sum_{d\geq 0} \frac{P^k}{\Pi_{\alpha=0}^m \Pi_{r=0}^d (P - \lambda_\alpha - r\hbar)} dP \tag{67}$$

simply computes the $\mathbb{C}^* \times \mathbf{T}$ equivariant push-forward to a point of the class $P^k \in H^*_{\mathbb{C}^* \times \mathbf{T}}(L'_d)$. We therefore see:

(a) for $k < (m+1)d+m$, (67) vanishes,

(b) for $k = (m+1)d+m$, (67) equals 1,

(c) for $k > (m+1)d + m$, (67) is an element of $\mathbb{Q}[\lambda, \hbar]$.

Expand the integrand of (65) in power series by $e^{Pz} = \sum_{k=0}^{\infty}(Pz)^k/k!$. Properties (a)-(c) then prove that the polynomiality of the coefficients of $E_d^Y = \sum_{k=0}^{(m+1)d+m} f_k(\lambda, \hbar)P^k$ is equivalent to the polynomiality of all coefficients of the terms $\{z^k q^d\}_{k=0}^{\infty}$ in $\Phi^Y(z,q)$. □

A set of correlators $Y_i \in \mathbf{R}[[\hbar^{-1}, q]]$ is defined to be of class $\mathcal{P}$ if the following three conditions are satisfied.

I. The rationality and regularity conditions hold.

II. The correlators y_i satisfy relations of the form (62).

III. Givental's polynomiality condition is met.

A suitable interpretation of II actually implies I, but we separate these conditions for clarity.

The most important property of class $\mathcal{P}$ is Givental's uniqueness result.

LEMMA 9. — *Let $Y_i, \overline{Y}_i \in \mathbf{R}[[\hbar^{-1}, q]]$ be two sets of correlators of class $\mathcal{P}$. If*

$$\forall i, \quad Y_i = \overline{Y}_i \quad \textit{modulo} \quad \hbar^{-2}, \tag{68}$$

then the sets of correlators agree identically: $Y_i = \overline{Y}_i$.

Proof. Let I_{id} and $\overline{I}_{id}$ be the respective initial data in the associated recursions (62). By the recursion formula (62) and the coefficient formula (53), we obtain the equality

$$Y_i = \sum_{d \geq 0} q^d (I_{id}^0 + \frac{I_{id}^1}{\hbar}) \text{ modulo } \hbar^{-2} \tag{69}$$

(and analogously for $\overline{Y}_i$). Assumption (68) therefore implies $I_{id}^0 = \overline{I}_{id}^0$ and $I_{id}^1 = \overline{I}_{id}^1$ for all i and d. In particular, $I_{i1} = \overline{I}_{i1}$.

To establish the Lemma, it is sufficient to prove $I_{id} = \overline{I}_{id}$ by induction. Assume $I_{ik} = \overline{I}_{ik}$ for all $0 \leq i \leq m$ and $k < d$. The equality $N_{ik} = \overline{N}_{ik}$ for $k < d$ then follows from the recursions. By (66), $\delta E_d = E_d^Y - E_d^{\overline{Y}}$ vanishes at $P = \lambda_i + r\hbar$ for all i and $1 \leq r \leq d-1$. Hence, the polynomial δE_d is divisible by $\Pi_{j=0}^m \Pi_{r=1}^{d-1}(P - \lambda_j - r\hbar)$. By (66) and the recursion (62), a computation shows:

$$\delta E_d(P = \lambda_i + d\hbar) = (m+1)\lambda_i \Pi_{j \neq i} \Pi_{r=1}^{d}(\lambda_i - \lambda_j + r\hbar)\ (I_{id} - \overline{I}_{id}).$$

By the polynomiality condition $\delta E_d \in \mathbb{Q}[P, \lambda, \hbar]$ and the above divisibility, we find $\hbar^{d-1}$ divides $I_{id} - \overline{I}_{id}$. Therefore, the initial data is allowed to differ only in the $\hbar^d$ and $\hbar^{d-1}$ coefficients. However, these coefficient are precisely the two appearing in (69) which agree by assumption (68). We have proven the equality $I_{id} = \overline{I}_{id}$. The inductive step is complete. □

By the results of Section 4.2, the correlators $Z_i(q, \hbar, m+1)$ are of class $\mathcal{P}$. Recall the hypergeometric correlators $Z_i^*(q, \hbar, m+1)$ defined by (47). A straightforward exercise

in algebra shows the correlators Z_i^* also to be of class $\mathcal{P}$. The polynomials $E_d^{Z^*}(P, \lambda, \hbar)$ associated to the correlators Z_i^* are:

$$E_d^{Z^*} = \Pi_{r=0}^{(m+1)d}((m+1)P - r\hbar).$$

The two sets of correlators Z_i, Z_i^* do not agree modulo $\hbar^{-2}$. The expansions modulo $\hbar^{-2}$ may be explicitly evaluated. From expression (51), the $\hbar^0$ term in Z_i is 1. The $\hbar^{-1}$ term in (51) vanishes since the classes in the relevant integrals over $\overline{M}_d$ are pull-backed via the map forgetting the first marking. Hence, $Z_i = 1$ modulo $\hbar^{-2}$. A direct computation yields:

$$Z_i^* = F(q) + \frac{\lambda_i(m+1)(G_{m+1}(q) - G_1(q)) + G_1(q)\sum_{\alpha=0}^m \lambda_\alpha}{\hbar} \quad \text{modulo} \quad \hbar^{-2},$$

where the functions $F(q)$ and $G_l(q)$ are defined by:

$$F(q) = \sum_{d=0}^{\infty} q^d \frac{((m+1)d)!}{(d!)^{m+1}}, \quad G_l(q) = \sum_{d=1}^{\infty} q^d \frac{((m+1)d)!}{(d!)^{m+1}} \Big(\sum_{r=1}^{ld} \frac{1}{r} \Big).$$

The last step in the proof of the Calabi-Yau case (iii) is the following. An explicit transformation $\overline{Z}_i$ of the correlator Z_i is found which satisfies:

(1) $\overline{Z}_i$ is of class $\mathcal{P}$,
(2) $\overline{Z}_i = Z_i^*$ modulo $\hbar^{-2}$.

Then, by Lemma 9, $\overline{Z}_i = Z_i^*$. This transformation will yield the Mirror prediction in the quintic 3-fold case.

4.4. **Transformations**

Let Y_i be a set of correlators of class $\mathcal{P}$. We define three transformations:

(a) $\overline{Y}_i(q, \hbar) = f(q)\ Y_i(q, \hbar)$,
(b) $\overline{Y}_i(q, \hbar) = \exp(\lambda_i g(q)/\hbar)\ Y_i(qe^{g(q)}, \hbar)$,
(c) $\overline{Y}_i(q, \hbar) = \exp(Cg(q)/\hbar)\ Y_i(q, \hbar)$,

where $f(q), g(q) \in \mathbb{Q}[[q]]$ satisfy $f(0) = 1$ and $g(0) = 0$, and $C \in \mathbf{R}$ is a homogeneous linear function of the λ's.

LEMMA 10. — *In each case* (a)-(c), $\overline{Y}_i$ *is a set of correlators of class* $\mathcal{P}$.

Proof. Since rational functions in $\lambda, \hbar$ satisfying the regularity condition of Lemma 5 form a subring, the correlators $\overline{Y}_i$ clearly satisfy condition I of class $\mathcal{P}$. A direct algebraic check shows the correlators $\overline{y}_i$ satisfy recursion relations of the form (62). The initial terms $\overline{I}_{id}$ change, but remain in $\mathbb{Q}(\lambda)[\hbar]$ of $\hbar$-degree at most d. The values $f(0) = 1$ and $g(0) = 0$ are needed for this verification. Condition II therefore holds for $\overline{Y}_i$.

Condition III of class $\mathcal{P}$ is checked via Lemma 8. The transformations (a)-(c) have the following effect on the double correlator:

(a) $\Phi^{\overline{Y}}(z, q) = f(qe^{z\hbar})f(q) \cdot \Phi^Y(z, q)$,

(b) $\Phi^{\overline{Y}}(z,q) = \Phi^Y(z + (g(qe^{z\hbar}) - g(q))/\hbar, qe^{g(q)})$,
(c) $\Phi^{\overline{Y}}(z,q) = \exp(C \cdot (g(qe^{z\hbar}) - g(q))/\hbar) \cdot \Phi^Y(z,q)$.

In each case, $\Phi^{\overline{Y}}$ is easily seen to remain in $\mathbb{Q}[\lambda,\hbar][[z,q]]$. Case (a) is clear. Since

$$\frac{g(qe^{z\hbar}) - g(q)}{\hbar} \in \mathbb{Q}[\lambda,\hbar][[z,q]],$$

the change of variables in case (b) and multiplication in case (c) preserve membership in $\mathbb{Q}[\lambda,\hbar][[z,q]]$. □

The transformation from $Z_i(q,\hbar,m+1)$ to $Z_i^*(q,\hbar,m+1)$ can now be established. Define the correlators $\overline{Z}_i$ by

$$\overline{Z}_i(q,\hbar) = F(q) \cdot \exp\left(\frac{(m+1)\lambda_i(G_{m+1}(q) - G_1(q)) + G_1(q)\sum_{\alpha=0}^m \lambda_\alpha}{\hbar F(q)}\right) \cdot$$

$$Z_i\left(q \cdot \exp\left(\frac{(m+1)(G_{m+1}(q) - G_1(q))}{F(q)}\right), \hbar, m+1\right).$$

By a composition of transformations established in Lemma 10, the correlators $\overline{Z}_i$ are of class $\mathcal{P}$. An explicit calculation using the results of Section 4.3 shows $\overline{Z}_i(q,h) = Z_i^*(q,\hbar,m+1)$ modulo $\hbar^{-2}$. By Lemma 9, $\overline{Z}_i(q,\hbar) = Z_i^*(q,\hbar,m+1)$.

Consider the change of variables defined by:

$$T = t + \frac{(m+1)(G_{m+1}(e^t) - G_1(e^t))}{F(e^t)}. \tag{70}$$

Exponentiating (70) yields

$$e^T = e^t \cdot \exp\left(\frac{(m+1)(G_{m+1}(e^t) - G_1(e^t))}{F(e^t)}\right). \tag{71}$$

Together (70) and (71) define a change of variables from formal series in T, e^T to formal series in t, e^t. This transformation is easily seen to be invertible.

Let $S_{\mathbf{T}}(T,\hbar,m+1) \in H_{\mathbf{T}}^*(\mathbf{P}^m)[[\hbar^{-1},T,e^T]]$ be the equivariant correlator (32) in the variable T. Let the correlator $\overline{S}_{\mathbf{T}}(t,\hbar) \in H_{\mathbf{T}}^*(\mathbf{P}^m)[[\hbar^{-1},t,e^t]]$ be obtained from $S_{\mathbf{T}}(T,\hbar,m+1)$ by the change of variables (70) followed by multiplication by the function

$$F(e^t) \cdot \exp\left(\frac{G_1(e^t)\sum_{\alpha=0}^\infty \lambda_\alpha}{\hbar F(e^t)}\right).$$

By (34) and the definition of $\overline{Z}_i$, we find

$$\langle \phi_i, \overline{S}_{\mathbf{T}}(t,\hbar)\rangle = e^{\lambda_i t/\hbar} l \lambda_i \overline{Z}_i(e^t,\hbar,l).$$

Consider the correlator $S_{\mathbf{T}}^*(t,\hbar,m+1) \in H_{\mathbf{T}}^*(\mathbf{P}^m)[[\hbar^{-1},t,e^t]]$ defined by (48). By equation (49), the equality $\overline{Z}_i(e^t,\hbar) = Z_i^*(e^t,\hbar)$, and property (31), we conclude $\overline{S}_{\mathbf{T}}(t,\hbar) = S_{\mathbf{T}}^*(t,\hbar,m+1)$.

After passing from equivariant to standard cohomology ($\lambda_i = 0$) and setting $\hbar = 1$, we obtain the Mirror result (case (iii) of Section 0). The series $S_X^*, S_X \in H^*(\mathbf{P}^m)[t][[e^t]]$ are determined by:

$$\begin{aligned} S_X^* &= \frac{1}{(m+1)H}\, S_{\mathbf{T}}^*(t, \hbar, m+1)|_{\lambda_i=0,\ \hbar=1} \\ &= \sum_{i=0}^{m-1} I_i(t) H^i. \\ S_X &= \frac{1}{(m+1)H}\, S_{\mathbf{T}}(T, \hbar, m+1)|_{\lambda_i=0,\ \hbar=1}. \end{aligned}$$

where $I_i(t) \in \mathbb{Q}[t][[e^t]]$ (see definitions (48) and (29)). The following equalities hold:

$$I_0(t) = F(e^t), \quad I_1/I_0\ (t) = t + \frac{(m+1)(G_{m+1}(e^t) - G_1(e^t))}{F(e^t)}.$$

We have shown S_X is obtained from $\sum_{i=0}^{m-1} I_i/I_0\ (t)$ by the change of variables $T = I_1/I_0\ (t)$. The proof of this explicit transformation between S_X^* and S_X completes case (iii) of Section 0.

4.5. **The quintic 3-fold**

Let $X \subset \mathbf{P}^4$ be a quintic 3-fold. The expected dimension of the moduli space of rational curves in X is 0 for all degrees. The correlator S_X is easily evaluated in terms of the Gromov-Witten invariants N_d of X directly from the definitions. Let $\mathcal{F} = 5T^3/6 + \sum_{d>0} N_d e^{dT}$. After setting $\hbar = 1$, we obtain from (29):

$$S_X = \frac{1}{5H} \cdot \sum_{d \geq 0} e^{(H+d)T} e_{2*}\left(\frac{c_{\text{top}}(E_d)}{1-\psi_2}\right).$$

It is necessary to calculate (for $d > 0$):

$$\begin{aligned} e_{2*}\left(\frac{c_{\text{top}}(E_d)}{1-\psi_2}\right) &= e_{2*}(c_{\text{top}}(E_d) + c_{\text{top}}(E_d)\psi_2 + c_{\text{top}}(E_d)\psi_2^2) \\ &= H^3 \cdot \langle \tau_0(1)\ \tau_1(H) \rangle_d^X + H^4 \cdot \langle \tau_0(1)\ \tau_2(1) \rangle_d^X \\ &= dN_d H^3 - 2N_d H^4. \end{aligned}$$

The expansion in the first line is truncated for dimension reasons. The first term vanishes. Finally, the string, dilaton, and divisor equations are applied to conclude the last line. This integral calculation appears in [LLY] and [Ki]. An algebraic calculation now yields:

$$S_X = 1 + TH + \frac{1}{5}\frac{d\mathcal{F}}{dT} H^2 + \left(\frac{1}{5} T \frac{d\mathcal{F}}{dT} - \frac{2}{5}\mathcal{F}\right) H^3.$$

After accounting for multiple covers by equation (3), S_X exactly equals the right side of (1). Equation (2) follows from the quantum differential equation obtained from the

$*_X$ product (for which S_X is part of a fundamental solution). The proven correspondence (iii) implies:

$$\mathcal{F}(T(t)) = \frac{5}{2}\Big(\frac{I_1}{I_0}(t)\frac{I_2}{I_0}(t) - \frac{I_3}{I_0}(t)\Big),$$

which is the standard form of the Mirror prediction for quintic 3-folds.

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