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Existence and Non-existence Results for Harmonic Maps between Spheres

VALTER PETTINATI - ANDREA RATTO

1. - Introduction

This paper deals with harmonic maps with prescribed symmetry; namely, the study of the existence of *equivariant* harmonic joins of two harmonic homogeneous polynomial maps of spheres.

This problem was first studied by Smith [14], who obtained existence results under the assumption of damping conditions; here we establish less restrictive damping conditions which are necessary and sufficient for the existence of equivariant harmonic joins.

As an application, we obtain new harmonic maps between spheres, as described in Section 4.

According to [14], the problem reduces to the qualitative study of solutions of an ordinary differential equation (equation (2.3) below); for this purpose we use comparison methods which can be successfully applied to the case of the Hopf construction [13]. Part of the results of this paper were announced in [10].

A new proof of our Main Theorem below has been given by Ding [4] using very different techniques: he uses a direct method of Calculus of variations which clarifies the variational aspects of equivariant theory.

2. - Notations and Main Theorem

Let $f_1 : S^p \longrightarrow S^r$ and $f_2 : S^q \longrightarrow S^s$ be harmonic homogeneous polynomial maps of degree k_1 and k_2 respectively.

Then f_1 and f_2 are harmonic maps with constant energy densities

$$e(f_1) = \frac{\lambda_1}{2}, \qquad e(f_2) = \frac{\lambda_2}{2},$$

where $\lambda_1 = k_1 \cdot (k_1 + p - 1)$ and $\lambda_2 = k_2 \cdot (k_2 + q - 1)$.

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Examples of such harmonic homogeneous polynomial maps are widely illustrated in [6] and [14].

We will study the (non-reduced) join of f_1 and f_2, denoted by $f_1 * f_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$.

In order to describe the join $f_1 * f_2$, we write every point $z \in S^{p+q+1} \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ in the form

$$z = \sin s \cdot x + \cos s \cdot y,$$

with $x \in S^p, y \in S^q$ and $s \in [0, \pi/2]$; we parametrize the range S^{r+s+1} similarly, so that the join map can be defined by

$$f_1 * f_2(z) = \sin B(s) \cdot f_1(x) + \cos B(s) \cdot f_2(y) \tag{2.1}$$

where $B : [0, \pi/2] \longrightarrow [0, \pi/2]$ is any differentiable function such that

$$\begin{aligned} &B(0) = 0, B\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \text{ and} \\ &B(s) \in \left(0, \frac{\pi}{2}\right) \text{ for all } s \in \left(0, \frac{\pi}{2}\right). \end{aligned} \tag{2.2}$$

Following Smith [14], the join map in (2.1) is harmonic if and only if the function $B(s)$ satisfies a second order ordinary differential equation; after the substitution $A(t) = B(\tan^{-1} e^t)$, $t \in \mathbb{R}$, this differential equation takes the form of a pendulum equation with variable gravity and damping

$$\begin{aligned} A''(t) + &\left[\frac{(p-1)e^{-t} - (q-1)e^t}{e^t + e^{-t}}\right] A'(t) \\ &+ \left[\frac{\lambda_2 e^t - \lambda_1 e^{-t}}{e^t + e^{-t}}\right] \sin A(t) \cos A(t) = 0 \end{aligned} \tag{2.3}$$

with boundary conditions (2.2) replaced by

$$\begin{aligned} &\lim_{t \to -\infty} A(t) = 0, \lim_{t \to +\infty} A(t) = \frac{\pi}{2} \text{ and} \\ &A(t) \in \left(0, \frac{\pi}{2}\right) \text{ for every } t \in \mathbb{R}. \end{aligned} \tag{2.4}$$

We will say that a solution $A(t)$ of (2.3) satisfying (2.4) is (or provides) an *equivariant harmonic join of f_1 and f_2*. We establish necessary and sufficient conditions for the existence of harmonic maps of the special form (2.1): namely,

MAIN THEOREM. *Let $f_1 : S^p \longrightarrow S^r$ and $f_2 : S^q \longrightarrow S^s$ be harmonic homogeneous polynomial maps of degree k_1 and k_2 respectively; and let*

$$\lambda_1 = k_1 \cdot (k_1 + p - 1), \qquad \lambda_2 = k_2 \cdot (k_2 + q - 1).$$

Then there exists an equivariant harmonic join $f_1 * f_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$ *if and only if the following generalized damping conditions* (G.D.C.) *hold:*

$$\text{G.D.C.} \begin{cases} \text{a)} & (q-1)^2 < 4\lambda_2 \quad \textit{or} \quad 2k_1 < (q-1) - \sqrt{(q-1)^2 - 4\lambda_2} \\ \textit{and} & \\ \text{b)} & (p-1)^2 < 4\lambda_1 \quad \textit{or} \quad 2k_2 < (p-1) - \sqrt{(p-1)^2 - 4\lambda_1}. \end{cases}$$

REMARK. In these notations Smith [14] proves the existence of an equivariant harmonic join provided that the following more restrictive damping conditions hold:

$$\text{D.C.} \begin{cases} \text{a)} & (q-1)^2 < 4\lambda_2 \\ \text{and} & \\ \text{b)} & (p-1)^2 < 4\lambda_1 \end{cases}$$

or $p = q$, $\lambda_1 = \lambda_2$.

It is worth noticing that D.C. a) depends upon f_2 only, but G.D.C. a) depends upon both f_1 and f_2; and similarly for D.C. b) and G.D.C. b).

3. - Proof of the Main Theorem

We have to prove that the assumptions G.D.C. are necessary and sufficient for the existence of a solution $\overline{A}(t)$ of (2.3) which satisfies (2.4).

First we prove the necessity: let us suppose that there exists a solution $\overline{A}(t)$ of (2.3) as in (2.4); we assume that G.D.C. a) does not hold, i.e.

$$\begin{cases} (q-1)^2 \geq 4\lambda_2 \\ 2k_1 \geq (q-1) - \sqrt{(q-1)^2 - 4\lambda_2} \end{cases} \tag{3.1}$$

and show that this leads to a contradiction.

We define

$$H(t) = \frac{[\tan \overline{A}(t)]'}{\tan \overline{A}(t)}. \tag{3.2}$$

By the definition of $H(t)$ there exist two constants $\bar{t}, c \in \mathbb{R}$ such that

$$\overline{A}(t) = \tan^{-1}\left[\exp\left(\int_{\bar{t}}^{t} H(s)\,ds + c\right)\right]. \tag{3.3}$$

In order to simplify the notations, we introduce functions $D(t)$, $G(t)$, $f(t)$ as follows:

$$D(t) = \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right], \quad G(t) = \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] \tag{3.4}$$

$$f(t) = \exp\left(\int_{\bar{t}}^{t} H(s)\,ds + c\right). \tag{3.5}$$

The direct substitution of the expression (3.3) for $\overline{A}(t)$ into equation (2.3) yields

$$f''(t) - \left[\frac{2f(t)f'^2(t)}{1+f^2(t)}\right] + D(t)f'(t) + G(t)f(t) = 0. \tag{3.6}$$

The function $f(t)$ being positive, equation (3.6) implies

$$H'(t) + H^2(t) + D(t)\cdot H(t) + G(t) \geq 0, \text{ for all } t \in \mathbb{R}. \tag{3.7}$$

We need the following fact:

$$\lim_{t\to-\infty} H(t) = k_1, \qquad \lim_{t\to+\infty} H(t) = k_2. \tag{3.8}$$

The proof of statement (3.8) follows easily from the expression of Smith's estimates (Lemmas 6.1, 6.2 of [14]) in terms of the function $H(t)$.

Now we show that (3.7) and (3.8) are not compatible.

Let $V_t(x)$ be the quadratic form

$$V_t(x) = x^2 + D(t)x + G(t). \tag{3.9}$$

We denote by $[a_t, b_t]$ the (possibly empty) interval where $V_t(x) \leq 0$. An elementary computation tells us that, under the assumptions (3.1), $k_2 < k_1$.

A long but straightforward analysis ([12] pp. 15-25) shows that statement (3.8), and the fact that $k_2 < k_1$, force the existence of a point $\bar{t} \in \mathbb{R}$ such that $\left[a_{\bar{t}}, b_{\bar{t}}\right]$ is not empty and

$$H'(\bar{t}) < 0, \qquad H(\bar{t}) \in \left[a_{\bar{t}}, b_{\bar{t}}\right]. \tag{3.10}$$

But (3.10) contradicts (3.7); that proves the necessity of assumption G.D.C. a). Similarly, one checks that G.D.C. b) is also necessary.

Now we prove that assumptions G.D.C. are sufficient for the existence of an equivariant harmonic join; for this purpose, we apply a refined version of the comparison argument introduced in [10].

More precisely, we use the following global comparison Lemma.

LEMMA 1. *Let $D(t)$, $G(t)$ be the damping and gravity functions introduced in* (3.4).

Suppose that there exist two differentiable functions $G_i(t)$, $i = 1, 2$, such that

a) $$G_1(t) < G(t) < G_2(t) \quad \textit{for every } t \in \mathbb{R}.$$

b) *The two differential equations*

$$A''(t) + D(t)A'(t) + G_i(t) \sin A(t) \cos A(t) = 0, \quad i = 1, 2,$$

admit a solution which satisfies (2.4).

Then equations (2.3) *admits a solution* $\overline{A}(t)$ *as in* (2.4).

The proof of Lemma 1 is based on standard arguments but it is rather lengthy and therefore omitted; it can be accomplished by introducing functions $\dot{\alpha}^+$, $\dot{\alpha}^-$ as in [10], [15], [16] and applying a standard comparison Theorem ([3] p. 210), as indicated in [10] and [12] pp. 6-10.

It is clear from Lemma 1 that the Main Theorem follows from the following two assertions:

i) If G.D.C. a) holds, then there exists a function $G_1(t)$ as in Lemma 1;

ii) If G.D.C. b) holds, then there exists a function $G_2(t)$ as in Lemma 1.

We only occupy ourselves with i), because the proof of ii) is similar.

It is easy to see that the existence of a function $G_1(t)$ as above is equivalent to the existence of a differentiable function $\overline{A} : \mathbb{R} \longrightarrow \left(0, \frac{\pi}{2}\right)$ such that

$$\lim_{t \to +\infty} \overline{A}(t) = \frac{\pi}{2}, \qquad \lim_{t \to -\infty} \overline{A}(t) = 0 \tag{3.11}$$

and

$$\overline{A}''(t) + D(t) \cdot \overline{A}'(t) + G(t) \sin \overline{A} \cos \overline{A}(t) > 0. \tag{3.12}$$

Let $H(t)$, $f(t)$ be functions associated to $\overline{A}(t)$ as in (3.2) and (3.5) respectively.

Similarly to (3.6), we have that (3.12) is equivalent to

$$f''(t) - \left[\frac{2f(t)f'^2(t)}{1 + f^2(t)}\right] + D(t) \cdot f'(t) + G(t) \cdot f(t) > 0. \tag{3.13}$$

And conditions (3.11) become

$$\lim_{t \to +\infty} f(t) = +\infty, \qquad \lim_{t \to -\infty} f(t) = 0, \tag{3.14}$$

Summarizing, the proof of our Main Theorem is reduced to check that, under the assumption G.D.C. a), there exists a differentiable function $f : \mathbb{R} \longrightarrow (0, +\infty)$ which satisfies (3.13) and (3.14).

We prove the existence of such a function in two Steps.

STEP 1. *Suppose that there exists a differentiable function* $\overline{f} : \mathbb{R} \longrightarrow (0, +\infty)$ *with the following properties:*

a) $\overline{f}(t)$ *has limits as in* (3.14);

b) $\overline{f}(t)$ *satisfies inequality* (3.13) *on* $(-\infty, t_0] \cup [t_1, +\infty)$ *for some* $t_0, t_1 \in \mathbb{R}$;

c) $\overline{f}''(t) + D(t) \cdot \overline{f}'(t) + G(t) \cdot \overline{f}(t) > 0$ *for all* $t \in \mathbb{R}$.

Then there exists the required function $f(t)$.

STEP 2. *Assume that* G.D.C. a) *holds. Then there exists a function* $\overline{f}(t)$ *as in Step 1.*

PROOF OF STEP 1. We take $f(t) = m\overline{f}(t)$, where $m \in (0, 1)$ is small enough to have

$$\overline{f}(t) - \left[\frac{2m^2\overline{f}(t)\overline{f}'^2(t)}{1 + m^2\overline{f}^2(t)}\right] + D(t) \cdot \overline{f}'(t) + G(t) \cdot \overline{f}(t) > 0, \qquad \text{for all } t \in [t_0, t_1]. \tag{3.15}$$

Notice that hypothesis c) ensures the existence of m as above. Now we observe that

$$\frac{m^2\overline{f}(t)\overline{f}'^2(t)}{\left[1 + m^2\overline{f}^2(t)\right]} < \frac{\overline{f}(t)\overline{f}'^2(t)}{\left[1 + \overline{f}^2(t)\right]}.$$

From this last fact and hypothesis b), we can conclude that inequality (3.15) holds for every $t \in \mathbb{R}$; thus, by construction, $f(t)$ has the required properties.

PROOF OF STEP 2. We assume G.D.C. a) and proceed to the explicit construction of $\overline{f}(t)$.

If $(q-1)^2 < 4\lambda_2$ the conclusion is well-known [14] and the construction of $\overline{f}(t)$ is elementary; so we assume $(q-1)^2 \geq 4\lambda_2$. The case $k_1 < k_2$ is elementary: in fact, it is easy to check that $\overline{f}(t) = e^{ht}$, with $h \in (k_1, k_2)$, satisfies properties a), b) and c) of Step 1.

The case $k_2 \geq k_1$ is more delicate: let

$$\Delta(t) \stackrel{\text{def}}{=} D^2(t) - 4G(t).$$

We define

$$T = \inf_{t \in \mathbb{R}} \{t : \Delta(t) = 0\}. \tag{3.16}$$

By using G.D.C. a), one shows that $T \in \mathbb{R}$.

Let h_1, h_2, δ be positive constants such that

$$\begin{cases} h_2 < k_2 \\ k_1 < h_1 < \frac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right] \end{cases} \tag{3.17}$$

and

$$\left[h_1^2 - (q-1)h_1 + \lambda_2\right] e^t + \left[h_1^2 + (p-1)h_1 - \lambda_1\right] e^{-t} > 0 \tag{3.18}$$

for every $t \in (-\infty, T+\delta]$.

A calculation shows that (3.17) and (3.18) are compatible, provided that δ is small and h_1 is close enough to

$$\frac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right].$$

Finally, we introduce a positive number ε defined by

$$\varepsilon \stackrel{\text{def}}{=} \inf_{\substack{t \geq T+\delta \\ x \in [h_2, h_1]}} \{V_t(x)\} \tag{3.19}$$

where $V_t(x)$ is the quadratic form (3.9).

By using G.D.C. a), one checks that $\varepsilon > 0$.

Now we are in the right position to define the function $\overline{f}(t)$; in fact, let $\overline{f}(t) = e^{H(t)}$, where $H(t)$ is any differentiable function with the following properties:

i)	$H(t) = h_1$	for all $t \in (-\infty, T+\delta]$;
ii)	$H(t) = h_2$	for all $t \in [\tilde{T}, +\infty)$, for some $\tilde{T} \in \mathbb{R}$;
iii)	$\lvert H'(t)\rvert < \varepsilon$	for all $t \in \mathbb{R}$;
iv)	$H(t) \in [h_2, h_1]$	for all $t \in \mathbb{R}$.

Then it is easy to check that $\overline{f}(t)$ satisfies a) and b) of Step 1; and a straightforward analysis ([12] pp. 28-31) proves that also condition c) is fulfilled, so ending the proof of the Main Theorem.

REMARKS. i) More generally, the form of the gravity $G(t)$ makes it reasonable to ask whether equation (2.3) admits special solutions $A(t)$ such that

$$\lim_{t \to -\infty} A(t) = 0, \qquad \lim_{t \to +\infty} A(t) = \frac{\pi}{2} + n\pi \tag{3.20}$$

for some $n \in \mathbb{N}$.

In fact, any such solution could be used to define an equivariant harmonic map of spheres.

However, a slight modification of our arguments proves that, if G.D.C. do not hold, then equation (2.3) does not have any solution as in (3.20).

ii) The discussion of regularity across the focal varieties of S^{p+q+1} has been omitted because one can repeat exactly the arguments of [14]; we merely limit ourselves to pointing out that Smith's treatment of regularity can be shorthened

by showing that our maps are globally weakly harmonic [8], continuous and belong to $L_1^2\left(S^{p+q+1},\ S^{r+s+1}\right)$: then they are smooth according to a regularity theorem of Hildebrandt ([6] p. 10).

4. - Applications of the Main Theorem

In this section we point out some consequences of the Main Theorem and illustrate some new examples of equivariant harmonic maps between spheres.

A good reference for homotopy theory is the book of Toda [17].

We are going to use the following examples of harmonic homogeneous polynomial maps of spheres (see Section 8 of [6]):

the identity map $\mathrm{id}^q : S^q \longrightarrow S^q$;

the k-fold rotation $i_k : S^1 \longrightarrow S^1$;

the Hopf fibrations $h_1 : S^3 \longrightarrow S^2$, $h_2 : S^7 \longrightarrow S^4$, $h_3 : S^{15} \longrightarrow S^8$;

maps $m_1 : S^{19} \longrightarrow S^{16}$, $m_2 : S^{33} \longrightarrow S^{32}$ obtained from orthogonal multiplications;

maps $c_1 : S^4 \longrightarrow S^4$, $c_2 : S^7 \longrightarrow S^7$, $c_3 : S^{13}$, $c_4 : S^{25} \longrightarrow S^{25}$, $d_1 : S^5 \longrightarrow S^5, d_2 : S^9 \longrightarrow S^9$ which are the gradient of isoparametric functions.

In particular, we recall that the maps i_2, h_1, h_2, m_1, m_2, c_1, c_2, c_3, c_4 have polynomial degree 2, while the maps i_3, d_1, d_2 have polynomial degree 3.

If the map f_2 of the Main Theorem is the identity map $\mathrm{id}^q : S^q \longrightarrow S^q$, then the join map $f_1 * \mathrm{id}^q$ as in (2.1) is homotopic to the $(q+1)$-suspension of f_1. We have

COROLLARY 1. *Let $f_1 : S^p \longrightarrow S^r$ be any harmonic homogeneous polynomial map of degree $k_1 \geq 2$.*

Then the homotopy class of the $(q+1)$-suspension of f_1 can be represented by an equivariant harmonic map $f_1 * \mathrm{id}^q$ if and only if $q = 0, \ldots, 5$.

PROOF. If $q \geq 1$, the Corollary follows immediately from the application of the Main Theorem to the case where f_2 is the identity map $\mathrm{id}^q : S^q \longrightarrow S^q$: for in this case $k_2 = 1$, $\lambda_2 = q$ and inspection of G.D.C. yields the required conclusion. The case $q = 0$ can be easily handled separately ([10], [15]).

REMARKS. i) There are examples where $q > 5$ and the homotopy class of $f_1 * \mathrm{id}^q$ can be represented by an equivariant harmonic map of the form $g_1 * g_2$ for some suitable harmonic homogeneous polynomial maps g_1, g_2 (see [7] and examples below).

ii) Our analysis has determined the precise combination of the parameters p, q, k_1, k_2 which separates existence from non-existence: it is geometrically interesting to notice that the join of a harmonic homogeneous polynomial map

of degree $k_1 = 2$ with $\mathrm{id}^6(q = \lambda_2 = 6)$ is exactly on the boundary of the non-existence area.

Now we list some examples of new harmonic maps: first we notice that the maps

$$h_3,\ c_3,\ c_4,\ m_1,\ m_2 \tag{4.1}$$

do not satisfy Smith's damping conditions D.C.; however we have

EXAMPLES 1. Corollary 1 can be applied to each of the maps in (4.1): harmonic suspensions of h_3 give a harmonic representative for the generator of $\pi_{n+7}(S^n) = \mathbb{Z}_{240}, n = 9, \ldots, 14$.

The map m_1 represents twice the generator of $\pi_{19}(S^{16}) = \mathbb{Z}_{24}$: thus we have a harmonic representative for twice the generator of $\pi_{n+3}(S^n) = \mathbb{Z}_{24}$, $n = 17, \ldots, 22$.

Harmonic suspensions of c_3, c_4 yield harmonic maps $f : S^n \longrightarrow S^n$ of Brouwer degree ± 2, $n = 14, \ldots, 19,\ 26, \ldots, 31$.

Harmonic suspensions of m_2 are homotopically trivial.

Inspection of the generalized damping conditions G.D.C. enables us to state

COROLLARY 2. *Let $f_1 : S^p \longrightarrow S^r, f_2 : S^q \longrightarrow S^s$ be two harmonic homogeneous polynomial maps of the same polynomial degree ($k_1 = k_2$). Then there exists an equivariant harmonic join $f_1 * f_2$.*

This Corollary has interesting applications to maps of polynomial degree 2. In fact, we have

EXAMPLES 2. Each map in (4.1) can be joined harmonically with any map among i_2, h_1, h_2, c_1, c_2.

For instance, we have harmonic maps $f : S^n \longrightarrow S^n$ of Brouwer degree ± 4, $n = 15,\ 21,\ 27,\ 33$. The harmonic join $h_2 * h_3$ represents the generator of $\pi_{23}(S^{13}) = \mathbb{Z}_6$.

EXAMPLES 3. One can join harmonically any two different maps in (4.1): for instance, we have harmonic maps $f : S^{39} \longrightarrow S^{39}$ of Brouwer degree ± 4.

EXAMPLES 4. The map c_3 does not satisfy Smith's damping conditions D.C.; however, inspection of G.D.C. tells us that c_3 can be harmonically joined with any map amongst i_3, d_1 and d_2: in particular, $i_3 * c_3 : S^{15} \longrightarrow S^{15}$ provides a harmonic map of Brouwer degree 6.

The generalized damping conditions G.D.C., and the consequent restrictions regarding the application of the equivariant method, interestingly contrast with the results of [11], where it is proved that every element of the groups $\pi_n(S^n)$, $n \in \mathbb{N}$, admits a harmonic representative, provided that the domain sphere is given a suitable riemannian metric.

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