Astérisque

GERD FALTINGS

Curves and their fundamental groups

Astérisque, tome 252 (1998), Séminaire Bourbaki,
exp. n° 840, p. 131-150

<http://www.numdam.org/item?id=SB_1997-1998__40__131_0>

© Société mathématique de France, 1998, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Séminaire BOURBAKI
50ème année, 1997-98, n° 840

Mars 1998

CURVES AND THEIR FUNDAMENTAL GROUPS
[following Grothendieck, Tamagawa and Mochizuki]

by **Gerd FALTINGS**

1. INTRODUCTION

The fundamental group of a topological space is one of the most elementary invariants in algebraic topology. In fact many interesting topological spaces (for example negatively curved) are $K(\pi, 1)$'s, and thus their fundamental groups classify them completely up to homotopy. Similarly hyperbolic spaces (constant curvature -1) of dimension at least three are up to isometry determined by their group (Mostow).

In SGA1, A. Grothendieck has defined the algebraic fundamental group of a scheme as the profinite group which classifies finite etale coverings. For a complex algebraic variety it is equal to the profinite completion of the topological fundamental group. However if the variety X is defined over a subfield K of $\mathbb{C}$ there is an additional structure. Namely the geometric fundamental group fits into an exact sequence

$$0 \longrightarrow \pi_1(X \otimes_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 0,$$

thus defining a homomorphism

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_K \overline{K})),$$

which determines the class of the extension if $\pi_1(X \otimes_K \overline{K})$ has trivial center.

We owe to A. Grothendieck the idea that this extension may encode many algebraic properties of X. Obviously one should restrict to varieties which classically would be $K(\pi, 1)$'s, so that for example projective spaces should be excluded. Grothendieck conjectures the existence of a "natural" class of anabelian schemes" (over fields K), which should have the following properties:

a) X is anabelian $\Longleftrightarrow$ $X \otimes_K \overline{K}$ is anabelian.

b) If $X \to Y$ is proper and smooth (a fibration) and Y as well as the fibers are anabelian, so is X.

c) Hyperbolic curves are anabelian.

d) The moduli stacks $\mathcal{M}_{g,n}$ (classifying curves of genus g with n punctures) are anabelian.

He then conjectures that for anabelian schemes X over a field K which is finitely generated over $\mathbb{Q}$, the functor which sends X to its fundamental group $\pi_1(X)$ is fully faithful. Here maps $\pi_1(X) \to \pi_1(Y)$ should be (and will be for the sequel of this talk) $\pi_1(Y \otimes_K \overline{K})$-conjugacy-classes of maps of extensions

$$0 \longrightarrow \pi_1(X \otimes_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 0.$$

Especially rational points $X(K)$ should correspond bijectively to conjugacy-classes of sections

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X).$$

In many cases the Mordell-Weil theorem already implies that $X(K)$ injects into the space of conjugacy-classes of sections. In this talk I want to report on recent progress about these conjectures. Firstly A.Tamagawa has shown that for (projective and irreducible) curves X over a finite field K, one can characterise the sections s coming from K-rational points entirely in terms of the groups. We present this here (in somehow modified form) to introduce a very important technique, namely to consider not only X but also an infinite number of coverings of it. This dates back to earlier work of H. Nakamura, and has been used by S. Mochizuki to show our main result:

Namely if K is a finitely generated extension of $\mathbb{Q}_p$, then for hyperbolic curves X_1 and X_2 over K, any open map of extensions

$$\pi_1(X_1) \longrightarrow \pi_1(X_2)$$

is induced from a (unique and dominant) map

$$X_1 \longrightarrow X_2.$$

This implies the corresponding result for number fields.

The proof uses p-adic Hodge theory, and we sketch the relevant facts. Also for simplicity we restrict ourselves to finite extensions of $\mathbb{Q}_p$ (local fields) and projective curves. Finally we do not discuss the vast amount of literature dealing with the problem how to recover a field from its absolute Galois group (for example [P]). I thank S. Mochizuki for his help. The talk is based on his ideas. However in some arguments I have used different approaches, to further scientific diversity.

2. PRELIMINARIES

For a connected scheme X and a geometric point x of X, P. Grothendieck defined in SGA1 the algebraic fundamental group $\pi_1(X,x)$. It is a profinite compact topological group, such that the category of finite etale coverings is equivalent to that of finite sets with continuous $\pi_1(X,x)$-action. To a finite etale covering

$$f : Y \longrightarrow X$$

one associates the set $f^{-1}(x)$. Here connected coverings correspond to homogeneous $\pi_1(X,x)$-sets, and connected pointed coverings to open subgroups of $\pi_1(X,x)$. The fundamental group is a covariant functor for pointed maps. In general for different base-points the corresponding fundamental groups are isomorphic, the isomorphism being unique up to conjugation. Hence for arbitrary maps we obtain a functor into the category of groups with conjugacy classes of homomorphisms. If we neglect base-points we use the notation $\pi_1(X)$. For a scheme X of finite type over the complex numbers $\mathbb{C}$ the algebraic fundamental group is the profinite completion of the topological fundamental group of $X(\mathbb{C})$. So for example for a smooth connected projective curve X of genus g the fundamental group $\pi_1(X,x)$ is (profinitely) generated by $2g$ elements

$$a_1, \cdots, a_g, b_1, \ldots, b_g,$$

subject to the single relation

$$a_1 \,.\, b_1 \,.\, a_1^{-1} \,.\, b_1^{-1} \,.\, \ldots \,.\, a_g \,.\, b_g \,.\, a_g^{-1} \,.\, b_g^{-1} = 1.$$

It is known that one obtains the same group for any such curve over an algebraically closed field of characteristic zero. In positive characteristic the fundamental group is a quotient of the above. As a warmup we prove

1. *Lemma.— Suppose X is a smooth projective irreducible curve over an algebraically closed field K, of genus $g \geq 2$. Then $\pi_1(X,x)$ has trivial center.*

Proof.— Suppose σ lies in the center. Then for any connected finite etale covering $Y \to X$, σ defines an automorphism of Y which acts on the fundamental group $\pi_1(Y,y)$ by inner automorphism. Especially σ acts trivially on the etale cohomology $H^1(Y,\mathbb{Z}_\ell)$, ℓ any prime invertible in K. As Y has also genus bigger than one σ must be the identity on Y, and as this holds for all Y we have $\sigma = 1$.

2. **COROLLARY.**— *Suppose X is a smooth projective geometrically irreducible curve over an arbitrary field K, of genus $g \geq 2$. Consider the extension* [SGA1]

$$0 \longrightarrow \pi_1(X \otimes_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 0.$$

The isomorphism class of this extension is then uniquely determined by the induced map

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_K \overline{K})) \quad (= \textit{automorphisms/inner automorphisms}).$$

More precisely for two such curves X and Y, any conjugacy-class of surjective homomorphisms $\pi_1(X \otimes_K \overline{K}) \to \pi_1(Y \otimes_K \overline{K})$ respecting these Galois actions extends to a homomorphism of extensions $\pi_1(X) \to \pi_1(Y)$, unique up to conjugation by $\pi_1(Y \otimes_K \overline{K})$.

Proof.— $\pi_1(X)$ is the fibered product of $\mathrm{Aut}(\pi_1(X \otimes_K \overline{K}))$ and $\mathrm{Gal}(\overline{K}/K)$, over $\mathrm{Out}(\pi_1(X \otimes_K \overline{K}))$. Here as topology on $\mathrm{Aut}(\pi_1(X \otimes_K \overline{K}))$ one uses uniform convergence.

3. CURVES OVER FINITE FIELDS

In this section we suppose that K is a finite field, thus $\mathrm{Gal}(\overline{K}/K)$ is free cyclic generated by Frobenius. Suppose X is a projective smooth geometrically irreducible curve over K. Then the number of K-rational points of X is equal to the alternating sum of the traces of (geometric) Frobenius on ℓ-adic etale cohomology. Especially this number can be read of from the action of Frobenius on $\pi_1(X \otimes_K \overline{K})$ (even the maximal abelian quotient suffices). Now suppose given a section

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

of the projection. For any characteristic open subgroup H of $G = \pi_1(X \otimes_K \overline{K})$ we obtain a subgroup $H \cdot s(\mathrm{Gal}(\overline{K}/K))$ of $\pi_1(X)$, which corresponds to an etale covering $Y = Y(H) \to X$. Here Y is also geometrically irreducible, and $H \cdot s(\mathrm{Gal}(\overline{K}/K))$ is equal to $\pi_1(Y)$. Y is a K-model of the Galois-cover of $X \otimes_K \overline{K}$ defined by H. If in fact the section s is defined by a rational point $x \in X(K)$, then chosing x as base point we see that Y is a pointed covering, that is x lifts to a point $y \in Y(K)$. There is a converse:

3. THEOREM (A. Tamagawa).— *Suppose each $Y(H)$ has a K-rational point. Then s is defined by a unique point $x \in X(K)$.*

Proof.— We may assume that X has genus ≥ 1. The non trivial part is to show that s comes from a section if each $Y(H)(K)$ is non empty which we now assume. For each integer n let $H_n \subset G$ the intersection of all kernels of homomorphisms from G to $\mathbb{Z}/n.\mathbb{Z}$, and

$$f_n : Y_n = Y(H_n) \longrightarrow X$$

the corresponding etale covering of X. Then the pullback in etale cohomology

$$f_n^* : H^1(X \otimes_K \overline{K}, \mathbb{Z}/n.\mathbb{Z}) \longrightarrow H^1(Y_n \otimes_K \overline{K}, \mathbb{Z}/n.\mathbb{Z})$$

vanishes, and by flat duality the same holds for the pushforward (now in flat cohomology)

$$f_{n,*} : H^1(Y_n \otimes_K \overline{K}, \mu_n) \longrightarrow H^1(X \otimes_K \overline{K}, \mu_n).$$

That means the trace-map on Jacobians $J_n \to J$ vanishes on K-valued n-torsion-points, and after composing with a high enough power of Frobenius the trace becomes divisible by n, as a homomorphism of abelian varieties defined over K. Now if y and z are two points in $Y_n(K)$ their difference $y - z$ is a divisor of degree zero and defines a point in $J_n(K)$. Its image under the trace-map is fixed by Frobenius, and thus lies in $n.J(K)$. It is the line bundle defined by the divisor $f_n(y) - f_n(z)$. As $J(K)$ is finite we can find an integer n which annihilates it. Then $f_n(y)$ and $f_n(z)$ define the same line bundle, thus must coincide. That is $f_n(Y_n(K)) \subset X(K)$ contains at most one element.

Also any $Y(H)$ is as good as $X = Y(G)$ for that argument. Thus we find a characteristic $H' \subset H$ with $Y(H')(K) \to Y(H)(K)$ constant. As each $Y(H)(K)$ was supposed to be non empty we derive that the projective limit of all $Y(H)(K)$ contains precisely one element, given by a coherent system of K-points $y(H)$ of $Y(H)$. In turn these define a system of sections

$$s(H) : \mathrm{Gal}(\overline{K}/K) \longrightarrow H \cdot s(\mathrm{Gal}(\overline{K}/K)),$$

well defined up to H-conjugation and (up to that) coherent. Especially for any H, $s(G)$ is modulo H G-conjugate to s . By compactness of G it follows that $s(G)$ is conjugate to s, so s comes from the (unique) point $y(G)$.

Without uniqueness the proof simplifies, as the filtering projective limit of non empty finite sets is non empty itself.

4. GALOIS COHOMOLOGY AND DIFFERENTIALS

Suppose V is a complete discrete valuation ring with fraction field K of characteristic 0, uniformiser π, and residue field $k = V/\pi \,.\, V$ perfect of characteristic $p > 0$. Suppose furthermore that R is a V-algebra of finite type etale over $V[u,v]/(u \,.\, v - \pi)$ (semistable singularity), or a strict henselisation of such, or an adic completion. We denote by $\omega_{R/V} = R \,.\, du/u$ its universal (finite) module of relative logarithmic differentials. For simplicity we also assume that $R \otimes_V \overline{K}$ is an integral domain. This always holds if k is algebraically closed, as the integral closure of V in R is unramified over V.

We denote the fundamental group of $\mathrm{Spec}(R \otimes_V \overline{K})$ by Δ. If $\overline{R}$ denotes the normalisation of R in the union of all integral finite etale coverings of $\mathrm{Spec}(R \otimes_V \overline{K})$, then Δ acts continuously on $\overline{R}$, and $\overline{R}$ contains the subextension R_∞ obtained by adjoining all p-power roots of u and v (the local parameters which were assumed to exist), and also of π. It is shown in [F] that $\overline{R}$ is almost etale over this subextension, which means the following:

R_∞ is the union of regular rings R_n. If $S_\infty \subset \overline{R}$ is the normalisation of R_∞ in a finite etale covering of $R_\infty \otimes_V K$, then for big n this covering is defined over $R_n \otimes_V K$, and we denote the corresponding normalisation by S_n. As R_n is regular and S_n is normal of dimension ≤ 2 we know that S_n is a projective R_n-module. Thus the determinant of the trace form on S_n is an invertible ideal in R_n which divides a power of π (or p). Then almost etale means that this power can be chosen arbitrarily small, provided n becomes big enough.

The notion of almost etale goes back to Tate, who used it to compute the continuous cohomology of $\mathrm{Gal}(\overline{K}/K)$ acting on Tate twists of $\overline{K}^\wedge$ (p-adic completion). Namely if V_∞ denotes the normalisation of V in a ramified $\mathbb{Z}_p$-extension of K, then $\overline{V}$ is almost etale over V_∞. As a consequence one gets that $H^i(\mathrm{Gal}(\overline{K}/K), \overline{K}^\wedge(n))$ vanishes if $i > 1$ or $n \neq 0$, and has dimension one over K if $n = 0$, and $i = 0, 1$. Similarly we can (almost) compute the Galois cohomologies $H^*(\Delta, \overline{R}/p^n \,.\, \overline{R})$ and $H^*(\Delta, \overline{R}^\wedge)$ (continuous cohomology of p-adic completion). If for simplicity we denote by "=" equality up to p-torsion annihilated by any p-power with exponent $> 1/(p-1)$, then

$$
\begin{aligned}
&H^0(\Delta, \overline{R}^\wedge) = (R \otimes_V \overline{V})^\wedge, \\
&H^1(\Delta, \overline{R}^\wedge) \text{ "=" } \omega_{R/V} \otimes_V \overline{V})^\wedge(-1) \quad \text{(Tate-twist)} \\
&H^i(\Delta, \overline{R}^\wedge) \text{ "=" } (0) \quad \text{for } i > 1.
\end{aligned}
$$

As a consequence one can give a proof for the Hodge-Tate decomposition. Suppose X is a projective smooth curve over K. Then

$$H^1(X \otimes_K \overline{K}, \mathbb{Q}_p) \otimes \overline{K}^\wedge \cong H^1(X, \mathcal{O}_X) \otimes \overline{K}^\wedge \oplus H^0(X, \omega_{R/K}) \otimes \overline{K}^\wedge(-1).$$

Namely for the proof we may replace K by a finite extension and assume that X has a semistable model over V. Then for each small enough open affine $\mathrm{Spec}(R)$ in this model the Galois-cohomology $H^*(\Delta, \overline{R}^\wedge \otimes_V K)$ is computed by a canonical bar-complex. Covering the model by such affines, one constructs out of these a double complex (with additional Cech-type differentials), whose cohomology $\mathcal{H}^*(X)$ is independent of choices. Also from the "local" calculations (plus technical remarks, like that the cohomology of a projective formal scheme over $\overline{V}^\wedge$ is equal to its algebraic cohomology, or that a spectral sequence degenerates because "different Tate-twists do not interact"), we have

4. THEOREM.—

$$\mathcal{H}^*(X) = H^*(X, \mathcal{O}_X) \otimes \overline{K}^\wedge \oplus H^{*-1}(X, \omega_{X/K}) \otimes \overline{K}^\wedge(-1).$$

Also as $\mathbb{Z}_p \subset \overline{R}$, there is a canonical map

$$H^*(X \otimes_K \overline{K}, \mathbb{Q}_p) \otimes \overline{K}^\wedge \longrightarrow \mathcal{H}^*(X),$$

which is then shown to be an isomorphism.

Finally for any smooth $\mathbb{Q}_p$-sheaf $\mathbb{L}$ on X we can define $\mathcal{H}^*(X, \mathbb{L})$ as above, replacing $\overline{R}^\wedge$ by $\overline{R}^\wedge \otimes \mathbb{L}$, and again there is a natural map

$$H^*(X \otimes_K \overline{K}, \mathbb{L}) \otimes \overline{K}^\wedge \longrightarrow \mathcal{H}^*(X, \mathbb{L}).$$

One can show that this is always an isomorphism. For unipotent $\mathbb{L}$'s (that is repeated extensions of sheaves induced from the base) this follows already from the result for $\mathbb{Q}_p$, and this will be all we need in the sequel.

Following Mochizuki we consider (for a projective geometrically irreducible curve X over K) the maximal pro-p quotient $\pi_1(X \otimes_K \overline{K})^{(p)}$ of the geometric fundamental group. It is (as all pro-p groups) topologically nilpotent, and has over $\mathbb{Q}_p$ an algebraic hull G. Here G is the projective limit of unipotent algebraic groups over $\mathbb{Q}_p$, and thus entirely determined by its Lie-algebra $\mathbf{g}$. Abstractly G can be generated p-adically

by $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ subject to the usual relation, and $\mathbf{g}$ by their logarithms (also subject to one relation). We denote by $Z^n(G)$ (respectively $Z^n(\mathbf{g})$) the descending central series, so

$$Z^0(G) = G, Z^0(G)/Z^1(G) = G^{ab} = T_p(J) \otimes \mathbb{Q}_p, \text{ etc.}$$

The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on G and $\mathbf{g}$ via outer automorphisms. These become a real action if we choose a section

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

of the projection, and anyway the quotients $\mathrm{gr}^n_Z(G) = \mathrm{gr}^n_Z(\mathbf{g})$ are canonical $\mathrm{Gal}(\overline{K}/K)$-modules. After base extension to $\overline{K}^\wedge$ these modules have a Hodge-Tate decomposition, with occuring Tate-twists by integers between 0 and n. As Galois linear extensions between vector spaces with different Tate-twists must split, this lifts for each n to a direct sum decomposition of $(\mathbf{g}/Z^n(\mathbf{g})) \otimes \overline{K}^\wedge$, which depends on the choice of s. However it defines a G-invariant filtration

$$E^m(\mathbf{g}/Z^n(\mathbf{g})) = \text{sum of all Tate-twists by integers } \geq m,$$

which is independent of the choice of s. Thus $\mathbf{g} \otimes \overline{K}^\wedge$ and $G(\overline{K}^\wedge)$ have canonical filtrations $E^m(\mathbf{g})$, respectively $E^m(G(\overline{K}^\wedge))$. The quotient $\mathbf{h} = \mathrm{gr}^0_E(\mathbf{g})$ can be easily shown (for example by our next arguments) to be a free pro-unipotent Lie-algebra in g generators, and also $H = \mathrm{gr}^0_E(G)$ is a free pro-unipotent group in g generators over $\overline{K}^\wedge$, on which $\mathrm{Gal}(\overline{K}/K)$ acts (depending on s). Mochizuki defines the section s to be Hodge-Tate if this action is base-extended from the trivial action (over K), that is if $\mathbf{h}$ is generated by its Galois-invariants. More generally s is called Hodge-Tate up to level n if this holds module $Z^n(\mathbf{h})$. He proves:

5. THEOREM.— *If s comes from a K-rational point, then it is Hodge-Tate.*

Proof.— Mochizuki uses a deformation argument to reduce to the ordinary case. Here we present an alternative, using the faithful representation of H (or $\mathbf{h}$) on the completed enveloping algebra of $\mathbf{h}$. As $\mathbf{h}$ is free in g generators this enveloping algebra is a free tensor-algebra.

Let $x \in X(K)$ induce s. Firstly $H^1(X, \mathcal{O}_X)$ classifies vector bundles on X which are trivialised in x, and extensions of $\mathcal{O}_X$ by $\mathcal{O}_X$. Thus if $\Omega = H^0(X, \omega_{X/K})$ denotes its dual, there exists a universal such extension

$$0 \longrightarrow \Omega \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

One computes that the map

$$H^1(X, \mathcal{H}\mathrm{om}(\mathcal{E}_1, \mathcal{O}_X)) \longrightarrow H^1(X, \mathcal{H}\mathrm{om}(\Omega \otimes_K \mathcal{O}_X, \mathcal{O}_X)) = (\Omega^{\otimes 2})^{\mathrm{dual}}$$

is an isomorphism, and gets a universal extension

$$0 \longrightarrow \Omega^{\otimes 2} \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow 0,$$

trivialised at x.

Continuing gives a projective system of extensions

$$0 \longrightarrow \Omega^{\otimes n} \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow 0.$$

Next assume that X admits a semistable model. We construct inductively $\overline{K}^\wedge$-valued smooth sheaves $\mathbb{L}_n$ on X, trivialised along the section x, such that for a small enough affine $\mathrm{Spec}(R)$ of the semistable model we have functorially in R

$$\mathbb{L}_n \otimes \overline{R}^\wedge \cong \mathcal{E}_n \otimes_R \overline{R}^\wedge.$$

The $\mathbb{L}_n$ correspond to continuous representations of $\pi_1(X, x)$ on finite dimensional $\overline{K}^\wedge$-vector spaces, whose restriction to $s(\mathrm{Gal}(\overline{K}/K))$ is the base-extension (from K to $\overline{K}^\wedge$) of the direct sum of all $\Omega^{\otimes m}$, $0 \leq m \leq n$. Furthermore they are extensions

$$0 \longrightarrow \Omega^{\otimes n} \otimes_K \overline{K}^\wedge \longrightarrow \mathbb{L}_n \longrightarrow \mathbb{L}_{n-1} \longrightarrow 0.$$

Start with the constant $\mathbb{L}_0 = \overline{K}^\wedge$. Given $\mathbb{L}_{n-1}$, the cohomology

$$H^1(X \otimes_K \overline{K}, \mathcal{H}\mathrm{om}(\mathbb{L}_{n-1}, \overline{K}^\wedge)) = \mathcal{H}^1(X, \mathcal{H}\mathrm{om}(\mathbb{L}_{n-1}, \overline{K}^\wedge))$$

has a Hodge-Tate decomposition, which (because of the isomorphisms $\mathbb{L}_n \otimes \overline{R}^\wedge = \mathcal{E}_n \otimes_R \overline{R}^\wedge$) is equal to

$$H^1(X, \mathcal{H}\mathrm{om}(\mathcal{E}_{n-1}, \mathcal{O}_X)) \otimes_K \overline{K}^\wedge \oplus H^0(X, \mathcal{H}\mathrm{om}(\mathcal{E}_{n-1}, \omega_{X/K})) \otimes_K \overline{K}^\wedge(-1).$$

If we identify the etale H^1 with isomorphism classes of smooth sheaves over $X \otimes_K \overline{K}$, the second direct summand measures the "local" extension obtained after tensoring with $\overline{R}^\wedge$. Thus to the first direct summand there corresponds a universal "locally trivial" extension $\mathbb{L}_n$ of the type as above, trivialisated at x. It is unique up to unique isomorphism, thus $\mathrm{Gal}(\overline{K}/K)$-invariant, and we are done. Now the $\mathbb{L}_n$

correspond to representations of the algebraic group G which must factor over H, and the corresponding representation of the Lie-algebra $\mathbf{h}$ on the projective limit is isomorphic to that on the completed enveloping algebra, hence faithful. As we know how $s(\mathrm{Gal}(\overline{K}/K))$ acts on it we derive that $\mathbf{h}$ is generated by its Galois invariants, so s is Hodge-Tate.

6. *Remark.*— a) Suppose

$$s' : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

is another section, and consider the extension

$$0 \longrightarrow \mathrm{gr}_Z^1(\mathbf{h}) \longrightarrow \mathbf{h}/Z^2(\mathbf{h}) \longrightarrow \mathrm{gr}_Z^0(\mathbf{h}) \longrightarrow 0,$$

with $\mathrm{Gal}(\overline{K}/K)$ acting via s'. As canonically

$$\mathrm{gr}_Z^0(\mathbf{h}) = \mathrm{Hom}_K(\Omega, \overline{K}^\wedge),$$
$$\mathrm{gr}_Z^1(\mathbf{h}) = \Lambda^2 \mathrm{gr}_Z^1(\mathbf{h}) = \mathrm{Hom}_K(\Lambda^2\Omega, \overline{K}^\wedge),$$

this corresponds to a class

$$c(s, s') \in \mathrm{Hom}_K(\Lambda^2 \Omega, \Omega) \otimes_K H^1(\mathrm{Gal}(\overline{K}/K, \overline{K}^\wedge)),$$

which must vanish if s' is also Hodge-Tate.

One can compute $c(s, s')$ as follows:

Computing modulo $Z^1(G)$ the difference of s and s' defines a cohomology-class in

$$H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{gr}_Z^0(G)) = H^1(\mathrm{Gal}(\overline{K}/K, T_p(J) \otimes \mathbb{Q}_p)).$$

By the Hodge-Tate decomposition of $T_p(J) \otimes \mathbb{Q}_p$ this maps to a class

$$c^*(s, s') \in \Omega^{\mathrm{dual}} \otimes_K H^1(\mathrm{Gal}(\overline{K}/K, \overline{K}^\wedge)).$$

Mochizuki shows (by an easy calculation) that up to a normalising factor two the second class maps to the first if we let Ω^{dual} act on $\Lambda^2 \Omega$ by interior multiplication. Especially if $g \geq 2$ then s' is Hodge-Tate up to level 2 if and only if $c^*(s, s')$ vanishes.

7. *Remark.*— b) For two such curves X and Y, and an open homomorphism of extensions

$$\alpha : \pi_1(X) \longrightarrow \pi_1(Y)$$

the composition $\alpha \circ s$ of α with a Hodge-Tate section

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

is again Hodge-Tate. This follows because the induced map on Lie-algebras $\mathbf{h}_X \to \mathbf{h}_Y$ is surjective.

8. *Remark.*— c) With similar methods one can show that for semistable X the Lie-algebra $\mathbf{g}$ is (with the action *via* a section s coming from a point) the projective limit of log-crystalline representations of $\mathrm{Gal}(\overline{K}/K)$, as defined by J.-M. Fontaine.

5. COHOMOLOGICAL COMPUTATIONS

Suppose X is a projective smooth geometrically irreducible curve over a field K of characteristic zero (or at least different from p). Assume also that the genus of X is at least one. The p-adic etale cohomologies $H^*(X, \mathbb{Z}_p)$ and $H^*(X \otimes_K \overline{K}, \mathbb{Z}_p)$ then coincide with the continuous cohomologies of the corresponding fundamental groups. They are related to the projective limits of cohomologies modulo p^n via short exact sequences

$$0 \to \mathrm{proj.lim}^{(1)} H^{m-1}(X, \mathbb{Z}_p/p^n.\mathbb{Z}) \to H^m(X, \mathbb{Z}_p) \to \mathrm{proj.lim}\, H^m(X, \mathbb{Z}/p^n.\mathbb{Z}) \to 0.$$

Here the derived projective limit $\mathrm{proj.lim}^{(1)}$ vanishes if the projective system satisfies a Mittag-Leffler condition, which will be the case in all our applications. As usual $\mathbb{Q}_p$-adic cohomology is defined by tensoring with that field.

Let $J^{(1)}$ denote the homogeneous space (under the Jacobian J of X) which classifies isomorphism classes of line-bundles of degree one on X. Note that a K-rational point of $J^{(1)}$ does not necessarily correspond to an actual line bundle of degree 1. (See below for some remarks about that topic.)

The fundamental group $\pi_1(J^{(1)})$ is the quotient of $\pi_1(X)$ under the commutator subgroup of $\pi_1(X \otimes_K \overline{K})$, with the surjection induced by the canonical map from X to $J^{(1)}$. As before denote by $\pi_1(J^{(1)})^{(p)}$ its quotient where the geometric fundamental group is replaced by its maximal pro-p-quotient. This is an extension of $\mathrm{Gal}(\overline{K}/K)$ by $T_p(J)$. Let us study sections s_{ab} of the projection $\pi_1(J^{(1)})^{(p)} \to \mathrm{Gal}(\overline{K}/K)$. Assume that X has genus $g \geq 2$, and note that the adjoint action of $\pi_1(X)$ on $H/Z^2(H)$ and $\mathbf{h}/Z^2(\mathbf{h})$ factors through $\pi_1(J^{(1)})^{(p)}$. Thus we define s_{ab} to be Hodge-Tate if $\mathbf{h}/Z^2(\mathbf{h})$ is generated by its invariants under $s_{ab}(\mathrm{Gal}(\overline{K}/K))$. This is the case if s is defined

by a K-rational point of X. In fact it suffices if s_{ab} comes from a K-rational point of $J^{(1)}$:

We may extend K and assume that s_{ab} is defined by a divisor of degree one on X, and that this divisor is a linear combination of K-rational points of X. But each such point defines a section s_{ab} which is Hodge-Tate. Also for one such Hodge-Tate section s_{ab}, another section is Hodge-Tate if and only if the difference is a Hodge-Tate class in $H^1(\mathrm{Gal}(\overline{K}/K), T_p(J))$, that is lies in the kernel of the map to $H^1(\mathrm{Gal}(\overline{K}/K), \Omega \otimes_K \overline{K}^{\wedge})$. This implies our claim, and also the well known fact that line-bundles of degree zero define Hodge-Tate classes in $H^1(\mathrm{Gal}(\overline{K}/K), T_p(J))$.

One may ask for a converse. Some approximation of such a converse holds if K is a local field, that is a finite extension of $\mathbb{Q}_p$. Namely if $J^{(m)}$ denotes the homogeneous J-space classifying line-bundles of degree n, multiplication by n induces

$$n. : J^{(1)} \longrightarrow J^{(n)}.$$

The induced map on geometric fundamental-groups is also multiplication by n.

9. PROPOSITION.— *Assume that K is a finite extension of $\mathbb{Q}_p$, X a curve of genus $g \geq 2$, and*

$$s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m)})^{(p)}$$

a section which is Hodge-Tate. Then there exists a positive integer n prime to p, such that

$$n \, . \, s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m \cdot n)})^{(p)}$$

is induced by a point in $J^{(m \cdot n)}(K)$.

Proof.— The key fact is of course the theorem of Bloch and Kato ([BK], example 3.11) that up to torsion $J(K)$ is isomorphic to the Hodge-Tate classes in $H^1(\mathrm{Gal}(\overline{K}/K), T_p(J))$. If $J^{(m)}$ has a K-rational point this implies the assertion, except for the fact that n can be chosen prime to p. However replacing m by a multiple we can find such a rational point. If n is divisible by p, we note that

$$p\cdot : J^{(n/p)} \longrightarrow J^{(n)}$$

is a principal homogeneous space with group $J[p]$ (p-division-points). The obstruction to lift a K-rational point $x \in J^{(n)}(K)$ to $J^{(n/p)}$ is a class in

$$H^1(\mathrm{Gal}(\overline{K}/K), J[p]) = H^1(\mathrm{Gal}(\overline{K}/K), T_p(J)/p \, . \, T_p(J)).$$

It coincides with the obstruction to lift the corresponding section

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(n)})$$

to $\pi_1(J^{(n/p)})$ and lifts of sections correspond uniquely to lifts of points (the geometric fibre over x is $\pi_1(J^{(n)})/\pi_1(J^{(n/p)})$, with $\mathrm{Gal}(\overline{K}/K)$-action *via* the section corresponding to x). Thus if we can lift sections we can lift points, and this allows to remove factors p from n.

The conclusion of the proposition merits its own name. It works for an arbitrary ground field K of characteristic zero.

10. DEFINITION.— *A section*

$$s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m)})^{(p)}$$

is called geometric up to torsion if there exists a positive integer n such that

$$n \,.\, s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m\cdot n)})^{(p)}$$

comes from a point in $J^{(m\cdot n)}(K)$.

¿From the previous it follows that we may always choose n prime to p, and that it suffices if the condition holds over a finite extension of K (use a trace argument).

In fact we would like this point in $J^{(m\cdot n)}(K)$ to be represented by an actual line-bundle $\mathcal{L}$ on X. There is an obstruction in the Brauer-group of K which has to vanish. Namely there exists over a Galois-extension of K a line-bundle $\mathcal{L}$ which represents the point in $J^{(m\cdot n)}$ and which is isomorphic to all its Galois-conjugates. However it may happen that these isomorphisms cannot be arranged to form a descent-datum, thus the obstruction. Nevertheless one checks that it is annihilated by the dimensions of cohomology-groups $H^i(X, \mathcal{L})$ and also by the Euler characteristic, that is by $m \cdot n + 1 - g$. Especially if p divides $g - 1$ and if m is prime to p, we may replace n by a further multiple (still prime to p) to get that $n \cdot s_{ab}$ is represented by an actual line-bundle $\mathcal{L}$, of degree $m \cdot n$.

The rest of the section becomes a little bit technical. We assume given two curves X_1 and X_2, of genus ≥ 2, and an open map of extensions

$$\alpha : \pi_1(X_1) \longrightarrow \pi_1(X_2),$$

inducing

$$\alpha_{ab} : \pi_1(J_1^{(m)})^{(p)} \longrightarrow \pi_1(J_2^{(m)})^{(p)}.$$

The base field will be an extension K^+ of a local p-adic field K, and the curves as well as the morphism α will be definable over a finite extension K_1 of K. In fact K^+ will also be a p-adic field with integers, V^+ a complete discrete valuation-ring dominating V but with residue field k^+ a function-field in one variable over k.

Recall that sections have the uniform p-adic topology.

11. PROPOSITION.— *Suppose*

$$s_{ab} : \mathrm{Gal}(\overline{K}^+/K^+) \longrightarrow \pi_1(J_1^{(1)})^{(p)}$$

is a section which is geometric up to torsion. Then

$$\alpha_{ab} \circ s_{ab} : \mathrm{Gal}(\overline{K}^+/K^+) \longrightarrow \pi_1(J_2^{(1)})^{(p)}$$

is the p-adic limit of such sections.

Proof.— We may pass to finite extensions of K and K^+ and assume that α, X_1, X_2 are defined over K, and the curves have semistable reduction. Furthermore the assertion holds if we replace K^+ by K, and as we may assume that X_1 has a K-rational point, we have to show the following claim:

α_{ab} preserves up to torsion the images of

$$J(L) \otimes \mathbb{Z}_p \longrightarrow H^1(\mathrm{Gal}(\overline{K}^+/K^+), T_p(J)), \quad (J = J_1, J_2).$$

For this we may replace J_i by the connected component of its Neron-model and then by the Raynaud-extensions G_i [see for example [C], Ch. II,1]. These are the extensions of abelian varieties by tori which define the same p-adic formal schemes as J_i. It is known that α_{ab} preserves $T_p(G_i) \subset T_p(J_i)$ (the quotient is maximal with the property that $\mathrm{Gal}(\overline{K}/K)$ acts *via* a finite unramified quotient), so it is enough to give a good characterisation of the images of $G_i(V^+) \otimes \mathbb{Z}_p$ in $H^1(\mathrm{Gal}(\overline{K}^+/K^+), T_p(G_i))$.

So suppose G is such a semiabelian variety, with associated p-divisible group G_∞. There is a map

$$\lambda : G(V^+) \longrightarrow \mathrm{Ext}_{V^+}(\mathbb{Q}_p/\mathbb{Z}_p, G_\infty) \quad (= \text{extensions of } p\text{-divisible groups over } V^+)$$

"which sends $g \in G(V^+)$ to its p^n-division-points". Also there is a variant of λ with V^+ replaced by k^+, and a reduction map (on both sides). It is classical that we get an isomorphism on the kernels of the reduction maps, that is extensions trivial

over k^+ correspond to elements of $G(V^+)$ which are trivial modulo the maximal ideal (these are also R-points of the associated formal group to G_∞). Thus the image of $G(V^+)$ consists of extensions lifting elements of $\lambda(G(k^+))$. Similarly the $\mathbb{Z}_p$-submodule generated by $G(V^+)$ consists of lifts of $\lambda(G(k^+) \otimes \mathbb{Z}_p)$. Now apply this to our situation. By a theorem of Tate α extends to a morphism of p-divisible groups (denoted for simplicity by the same name)

$$\alpha : G_{1,\infty} \longrightarrow G_{2,\infty}.$$

Its reduction modulo the maximal ideal is a homomorphism of p-divisible groups over k. By a variant of a (different) theorem of Tate it lies in $\mathrm{Hom}_k(G_1, G_2) \otimes \mathbb{Z}_p$. Thus

$$\alpha(G_1(k^+) \otimes \mathbb{Z}_p) \subset G_2(k^+) \otimes \mathbb{Z}_p),$$

and α also respects the $\mathbb{Z}_p$-lattices generated by $\lambda(G_i(V^+))$. Finally apply the obvious maps from $\mathrm{Ext}_W(\mathbb{Q}_p/\mathbb{Z}_p, G)$ to $H^1(\mathrm{Gal}(\overline{K}^+/K^+), T_p(G))$ to obtain the assertion.

Let us sketch a proof for the mentioned variant of Tate's theorem:

12. THEOREM (Tate).— *Suppose k is a finite field of characteristic p, G_1 and G_2 semiabelian varieties over k. Then the map*

$$\mathrm{Hom}_k(G_1, G_2) \otimes \mathbb{Z}_p \longrightarrow \mathrm{Hom}_k(G_1(p^\infty), G_2(p^\infty))$$

is an isomorphism.

Proof.— One can check that any homomorphism of p-divisible groups respects the Tate-modules of the subtori (eigenvalues of Frobenius on crystalline cohomology), and that the result holds for tori. Also extensions of an abelian variety A by $\mathbb{G}_m$ are classified by $A^t(k)$ (dual abelian variety) which is a finite abelian group so that any such extension splits up to isogeny. These two remarks allow us to reduce to the abelian case, so that from now on G_1 and G_2 are abelian varieties over k. If

$$\alpha_n : G_1(p^n) \longrightarrow G_2(p^n)$$

denotes the n-th stage of α (a map of p-divisible groups over k), consider the quotient

$$H_n = (G_1 \times G_2)/\mathrm{graph}(\alpha_n)$$

which is also an abelian scheme over k. One then knows that infinitely many H_n's are isomorphic and derives the result (see for example [CF], Ch. V, proof of th. 4.7). One can even adapt that proof to extend theorem 12 to function-fields).

6. PROOF OF THE MAIN RESULT

As before K is a finite extension of $\mathbb{Q}_p$, (with V, π, k, etc.), K^+ an extension, with integers V^+ etc., such that k^+ is a function field in one variable over k. The universal (finite) module of differentials of V^+ is denoted by ω_{V^+}, and let $\omega_{K^+} = \omega_{V^+} \otimes_{V^+} K^+$. Furthermore X should be a geometrically irreducible projective curve over K, of genus $g \geq 2$. A point $x \in X(K^+)$ is called non-constant if it does not lie in $X(\overline{K})$. An equivalent condition is the following:

The pullback (by x) of any global differential form on X defines an element of ω_{K^+}, thus a linear map (the derivative of x) from $\Omega = H^0(X, \omega_{X/K})$ to ω_{K^+}. x is non-constant precisely if this map is not identically zero.

If this is the case, it determines a K^+-point in the projective space $\mathbb{P}(\Omega) = \mathbb{P}^{g-1}$, which is the image of x under the canonical map (embedding if X is not hyperelliptic)

$$X \longrightarrow \mathbb{P}(\Omega).$$

We can recover the derivative from Galois-cohomology, as follows: Let

$$\Delta = \mathrm{Gal}(\overline{K}^+/L) \subset \mathrm{Gal}(\overline{K}^+/K^+)$$

denote the subgroup fixing the compositum L of $\overline{K}$ and the maximal unramified extension of K^+. Note that L contains the maximal tame extension of K^+, so that Δ is a pro-p-group. Then the Galois-cohomology of Δ with values in the p-adic completion of $\overline{V}^+$ is (up to a finite p-power) equal to the tensor product of ω_{V^+} with $W^\wedge(-1)$ (X = integers in L). Furthermore the induced map on fundamental groups $\mathrm{Gal}(\overline{K}^+/K^+) \to \pi_1(X)$ sends Δ into $\pi_1(X \otimes_K \overline{K})$, and induces by pullback a map

$$H^1(X \otimes_K \overline{K}, \overline{K}^\wedge) = \Omega^{\mathrm{dual}} \otimes_K \overline{K}^\wedge \oplus \Omega \otimes_K \overline{K}^\wedge(-1) \longrightarrow \omega_{V^+} \otimes_{K^+} L^\wedge(-1).$$

As it is Galois-invariant it must vanish on the first direct summand $\Omega^{\mathrm{dual}} \otimes_K \overline{K}^\wedge$, and on the second it is the derivative of x. Thus we can recover the derivative of any L-valued point from its induced map on fundamental groups. We also remark that both sides have natural integral structures, and then the p-powers in the denominators of these maps have an upper bound only depending on X, and not even that if X has semistable reduction. Also the association

$$\{\text{maps of extensions } \mathrm{Gal}(\overline{K}^+/K^+) \longrightarrow \pi_1(X)\} \longrightarrow \mathrm{Hom}_K(\Omega, \omega_{V^+} \otimes_{V^+} L^\wedge)$$

is continuous, where on the left the topology is uniform convergence, and on the right the p-adic one. Especially if for a sequence of points $x_n \in X(L)$ the induced maps

on fundamental groups converge, and the limit has non-trivial derivative, then the images of the points in $\mathbb{P}(\Omega)$ points converge p-adically to a non-constant limit, and so do the points themselves if X is not hyperelliptic.

13. **Remark.**— If $X \to Y$ is an etale Galois-covering with group G, and if X is hyperelliptic, then so is Y, and G is abelian of exponent 2:

We have to show that any element in G has exponent 2, thus may assume that G is cyclic and thus abelian. The hyperelliptic involution on X is unique, thus commutes with G and induces a hyperelliptic involution on Y. However this involution acts as $-\mathrm{id}$ on the Jacobian of Y and thus on any covering group of an abelian cover. Hence $\mathrm{id} = -\mathrm{id}$ on G.

This remark will be helpful to exclude hyperelliptic curves.

Finally we can formulate and prove the main result:

14. THEOREM.— *Suppose K is a finite extension of $\mathbb{Q}_p$, X and Y smooth geometrically irreducible projective curves over K,*

$$\alpha : \pi_1(X) \longrightarrow \pi_1(Y)$$

an open homomorphism of extensions (of $\mathrm{Gal}(\overline{K}/K)$). Then α is induced from a unique dominant morphism of curves $X \to Y$.

Proof.— We start with some general remarks. Obviously we may assume that α is surjective. Then α is uniquely determined by its restriction to the geometric fundamental group $\pi_1(X \otimes_K \overline{K})$, as the geometric π_1's have trivial centers. Hence we can always pass to finite extensions of K, and for example assume that X and Y have semistable models. Furthermore passing to (two disjoint) cyclic etale coverings of Y we may assume that Y is not hyperelliptic, and that $g(Y) - 1$ is divisible by p. Also Hodge-Tate theory defines an injective "pullback"

$$\alpha^* : \Omega_Y \longrightarrow \Omega_X$$

(which will be the differential of our map $X \to Y$).

Choose an extension V^+ as before, and a non-constant point $x \in X(K^+)$ such that the pullback of some differential on Y does not vanish in x. If $X^+ = X \otimes_K K^+$ and Y^+ denote the base-changes, then the point x defines a section

$$\mathrm{Gal}(\overline{K}^+/\overline{K}) \longrightarrow \pi_1(X^+) = \text{fibered product of } \pi_1(X) \text{ and } \mathrm{Gal}(\overline{K}^+/K^+),$$

and its composition with α a section s into $\pi_1(Y^+)$. If we project further to $\pi_1(J_Y^{(1)})^{(p)}$ the corresponding section is geometric up to torsion. It thus follows that Y^+ admits a line-bundle of degree prime to p. Of course this already follows from the existence of a semistable model.

But any characteristic open subgroup $H \subset \pi_1(Y \otimes_K \overline{K})$ extends to an open subgroup $H \cdot s(\mathrm{Gal}(\overline{K}^+/K))$ of $\pi_1(Y^+)$, which is the fundamental group of a cover $Y^+(H)$ of Y^+. Then s defines a section $s_{ab}(H)$ for $J^{(1)}$ which is limit of sections geometric up to torsion :

If $X^+(H)$ denotes the covering of X^+ corresponding to the subgroup $\alpha^{-1}(H) \cdot s(\mathrm{Gal}(\overline{K}^+/K^+))$ of $\pi_1(X^+)$, then x lifts to a rational point $x(H) \in X^+(H)(K^+)$ (which in turn defines s). Furthermore everything is definable over a finite extension of K, thus the assertion by proposition 11.

Especially each $Y(H)$ admits a line-bundle $\mathcal{L}(H)$ of degree prime to p, and thus contains a rational point in an extension field of K^+ of degree prime to p which is contained in L. Thus a point $y^*(H) \in Y^+(H)(L)$, and a corresponding section (well defined up to H-conjugation)

$$s^*(H) : \mathrm{Gal}(\overline{L}/L) \longrightarrow H \cdot s(\mathrm{Gal}(\overline{K}^+/K^+)) \subset \pi_1(Y^+)$$

lifting s. As H decreases the $s^*(H)$ must converge to s, and thus the Y-projections of the $y^*(H)$ converge to a point $y \in Y(L^\wedge)$ (considering the induced map on Tate-modules, which determines the points as Y is not hyperelliptic). Also y will take values in K^+. Similarly for fixed H we consider the $Y(H)$-projections of all $y^*(H')$ (for $H' \subset H$), which have a limit $y(H)$. These $y(H) \in Y(H)(L^\wedge)$ form a compatible projective system of K^+-points. Considering induced maps on Tate-modules one sees that on $\mathrm{Gal}(\overline{K}^+/K^+)$ they induce the section s (as the geometric fundamental group has trivial center), and are non-constant. Especially Y is equal to the smallest K-scheme containing the point $y(H) \in \mathbb{P}^{g-1}(K^+)$.

By Hodge-Tate theory the map α induces an injective

$$\alpha^* : H^0(Y, \omega_Y) \longrightarrow H^0(X, \omega_X)$$

which maps the linear form given by evaluation on x to a multiple for the corresponding form for y. For any element f of the symmetric algebra of $H^0(Y, \omega_Y)$ which vanishes on Y and thus on $y \in Y(K^+)$, its pullback $\alpha^*(f)$ thus vanishes on x and also X. That is α^* defines a map (first birational, but then regular) of (canonically embedded) curves $X \to Y$.

By the same reasoning we lift to a compatible system of pointed maps from $X^+(H)$ to $Y^+(H)$. On fundamental groups the induced transformation α' : $\pi_1(X^+) \to \pi_1(Y^+)$ thus has the property that it sends $\alpha^{-1}(H)$ into H, for all characteristic open $H \subset \pi_1(Y \otimes_K \overline{K})$. It is thus equal to the composition of α with an endomorphism of $\pi_1(Y)$ which respects all such H, and which induces (Hodge-Tate) the identity on global differentials of $Y(H)$. As it also induces the identity on the covering group (over $\overline{K}^+$) of $Y^+(H) \to Y$ it is the identity on $\pi_1(Y \otimes_K \overline{K})$ and thus on all of $\pi_1(Y)$.

15. *Remark.*— a) The proof needs only the maximal pro-p-quotient of the geometric fundamental group, that is any open homomorphism of the extensions of $\mathrm{Gal}(\overline{K}/K)$ of these groups comes from a dominant morphism. Even quotients by suitable commutator-subgroups suffice (but not in the descending central series).

¿From this one derives again the previous result, by applying it to all coverings like $X(H)$.

b) One can treat open hyperbolic curves: Suppose U is such a curve, with compactification X. Replace the Jacobian J_X by the generalised Jacobian which classifies line-bundles on X with a trivialisation on the divisor at infinity $X - U$, and ω_X by differentials with simple poles along this divisor. Then for two curves U_1, U_2 any open homomorphism (of extensions) between $\pi_1(U)$'s comes first from a map between compactifications X. As etale coverings of the open curve U_2 (allowed to be ramified along the boundary) pull back to such coverings of U_1, this maps U_1 to U_2.

c) For complete curves (but not for affines) one may replace the condition "α is open" by "α has non-zero degree", that is the induced map

$$\alpha^* : H^2(Y \otimes_K \overline{K}, \mathbb{Q}_p) = H^2(\pi_1(Y \otimes_K \overline{K}, \mathbb{Q}_p)) \longrightarrow H^2(X \otimes_K \overline{K}, \mathbb{Q}_p)$$

is an isomorphism. This property is inherited by all finite coverings.

d) The result over K implies it over any subfield. Namely suppose one has found a K-rational point of the Hom-scheme $\mathbf{Hom}(X, Y)$. If the induced map on Tate-modules is defined over this subfield, then so is the map. Especially this applies to number fields.

e) The theorem extends to finitely generated extensions of $\mathbb{Q}_p$.

REFERENCES

[BK] S. BLOCH, K. KATO - *L-functions and Tamagawa numbers of motives*, Grothendieck Festschrift I, 333-400, Birkhäuser, Boston 1990.

[CF] C.L. CHAI, G. FALTINGS - *Degeneration of abelian varieties*, Springer-Verlag, Berlin 1990.

[F] G. FALTINGS - *Hodge-Tate structures and modular forms*, Math. Annalen **278** (1987), 133-149.

[G] A. GROTHENDIECK - Letter to G. Faltings (June 27th, 1983).

[M] S. MOCHIZUKI - *The Local pro-p anabelian geometry of curves*, preprint RIMS, 1996.

[P] F. POP - *On Grothendieck's conjecture of birational anabelian geometry*, Ann. of Math. **138** (1994), 145-182.

[T] A. TAMAGAWA - *The Grothendieck conjecture for affine curves*, Compositio Math. **109** (1997), 135-194.

Gerd FALTINGS

Max-Planck-Institut für Matematik
Gottfried-Claren-Straße 26
D-53225 BONN (Germany)
E-mail : gerd at mpim-bonn.mpg.de