ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 5, n° 2 (1978), p. 381-404

<http://www.numdam.org/item?id=ASNSP_1978_4_5_2_381_0>

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Asymptotic Properties of Steady Plane Solutions of the Navier-Stokes Equations with Bounded Dirichlet Integral (*).

D. GILBARG (**) - H. F. WEINBERGER (***)

dedicated to Jean Leray

1. – Introduction.

It was shown in an earlier paper [1] that the solution of the two-dimensional exterior problem for the Navier-Stokes equations which was constructed in the fundamental work [2] of Jean Leray actually converges to a constant velocity at infinity in a mean square sense, while the pressure converges pointwise.

The proof was done in two parts. In the first part it was established that the velocity, which Leray had constructed so that it was Dirichlet integrable, was also uniformly bounded. We then showed that in any bounded Dirichlet integrable solution the velocity has a limit in mean and the pressure has a pointwise limit at infinity. We were, however, unable to show that the limit in mean is equal to the assigned velocity at infinity.

In the present paper we investigate the properties of an arbitrary solution $\{\boldsymbol{w}, p\}$ of the two-dimensional Navier-Stokes equations

$$\begin{aligned} \Delta \boldsymbol{w} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} &= \nabla p \\ \nabla\cdot\boldsymbol{w} &= 0\,, \end{aligned} \tag{1.1}$$

(*) This work was supported in part through NSF grants MCS-75-23332-A02 and 37660-X.

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Pervenuto alla Redazione il 15 Giugno 1977.

in a neighborhood of infinity, which has the property that

$$\int_{r>r_0} |\nabla \boldsymbol{w}|^2 \, dx \, dy < \infty .$$

We shall show that while $\boldsymbol{w}$ may not be bounded under these assumptions, it grows more slowly than $(\log r)^{\frac{1}{2}}$. The pressure has a finite limit at infinity. The velocity either has a limit in the mean w_∞ or

$$\int_0^{2\pi} |\boldsymbol{w}(r, \theta)|^2 \, d\theta$$

approaches infinity as $r \to \infty$. The vorticity $\omega = u_y - v_x$, where $\boldsymbol{w} = (u, v)$, approaches zero more rapidly than $r^{-\frac{3}{4}}(\log r)^{\frac{1}{8}}$ and the first derivatives of the velocity decay more rapidly than $r^{-\frac{3}{4}}(\log r)^{\frac{9}{8}}$ at infinity.

A particular application of our results is a Liouville theorem, which states that if $\{\boldsymbol{w}, p\}$ is a solution of (1.1) in the entire x, y plane and $|\nabla \boldsymbol{w}|$ is square integrable, then $\boldsymbol{w}$ and p are constant.

We remark that while our bounds allow the velocity to grow at infinity, there is no known example of a Dirichlet integrable solution of the Navier-Stokes equations which is unbounded at infinity. However, only very few solutions which are not potential flows are known.

Under the different hypothesis that the flow is *PR* (physically reasonable) in the sense of Finn, namely, that the velocity approaches a limit asymptotically as $O(r^{-1-\varepsilon})$, $\varepsilon > 0$, Smith [3] has shown that the Dirichlet integral is bounded, and has derived much more precise asymptotic estimates on the velocity and its derivatives. The present work shows that the Liouville theorem (Theorem 2) and the pointwise convergence of the pressure (Theorem 3) can be obtained under the weaker hypothesis of bounded Dirichlet integral of the velocity.

The contrast with the three-dimensional Navier-Stokes equations is quite marked. There the finiteness of the Dirichlet integral is a more restrictive condition and, as Babenko [4] has shown, it implies that the velocity approaches a pointwise limit as $O(r^{-1})$ and converges even more rapidly outside a wake region. Whether an analogous result is true for two-dimensional flows is an open question.

Throughout this paper we shall use the same letter C to represent different constants. The dependence of C on the given data will be visible in context.

2. – Preliminary estimates.

We first establish several lemmas required in the later developments.

LEMMA 2.1. *Let $f \in C^1$ in $r \geqslant r_0$ and have finite Dirichlet integral*

$$\int_{r>r_0} |\nabla f|^2 \, dx\, dy < \infty . \tag{2.1}$$

Then

$$\lim_{r\to\infty} \frac{1}{\log r} \int_0^{2\pi} f(r, \theta)^2 \, d\theta = 0 . \tag{2.2}$$

PROOF. By the Schwarz inequality we have

$$\frac{d}{dr}\left\{\int_0^{2\pi} f(r, \theta)^2 \, d\theta\right\}^{\frac{1}{2}} = \left\{\int_0^{2\pi} f^2 \, d\theta\right\}^{-\frac{1}{2}} \int_0^{2\pi} f f_r \, d\theta \leqslant \left\{\int_0^{2\pi} f_r^2 \, d\theta\right\}^{\frac{1}{2}} .$$

Integrating between $r_1 (\geqslant r_0)$ and r and again applying the Schwarz inequality, we obtain

$$\left\{\int_0^{2\pi} f(r, \theta)^2 \, d\theta\right\}^{\frac{1}{2}} - \left\{\int_0^{2\pi} f(r_1, \theta)^2 \, d\theta\right\}^{\frac{1}{2}} \leqslant \int_{r_1}^{r} \left\{\int_0^{2\pi} f_r(\varrho, \theta)^2 \, d\theta\right\}^{\frac{1}{2}} d\varrho$$

$$\leqslant \left\{\int_0^{2\pi r} \int_{r_0} f_r(\varrho, \theta)^2 \, \varrho \, d\varrho \, d\theta\right\}^{\frac{1}{2}} \left(\log \frac{r}{r_1}\right)^{\frac{1}{2}} .$$

Thus,

$$\int_0^{2\pi} f(r, \theta)^2 \, d\theta \leqslant 2 \int_0^{2\pi} f(r_1, \theta)^2 \, d\theta + 2 \left\{\int_{r_1}^{r} \int_0^{2\pi} f_r^2 \, \varrho \, d\varrho \, d\theta\right\} \log \frac{r}{r_1} ,$$

so that

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_0^{2\pi} f(r, \theta)^2 \leqslant 2 \int_{r>r_1} |\nabla f|^2 \, dx\, dy .$$

Letting $r_1 \to \infty$, we obtain (2.2).

LEMMA 2.2. *Let* $\boldsymbol{f} = (f_1, f_2)$, *where* f_1 *and* f_2 *satisfy the hypotheses of* Lemma 2.1. *Then there is a sequence* $\{r_n\}$, $r_n \in (2^n, 2^{n+1})$, *such that*

$$\lim_{n\to\infty} |\boldsymbol{f}(r_n, \theta)|^2/\log r_n = 0 \tag{2.3}$$

uniformly in θ.

PROOF. We suppose $2^n > r_0$, and let A_n denote the annulus $2^n < r < 2^{n+1}$. Since

$$\int_{2^n}^{2^{n+1}} \left\{ \int_0^{2\pi} |\boldsymbol{f}_\theta(r, \theta)|^2 \, d\theta \right\} \frac{dr}{r} \leqslant \int_{A_n} |\nabla \boldsymbol{f}|^2 \, dx \, dy ,$$

it follows from the integral theorem of the mean that for some $r_n \in (2^n, 2^{n+1})$

$$\int_0^{2\pi} |\boldsymbol{f}_\theta(r_n, \theta)|^2 \, d\theta \leqslant \frac{1}{\log 2} \int_{A_n} |\nabla \boldsymbol{f}|^2 \, dx \, dy . \tag{2.4}$$

By Schwarz's inequality

$$|\boldsymbol{f}(r_n, \theta)| \leqslant |\boldsymbol{f}(r_n, \varphi)| + \left\{ \pi \int_0^{2\pi} |\boldsymbol{f}_\theta(r_n, \theta')|^2 \, d\theta' \right\}^{\frac{1}{2}}$$

for any θ and φ in $[0, 2\pi]$. Hence

$$\boldsymbol{f}(r_n, \theta)|^2 \leqslant 2|\boldsymbol{f}(r_n, \varphi)|^2 + 2\pi \int_0^{2\pi} |\boldsymbol{f}(r_n, \theta')|^2 \, d\theta' .$$

Integrating this inequality with respect to φ, we find

$$2\pi |\boldsymbol{f}(r_n, \theta)|^2 \leqslant 2 \int_0^{2\pi} |\boldsymbol{f}(r_n, \varphi)|^2 d\varphi + 4\pi^2 \int_0^{2\pi} |\boldsymbol{f}_\theta(r_n, \theta')|^2 d\theta' .$$

The conclusion (2.3) now follows from (2.4) and Lemma 2.1.

LEMMA 2.3. *Let* $\omega = u_y - v_x$ *be the vorticity of a velocity vector* $\boldsymbol{w} = (u, v)$ *satisfying the Navier-Stokes equations* (1.1) *in* $r \geqslant r_0$ *and having finite Dirichlet integral*

$$\int_{r>r_0} |\nabla \boldsymbol{w}|^2 \, dx \, dy < \infty . \tag{2.5}$$

Then

$$\int_{r>r_0} |\nabla\omega|^2\,dx\,dy < \infty\,. \tag{2.6}$$

PROOF. By taking the curl of the Navier-Stokes equations, we find that

$$\Delta\omega - \boldsymbol{w}\cdot\nabla\omega = 0\,. \tag{2.7}$$

Let $\eta(r)$ be a smooth function which vanishes near $r = r_0$ and near $r = \infty$. Let $h(\omega)$ be a function of one variable which is C^1 and piecewise C^2. An easy computation which uses the fact that $\operatorname{div}\boldsymbol{w} = 0$ shows that

$$\operatorname{div}[\eta(r)\nabla h(\omega) - h(\omega)\nabla\eta(r) - \eta h(\omega)\boldsymbol{w}] = \eta h''(\omega)|\nabla\omega|^2 - h(\omega)[\Delta\eta + \boldsymbol{w}\cdot\nabla\eta] + \eta h'(\omega)[\Delta\omega - \boldsymbol{w}\cdot\nabla\omega]\,.$$

Since ω satisfies (2.7) and η vanishes near $r = r_0$ and $r = \infty$, integration over the domain $r > r_0$ yields the identity

$$\int \eta h''(\omega)|\nabla\omega|^2\,dx\,dy = \int h(\omega)[\Delta\eta + \boldsymbol{w}\cdot\nabla\eta]\,dx\,dy\,. \tag{2.8}$$

To use this identity, we choose $R > r_1 > r_0$ and non-negative C^2 cut-off functions ξ_1 and ξ_2 such that

$$\xi_1(r) = \begin{cases} 0\,, & r \leqslant \tfrac{1}{2}(r_0 + r_1) \\ 1\,, & r \geqslant r_1 \end{cases}\,, \qquad \xi_2(r) = \begin{cases} 1\,, & r \leqslant 1 \\ 0\,, & r \geqslant 2 \end{cases} \tag{2.9}$$

and set

$$\eta(r) = \xi_1(r)\xi_2(r/R)\,.$$

We choose a positive constant ω_0 and set

$$h(\omega) = \begin{cases} \omega^2\,, & |\omega| \leqslant \omega_0\,. \\ \omega_0(2|\omega| - \omega_0)\,, & |\omega| \geqslant \omega_0\,. \end{cases}$$

Then the identity (2.8) shows that

$$\int_{\substack{|\omega|<\omega_0 \\ r_1<r<R}} |\nabla\omega|^2\,dx\,dy \leqslant \int_{|\omega|<\omega_0} \eta|\nabla\omega|^2\,dx\,dy = \int_{\substack{\{r_0<r<r_1\} \\ \cup\{R<r<2R\}}} h(\omega)[\Delta\eta + \boldsymbol{w}\cdot\nabla\eta]\,dx\,dy\,. \tag{2.10}$$

Consider the portion of the right integral over the annulus $R < r < 2R$. We have $|\Delta\eta| \leqslant C/R^2$ and $|\nabla\eta| \leqslant C/R$ for a constant C independent of R.

Clearly $h(\omega) \leqslant \omega^2$ and $h(\omega) \leqslant 2\omega_0|\omega|$. Therefore

$$\left| \int_{R<r<2R} h\,\Delta\eta\,dx\,dy \right| \leqslant \frac{C}{R^2} \int_{R<r<2R} \omega^2\,dx\,dy \leqslant \frac{C}{R^2} \int_{R<r<2R} |\nabla \boldsymbol{w}|^2\,dx\,dy \to 0 \quad \text{as } R \to \infty. \tag{2.11}$$

Writing

$$\bar{\boldsymbol{w}}(r) = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{w}(r, \theta)\,d\theta,$$

we have for the other part of the integral over $R < r < 2R$

$$\left| \int h\boldsymbol{w} \cdot \nabla\eta\,dx\,dy \right| \leqslant \int |h(\boldsymbol{w} - \bar{\boldsymbol{w}}) \cdot \nabla\eta|\,dx\,dy + \int |h\bar{\boldsymbol{w}} \cdot \nabla\eta|\,dx\,dy \leqslant 2\omega_0 \int |\omega||\nabla\eta||\boldsymbol{w} - \bar{\boldsymbol{w}}|\,dx\,dy + \int \omega^2 |\nabla\eta||\bar{\boldsymbol{w}}|\,dx\,dy. \tag{2.12}$$

From Wirtinger's inequality,

$$\int_0^{2\pi} |\boldsymbol{w} - \bar{\boldsymbol{w}}|^2\,d\theta \leqslant \int_0^{2\pi} |\boldsymbol{w}_\theta|^2\,d\theta, \tag{2.13}$$

we obtain the estimate

$$\int_{R<r<2R} |\omega||\nabla\eta||\boldsymbol{w} - \bar{\boldsymbol{w}}|\,dx\,dy \leqslant \left\{ \int_{R<r<2R} \omega^2\,dx\,dy \right\}^{\frac{1}{2}} \left\{ C \int_R^{2R} \frac{1}{r^2} \left(\int_0^{2\pi} |\boldsymbol{w}_\theta|^2\,d\theta \right) r\,dr \right\}^{\frac{1}{2}} \leqslant C \left\{ \int_{r>R} \omega^2\,dx\,dy \right\}^{\frac{1}{2}} \left\{ \int_{R<r<2R} |\nabla \boldsymbol{w}|^2\,dx\,dy \right\}^{\frac{1}{2}} \to 0 \quad \text{as } R \to \infty. \tag{2.14}$$

From Lemma 2.1 we infer that

$$\bar{\boldsymbol{w}}(r) = o(\sqrt{\log r})$$

and hence

$$\int_{R<r<2R} \omega^2 |\nabla\eta||\bar{\boldsymbol{w}}|\,dx\,dy \leqslant \frac{C(\log R)^{\frac{1}{2}}}{R} \int_{r>R} \omega^2\,dx\,dy \to 0 \quad \text{as } R \to \infty. \tag{2.15}$$

Inserting (2.14) and (2.15) into (2.12) and combining the result with (2.11), we conclude from (2.10) that

$$\lim_{R\to\infty} \int\limits_{\substack{|\omega|<\omega_0 \\ r_1<r<R}} |\nabla\omega|^2\,dx\,dy \leqslant \int\limits_{r_0<r<r_1} \omega^2(|\Delta\eta| + |\boldsymbol{w}||\nabla\eta|)\,dx\,dy \leqslant K$$

where K is a constant independent of ω_0. Letting $\omega_0 \to \infty$, we infer that

$$\int\limits_{r>r_1} |\nabla\omega|^2\,dx\,dy \leqslant K$$

and (2.6) follows.

LEMMA 2.4. *Under the hypotheses of Lemma* 2.3 *we have*

$$\lim_{r\to\infty} r^{\frac{1}{2}}|\omega(r,\theta)| = 0\,, \tag{2.16}$$

uniformly in θ.

PROOF. For $2^n > r_0$,

$$\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} (r^2\omega^2 + 2r|\omega\omega_\theta|)\,d\theta \leqslant \int\limits_{2^n<r<2^{n+1}} (\omega^2 + 2|\omega||\nabla\omega|)\,dx\,dy$$
$$\leqslant \int\limits_{r>2^n} (2\omega^2 + |\nabla\omega|^2)\,dx\,dy\,.$$

Hence by the integral theorem of the mean, there is an $r_n \in (2^n, 2^{n+1})$ such that

$$\int_0^{2\pi} [r_n^2\omega(r_n,\theta)^2 + 2r_n|\omega(r_n,\theta)\omega_\theta(r_n,\theta)|]\,d\theta \tag{2.17}$$
$$\leqslant \frac{1}{\log 2} \int\limits_{r>2^n} (2\omega^2 + |\nabla\omega|^2)\,dx\,dy\,.$$

One sees that

$$\omega(r_n,\theta)^2 - \frac{1}{2\pi}\int_0^{2\pi} \omega(r_n,\theta')^2\,d\theta' \leqslant \int_0^{2\pi} \left|\frac{\partial}{\partial\theta'}\,\omega(r_n,\theta')^2\right| d\theta'$$
$$= 2\int_0^{2\pi} |\omega(r_n,\theta')\,\omega_\theta(r_n,\theta')|\,d\theta'\,,$$

and hence from (2.17) and Lemma 2.3

$$\lim_{n\to\infty} [r_n \max_\theta \omega(r_n, \theta)^2] = 0 . \tag{2.18}$$

Since ω is a solution of the elliptic equation (2.7), it satisfies the maximum principle. Noting that $r_{n+1} \leqslant 4r_n$, we infer that for $r \in (r_n, r_{n+1})$

$$r \max_\theta \omega(r, \theta)^2 \leqslant \max [4r_n \max_\theta \omega(r_n, \theta)^2, r_{n+1} \max_\theta \omega(r_{n+1}, \theta)^2] .$$

From (2.18) we now conclude the desired result (2.16).

The preceding lemmas yield the following growth estimate for the velocity.

THEOREM 1. *If $\boldsymbol{w}$ is a solution in $r \geqslant r_0$ of the Navier-Stokes equations* (1.1) *and*

$$\int_{r>r_0} |\nabla \boldsymbol{w}|^2 \, dx\, dy < \infty ,$$

then

$$\lim_{r\to\infty} |\boldsymbol{w}(r, \theta)|^2 / \log r = 0 \tag{2.19}$$

uniformly in θ.

PROOF. Let $r = |z| \geqslant 8 \max(r_0, 1)$ and choose the integer n so that $r \in [2^n, 2^{n+1})$. Let A_n now denote the annulus $r_{n-2} < |\zeta| < r_{n+2}$, with $r_n \in (2^n, 2^{n+1})$ such that (2.3) holds for $\boldsymbol{f} = \boldsymbol{w}$. Then by the Cauchy integral formula representation of $w = u - iv$ in A_n (see, for example, [5] p. 3), we have

$$\left\{ \begin{aligned} w(z) &= \frac{1}{2\pi i} \oint_{\partial A_n} \frac{w(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{\pi} \int_{A_n} \frac{w_{\bar\zeta}(\zeta)}{\zeta - z} \, d\xi\, d\eta , \qquad z \in A_n,\ \zeta = \xi + i\eta \\ &= \frac{1}{2\pi i} \left(\oint_{\partial A_n} \frac{w(\zeta)}{\zeta - z} \, d\zeta + \int_{A_n} \frac{\omega(\zeta)}{\zeta - z} \, d\xi\, d\eta \right); \end{aligned} \right. \tag{2.20}$$

the latter equality results from the fact that

$$w_{\bar z} = \frac{1}{2}(w_x + i w_y) = \frac{i}{2}(u_y - v_x) + \frac{1}{2}(u_x + v_y) = -\frac{\omega}{2i} .$$

Since $|z| \in [2^n, 2^{n+1})$, $\operatorname{dist}(z, \partial A_n) \geqslant 2^{n-1} \geqslant |z|/4 = r/4$. It follows from Lemma 2.2 that the line integral in (2.20) is $o(\sqrt{\log r})$. To estimate the other

integral, we write

$$\left| \int_{A_n} \frac{\omega(\zeta)}{\zeta - z} \, d\xi \, d\eta \right| \leqslant \int_D \left| \frac{\omega(\zeta)}{\zeta - z} \right| d\xi \, d\eta + \int_{A_n - D} \left| \frac{\omega(\zeta)}{\zeta - z} \right| d\xi \, \eta d$$

where D is the disc $|\zeta - z| < 1$ with center z. By Lemma 2.4 the first member on the right is bounded by $Cr^{-\frac{1}{2}}$; and since A_n is contained in the disc $|\zeta - z| < 5r$, we have

$$\int_{A_n - D} \left| \frac{\omega(\zeta)}{\zeta - z} \right| d\xi \, d\eta \leqslant \left(\int_{A_n} \omega^2 \, d\xi \, d\eta \right)^{\frac{1}{2}} \left(\int_{1 < |\zeta - z| < 5r} |\zeta - z|^{-2} \, d\xi \, d\eta \right)^{\frac{1}{2}} \leqslant \varepsilon_n (\log r)^{\frac{1}{2}},$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Combining these estimates, we see that $|w(z)| = o(\sqrt{\log r})$ as $r \to \infty$, and this proves the theorem.

3. – A Liouville theorem.

The results of the preceding section imply the following Liouville theorem.

THEOREM 2. *Let $\{\boldsymbol{w}, p\}$ be a solution of the Navier-Stokes equations* (1.1) *defined over the entire plane and assume*

$$\int_{R^2} |\nabla \boldsymbol{w}|^2 < \infty .$$

Then $\boldsymbol{w}$ and p are constant.

PROOF. Since ω is a solution of the elliptic equation

$$\Delta \omega - \boldsymbol{w} \cdot \nabla \omega = 0 ,$$

it satisfies the maximum principle. Lemma 2.4 shows that $\omega \to 0$ at infinity and hence $\omega \equiv 0$. Thus we have both $u_x + v_y = 0$ and $u_y - v_x = 0$ over the entire plane and accordingly $w = u - iv$ is an entire analytic function satisfying

$$\int_{R^2} |w'(z)|^2 \, dx \, dy < \infty .$$

It follows, for example by considering the Taylor series of $w'(z)$, integrating over $|z| < R$, and letting $R \to \infty$ that $w'(z) \equiv 0$ and hence w is constant.

4. – Convergence of the pressure.

We now show that the pressure p has a pointwise limit at infinity.

THEOREM 3. *Let $\{\boldsymbol{w}, p\}$ be a solution of the Navier-Stokes equations* (1.1) *in $r \geqslant r_0$, with finite Dirichlet integral*

$$\int_{r>r_0} |\nabla \boldsymbol{w}|^2 \, dx\, dy < \infty . \tag{4.0}$$

Then the pressure p has a finite limit at infinity.

This result is a consequence of the following lemmas, which are proved under the hypotheses of the theorem.

LEMMA 4.1. *The average pressure*

$$\overline{p}(r) = \frac{1}{2\pi} \int_0^{2\pi} p(r, \theta)\, d\theta$$

has a limit at infinity

$$\lim_{r\to\infty} p(r) = p_\infty < \infty .$$

PROOF. Since $\Delta u = \omega_y$ and $\Delta v = -\omega_x$, the Navier-Stokes equations can be written in the form

$$\begin{aligned} \omega_y + uv_y - vu_y &= p_x \\ -\omega_x - uv_x + vu_x &= p_y \\ u_x + v_y &= 0 . \end{aligned} \tag{4.1}$$

It follows that

$$p_r = \frac{1}{r}(\omega_\theta + uv_\theta - vu_\theta) . \tag{4.2}$$

We average this equation to find that

$$\overline{p}'(r) = \frac{1}{2\pi r} \int_0^{2\pi} [(u - \overline{u})v_\theta - (v - \overline{v})u_\theta]\, d\theta .$$

Hence by the Schwarz and Wirtinger inequalities, we have for any $r_2 \geqslant r_1 \geqslant r_0$

$$4\pi^2|\overline{p}(r_2) - \overline{p}(r_1)|^2 = \left| \int_{r_1}^{r_2}\int_0^{2\pi} \frac{1}{r}[(u-\overline{u})v_\theta - (v-\overline{v})u_\theta]\,d\theta\,dr \right|^2$$

$$\leqslant \int_0^{2\pi}\int_{r_1}^{r_2} \frac{|\boldsymbol{w}-\overline{\boldsymbol{w}}|^2}{r}\,dr\,d\theta \cdot \int_0^{2\pi}\int_{r_1}^{r_2} \frac{1}{r}\,|\boldsymbol{w}_\theta|^2\,dr\,d\theta$$

$$\leqslant \left(\int_0^{2\pi}\int_{r_1}^{r_2} \frac{1}{r}\,|\boldsymbol{w}_\theta|^2\,dr\,d\theta \right)^2 \leqslant \left(\int_{r>r_1} |\nabla \boldsymbol{w}|^2\,dx\,dy \right)^2 .$$

Since the right member of this inequality tends to zero as $r_1 \to \infty$, it follows that $\overline{p}(r)$ has a limit p_∞, as asserted.

LEMMA 4.2. *There is a sequence* $\{R_n\}$, $R_n \in (2^{2^n}, 2^{2^{n+1}})$, *such that*

$$\lim_{n\to\infty} \int_0^{2\pi} |p(R_n,\theta) - \overline{p}(R_n)|^2\,d\theta = 0 . \tag{4.3}$$

PROOF. It follows from (4.1), (4.0), (2.6) and (2.19) that for any $r_1 > \max(r_0, 1)$

$$\int_{r<r_1} \frac{|\nabla p|^2}{\log r}\,dx\,dy < \infty .$$

By the integral theorem of the mean and Wirtinger's inequality there is an $R_n \in (2^{2^n}, 2^{2^{n+1}})$ such that

$$\log 2 \int_0^{2\pi} |p(R_n,\theta) - \overline{p}(R_n)|^2\,d\theta = \int_{2^{2^n}}^{2^{2^{n+1}}} \int_0^{2\pi} \frac{|p(r,\theta) - \overline{p}(r)|^2}{r \log r}\,d\theta\,dr$$

$$\leqslant \int_{2^{2^n}}^{2^{2^{n+1}}} \int_0^{2\pi} \frac{p_\theta^2}{r \log r}\,d\theta\,dr \leqslant \int_{2^{2^n}<r<2^{2^{n+1}}} \frac{|\nabla p|^2}{\log r}\,dx\,dy \to 0 \quad \text{as } n \to \infty .$$

LEMMA 4.3. *Let* p_∞ *be as in Lemma* 4.1. *Then*

$$\lim_{r\to\infty} \int_0^{2\pi} |p(r,\theta) - p_\infty|^2\,d\theta = 0 . \tag{4.4}$$

PROOF. Taking the divergence of ∇p in the Navier-Stokes equations (4.1), we find that

$$\Delta p = 2(u_x v_y - u_y v_x)\,.$$

The right member is absolutely integrable in $r > r_0$. It follows that $\Delta \bar{p}$ is also absolutely integrable and hence

$$H \equiv \Delta(p - \bar{p}) \in L_1 \quad \text{in } r > r_0\,. \tag{4.5}$$

Let A_{nm} denote the annulus $R_n < r < R_m$, the sequence of radii R_n being defined as in Lemma 4.2. We have the representation

$$\begin{aligned} p(r,\theta) - \bar{p}(r) = &-\int_{A_{nm}} G(r,\theta;\varrho,\varphi) H(\varrho,\varphi)\,\varrho\,d\varrho\,d\varphi \\ &+ \oint_{\varrho=R_n} \frac{\partial G}{\partial \varrho}(r,\theta;\varrho,\varphi)(p(\varrho,\varphi) - \bar{p}(\varrho))\,\varrho\,d\varphi \\ &- \oint_{\varrho=R_m} \frac{\partial G}{\partial \varrho}(r,\theta;\varrho,\varphi)(p(\varrho,\varphi) - \bar{p}(\varrho))\,\varrho\,d\varphi \end{aligned} \tag{4.6}$$

where $G = G(r,\theta;\varrho,\varphi)$ is the harmonic Green's function for the annulus A_{nm}. G can be written in the form (see, for example, [7, p. 140, problem 2 with answer on p. 417])

$$\begin{aligned} G(r,\theta;\varrho,\varphi) = &-\sum_{k=1}^{\infty} \frac{1}{2\pi k(R_m^{2k} - R_n^{2k})}(r^k - R_n^{2k}/r^k)\cdot \\ &\cdot(\varrho^k - R_m^{2k}/\varrho^k)\cos k(\theta - \varphi) + \frac{\log(r/R_n)\log(R_m/\varrho)}{2\pi \log(R_m/R_n)}, \qquad r < \varrho\,, \end{aligned} \tag{4.7}$$

with r and ϱ interchanged when $r > \varrho$. Since $p - \bar{p}$ and H have average value zero for each r, the last term in G does not contribute to the representation (4.6) and it will be omitted in the following. Writing $\hat{G}$ for G minus the last term, we set

$$\begin{aligned} \hat{G}^{(2)}(r;\varrho_1,\varphi_1;\varrho_2,\varphi_2) &\equiv \int_0^{2\pi} \hat{G}(r,\theta,\varrho_1,\varphi_1)\,\hat{G}(r,\theta;\varrho_2,\theta_2)\,d\theta \\ &= \sum_{k=1}^{\infty} \frac{(r^k - R_n^{2k}/r^k)^2(\varrho_1^k - R_m^{2k}/\varrho_1^k)(\varrho_2^k - R_m^{2k}/\varrho_2^k)\cos k(\varphi_1 - \varphi_2)}{4\pi k^2 (R_m^{2k} - R_n^{2k})^2} \end{aligned}$$

when $r < \varrho_1$, $r < \varrho_2$, with similar expressions for the other cases. The maximum with respect to ϱ_1 and ϱ_2 of this expression occurs when $\varrho_1 = \varrho_2 = r = (R_m R_n)^{\frac{1}{2}}$ and $\varphi_1 = \varphi_2$. Thus

$$|\hat{G}^{(2)}(r;\, \varrho_1, \varphi_1;\, \varrho_2, \varphi_2)| \leqslant \sum_{k=1}^{\infty} \frac{(R_m^k - R_n^k)^2}{4\pi k^2 (R_m^k + R_n^k)^2} \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} \equiv C_1 .$$

Also,

$$\frac{\partial \hat{G}}{\partial \varrho}(r, \theta;\, \varrho, \varphi)\bigg|_{\varrho = R_m} = -\sum_{k=1}^{\infty} \frac{(r^k - R_n^{2k}/r^k) R_m^{k-1}}{\pi (R_m^{2k} - R_n^{2k})} \cos k(\theta - \varphi)$$

and thus, if $R_n < r \leqslant R_{m-2}$, we have

$$\int_0^{2\pi} |G_\varrho(r, \theta;\, R_m, \varphi)|^2 R_m^2 \, d\varphi = \sum_{k=1}^{\infty} \frac{(r^k - R_n^{2k}/r^k)^2 R_m^{2k}}{\pi (R_m^{2k} - R_n^{2k})^2}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{(R_{m-2}/R_m)^{2k}}{\pi [1 - (R_n/R_m)^{2k}]^2} \leqslant \frac{\pi \sum_{k=1}^{\infty} 2^{-2^m k}}{\pi [1 - (R_n/R_m)^2]^2} \leqslant C_2 .$$

Similarly, we see that if $R_{n+2} \leqslant r < R_m$,

$$\int_0^{2\pi} |\hat{G}_\varrho(r, \theta;\, R_n, \varphi)|^2 R_n^2 \, d\varphi \leqslant C_3 .$$

From (4.6) it follows that for $r \in [R_{n+2}, R_{m-2}]$, $m \geqslant n + 5$,

$$\frac{1}{3} \int_0^{2\pi} |p(r, \theta) - \overline{p}(r)|^2 \, d\theta \leqslant C_1 \left(\int_{R_n < r < R_m} H \, dx \, dy \right)^2$$

$$+ 2\pi C_2 \int_0^{2\pi} |p(R_m, \varphi) - \overline{p}(R_m)|^2 \, d\varphi + 2\pi C_3 \int_0^{2\pi} |p(R_n, \varphi) - \overline{p}(R_n)|^2 \, d\varphi .$$

By letting $m \to \infty$ and using (4.3) and (4.5), we obtain an upper bound on the left member for $r > 2^{2^{n+3}}$, and this bound approaches zero as $n \to \infty$. From this we infer

$$\lim_{r \to \infty} \int_0^{2\pi} |p(r, \theta) - \overline{p}(r)|^2 \, d\theta = 0 .$$

Since $\overline{p}(r)$ has the limit p_∞, we obtain (4.4).

LEMMA 4.4. *Suppose that* $p_\infty = 0$. *Then*

$$\lim_{(x,y)\to\infty} p(x, y) = 0 . \tag{4.7}$$

PROOF. Let the point $P(2R, \theta)$ be the origin of a new system of polar coordinates (r', θ') and suppose that $R > r_0$. In these new coordinates we still have

$$p_{r'} = \frac{1}{r'}(\omega_{\theta'} + uv_{\theta'} - vu_{\theta'})$$

from the Navier-Stokes equations. Integrating with respect to r' and θ', we find that

$$p(P) = \frac{1}{2\pi}\int_0^{2\pi} p(r', \theta')\, d\theta' + \frac{1}{2\pi}\int_0^{r'}\int_0^{2\pi}\frac{1}{\varrho}\{[u(\varrho, \theta') - \tilde{u}(\varrho)]v_{\theta'}(\varrho, \theta') - [v(\varrho, \theta') - \tilde{v}(\varrho)]\, u_{\theta'}(\varrho, \theta')\}\, d\varrho\, d\theta'$$

where

$$\tilde{u}(r') = \frac{1}{2\pi}\int_0^{2\pi} u(r', \theta')\, d\theta' ,$$

$$\tilde{v}(r') = \frac{1}{2\pi}\int_0^{2\pi} v(r', \theta')\, d\theta' .$$

We multiply this relation by r' and integrate from 0 to R to find that

$$p(P) = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} p(r', \theta')\, r'\, dr'\, d\theta' + \frac{1}{\pi R^2}\int_0^R\int_0^{r'}\int_0^{2\pi}\frac{1}{\varrho}\{(u - \tilde{u})v_{\theta'} - (v - \tilde{v})u_{\theta'}\}\, r'\, d\theta'\, d\varrho\, dr' . \tag{4.8}$$

We again see from the Wirtinger and Schwarz inequalities that

$$\left|\int_0^{2\pi}\{(u - \tilde{u})v_{\theta'} - (v - \tilde{v})u_{\theta'}\}\, d\theta'\right| \leqslant \int_0^{2\pi}|w_{\theta'}|^2 d\theta' \leqslant \int_0^{2\pi}\varrho^2|\nabla \boldsymbol{w}|^2 d\theta' .$$

Since the disc $r' < R$ is contained in the annulus $R < r < 3R$, the second term in (4.8) is bounded by

$$\frac{1}{\pi R^2}\int_0^R\int_0^{r'}\int_0^{2\pi} |\nabla \boldsymbol{w}(\varrho, \theta')|^2 \varrho\, d\varrho\, d\theta'\, r'\, dr' \leqslant \frac{1}{2\pi}\int_{R<r<3R} |\nabla \boldsymbol{w}|^2\, dx\, dy\,.$$

To estimate the first term on the right of (4.8), we note that by Schwarz's inequality

$$\left|\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} p r'\, dr'\, d\theta'\right|^2 \leqslant \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} p^2 r'\, dr'\, d\theta'$$

$$\leqslant \frac{1}{\pi R^2}\int_{R<r<3R} p^2\, dx\, dy \leqslant \frac{4}{\pi}\max_{R<r<3R}\int_0^{2\pi} p(r, \theta)^2\, d\theta\,.$$

Thus we see from (4.8) that

$$|p(2R, \theta)| \leqslant \left\{\frac{4}{\pi}\max_{R<r<3R}\int_0^{2\pi} p(r, \theta)^2\, d\theta\right\}^{\frac{1}{2}} + \frac{1}{2\pi}\int_{R<r<3R} |\nabla \boldsymbol{w}|^2\, dx\, dy\,.$$

We see from (4.4) with $p_\infty = 0$ and from (4.0) that the right-hand side approaches zero as $R \to \infty$. Thus we have (4.7).

Theorem 3 follows immediately from this lemma and the observation that $\{\boldsymbol{w}, p - p_\infty\}$ is also a solution of the Navier-Stokes equations.

5. – Mean convergence of the velocity.

It is an immediate consequence of the Navier-Stokes equations that the quantity

$$\Phi = p + \tfrac{1}{2}|\boldsymbol{w}|^2$$

satisfies the equation

$$\Delta\Phi - \boldsymbol{w}\cdot\nabla\Phi = \omega^2\,.$$

Since the right-hand side is non-negative, Φ cannot have an interior maximum unless it is constant. It follows that the quantity

$$\max_\theta \Phi(r, \theta)$$

also has no maxima. Hence it must be monotone for sufficiently large r. Therefore this quantity has a limit in the extended sense:

$$\lim_{r\to\infty} \max_{\theta} \Phi(r, \theta) = A \in [-\infty, \infty] .$$

Since $p(r, \theta)$ has a limit p_∞, it follows that the limit

$$\lim_{r\to\infty} \max_{\theta} |\boldsymbol{w}(r, \theta)| = L , \qquad L \in [0, \infty] , \tag{5.1}$$

exists.

If L is finite, then $|\boldsymbol{w}|$ is bounded so that the conclusions of [1] are valid. For the sake of completeness we shall state and prove our theorem on mean convergence of the velocity in such a way that it contains these results. The proof will require the following strengthened form of Lemma 2.3.

LEMMA 5.1. *Under the hypotheses of Lemma* 2.3, *we have*

$$\int_{r>r_1} \frac{r}{(\log r)^{\frac{1}{2}}} |\nabla\omega|^2 \, dx\, dy < \infty \qquad (r_1 > \max(r_0, 2 - r_0)) . \tag{5.2}$$

PROOF. Choose $R > r_1 > \max(r_0, 2 - r_0)$ and non-negative C^2 functions ξ_1 and ξ_2 with the properties (2.9). Letting

$$\eta = \xi_1(r)\, \xi_2(r/R)\, \frac{r}{(\log r)^{\frac{1}{2}}} ,$$

we insert this function and $h(\omega) = \omega^2$ into (2.8) to obtain

$$2\int \eta |\nabla\omega|^2 \, dx\, dy = \int \omega^2 (\Delta\eta + \boldsymbol{w}\cdot\nabla\eta) \, dx\, dy . \tag{5.3}$$

One verifies easily that there is a constant C independent of R such that

$$|\Delta\eta| \leqslant C , \qquad |\nabla\eta| \leqslant C/(\log r)^{\frac{1}{2}}$$

and hence from (5.3) and (2.19)

$$\int_{r_1<r<R} \frac{r}{(\log r)^{\frac{1}{2}}} |\nabla\omega|^2 \leqslant \int \eta |\nabla\omega|^2 \, dx\, dy \leqslant C \int_{r>r_1} \omega^2 \, dx\, dy . \tag{5.4}$$

Letting $R \to \infty$, we obtain (5.2).

We again define the averaged quantity

$$\bar{\boldsymbol{w}}(r) = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{w}(r, \theta)\, \mathrm{d}\theta\,,$$

and put

$$\psi(r) = \arg\big(\bar{u}(r) + i\bar{v}(r)\big)\,.$$

THEOREM 4. *Let* $\boldsymbol{w} = (u, v)$ *be a solution of the Navier-Stokes equations* (1.1) *in* $r \geqslant r_0$ *and let*

$$\int_{r>r_0} |\nabla \boldsymbol{w}|^2\, dx\, dy < \infty\,.$$

Then

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{w}(r, \theta) - \bar{\boldsymbol{w}}(r)|^2\, d\theta = 0 \tag{5.5}$$

and

$$\lim_{r\to\infty} |\bar{\boldsymbol{w}}| = L\,, \tag{5.6}$$

where L *is defined by* (5.1). *If* $0 < L < \infty$, *then*

$$\lim_{r\to\infty} \psi(r) = \psi_\infty \tag{5.7}$$

exists and

$$\lim_{r\to\infty} \int_0^{2\pi} \left[(u(r, \theta) - L\cos\psi_\infty)^2 + (v(r, \theta) - L\sin\psi_\infty)^2\right] d\theta = 0\,,$$

while if $L = +\infty$

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{w}(r, \theta)|^2\, d\theta = \infty\,. \tag{5.8}$$

PROOF. To prove (5.5) we note that because of Wirtinger's inequality

$$\frac{d}{dr}\int_0^{2\pi} |\boldsymbol{w} - \bar{\boldsymbol{w}}|^2\, d\theta = \int_0^{2\pi} 2\boldsymbol{w}_r \cdot (\boldsymbol{w} - \bar{\boldsymbol{w}})\, d\theta$$

$$\leqslant \int_0^{2\pi} \left[r|\boldsymbol{w}_r|^2 + \frac{|\boldsymbol{w} - \bar{\boldsymbol{w}}|^2}{r}\right] d\theta \leqslant \int_0^{2\pi} |\nabla \boldsymbol{w}|^2\, r\, d\theta\,.$$

Since the right-hand side is integrable with respect to r, we find that $\int_0^{2\pi} |\boldsymbol{w} - \bar{\boldsymbol{w}}|^2 \, d\theta$ has a limit as $r \to \infty$. On the other hand, again by Wirtinger's inequality,

$$\int_0^{\infty} \int_{r_0}^{2\pi} |\boldsymbol{w} - \bar{\boldsymbol{w}}|^2 \, d\theta \, \frac{dr}{r} < \infty .$$

It follows that the limit must be zero, so that (5.5) holds.

To prove (5.6), we recall that there is a sequence $\{r_n\}$ with $2^n < r_n < 2^{n+1}$ such that (2.4) holds with $\boldsymbol{f} = \boldsymbol{w}$. Then

$$\lim_{n\to\infty} \int_0^{2\pi} |\boldsymbol{w}_\theta(r_n, \theta)|^2 \, d\theta = 0 .$$

Since for any θ and φ

$$|\boldsymbol{w}(r_n, \theta) - \boldsymbol{w}(r_n, \varphi)|^2 \leqslant \pi \int_0^{2\pi} |\boldsymbol{w}_\theta(r_n, \theta)|^2 \, d\theta ,$$

it follows from the definition (5.1) of L that

$$\lim_{n\to\infty} |\boldsymbol{w}(r_n, \theta)| = L ,$$

uniformly in θ. In particular,

$$\lim_{n\to\infty} |\bar{\boldsymbol{w}}(r_n)| = L .$$

But for $r \in [r_n, r_{n+1}]$

$$|\bar{\boldsymbol{w}}(r) - \bar{\boldsymbol{w}}(r_n)|^2 = \left| \frac{1}{2\pi} \int_{r_n}^{r} \int_0^{2\pi} \boldsymbol{w}_r(\varrho, \theta) \, d\varrho \, d\theta \right|^2$$

$$\leqslant \frac{1}{2\pi} \int_{r>r_n} |\nabla \boldsymbol{w}|^2 \, dx \, dy \int_{r_n}^{r} \frac{d\varrho}{\varrho} \leqslant \frac{\log 4}{2\pi} \int_{r>r_n} |\nabla \boldsymbol{w}|^2 \, dx \, dy ,$$

which goes to zero with as n approaches infinity. Thus (5.6) is valid.

To prove (5.7) we note that if $L > 0$ and $\gamma \in (0, L)$, there is an $\hat{r}$ such that $|\bar{\boldsymbol{w}}| \geqslant \gamma$ for $r > \hat{r}$. We see from the form (4.1) of the Navier-Stokes equa-

tions that

$$\omega_r + uv_r - vu_r + \frac{1}{r} p_\theta = 0 .$$

Averaging this equation and dividing by $|\overline{\boldsymbol{w}}|^2$, we have

$$\frac{\bar{\omega}'}{|\overline{\boldsymbol{w}}|^2} + \psi' + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|\overline{\boldsymbol{w}}|^2} [(u - \bar{u}) v_r - (v - \bar{v}) u_r] \, d\theta = 0 .$$

Hence for $\varrho_2 > \varrho_1 > \hat{r}$ we have

$$\psi(\varrho_2) - \psi(\varrho_1) = - \frac{1}{2\pi} \int_{\varrho_1}^{\varrho_2} \int_0^{2\pi} \frac{1}{|\overline{\boldsymbol{w}}(r)|^2} \cdot \{\omega_r(r, \theta) + [u(r, \theta) - \bar{u}(r)] v_r - [v(r, \theta) - \bar{v}(r)] u_r\} \, dr \, d\theta .$$

Since $|\overline{\boldsymbol{w}}| \geqslant \gamma$,

$$|\psi(\varrho_2) - \psi(\varrho_1)| \leqslant \frac{1}{4\pi\gamma^2} \int_{\varrho_1}^{\varrho_2} \int_0^{2\pi} \left\{ \frac{r}{\log r} \omega_r^2 + \frac{\log r}{r^3} + \frac{1}{r^2} |\boldsymbol{w} - \overline{\boldsymbol{w}}|^2 + |\boldsymbol{w}_r|^2 \right\} r \, dr \, d\theta ,$$

and by Wirtinger's inequality

$$|\psi(\varrho_2) - \psi(\varrho_1)| \leqslant \frac{1}{4\pi\gamma^2} \left\{ \int_{\varrho_1 < r < \varrho_2} \left(\frac{r}{\log r} |\nabla\omega|^2 + |\nabla \boldsymbol{w}|^2 \right) dx \, dy + 2\pi \int_{\varrho_1}^{\varrho_2} \frac{\log r}{r^2} \, dr \right\} .$$

Since, by virtue of Lemma 5.1, the right-hand side approaches zero as ϱ_1, $\varrho_2 \to \infty$, it follows that $\psi(r)$ has a limit ψ_∞.

If $L = \infty$, (5.8) follows from (5.5), (5.6), and the triangle inequality.

6. – Some estimates for the derivatives of the velocity.

By using Lemma 5.1 we can improve the result of Lemma 2.4.

THEOREM 5.

$$\lim_{r \to \infty} \frac{r^{\frac{3}{4}}}{(\log r)^{\frac{1}{8}}} |\omega(r, \theta)| = 0 \tag{6.1}$$

uniformly in θ.

PROOF. We note that for $2^n > r_0$

$$\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} \left(r^2\omega^2 + 2\frac{r^{\frac{3}{2}}}{(\log r)^{\frac{1}{4}}} |\omega\omega_\theta| \right) d\theta \leqslant \int_{2^n<r<2^{n+1}} \left(\omega^2 + 2\frac{r^{\frac{1}{2}}}{(\log r)^{\frac{1}{4}}} |\omega||\nabla\omega| \right) dx\,dy$$

$$\leqslant \int_{r>2^n} \left(2\omega^2 + \frac{r}{(\log r)^{\frac{1}{2}}} |\nabla\omega|^2 \right) dx\,dy .$$

Using (5.2) and proceeding exactly as in the proof of Lemma 2.4, we obtain (6.1).

If L is finite in (5.1) so that $|\boldsymbol{w}|$ is bounded, we may suppress the logarithmic term in both the statement and the proof of Lemma 5.1. We may then do the same in Theorem 5. Thus we establish the following result.

THEOREM 6. *If $L < \infty$ in* (5.1), *so that $|\boldsymbol{w}|$ is bounded, then*

$$\int_{r<r_0} r|\nabla\omega|^2 < \infty \tag{6.2}$$

and

$$\lim_{r\to\infty} r^{\frac{3}{2}}|\omega(r,\theta)| = 0 \tag{6.3}$$

uniformly in θ.

The results in Theorems 5 and 6 on the decay of the vorticity can be extended to the first derivatives of the velocity. For this purpose we prove the following lemma.

LEMMA 6.1. *The vorticity ω satisfies a Hölder condition*

$$|\omega(z_1) - \omega(z_2)| \leqslant C\mu(R)|z_1 - z_2|^{\frac{1}{2}}, \quad |z_1|,\ |z_2| > R + 2,\ |z_1 - z_2| \leqslant 1, \tag{6.4}$$

where C is a constant independent of R and

$$\begin{cases} \lim\limits_{R\to\infty} R^{\frac{3}{4}}(\log R)^{-3/8}\mu(R) = 0 , \\ \lim\limits_{R\to\infty} R^{\frac{3}{4}}\mu(R) = 0 \qquad \textit{if } |\boldsymbol{w}| \textit{ is bounded} . \end{cases} \tag{6.5}$$

PROOF. We define

$$\mu(R) = \sup_{r\geqslant R} |\omega(r,\theta)|[1 + |\boldsymbol{w}(r,\theta)|^{\frac{1}{2}}] , \tag{6.6}$$

so that (6.5) is a consequence of (2.19), (6.1), and (6.3). We need to prove (6.4).

Let $C_{r'} = C_{r'}(z_0)$ denote the disc of radius r' and center z_0, with $|z_0| > R + 2$. We set

$$D(r') = D(r'; z_0) = \int_{C_{r'}} |\nabla\omega|^2 \, dx\, dy$$

and show first that

$$D(1) \leqslant C\mu^2(R) \tag{6.7}$$

for an absolute constant C. Namely let η be a non-negative C^2 cut-off function such that $\eta(r) = 1$ for $r \leqslant 1$, $\eta = 0$ for $r \geqslant 2$. Inserting $\eta = \eta(|z - z_0|)$ and $h(\omega) = \omega^2$ into (2.8), we obtain

$$2\int_{C_2} \eta |\nabla\omega|^2 \, dx\, dy = \int_{C_2} \omega^2 (\Delta\eta + \boldsymbol{w}\cdot\nabla\eta) \, dx\, dy \,,$$

and since $|\Delta\eta| \leqslant C$, $|\nabla\eta| \leqslant C$ in C_2 for constants C depending on the choice of η, we have

$$D(1) = \int_{C_1} |\nabla\omega|^2 \, dx\, dy \leqslant C\mu^2(R) \,.$$

We now derive a growth estimate for $D(r)$, from which (6.4) will follow. Multiplying the vorticity equation (2.7) by ω, integrating by parts, and using the fact that $\nabla\cdot\boldsymbol{w} = 0$, we find

$$\int_{C_{r'}} |\nabla\omega|^2 \, dx\, dy = \int_{\partial C_{r'}} \omega\omega_{r'} r' \, d\theta' - \tfrac{1}{2}\int_{\partial C_{r'}} \omega^2 \boldsymbol{w}\cdot\boldsymbol{r}' \, d\theta' \,, \tag{6.8}$$

where r', θ' are now polar coordinates with respect to z_0 as origin.

Setting

$$\bar{\omega}(r') = \frac{1}{2\pi}\int_{\partial C_{r'}} \omega(r', \theta') \, d\theta' \,,$$

we have in (6.8)

$$\int_{\partial C_{r'}} \omega\omega_{r'} r' \, d\theta' = \int_{\partial C_{r'}} (\omega - \bar{\omega})\omega_{r'} r' \, d\theta' + \bar{\omega}\int_{\partial C_{r'}} \omega_{r'} r' \, d\theta'$$

$$\leqslant \frac{r'}{2}\int_{\partial C_{r'}} \left[\frac{(\omega - \bar{\omega})^2}{r'^2} + \omega_{r'}^2\right] r' \, d\theta + \bar{\omega}\int_{C_{r'}} \Delta\omega \, dx\, dy \,.$$

From Wirtinger's inequality and (2.7) we see that the right member is at most

$$\frac{r'}{2}\int_{\partial C_{r'}}|\nabla\omega|^2 r'\,d\theta+\bar{\omega}\int_{\partial C_{r'}}\omega\boldsymbol{w}\cdot\boldsymbol{r}'\,d\theta\leqslant\frac{r'}{2}D'(r')+C\mu^2(R)r'\,.$$

We then see from (6.8) that

$$D(r')\leqslant\frac{r'}{2}D'(r')+Ar'\,,\qquad A=C\mu^2(R)\,.$$

It follows that

$$\left(D(r')/r'^2\right)'\geqslant-2A/r'^2\,.$$

Integrating this inequality between r' and 1, we obtain

$$D(1)-D(r')/r'^2\geqslant 2A(1-1/r')\geqslant-2A/r'\,,$$

and hence by (6.7)

$$D(r')\leqslant D(1)r'^2+2Ar'\leqslant C\mu^2(R)r'\qquad(r'\leqslant 1)\,. \tag{6.9}$$

Since this estimate is valid for all discs $|z-z_0|\leqslant r'\leqslant 1$ contained in $|z|>R+2$, it follows from Morrey's lemma (see [6], for example) that

$$|\omega(z_1)-\omega(z_2)|\leqslant C\mu(R)|z_1-z_2|^{\frac{1}{2}}$$

for all z_1, z_2 such that $|z_1|$, $|z_2|>R+2$ and $|z_1-z_2|\leqslant 1$; the constant C depends only on the constant in (6.9) and is therefore independent of R. This proves the lemma.

THEOREM 7. *If a solution $\boldsymbol{w}=(u,v)$ of the Navier-Stokes equations has bounded Dirichlet integral in $r\geqslant r_0$ then*

$$\lim_{r\to\infty} r^{3/4}(\log r)^{-9/8}|\nabla\boldsymbol{w}(r,\theta)|=0 \tag{6.10}$$

uniformly in θ.

PROOF. We apply the Cauchy integral formula (2.20) in a disc $C_R=C_R(z)$ of radius R and center z with $|z|=2R>2\max(r_0,2)$. For $w=u-iv$ this gives

$$w(z')=\frac{1}{2\pi i}\left\{\oint_{\partial C_R}\frac{w(\zeta)}{\zeta-z'}\,d\zeta+\int_{C_R}\frac{\omega(\zeta)}{\zeta-z'}\,d\xi\,d\eta\right\},\qquad z'\in C_R,\ \zeta=\xi+i\eta\,.$$

We rewrite this formula as

$$w(z') = \frac{1}{2\pi i}\left\{ \oint_{\partial C_R} \frac{w(\zeta)}{\zeta - z'}\, d\zeta + \int_{C_R} \frac{\omega(\zeta) - \omega(z)}{\zeta - z'}\, d\xi\, d\eta + \omega(z) \int_{C_R} \frac{d\xi\, d\eta}{\zeta - z'} \right\}$$

and since

$$\int_{C_R} \frac{d\xi\, d\eta}{\zeta - z'} = \pi(\bar{z} - \bar{z}')\,,$$

we may differentiate this representation to find that

$$w_z(z) = \frac{1}{2\pi i}\left\{ \oint_{\partial C_R} \frac{w(\zeta)}{(\zeta - z)^2}\, d\zeta + \int_{C_R} \frac{\omega(\zeta) - \omega(z)}{(\zeta - z)^2}\, d\xi\, d\eta \right\}$$

where $w_z = \frac{1}{2}(w_x - i w_y)$. One sees from (2.19) that the line integral is $o(\sqrt{\log R}/R)$. We estimate the area integral by using (6.4) and the definition (6.6) of $\mu(R)$:

$$\left| \int_{C_R} \frac{\omega(\zeta) - \omega(z)}{(\zeta - z)^2}\, d\xi\, d\eta \right| \leqslant \int_{|\zeta - z| < 1} + \int_{1 < |\zeta - z| < R} \frac{|\omega(\zeta) - \omega(z)|}{|\zeta - z|^2}\, d\xi\, d\eta$$

$$\leqslant C\mu(R) \int_0^{2\pi} \int_0^1 \frac{\varrho^{\frac{1}{2}}}{\varrho^2}\, \varrho\, d\varrho\, d\varphi + 2 \sup_{|\zeta| > R} |\omega| \int_0^{2\pi} \int_1^R \frac{\varrho\, d\varrho\, d\varphi}{\varrho^2}$$

$$\leqslant C[\mu(R) + \log R \sup_{\varrho \geqslant R} |\omega(\varrho, \varphi)|]$$

for a suitable constant C. We recall that $R = \frac{1}{2}|z|$ and that $|w_{\bar{z}}| = |-\omega/2i| \leqslant \mu(R)$. Combining these estimates, we see that

$$|w_z(z)| + |w_{\bar{z}}(z)| \leqslant C\mu(\tfrac{1}{2}r) + \log \tfrac{1}{2}r \sup_{\varrho \geqslant \frac{1}{2}r} |\omega(\varrho, \varphi)|]\,.$$

(6.10) now follows from (6.5) and (6.1).

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