COMPOSITIO MATHEMATICA

Y. GORDON

p-local unconditional structure of Banach spaces

Compositio Mathematica, tome 41, n° 2 (1980), p. 189-205

<http://www.numdam.org/item?id=CM_1980__41_2_189_0>

© Foundation Compositio Mathematica, 1980, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

COMPOSITIO MATHEMATICA, Vol. 41, Fasc. 2, 1980, pag. 189–205
© 1980 Sijthoff & Noordhoff International Publishers – Alphen aan den Rijn
Printed in the Netherlands

p-LOCAL UNCONDITIONAL STRUCTURE OF BANACH SPACES*

Y. Gordon

Abstract

There are Banach spaces which fail to have p-local unconditional structure (p-l.u.st.) for any p, $\infty > p > 0$. In particular, there exist n-dimensional Banach spaces E_n, $n = 1, 2, \ldots$, whose p-l.u.st. constants are "almost" the largest possible theoretical value $\min\{n^{1/2}, n^p\}$. The p-l.u.st. constant is smaller and not equivalent to the usual l.u.st. constant.

1. Introduction

Given any $\infty > p \geq 0$, let η_p be the ideal norm defined in the following manner: If $T \in L(E, F)$ is a bounded operator from a Banach space E to a Banach space F which can be written the form $Tx = \Sigma_{i \geq 1} A_i x$ $(x \in E)$, where A_i $(i = 1, 2, \ldots)$ are in the class $\mathcal{F}(E, F)$ of the finite-rank operators from E to F, then

$$\eta_p(T) = \inf \sup \left\| \sum_{i \leq N} \pm (r(A_i))^p A_i \right\|$$

where $r(A)$ denotes the rank of an operator A, the supremum ranges over all choices of $\pm$ signs and integers N, and the infimum is taken over all the possible representations of the operator T.

$\eta_p(T)$ is a non-decreasing function of p, and η_p is a Banach ideal norm, that is has the following properties:

(1) η_p is a norm and $\eta_p(E, F) = \{T \in L(E, F); \eta_p(T) < \infty\}$ is a Banach space under the norm η_p.

* Supported in part by NSF-MCS 77-04174.

0010–437X/80/05/0189–18$00.20/0

(2) $\eta_p(T) = \|T\|$ whenever $r(T) = 1$.

(3) If $u \in L(G, E)$, $T \in \eta_p(E, F)$, $v \in L(F, H)$, then $vTu \in \eta_p(G, H)$ and $\eta_p(vTu) \leq \|v\|\|u\|\eta_p(T)$.

We recall some well known facts about a general Banach ideal norm α which may be found in [9]. If $T \in L(E, F)$, the adjoint ideal norm $\alpha^*(T)$ is defined as the least C such that the inequality

$$\operatorname{trace}(SvTu) \leq C\|v\|\|u\|\alpha(S)$$

holds for any finite-dimensional normed spaces X and Y, $u \in L(X, E)$, $v \in L(F, Y)$ and $S \in L(Y, X)$. If X and Y are finite-dimensional normed spaces, the dual space $(\alpha(X, Y))'$ can be naturally identified with $\alpha^*(Y, X) = (L(Y, X), \alpha^*)$ via the identity $\langle T, S\rangle = \operatorname{trace}(ST)$ for $T \in \alpha(X, Y)$, $S \in \alpha^*(Y, X)$. Hence, if $T \in L(E, F)$, $\alpha^{**}(T) = \sup \alpha(vTu)$, where the supremum ranges over all finite-dimensional normed spaces X and Y, $u \in L(X, E)$ and $v \in L(F, Y)$ with $\|u\| = \|v\| = 1$. From this we get immediately that $\eta_p^{**}(T) = \eta_p^{**}(T')$ for every operator $T \in L(E, F)$.

If in the definition of $\eta_p(T)$, T is further restricted only to representations for which $r(A_i) = 1$ for all i, then the corresponding resulting norm which is independent of p was called in [8] the weakly nuclear norm of T and denoted by $\eta(T)$. It follows that $\|\cdot\| \leq \eta_p \leq \eta_q \leq \eta$ for $0 \leq p < q < \infty$, and since $\eta_0 = \|\cdot\|$ on finite dimensional spaces, taking double adjoints we obtain $\eta_0^{**} = \|\cdot\|^{**} = \|\cdot\| \leq \eta_p^{**} \leq \eta_q^{**} \leq \eta^{**}$.

Using ultraproducts it can be shown (see for example [16]) that $T \in \eta^{**}(E, F)$ if and only if $j_F T$ factors through some Banach lattice, more precisely, $\eta^{**}(T) = \inf\|v\|\|u\|$, where the infimum ranges over all Banach lattices L and $u \in L(E, L)$, $v \in L(L, F'')$, satisfying $j_F T = vu$, where $j_F : F \to F''$ is the canonical inclusion. Thus, if T is a map on, or, into, a norm one complemented subspace of a Banach lattice, then $\|T\| = \eta_p^{**}(T) = \eta^{**}(T)$. If $T = I_E$ the identity operator on a Banach space E, $\eta^{**}(I_E)$ is generally better known as the local unconditional structure (l.u.st.) constant of E which is usually denoted by $\chi_u(E)$ [7]. If α is an ideal norm, $\alpha(E)$ denotes $\alpha(I_E)$. For $p > 0$, $\eta_p^{**}(E)$ will be called the p-l.u.st. constant of E. $\chi(E)$ will denote the unconditional basis constant of E.

If $\dim(E) = n$ and $0 \leq p < q < \infty$, then trivially we get from the definitions

$$1 \leq \eta_p(E) \leq \eta_q(E) \leq \eta(E) = \chi_u(E) \leq \chi(E) \leq d(E, \ell_2^n) \leq \sqrt{n},$$

and also, since $r(A_i) \leq n$, $\eta_q(E) \leq n^{q-p}\eta_p(E)$. Moreover, the represen-

tation $I_E = I_E$ shows that $\eta_p(E) \leq n^p$, thus $\eta_p(E) \leq \min\{n^{1/2}, n^p\}$ for all $p \geq 0$.

The main result here shows that the last inequality is asymptotically "almost" the best possible. There exists a sequence E_n, $n = 1, 2, \ldots$, of n-dimensional spaces for which $\eta_p(E_n) \geq \min\{an^{1/2}, an^p\} \exp(-\sqrt{\log n})$ where a is an absolute positive constant. Since the exponential factor tends to zero more slowly than any negative power of n, this implies that if $p \neq q$ and $0 < p < \frac{1}{2}$, then η_p and η_q are not equivalent ideal norms, and in particular η_p and η are not equivalent ideal norms. Since $\eta_p(E_n) \to \infty$ as $n \to \infty$, this also implies there exists a reflexive separable Banach space which fails to have p-l.u.st. for all $p > 0$.

Regarding x_u it was proved in [3] that there is an absolute constant $c > 0$ and a sequence of spaces F_n, $\dim(F_n) = n$, such that $x_u(F_n) \geq c\sqrt{n}$. Our result therefore is of interest for the smaller p-l.u.st. constants η_p. We do not know if $\eta_p(E)$ can be asymptotically equivalent to $\min\{n^{1/2}, n^p\}$ for a sequence of spaces E_n, $\dim(E_n) = n$. It is also an open question whether for $q > p \geq \frac{1}{2}$ $n_p(E)$ and $\eta_q(E)$ are always equivalent when $\dim(E) < \infty$; the same question is also open for the constants $x_u(E)$ and $x(E)$. It was proved recently by Johnson, Lindenstrauss and Schechtman, that there exists a Banach spaces E with $x_u(E) = \infty$, that is E does not have local unconditional structure, yet E has an unconditional Schauder decomposition into 2-dimensional spaces. This fact implies that $x_u(E)$ and $\eta_p(E)$ are not equivalent since $\eta_p(E)$ is finite for such spaces. Also unknown is whether many of the spaces which fail l.u.st. also fail p-l.u.st. for some $p > 0$. Does $L_q(\infty > q \geq 1)$ have a subspace without p-l.u.st.? G. Pisier proved that if $p > 2$, L_q has a subspace without l.u.st. (See [15] for $q > 4$; for $2 < q$ we know of an unpublished proof).

To obtain the lower estimates for $\eta_p(E_n)$ we use the characterization of the adjoint norm η_p^* proved in the next section and an inequality due to S. Chevet which was communicated to us by G. Pisier who has used the inequality to prove that l.u.st. constant x_u of $\underbrace{\ell_1^n \hat{\otimes}_\epsilon \ell_1^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_1^n}_{2^k}$ is bigger than $C_\alpha N^\alpha$, where $N = n^{2^k}$ and n, k are chosen in some appropriate relation to N, and where α is any scalar $< \frac{1}{2}$, and $c_\alpha > 0$ is a constant depending only on α.

2. p-local unconditional structure

A characterization of η_p^* is given by the following proposition.

PROPOSITION 1: *If p, C are non-negative constants and $T \in L(E, F)$, then the following statements are equivalent*:

(1) $\eta_p^*(T) \le C$.

(2) $\Sigma_{i=1}^n \operatorname{trace}(TA_i) \le C \max_{\pm} \|\Sigma_{i=1}^n \pm (r(A_i))^p A_i\|$ *for any choice of* $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$.

(3) *If $K_{E'}$ denotes the w*-closure of the extreme points of the unit ball of E' equipped with the w* topology, there exists a probability measure on the compact topological product space $K = K_{E'} \times K_{F''}$ such that for every $A \in \mathcal{F}(F, E)$ holds the inequality*

$$\operatorname{trace}(TA) \le C(r(A))^p \int_K |\langle A'(x'), y''\rangle| \, d\mu(x', y'').$$

PROOF: Let C_i $(i = 1, 2, 3)$ denote a constant C which appears in the inequality of statement (i). Let X, Y be finite-dimensional spaces and $\epsilon > 0$, and let $S = \Sigma_{i=1}^n A_i$ where $A_i \in L(Y, X)$ are chosen to satisfy $(1+\epsilon)\eta_p(S) \ge \max_{\pm} \|\Sigma \pm (r(A_i))^p A_i\|$. Then, for any $u \in L(X, E)$, $v \in L(F, Y)$, we get

$$\begin{aligned}
\operatorname{trace}(SvTu) &= \sum_{i=1}^n \operatorname{trace}(A_i v T u) = \sum_{i=1}^n \operatorname{trace}(u A_i v T) \\
&\le C_2 \max_{\pm} \left\| \sum_{i=1}^n \pm (r(uA_iv))^p u A_i v \right\| \\
&\le C_2 \|u\| \|v\| \max_{\pm} \left\| \sum_{i=1}^n \pm (r(A_i))^p A_i \right\| \\
&\le C_2 \|u\| \|v\| (1+\epsilon) \eta_p(S),
\end{aligned}$$

this implies $\inf C_1 \le C_2(1+\epsilon)$, therefore $\inf C_1 \le \inf C_2$.

Given arbitrary $B_i \in \mathcal{F}(F, E)$, $i = 1, 2, \ldots, n$,

$$\begin{aligned}
\sum_{i=1}^n \operatorname{trace}(TB_i) &\le C_3 \int_K \sum_{i=1}^n (r(B_i))^p |\langle B_i'(x'), y''\rangle| \, d\mu \\
&\le C_3 \sup\left\{ \sum_{i=1}^n (r(B_i))^p |\langle B_i'(x'), y''\rangle| ; \|x'\| = \|y''\| = 1 \right\} \\
&= C_3 \max_{\pm} \left\| \sum_{i=1}^n \pm (r(B_i))^p B_i \right\|,
\end{aligned}$$

hence $\inf C_2 \le C_3$.

If $A \in \mathcal{F}(F, E)$, let $\tilde{A} \in C(K)$ be the function defined by: $\tilde{A}(x', y'') = \langle A'(x'), y'' \rangle (r(A))^p$, and denote by M the convex hull of the set $\{C_2\tilde{A};\ A \in \mathcal{F}(F, E), \operatorname{trace}(TA) = 1\}$. Statement (2) implies that M is disjoint from the set $N = \{f \in C(K),\ f < 1\}$ which is also convex and contains the open unit ball of $C(K)$, therefore there exists a probability measure $\mu \in M(K) = (C(K))'$ such that $\mu(g) \geq 1$ for all $g \in M$, this shows that $\inf C_3 \leq C_2$.

Let now $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$, and consider the space $X = \operatorname{span}\{A_i(y);\ y \in F,\ i = 1, 2, \ldots, n\}$. Let $u: X \to E$ be the inclusion map, and $S = \Sigma_{i=1}^n A_i$ be the map of F into X, S' maps X' to F' and $(S')_a$ will denote the map S' of X' onto $S'(X')$. Let j be the inclusion of $S'(X')$ in F', then $v = j'j_F$ maps F to $Y = (S'(X'))'$. Both X and Y are now finite-dimensional spaces, and

$$\begin{aligned}\sum_{i=1}^{n} \operatorname{trace}(TA_i) &= \operatorname{trace}(vTu(S')_a') \\ &\leq C_1 \|u\| \|v\| \eta_p((S')_a') \\ &\leq C_1 \max_{\pm} \left\| \sum_{i=1}^{n} \pm A_i (r(A_i))^p \right\|\end{aligned}$$

the last inequality is because $\|u\| = \|v\| = 1$ and the fact that if we denote by $\tilde{A}_i$ the operator A_i'' considered as a map of Y to X, then

$(S')_a' = \Sigma_{i=1}^n \tilde{A}_i$ and so

$$\eta_p((S')_a') \leq \max_{\pm} \left\| \sum_{i=1}^{n} \pm (r(\tilde{A}_i))^p \tilde{A}_i \right\| = \max_{\pm} \left\| \sum_{i=1}^{n} \pm (r(A_i))^p A_i \right\|.$$

Therefore, $\inf C_2 \leq C_1$, and the proof is complete. □

We need a preliminary lemma which was used in [6].

LEMMA 2: *If x_i, y_i, $i = 1, 2, \ldots, m$, are arbitrary vectors in ℓ_2^n, then $n \Sigma_{i=1}^m \Sigma_{j=1}^m \langle x_i, y_j \rangle^2 \geq (\Sigma_{i=1}^m \langle x_i, y_i \rangle)^2$.*

PROOF: Without loss of generality we can assume $\{y_i\}_{i=1}^m$ are fixed such that the operator $T = \Sigma_{i=1}^m y_i \otimes y_i$ has rank n. We shall maximize the function $f(x_1, x_2, \ldots, x_m) = \Sigma_{i=1}^m \langle x_i, y_i \rangle$ subject to the constraint $\Sigma_{i=1}^m \Sigma_{j=1}^m \langle x_i, y_j \rangle^2 = 1$. At the maximum point, the function

$$\varphi \equiv \sum_{i=1}^{m} \langle x_i, y_i \rangle - \lambda \left(\sum_{i=1}^{m} \sum_{j=1}^{m} \langle x_i, y_j \rangle^2 - 1 \right)$$

satisfies $\partial\varphi/\partial x_{ik} = 0$, where $x_i = (x_{ik})_{k=1}^n$. This yields $y_i = 2\lambda \sum_{j=1}^m \langle x_i, y_j\rangle y_j = 2\lambda T(x_i)$, hence $f = \sum_{i=1}^m \langle x_i, y_i\rangle = 2\lambda \sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j\rangle^2 = 2\lambda \cdot T = \sum_{i=1}^m y_i \otimes y_i = 2\lambda \sum_{i=1}^m y_i \otimes Tx_i$, therefore $I_{\ell_2^n} = 2\lambda \sum_{i=1}^m x_i \otimes y_i$ and taking trace, $n = 2\lambda \sum_{i=1}^m \langle x_i, y_i\rangle = 4\lambda^2$, so that $2\lambda = \sqrt{n}$. □

Given Banach spaces E and F let $E\hat{\otimes}_\epsilon F$ denote the completion of the tensor product space $E \otimes F$ under the ϵ-norm, that is the ordinary norm induced on it as a subspace of $L(E', F)$. $E \otimes_\pi F$ denotes the completion of $E \otimes F$ under the π-norm, that is, on $E \otimes F$ the norm $|\cdot|_\pi$ is defined as

$$\left|\sum_{i=1}^n x_i \otimes y_i\right|_\pi = \sup\left\{\sum \langle y_i, Bx_i\rangle;\ \|B\| \le 1,\ B \in L(E, F')\right\}.$$

If k is a positive integer, E_ϵ^k will denote the space $\underbrace{E\hat{\otimes}_\epsilon E\hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E}_{k}$, and for $\{x_i\}_{i=1}^k \subset E$, $\vec{x} = \otimes_{i=1}^k x_i = x_1 \otimes x_2 \otimes \cdots \otimes x_k$ will be a k tensor in E_ϵ^k. If $u_i \in L(E, E)$ are isometries on E $(i = 1, 2, \ldots, k)$, let $\vec{u} = \otimes_{i=1}^k u_i = u_1 \otimes u_2 \otimes \cdots \otimes u_k$ be the isometry of E_ϵ^k defined by: $\vec{u}(\vec{x}) = \otimes_{i=1}^k u_i(x_i)$. This definition makes $\vec{u}$ also an isometry on $E_\pi^k = \underbrace{E\hat{\otimes}_\pi \otimes \cdots \hat{\otimes}_\pi E}_{k}$. We shall denote by π_p $(1 \le p < \infty)$ the p-absolutely summing ideal norm [14].

LEMMA 3: *Let E have a normalized symmetric basis $\{e_i\}_{i=1}^n$, and let $T: E \to \ell_2^n$ be the basis to basis map, $T(e_i) = (\underbrace{0, \ldots, 0, 1}_{i}, 0, \ldots, 0)$, $i = 1, 2, \ldots, n$. Let $A = \sum_{i_1=1}^n \ldots \sum_{i_k=1}^n a_{i_1, i_2, \ldots, i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ be any norm-one element in E_ϵ^k, then*

$$\eta_{1/2}^*(E_\epsilon^k) \sum |a_{i_1, i_2, \ldots, i_k}| \le (n\sqrt{2}\pi_1(T))^k.$$

PROOF: By Pietsch [14] there exists a probability measure μ on $K_{E'}$ such that for every $x = \sum_{i=1}^n \xi_i e_i \in E$

$$\|Tx\|_2 = \left(\sum_{i=1}^n \xi_i^2\right)^{1/2} \le (\pi_1(T) \int_{K_{E'}} |\langle x, x'\rangle| d\mu(x').$$

Let $\vec{d\mu} = \underbrace{d\mu \times d\mu \times \cdots \times d\mu}_{k}$ be the product measure on the set of extreme points of the unit ball of $(E_\epsilon^k)' = (E')_\pi^k$,

$$K_{(E')^k_\pi} = \left\{ \vec{x}' = \bigotimes_{i=1}^{k} x'_i;\ x'_i \in K_{E'} \right\}.$$

Let $u = \Sigma_{i=1}^{m} A_i \otimes B_i$ be any rank-m operator in $L(E^k_\epsilon, E^k_\epsilon)$, where $A_i \in (E^k_\epsilon)'$ and $B_i \in E^k_\epsilon$. Suppose A_i and B_i have the representations

$$A_i = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a^{(i)}_{i_1,\ldots,i_k} e'_{i_1} \otimes \cdots \otimes e'_{i_k}, \quad \text{and}$$

$$B_i = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} b^{(i)}_{i_1,\ldots,i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

Let $\epsilon^{(i)} = (\epsilon^{(i)}_j)_{j=1}^{n}$, $\epsilon^{(i)}_j = \pm 1$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$, and $g_{\epsilon^{(i)}}$ be the isometry of E defined by $g_{\epsilon^{(i)}}(e_j) = \epsilon^{(i)}_j e_j$. Let $\pi^{(i)}$ be any permutation of the integers $\{1, 2, \ldots, n\}$, and $g_{\pi^{(i)}}$ be the isometry of E defined by $g_{\pi^{(i)}}(e_j) = e_{\pi^{(i)}(j)}$. Set $g_i = g_{\epsilon^{(i)}} g_{\pi^{(i)}}$, and let $\vec{g}_{\epsilon,\pi} = \bigotimes_{i=1}^{k} g_i$ be the isometry of E^k_ϵ.

Denote by Av_ϵ and Av_π the averages with respect to signs and permutations, that is, if $f(\epsilon^{(1)}, \ldots, \epsilon^{(k)})$ is a real function then

$$Av_\epsilon(f) = 2^{-nk} \sum_{\epsilon} f(\epsilon^{(1)}, \ldots, \epsilon^{(k)})$$

where the sum is taken over all possible distinct elements $(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)})$; and similarly for a function $h(\pi^{(1)}, \ldots, \pi^{(k)})$

$$Av_\pi(h) = (n!)^{-k} \sum_{\pi} h(\pi^{(1)}, \ldots, \pi^{(k)})$$

where the sum ranges over all $(n!)^k$ possible choices of $(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})$.

We shall find a lower bound for the integral

$$I = Av_\epsilon Av_\pi \int_{K_{(E')^k_\pi}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\vec{\mu}(\vec{x}')$$

which will give the claim of the lemma. First observe that

$$\begin{aligned} &|\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| = \\ &= \Bigg| \sum_{\ell_1,\ldots,\ell_k=1}^{n} \langle e_{\ell_1}, x'_1\rangle \ldots \langle e_{\ell_k}, x'_k\rangle \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{n} \langle g_1(e_{i_1}), e'_{j_1}\rangle \\ &\quad \ldots \langle g_k(e_{i_k}), e'_{j_k}\rangle a_{i_1,\ldots,i_k} \sum_{i=1}^{m} a^{(i)}_{j_1}, \ldots, {}_{j_k} b^{(i)}_{\ell_1}, \ldots, {}_{\ell_k} \Bigg|. \end{aligned}$$

Integrating with respect to $d\mu(x_1')$ first we get

$$\int_{K_{E'}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\mu(x_1') \geq (\pi_1(T)^{-1}) \left[\sum_{\ell_1=1}^{n} \left(\sum_1\right)^2\right]^{1/2}$$

where

$$\sum_1 = \sum_{\ell_2,\ldots,\ell_k=1}^{n} \langle e_{\ell_2}, x_2'\rangle \ldots \langle e_{\ell_k}, x_k'\rangle \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{n} \langle g_1(e_{i_1}), e_{j_1}'\rangle$$
$$\ldots \langle g_k(e_{i_k}), e_{j_k}'\rangle a_{i_1,\ldots,i_k} \sum_{i=1}^{m} a_{j_1,\ldots,j_k}^{(i)} b_{\ell_1,\ldots,\ell_k}^{(i)}.$$

Next, integrating with respect to $d\mu(x_2')$ and using the fact, which we shall also use throughout, that

$$\int \left[\sum_{\ell_1=1}^{n} \left(\sum_1\right)^2\right]^{1/2} d\mu(x_2') \geq \left[\sum_{\ell_1=1}^{n} \left(\int \left|\sum_1\right| d\mu(x_2')\right)^2\right]^{1/2},$$

we obtain

$$\int_{K_{E'}} \int_{K_{E'}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\mu(x_1') d\mu(x_2') \geq (\pi_1(T))^{-2} \left[\sum_{\ell_1,\ell_2=1}^{n} \left(\sum_2\right)^2\right]^{1/2}$$

where $\Sigma_2 = \Sigma_{\ell_3,\ldots,\ell_k=1}^{n} \langle e_{\ell_3}, x_3'\rangle \ldots$ (the $\ldots$ represent the same terms which appear in Σ_1).

If we continue to integrate with respect to $d\mu(x_3')$ and so on, finishing with $d\mu(x_k')$ we obtain

$$\int_{K_{(E')_\pi^k}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon,\pi}(A)\rangle| d\vec{\mu}(\vec{x}') \geq (\pi_1(T))^{-k} \left[\sum_{\ell_1,\ldots,\ell_k=1}^{n} \left(\sum_k\right)^2\right]^{1/2}$$

where

$$\sum_k = \sum_{i_1,\ldots,i_k=1}^{n} \sum_{j_1,\ldots,j_k=1}^{n} \langle g_1(e_{i_1}), e_{j_1}'\rangle \ldots \langle g_k(e_{i_k}), e_{j_k}'\rangle a_{i_1,\ldots,i_k}$$
$$\cdot \sum_{i=1}^{m} a_{j_1}^{(i)}, \ldots, {}_{j_k} b_{\ell_1}^{(i)}, \ldots, {}_{\ell_k}.$$

Let $\sigma^{(i)} = \pi^{(i)^{-1}}$, i.e. $\sigma^{(i)}(r) = j$ iff $\pi^{(i)}(j) = r$, then

$$\langle g_1(e_{i_1}), e_{j_1}'\rangle = \langle \epsilon_{j_1}^{(1)} e_{\pi^{(1)}(i_1)}, e_{j_1}'\rangle = \begin{cases} 0; & i \neq \sigma^{(1)}(j_1) \\ \epsilon_{j_1}^{(1)}; & i_1 = \sigma^{(1)}(j_1) \end{cases}.$$

Using Khintchine's inequality [17]

$$A v_{\epsilon^{(1)}} \left| \sum_{i_1 j_1} c_{i_1, j_1} \epsilon_{j_1}^{(1)} \right| \geq 2^{-1/2} \left(\sum_{j_1=1}^{n} \left(\sum_{i_1=1}^{n} c_{i_1, j_1} \right)^2 \right)^{1/2}$$

and averaging over all $\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)}$, we get

$$2^{k/2} A v_{\epsilon} \left| \sum_{k} \right| \geq \left[\sum_{j_1, \ldots, j_k=1}^{n} (a_{\sigma^{(1)}(j_1), \ldots, \sigma^{(k)}(j_k)})^2 \left(\sum_{i=1}^{m} a^{(i)}_{j_1, \ldots, j_k} b^{(i)}_{\ell_1, \ldots, \ell_k} \right)^2 \right]^{1/2}.$$

Hence,

$$\begin{aligned} & A v_{\epsilon} \int_{K_{(E')^k_{\pi}}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon, \pi}(A) \rangle| d\vec{\mu}(\vec{x}') \geq \\ & \geq (\sqrt{2} \pi_1(T))^{-k} \left[\sum_{\ell_1, \ldots, \ell_k=1}^{n} \sum_{j_1, \ldots, j_k=1}^{n} (a_{\sigma^{(1)}(j_1), \ldots, \sigma^{(k)}(j_k)})^2 \right. \\ & \quad \left. \times \left(\sum_{i=1}^{m} a^{(i)}_{j_1, \ldots, j_k} b^{(i)}_{\ell_1, \ldots, \ell_k} \right)^2 \right]^{1/2}. \end{aligned}$$

Now we shall average over all permutations, and use the fact that

$$A v_{\pi} |a_{\sigma^{(1)}(j_1), \ldots, \sigma^{(k)}(j_k)}| = n^{-k} \sum_{i_1, \ldots, i_k}^{n} |a_{i_1, \ldots, i_k}|,$$

this gives the following estimate for I

$$\begin{aligned} & (n \sqrt{2} \pi_1(T))^k I \geq \\ & \geq \sum_{i_1, \ldots, i_k=1}^{n} |a_{i_1, \ldots, i_k}| \left[\sum_{\ell_1, \ldots, \ell_k=1}^{n} \sum_{j_1, \ldots, j_k=1}^{n} \left(\sum_{i=1}^{m} a^{(i)}_{j_1, \ldots, j_k} \right. \right. \\ & \left. \left. \times b^{(i)}_{\ell_1, \ldots, \ell_k} \right)^2 \right]^{1/2} \geq \sum |a_{i_1, \ldots, i_k}| m^{-1/2} |\mathrm{trace}(u)| \end{aligned}$$

the last inequality follows from Lemma 2. The proof is completed by applying Proposition 1 for $p = \frac{1}{2}$ while noting that $\vec{g}_{\epsilon, \pi}(A)$ are norm-one elements of E^k_{ϵ} and the $\vec{x}'$ which appear in I are norm-one elements of the dual space $(E')^k_{\pi}$. □

LEMMA 4: *With the notation of Lemma* 3,

$$(\sqrt{2} \pi_1(T))^k \eta_{1/2}(E^k_{\epsilon}) \geq \sum |a_{i_1, i_2, \ldots, i_k}|.$$

PROOF: This follows immediately from Lemma 3 and the obvious inequality $\alpha(F)\alpha^*(F) \geq \dim(F)$ for any ideal norm α and finite-dimensional space F. □

We shall next use the following inequality due to S. Chevet [1], for the sake of completeness we include the proof. If $\{x_i\}_{i=1}^n \subset E$, we shall denote by $\epsilon_2(\{x_i\}) = \sup\{(\Sigma_{i=1}^n |\langle x_i, x'\rangle|^2)^{1/2};\ x' \in E',\ \|x'\| = 1\}$.

LEMMA 5: *If $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ are elements in Banach spaces E and F respectively, and $g_{i,j}$ $(i, j = 1, 2, \ldots, n)$ is a sequence of equidistributed, independent, orthonormal random Gaussian variables, then*

$$(*) \qquad \frac{\Lambda}{2} \leq \mathbf{E}\left(\left\|\sum_{i,j} g_{i,j} x_i \otimes y_j\right\|_{E \hat{\otimes}_\epsilon F}\right) \leq \sqrt{2}\Lambda$$

where $\Lambda = \epsilon_2(\{y_i\})\mathbf{E}(\|\Sigma_i g_{i,1} x_i\|) + \epsilon_2(\{x_i\})\mathbf{E}(\|\Sigma_j g_{j,1} y_j\|)$.

PROOF: Let $T = \{(\xi, \eta);\ \xi \in E',\ \eta \in F',\ \|\xi\| = \|\eta\| = 1\}$. For each $t = (\xi, \eta) \in T$, define the random variables

$$X_t = \sum_{i,j} g_{i,j}\langle x_i, \xi\rangle\langle y_j, \eta\rangle \quad \text{and}$$

$$Y_t = \alpha \sum_j g_{j,1}\langle y_j, \eta\rangle + \beta \sum_i g_{i,2}\langle x_i, \xi\rangle$$

where $\alpha = \epsilon_2(\{x_i\})$, $\beta = \epsilon_2(\{y_j\})$. It is easy to see that if $s = (\xi_1, \eta_1) \in T$, then

$$\begin{aligned}
\mathbf{E}(|X_t - X_s|^2) &= \sum_{i,j} (\langle x_i, \xi\rangle\langle y_j, \eta\rangle - \langle x_i, \xi_1\rangle\langle y_j, \eta_1\rangle)^2 \\
&= \sum_{i,j} (\langle x_i, \xi - \xi_1\rangle\langle y_j, \eta\rangle + \langle x_i, \xi_1\rangle\langle y_j, \eta - \eta_1\rangle)^2 \\
&\leq 2 \sum_{i,j} (\langle x_i, \xi - \xi_1\rangle\langle y_j, \eta\rangle)^2 + (\langle x_i, \xi_1\rangle\langle y_j, \eta - \eta_1\rangle)^2 \\
&= 2\mathbf{E}(|Y_t - Y_s|^2)
\end{aligned}$$

hence $\mathbf{E}(|X_t - X_s|^2) \leq 2\mathbf{E}(|Y_t - Y_s|^2)$. By a result due to Sudakov ([2], Corollaire 2.1.3) this implies $\mathbf{E}(\vee_T X_t) \leq \sqrt{2}\mathbf{E}(\vee_T Y_t)$, from which the right hand side of (*) follows.

For the other side, pick $\xi_0 \in E'$, $\|\xi_0\| = 1$, such that $\alpha = \epsilon_2(\{x_i\})$, and define the random variables $Z_\eta = \Sigma_{i,j} g_{i,j}\langle x_i, \xi_0\rangle\langle y_j, \eta\rangle$ and $W_\eta = \alpha \Sigma_{j=1}^n g_{j,1}\langle y_j, \eta\rangle$. Then

$$\mathbf{E}(|Z_\eta - Z_{\eta_1}|^2) = \alpha^2 \sum_j \langle y_j, \eta - \eta_1\rangle^2 = \mathbf{E}(|W_\eta - W_{\eta_1}|^2),$$

so again $\mathbf{E}(\sup_{\|\eta\|=1} Z_\eta) = \mathbf{E}(\sup_{\|\eta\|=1} W_\eta)$, but

$$\begin{aligned}\mathbf{E}\left(\left\|\sum_{i,j} g_{i,j} x_i \otimes y_j\right\|_{E\hat{\otimes}_\epsilon F}\right) &\geq \mathbf{E}\left(\left\|\sum_{i,j} g_{i,j}\langle x_i, \xi_0\rangle y_j\right\|\right)\\ &= \mathbf{E}\left(\sup_{\|\eta\|=1} Z_\eta\right) = \mathbf{E}\left(\sup_{\|\eta\|=1} W_\eta\right)\\ &= \alpha\mathbf{E}\left(\left\|\sum_i g_{i,1} y_i\right\|\right), \quad \text{and similarly, } \geq \beta\mathbf{E}\left(\left\|\sum_i g_{i,1} x_i\right\|\right),\end{aligned}$$

hence the left side of (*). □

THEOREM 6: *Let $\{e_i\}_{i=1}^n$ be a symmetric basis for a Banach space E, and let $T: E \to \ell_2^n$ be the natural basis to basis map. Then, if $E_\epsilon^{2^k} = \underbrace{E\hat{\otimes}_\epsilon E\hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E}_{2^k}$*

$$\eta_{1/2}(E_\epsilon^{2^k})(\|T^{-1}\|\pi_1(T))^{2^k} \geq \sqrt{\frac{2}{\pi n}}\, n^{2^k} 2^{-3k/2-2^{k-1}}.$$

PROOF: For each integer $k = 1, 2, \ldots$, let I_k denote a set consisting of n^{2^k} elements, and let $\{b_\nu\}_{\nu\in I_k}$ denote the natural basis of $E_\epsilon^{2^k}$ (each b_ν has the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$, where $1 \leq i_j \leq n$). Consider the random vectors of $E_\epsilon^{2^k} = E_\epsilon^{2^{k-1}} \hat{\otimes}_\epsilon E_\epsilon^{2^{k-1}}$ of the form $A^k = \Sigma_{\alpha,\beta\in I_{k-1}} g_{\alpha,\beta} b_\alpha \otimes b_\beta$. By Lemma 5

$$\mathbf{E}(\|A^k\|_{E_\epsilon^{2^k}}) \leq 2\sqrt{2}\epsilon_2(\{b_\alpha\}_{\alpha\in I_{k-1}})\mathbf{E}(\|A^{k-1}\|_{E_\epsilon^{2^{k-1}}}).$$

Since $\{\bigotimes_{i=1}^{2^{k-1}} x_i';\ x_i' \in K_{E'}\}$ is the set of extreme points of the unit ball of $(E_\epsilon^{2^{k-1}})' = (E')_\pi^{2^{k-1}}$, we have

$$\epsilon_2(\{b_\alpha\}_{\alpha\in I_{k-1}}) = \sup\left\{\left(\sum_{i_1,\ldots,i_{2^{k-1}}=1}^n (\langle e_{i_1}, x_1'\rangle \ldots \langle e_{i_{2^{k-1}}}, x_{2^{k-1}}'\rangle\right)^2;$$

$$x_i' \in K_{E'}\} = \left[\sup\left\{\left(\sum_{i=1}^n \langle e_i, x'\rangle^2\right)^{1/2};\ x' \in K_{E'}\right\}\right]^{2^{k-1}} = \|T^{-1}\|^{2^{k-1}},$$

hence we get the reduction formula

$$\mathbf{E}(\|A^k\|_{E_\epsilon^{2^k}}) \leq 2\sqrt{2}\|T^{-1}\|^{2^{k-1}}\mathbf{E}(\|A^{k-1}\|_{E_\epsilon^{2^{k-1}}}),$$

and so

$$\mathbf{E}(\|A^k\|_{E_\epsilon^{2^k}}) \leq (2\sqrt{2})^k\|T^{-1}\|^{2^{k-1}+2^{k-2}+\cdots+2^0}\mathbf{E}\left(\left\|\sum_{i=1}^n g_i e_i\right\|_E\right)$$

$$= (2\sqrt{2})^k \|T^{-1}\|^{2k-1} \mathbf{E}\left(\left\|\sum_{i=1}^{n} g_i e_i\right\|\right)$$

$$\leq (2\sqrt{2})^k \|T^{-1}\|^{2k} \mathbf{E}\left(\left(\sum g_i^2\right)^{1/2}\right)$$

$$\leq (2\sqrt{2})^k \|T^{-1}\|^{2k} \sqrt{n}.$$

On the other hand, $\mathbf{E}(\Sigma_{\alpha,\beta\in I_{k-1}} |g_{\alpha,\beta}|) = \sqrt{2/\pi}\, n^{2^k}$, hence

$$\sup\left(\sum |a_{i_1,\ldots,i_{2^k}}| \Big/ \left\|\sum a_{i_1,\ldots,i_{2^k}} e_{i_1} \otimes \cdots \otimes e_{i_{2^k}}\right\|_{E_\epsilon^{2^k}}\right)$$

$$\geq \mathbf{E}\left(\sum_{\alpha,\beta\in I_{k-1}} |g_{\alpha,\beta}|\right) \Big/ \mathbf{E}(\|A^k\|)$$

$$\geq \sqrt{\frac{2}{\pi}}\, n^{2^k} / (2/\sqrt{2})^k \sqrt{n} \|T^{-1}\|^{2^k},$$

and the inequality of Lemma 4 establishes the Theorem. □

REMARK: More generally, the same proof can show that if E_i $(i = 1, 2, \ldots, 2^k)$ are n_i-dimensional spaces with symmetric bases, and if $T_i : E_i \to \ell_2^{n_i}$ are the natural basis to basis maps, then for $F = E_1 \hat{\otimes}_\epsilon E_2 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E_{2^k}$,

$$\eta_{1/2}(F) \prod_{j=1}^{2^k} \|T_j^{-1}\| \pi_1(T_j) \geq \sqrt{2/\pi} \prod_{j=1}^{2^k} n_j / 2^{k/2+2^k-1} \left(\sum_{j=1}^{2^k} \sqrt{n_j}\right).$$

It is not essential that 2^k spaces appear in F, however, if the number is not 2^k then the bottom line on the right hand side of the inequality will be different.

EXAMPLE: If $E = \ell_2^n$, then $\|T^{-1}\| = 1$, $\pi_1(T) \leq \sqrt{\pi n/2}$ [5], hence if $E_N = \underbrace{\ell_2^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_2^n}_{2^k}$ where $N = n^{2^k}$,

$$\eta_{1/2}(E_N) \geq \sqrt{2/\pi}\, \sqrt{n^{2^k}} / (\sqrt{\pi})^{2^k} 2^{3k/2} \sqrt{n} \geq a \sqrt{N} e^{-\sqrt{\log N}}$$

where $a > 0$ is constant, and k, n are chosen in an appropriate relation to N. If $0 < p < \frac{1}{2}$, then $\eta_p(E_N) \geq N^{p-1/2} \eta_{1/2}(E_N) \geq a N^p e^{-\sqrt{\log N}} \to \infty$ as $N \to \infty$.

COROLLARY 7: *If E is an n-dimensional normed space such that $d(E, \ell_2^n) < (n/\pi)^{1/2}$, then $\eta_{1/2}(E_\epsilon^k) \to \infty$ as $k \to \infty$.*

PROOF: First observe that if $k > \ell$ then E_ϵ^ℓ is isometric to a norm one-complemented subspace of E_ϵ^k, therefore the ideal property of the norm $\eta_{1/2}$ implies that $\eta_{1/2}(E_\epsilon^k)$, $k = 1, 2, \ldots$, is a nondecreasing sequence. Clearly $d(E_\epsilon^k, (\ell_2^n)_\epsilon^k) \le (d(E, \ell_2^n))^k$, hence using the estimates for $\eta_{1/2}((\ell_2^n)_\epsilon^{2^k})$ we obtain

$$\eta_{1/2}(E_\epsilon^{2^k}) \ge \eta_{1/2}((\ell_2^n)_\epsilon^{2^k})/(d(E, \ell_2^n))^{2^k} \ge \sqrt{2/\pi}\left(\frac{\sqrt{n/\pi}}{d(E, \ell_2^n)}\right)^{2^k}/2^{3k/2}\sqrt{n}$$

which tends to ∞ with k.

REMARKS: (1) It may be true that $\eta_{1/2}(E_\epsilon^k) \to \infty$ as $k \to \infty$ whenever $d(E, \ell_2^n) < \sqrt{n}$. It is obviously false if $d(E, \ell_2^n) = \sqrt{n}$, as in the case $E = \ell_\infty^n$.

(2) If $1 < q < \infty$, and $c > |\frac{1}{2} - 1/q|$, there exists N such that if $n \ge N$ and E is any m-dimensional subspace of $L_q(\mu)$, then $\eta_c(E_\epsilon^k) \to \infty$ as $k \to \infty$. The reason for this is that $d(E, \ell_2^n) \le n^{|1/2 - 1/q|}$ by [13], so if $c \le \frac{1}{2}$

$$\eta_c(E_\epsilon^{2^k}) \ge {}^{(c-1/2)2^k}\eta_{1/2}(E_\epsilon^{2^k})$$

which tends to ∞ by applying the inequality in the proof of Corollary 7.

THEOREM 8: *There exists a reflexive separable Banach space E with both $\eta_p(E)$ and $\eta_p^{**}(E)$ infinite for all values of $p > 0$.*

PROOF: Let E_N be the space in the Example, and let $E = (\Sigma_{i=1}^\infty \oplus E_{N_i})_{\ell_2}$ where $N_i \to \infty$, $N_i = (n_i)^{2^{k_i}}$, which the proper relation maintained between n_i and k_i with N_i. Since E_{N_i} is norm one complemented in E, it follows that

$$\eta_p(E) \ge \eta_p^{**}(E) \ge \eta_p^{**}(E_{N_i}) = \eta_p(E_{N_i}) \ge aN_i^p \exp(-\sqrt{\log N_i}) \xrightarrow[i\to\infty]{} \infty.$$

□

In order to estimate $\pi_1(T)$ for general spaces it may be useful to apply the following proposition.

PROPOSITION 9: *For any $0 < p, q, r < \infty$ there exist constants $a_{r,q}$, $b_{p,q} > 0$ such that for any Banach space E and any operator $T: E \to \ell_2^n$*

(1) $b_{p,q}^{-1}\pi_p(T) \le \sqrt{n}(\int_{S_n} \|T'x\|^q dm(x))^{1/q} \le a_{r,q}^{-1}\pi_r(T')$, *where* $S_n = \{x \in \ell_2^n;\ \|x\|_2 = 1\}$ *and $dm(x)$ is the rotation invariant normalized measure on S_n.*

(2) *If E' is a subspace of an L_s-space, $1 \le s < \infty$, and if $0 < p \le s \le r < \infty$ and $0 < q < \infty$, the inequalities of* (1) *becomes equivalence relations and the constants of equivalence are independent of n, T and E.*

(3) *If $\dim(E) = n$ and E has a symmetric basis and $T : E \to \ell_2^n$ is the basis to basis map, then all the values $I_q = (\int_{S_n} \|T'x\|^q dm(x))^{1/q}$ $(0 < q < \infty)$ are equivalent and the constants of equivalence are independent of n and E.*

PROOF: (1) By [14] there exists a probability measure μ on S_n such that for all $x \in \ell_2^n$

$$\|T'x\| \le \pi_q(T') \left(\int_{S_n} |\langle x, x' \rangle|^q d\mu(x') \right)^{1/q}$$

hence by integrating with respect to dm

$$I_q \le \pi_q(T') \left(\int_{S_n} \int_{S_n} |\langle x, x' \rangle|^q d\mu(x') dm(x) \right)^{1/q} = \pi_q(T') / \pi_q(\ell_2^n)$$

since $(\int_{S_n} |\langle x, x' \rangle|^q dm(x))^{1/q} = \|x'\|_2 (\pi_q(\ell_2^n))^{-1}$ [5]. Since $\pi_q(\ell_2^n) \sim \sqrt{n}$, and $\pi_q(T')$ is a non-increasing function of q, and I_q is a non-decreasing function of q, the right hand side of (1) readily follows.

Without loss of generality we may assume that T' is a $1-1$ map, and define the probability measure ν on the unit ball $B_{E'}$ of E' by

$$\int_{B_{E'}} f d\nu = \int_{S_n} f(T'x/\|T'x\|) \|T'x\|^q dm(x) \Big/ \int_{S_n} \|T'x\|^q dm(x)$$

for $f \in C(B_{E'})$. Taking $f = |\langle \xi, \cdot \rangle|^q$ where $\xi \in E$, we obtain

$$\begin{aligned} \int_{B_{E'}} |\langle \xi, x' \rangle|^q d\nu(x') &= \int_{S_n} |\langle T\xi, x \rangle|^q dm(x) \Big/ \int_{S_n} \|T'x\|^q dm(x) \\ &= \|T\xi\|^q \Big/ (\pi_q(\ell_2^n))^q \int_{S_n} \|T'x\|^q dm(x), \end{aligned}$$

therefore, $\pi_q(T) \le \pi_q(\ell_2^n)(\int_{S_n} \|T'x\|^q dm(x))^{1/q}$, and as above this proves the left hand side of (1).

(2) Let $j : E' \to L_s$ be an isometric embedding, then by [12] $\pi_s(jT') \le \pi_s((jT')')$, that is

$$\pi_s(T') = \pi_s(jT') \le \pi_s(T''j') \le \pi_s(T'') = \pi_s(T),$$

and (2) follows from the inequalities

$$\pi_s(T') \le \pi_s(T) \le \pi_p(T) \le b_{p,q}\sqrt{n} I_q \le b_{p,q} a_{r,q}^{-1} \pi_r(T') \le b_{p,q} a_{r,q}^{-1} \pi_s(T').$$

(3) Set $|x|_{E'} = \|T'x\|$ for vectors $x \in \ell_2^n = (R^n, \|\cdot\|_2)$. Since E is symmetric, by [10] $\|T\|\|T^{-1}\| = d(E, \ell_2^n)$, and $d(E, \ell_2^n) \le \sqrt{n}$, so $a\|x\|_2 \le |x|_{E'} \le b\|x\|_2$ for all $x \in \ell_2^n$, where $b/a \le \sqrt{n}$. From the remark following Lemma 2.7 in [4], the values $I_q = (\int_{S_n} |x|_{E'}^q dm(x))^{1/q}$ $(0 < q < \infty)$ are all equivalent to the Levy mean M^*, which is by definition the unique number such that $m(\{x \in S_n;\ |x|_{E'} \le M^*\}) \le \frac{1}{2}$ and $m(\{x \in S_n;\ |x|_{E'} \ge M^*\}) \le \frac{1}{2}$, that is, there exist absolute positive constants a_q, b_q such that $a_q M^* \le I_q \le b_q M^*$. □

COROLLARY 10: *If* $\dim(E) = n$, *there are absolute constants* $a, b > 0$ *such that for any* $T: E \to \ell_2^n$

$$a\pi_1(T) \le \sqrt{n} \int_{S_n} \|T'x\| dm(x) \le b\pi_1(T)\sqrt{\ln n}\, x_u(E).$$

PROOF: Let γ_p denote the best factorization through an L_p-space norm [9]. Interpolation technique as in Theorem 7 [11] shows that $\pi_p(T') \le n^{1/p}\gamma_\infty(T')$. Since $\gamma_\infty(T') = \gamma_1(T) \le x_u(E)\pi_1(T)$ [7], and $\pi_p(\ell_2^n) \ge c\sqrt{n/p}$ [5], we obtain

$$\int_{S_n} \|T'x\| dm(x) \le \left(\int_{S_n} \|T'x\|^p dm(x)\right)^{1/p} \le \pi_p(T')/\pi_p(\ell_2^n) \le \sqrt{p/n}\, c^{-1} n^{1/p} \pi_1(T) x_u(E)$$

and the estimate follows by taking $p = \ln n$, and from Proposition 9. □

COROLLARY 11: *Let* $E_N = \underbrace{\ell_p^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_p^n}_{2^k}$, $N = n^{2^k}$, *where* $1 \le p \le 2$. *There exists* $b > 0$, *such that* $\eta_{1/2}(E_N) \ge b\sqrt{N} e^{-\sqrt{\log N}}$ *for the proper relation between* n, k *with* N.

PROOF: Factor $T: \ell_p^n \xrightarrow{A} \ell_1^n \xrightarrow{B} \ell_2^n$, where A, B are the inclusions. Then $\|T^{-1}\| = n^{1/p - 1/2}$, and $\pi_1(T) \le \|A\|\pi_1(B) = \sqrt{2} n^{1-1/p}$, since $\pi_1(B) = \sqrt{2}$ is the Khintchine constant. Therefore, $\|T^{-1}\|\pi_1(T) \le \sqrt{2n}$. The proof is concluded as in the Example following Theorem 6. □

REMARK: Since the distance between E_ϵ^k and $(\ell_p^n)_\epsilon^k$ is $\leq (d(E, \ell_p^n))^k$, it follows from the estimates of Corollary 10, Theorem 6 and the inequality

$$\eta_{1/2}(E_\epsilon^k) \geq \eta_{1/2}((\ell_p^n)_\epsilon^k)(d(E, \ell_p^n)^{-k},$$

that if E is any n-dimensional Banach space such that $\inf_{1\leq p\leq 2} d(E, \ell_p^n) < \sqrt{n}/2$, then $\eta_{1/2}(E_\epsilon^{2^k}) \to \infty$ as $k \to \infty$. □

Denote by $r_i(t)$, the i-th Rademacher function on $[0, 1]$.

COROLLARY 12: *Let $1 < p \leq 2 \leq q < \infty$, $1/p < 1/q + \frac{1}{2}$. Assume F is a Banach space of type p and cotype q. Then, for any c satisfying $c > 1/p - 1/q$, there exists an integer N such that if $n > N$ and if E is any n-dimensional symmetric subspace of F, then $\eta_c(E_\epsilon^k) \to \infty$ as $k \to \infty$.*

PROOF: Let α be the type-p constant and β be the cotype-q constant of F respectively. Let $\{e_i\}_{i=1}^n$ denote the symmetric basis of a subspace $E \subset F$, and $\{e_i'\}_1^n$ be the biorthogonal functionals. The inequality

$$\left(\sum_i^n |\xi_i|^q\right)^{1/q} \leq \beta \int_0^1 \left\|\sum_{i=1}^n \xi_i r_i(t) e_i\right\| dt = \beta \left\|\sum_{i=1}^n \xi_i e_i\right\|$$

implies that $\|\Sigma \xi_i e_i'\| \leq \beta \|\xi\|_{q'}$. By Proposition 9

$$b_{1,q'}^{-1} n^{-1/2} \pi_1(T) \leq \left(\int_{S_n} \|T'x\|^{q'} dm(x)\right)^{1/q'} \leq \beta \left(\int_{S_n} \|\xi\|_{q'}^{q'} dm(\xi)\right)^{1/q'}$$
$$= \beta n^{1/q'} (\pi_{q'}(\ell_2^n))^{-1},$$

hence $\pi_1(T) \leq \beta c_q n^{1/q'}$, where $c_q > 0$ is constant. Select scalars $\{x_i\}_{i=1}^n$ so that $\Sigma x_i^2 = 1$ and $\|T^{-1}\| = \|\Sigma x_i e_i\|$. Then,

$$\|T^{-1}\| = \int_0^1 \left\|\sum x_i r_i(t) e_i\right\| dt \leq \alpha \|x\|_p \leq \alpha n^{1/p - 1/2},$$

therefore $\|T^{-1}\| \pi_1(T) \leq \alpha \beta c_q n^{1/p - 1/q + 1/2}$. Combining this with the inequality $\eta_c(E_\epsilon^{2^k}) \geq \eta_{1/2}(E_\epsilon^{2^k})(n^{2^k})^{c-1/2}$, and Theorem 6, establishes that for $c > 1/p - 1/q$ and n sufficiently large, $\eta_{1/2}(E_\epsilon^{2^k})(n^{2^k})^{c-1/2} \to \infty$ as $k \to \infty$. □

REFERENCES

[1] S. CHEVET: Series de variables aleatoires Gaussiens a valeurs dans $E\hat{\otimes}_\epsilon F$. Applications aux produits d'espaces de Wiener abstraits. Seminaire Maurey-Schwartz (1977–78) expose XIX.

[2] X.M. FERNIQUE: Regularite des trajectoires des fonctions aleatoires Gaussiens, Lecture Notes in Math. 480 (1975), Springer Verlag, 1–96.

[3] T. Figiel, S. KWAPIEN and A. PELCYNSKI: Sharp estimates for the constant of local unconditional structure of Minkowsky spaces, *Bull. Acad. Polon. Sci. 25* (1977) 1221–1226.

[4] T. FIGIEL, J. LINDENSTRAUSS and V.D. MILMAN: The dimension of almost spherical sections of convex bodies. *Acta. Math. 139* (1977) 53–94.

[5] Y. GORDON: On p-absolutely summing constants of Banach spaces, *Israel J. Math. 7* (1969) 151–163.

[6] Y. GORDON: Unconditional Schauder decomposition of normed ideals of operators between some ℓ_p spaces, *Pacific J. Math. 60* (1975), 71–82.

[7] Y. GORDON and D.R. LEWIS: Absolutely summing operators and local unconditional structures, *Acta Math. 133* (1974) 27–48.

[8] Y. GORDON and D.R. LEWIS: Banach ideals on Hilbert spaces, *Studia Math. 54* (1975) 161–172.

[9] Y. GORDON, D.R. LEWIS and J.R. RETHERFORD,, Banach ideals of operators with applications, *J. Funct. Anal. 14* (1973) 85–129.

[10] W.B. JOHNSON, B. MAUREY, G. SCHECHTMAN and L. TZAFRIRI: Symmetric structures in Banach spaces, *Memoirs Amer. Math. Soc.*, 1979, Vol. 19, Number 217.

[11] J.L. KRIVINE: Theorems des factorisation dans les espaces reticules, Seminaire Maurey-Schwartz, Expose XXII et XXIII, 1974.

[12] S. KWAPIEN: On a theorem of L. Schwartz and its applications to analysis, *Studia Math. 38* (1969) 193–201.

[13] D.R. LEWIS: Finite-dimensional subspaces of L_p, *Studia Math. 63* (1978) 207–212.

[14] A. PIETSCH: Absolut p-summierende Abbildungen in normierten Räumen, *Studia Math. 28* (1967) 333–353.

[15] G. PISIER: Some results on Banach spaces without local unconditional structure, Seminaire Maurey-Schwartz.

[16] S. RISNER: The G.L. property (Technion preprint, 1977).

[17] S.J. SZAREK: On the best constants in the Khintchine inequality, *Studia Math. 58* (1976) 197–208.

(Oblatum 27-VII-1978 & 19-III-1979)

The Ohio State University
Columbus, Ohio 43210
and
The Technion-Israel Institute
of Technology
Haifa, Israel