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Boundary values for Sobolev-spaces with weights. Density of $D(\Omega)$ in $W^s_{p,\gamma_0,\ldots,\gamma_r}(\Omega)$ and in $H^s_{p,\gamma_0,\ldots,\gamma_r}(\Omega)$ for $s>0$ and $r=\left[s-\frac{1}{p}\right]^-$

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BOUNDARY VALUES FOR SOBOLEV - SPACES WITH WEIGHTS. DENSITY OF $D(\Omega)$ IN $W^s_{p,\gamma_0,\dots,\gamma_r}(\Omega)$ AND IN $H^s_{p,\gamma_0,\dots,\gamma_r}(\Omega)$ FOR $s>0$ AND $r=\left[s-\frac{1}{p}\right]^-$

by HANS TRIEBEL

1. Introduction and results.

Let Ω be a bounded domain in the Euclidean n-space R_n with smooth boundary. $\partial\Omega \in C^\infty$. We consider the SOBOLEV-SLOBODECKIJ-spaces $W^s_p(\Omega)$; $s>0$; $1<p<\infty$;

$$W^s_p(\Omega)=\{f\,|\,f\in D'(\Omega),\ \|f\|_{W^s_p}=\sum_{|\alpha|\leqq s}\|D^\alpha f\|_{L_p}<\infty\}$$

for $s=$ integer,

$$W^s_p(\Omega)=\Big\{f\,|\,f\in D'(\Omega),\ \|f\|_{W^s_p}=$$

$$=\|f\|_{W^{[s]}_p}+\sum_{|\alpha|=[s]}\left(\int_{\Omega\times\Omega}\frac{|D^\alpha f(x)-D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}}\,dx\,dy\right)^{\frac{1}{p}}<\infty\Big\}$$

for $s\neq$ integer, $s=[s]+\{s\}$ with $[s]$ integer, $0<\{s\}<1$. $D'(\Omega)$ denotes the complex distributions over Ω. We have a similar definition when we replace Ω by R_n or an other bounded or unbounded domain. It is well-known that $W^s_p(\Omega)$ is the restriction of $W^s_p(R_n)$ to Ω, and the norms $\|f\|_{W^s_p(\Omega)}$ and

$$\inf_{\tilde f\in W^s_p(R_n)}\|\tilde f\|_{W^s_p(R_n)}$$

$$\tilde f(x)=f(x)\quad\text{for}\quad x\in\Omega$$

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are equivalent. Further we consider the LEBESGUE — spaces or BESSEL — potential-spaces $H_p^s(\Omega)$. The definition of $H_p^s(R_n)$, $s > 0$, $1 < p < \infty$, is

$$H_p^s(R_n) = \{f \mid f \in S'(R_n),\ g = F^{-1}(1 + |\xi|^2)^{\frac{s}{2}} Ff \in L_p(R_n)\}$$

with

$$\|f\|_{H_p^s(R_n)} = \|g\|_{L_p(R_n)}.$$

S' is the set of tempered distributions. F is the Fouriertransformation. F^{-1} is the inverse Fouriertransformation. $H_p^s(\Omega)$ is defined as the restriction of $H_p^s(R_n)$ to Ω,

$$\|f\|_{H_p^s(\Omega)} = \inf_{\tilde{f} \in H_p^s(R_n)} \|\tilde{f}\|_{H_p^s(R_n)}$$

$$f(x) = \tilde{f}(x) \quad \text{for} \quad x \in \Omega.$$

If s an integer, so holds $H_p^s(\Omega) = W_p^s(\Omega)$.

Let $f \in W_p^s(\Omega)$ or $f \in H_p^s(\Omega)$, $s > \frac{1}{p}$. Then

$$\gamma_j f|_{\partial\Omega} = \frac{\partial^j f}{\partial \nu^j}\Big|_{\partial\Omega} \in L_p(\partial\Omega) \quad \text{for} \quad 0 \leqq j < s - \frac{1}{p}$$

$\nu = \nu_y$ denotes the normal vector in $y \in \partial\Omega$ (See [10] or [12]). For a real number α we set

$$\alpha = [\alpha]^- + \{\alpha\}^+ \text{ with } [\alpha]^- \text{ integer}, \quad 0 < \{\alpha\}^+ \leqq 1.$$

For $s > \frac{1}{p}$ we define

$$W_{p,\gamma_0,\ldots,\gamma_r}^s(\Omega) = \left\{f \mid f \in W_p^s(\Omega),\ \gamma_j f|_{\partial\Omega} = 0;\ 0 \leqq j \leqq \left[s - \frac{1}{p}\right]^- = r\right\},$$

and in the same way $H_{p,\gamma_0,\ldots,\gamma_r}^s(\Omega)$. Further we write $\overset{\circ}{W}{}_p^s(\Omega)$ ($\overset{\circ}{H}{}_p^s(\Omega)$) for the completion of $D(\Omega)$ in $W_p^s(\Omega)$ ($H_p^s(\Omega)$). $D(\Omega)$ is the set of all complex infinitely differentiable functions with compact support in Ω.

THEOREM 1. *Let* $s > \frac{1}{p}$ *and* $r = \left[s - \frac{1}{p}\right]^-$. *Then holds*

$$W_{p,\gamma_0,\ldots,\gamma_r}^s(\Omega) = \overset{\circ}{W}{}_p^s(\Omega)$$

and

$$H^s_{p,\gamma_0,\ldots,\gamma_r}(\Omega) = \overset{\circ}{H}{}^s_p(\Omega).$$

For $s \leqq \dfrac{1}{p}$ *holds*

$$W^s_p(\Omega) = \overset{\circ}{W}{}^s_p(\Omega) \text{ and } H^s_p(\Omega) = \overset{\circ}{H}{}^s_p(\Omega).$$

For $p = 2$ the result is known and proved by LIONS and MAGENES [6]. For $1 < p < \infty$, $p \neq 2$, the result for the W-spaces is also known for the « non-singular » cases $s - \dfrac{1}{p} \neq$ integer, see LIONS-MAGENES [8]. For the H-spaces in the non-singular cases see SHANIR [14]. The density of $D(\Omega)$ in $W^s_p(\Omega)$ and $H^s_p(\Omega)$, $s < \dfrac{1}{p}$, is also known and proved by LIONS and MAGENES in [7]. In [7] is also a proof for $W^{1/p}_p(\Omega) = \overset{\circ}{W}{}^{1/p}_p(\Omega)$. The author is unknown if the problem for the singular cases $s - \dfrac{1}{p} =$ integer is solved. In the book of LIONS and MAGENES is it remarked as a problem ([6], problem 18.3, p. 116). We give a proof including the singular cases. The considerations show that the main part of this note is concerned with the singular cases, the considerations for the non-singular cases are simple and more or less an appendix to the singular cases. Our main tool is a comparsion of the W spaces and the H-spaces with special SOBOLEV-spaces with weights on the background of interpolation theory. So we carry over the singular cases for W-spaces and H-spaces to singular cases for SOBOLEV-spaces with weights. After solving the problem for these spaces we return to the W-spaces and H-spaces. Now we describe the needed SOBOLEV-spaces with weights.

We set

$$M = \{x \mid x = (x_1, \ldots, x_n) \in R_n,\ 0 < x_n < 1\}.$$

For $1 < p < \infty$, an integer l; $l = 1, 2, \ldots$; and a real number α; $0 \leqq \alpha \leqq lp$; we define

$$P_{l,\alpha,p} = \left\{ f \mid f \in D'(M),\ \|f\|_{P_{l,\alpha,p}} = \left\| x_n^{\alpha/p} \frac{\partial^l f}{\partial x_n^l} \right\|_{L_p} + \|f\|_{L_p} < \infty \right\}.$$

$P_{l,\alpha,p}$ is a BANACH-space. With spaces of such or similar type are concerned many papers in the last years, see the collected papers in [20], especially the papers of DŽABRAILOV, JU. S. NIKOL'SKIJ and USPENSKIJ. We refer also to [1]. The boundary values on the hyperplane $\{x \mid x_n = 0\}$ are known.

But for selfcontainedness we shall develop the needed results. We set

$$M^+ = \{x \mid x \in R_n,\ 0 < x_n \leqq 1\}$$

and denote with $\overset{\circ}{P}_{l,\alpha,p}$ the closure of all functions $f \in C^\infty(\bar{M})$ with compact support in M^+. Further we need an interpolation method. We use the K-method, developed by LIONS-PEETRE [9] and PEETRE [13] (See also [2]). Let B_0 and B_1 be two BANACH-spaces with $B_1 \subset B_0$. Then we set for $u \in B_0$ and $t > 0$

$$K(t, u) = K(t, u, B_0, B_1) = \inf_{\substack{u=u_0+u_1 \\ u_1 \in B_1}} (\| u_0 \|_{B_0} + t \| u_1 \|_{B_1}),$$

and for $0 < \theta < 1$, and $1 \leqq p \leqq \infty$,

$$(B_0, B_1)_{\theta,p} = \left\{u \mid u \in B_0,\ \| u \|_{\theta,p} = \left(\int_0^\infty (t^{-\theta} K(t, u))^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \right\}.$$

(For $p = \infty$ we have to change the definition in the usual way). $(B_0, B_1)_{\theta,p}$ is a BANACH-space and $\| u \|_{\theta,p}$ is a norm.

THEOREM 2. (a) *Let* $0 < \theta < 1$, *and* $0 \leqq \alpha_1 \leqq \alpha_0 \leqq lp$. *Then holds*

$$(P_{l,\alpha_0,p} \cdot P_{l,\alpha_1,p})_{\theta,p} = P_{l,\alpha_0(1-\theta)+\alpha_1\theta,p}.$$

(b) *Let* $0 \leqq \alpha < lp - 1$. *For a function* $f \in P_{l,\alpha,p}$ *the expression* $\frac{\partial^j f}{\partial x_n^j}(x)$; $0 \leqq j \leqq l - \left[\frac{\alpha+1}{p}\right] - 1$; *has boundary values on the plane* $\{x \mid x_n = 0\}$ *lying in* $L_p(R_{n-1})$, *and*

$$\sum_{j=0}^{l-\left[\frac{\alpha+1}{p}\right]-1} \left\| \frac{\partial^j f}{\partial x_n^j}(x_1, \ldots, x_{n-1}, 0) \right\|_{L_p(R_{n-1})} \leqq c \| f \|_{P_{l,\alpha,p}}$$

(*c does not depend on* f).

(c) *Let* $0 \leqq \alpha < lp - 1$. *Then holds*

$$\overset{\circ}{P}_{l,\alpha,p} = \left\{ f \mid f \in P_{l,\alpha,p},\ \frac{\partial^j f}{\partial x_n^j}\Big|_{x_n=0} = 0 \text{ for } 0 \leqq j \leqq l - \left[\frac{\alpha+1}{p}\right] - 1 \right\}.$$

For $lp - 1 \leqq \alpha \leqq lp$ *holds*

$$\overset{\circ}{P}_{l,\alpha,p} = P_{l,\alpha,p}.$$

The most difficult part is the proof of (*c*) for the singular cases $\frac{\alpha}{p} =$ integer $-\frac{1}{p}$.

First we prove theorem 2. On the basis of this result we prove theorem 1.

Interpolation theory and the method for the proof of theorem 1 lead to a sharper result than theorem 1. For description of this result we introduce the BESOV-spaces

$$B^s_{pq}(\Omega) = (L_p(\Omega),\ W^l_p(\Omega))_{\theta,q},$$

l integer; $l = 1, 2, \ldots$; $0 < \theta < 1$; $s = \theta l$; $1 < p < \infty$; $1 \leqq q \leqq \infty$. It is possible to describe the norms of $B^s_{pq}(\Omega)$ explicitly, but we do not make it, see [11] or [19]. (For the domain M, $q = 2$, and $s \neq$ integer see formula (33). In the general case the norms have a similar structure). Now we can formulate a result which is sharper than theorem 1. We denote with $\overset{\circ}{B}{}^s_{pq}(\Omega)$ the completion of $D(\Omega)$ in $B^s_{pq}(\Omega)$.

THEOREM 3. (a) *Let* $1 < p < \infty$; $1 < q \leqq \infty$. *For* $s > \frac{1}{p}$ *holds*

$$\overset{\circ}{B}{}^s_{pq}(\Omega) = \left\{ f \,|\, f \in B^s_{pq}(\Omega),\ \left.\frac{\partial^j f}{\partial \nu^j}\right|_{\partial\Omega} = 0;\ j = 0, \ldots, \left[s - \frac{1}{p}\right]^- \right\}.$$

For $0 < s \leqq \frac{1}{p}$ *holds*

$$\overset{\circ}{B}{}^s_{pq}(\Omega) = B^s_{pq}(\Omega).$$

(b) *Let* $1 < p < \infty$. *For* $s \geqq \frac{1}{p}$ *holds*

$$\overset{\circ}{B}{}^s_{p,1}(\Omega) = \left\{ f \,|\, f \in B^s_{p,1}(\Omega),\ \left.\frac{\partial^j f}{\partial \nu^j}\right|_{\partial\Omega} = 0;\ j = 0, \ldots, \left[s - \frac{1}{p}\right] \right\}.$$

For $0 < s < \frac{1}{p}$ *holds*

$$\overset{\circ}{B}{}^s_{p,1}(\Omega) = B^s_{p,1}(\Omega).$$

Theorem 3 is sharper than theorem 1 because

$$B^s_{p,\min(2,p)}(\Omega) \subset H^s_p(\Omega) \subset B^s_{p,\max(2,p)}(\Omega)$$

and

$$B^s_{p,\min(2,p)}(\Omega) \subset W^s_p(\Omega) \subset B^s_{p,\max(2,p)}(\Omega).$$

The singular cases $s = \text{integer} + \frac{1}{p}$ are the most interesting cases. The results for the non-singular cases follow immediately from theorem 1. For fixed p the spaces B^s_{pq} with $1 \leqq q \leqq \infty$ are very « near » to each other in the sense of interpolation theory. From this point of view the difference in (a) and (b) for the singular cases makes clear that the question of boundary values and approximation in these cases is delicate.

The motive for the considerations in this paper is the following. In [18] we show that the spaces $W^s_p(\Omega)$ and $H^s_p(\Omega)$, $s > 0$, are isomorphic to l_p or $L_p((0,1))$. Especially they have a SCHAUDER-basis. With help of theorem 1 follows in an easy way that the spaces $\overset{\circ}{W}{}^s_p(\Omega)$ (and $\overset{\circ}{H}{}^s_p(\Omega)$), $s > 0$, are complemented subspaces of $W^s_p(\Omega)$ (and $H^s_p(\Omega)$). So they are also isomorphic to l_p or $L_p((0,1))$, and they have also a SCHAUDER basis.

2. Proofs.

2.1. *Density property for the spaces* $P_{l,\alpha,p}$. We want to show that the C^∞-functions with compact support in $\bar{M}$ are dense in $P_{l,\alpha,p}$; $l = 1, 2, \ldots$; $0 \leqq \alpha \leqq lp$; $1 < p < \infty$. We choose a function $\chi(t)$ with

$$\chi(t) \in C^\infty([0,1]); \quad 0 \leqq \chi(t) \leqq 1; \quad \chi(t) = 1 \quad \text{for} \quad 0 \leqq t \leqq \frac{1}{2}.$$

We set $\psi(t) = 1 - \chi(t)$. Let be $u \in P_{l,\alpha,p}$. Then holds

$$\chi(x_n)\, u(x) \in P_{l,\alpha,p} \quad \text{and} \quad \psi(x_n)\, u(x) \in W^l_{p,x_n}(M).$$

By this holds

$$W^l_{p,x_n}(M) = \left\{ f \,|\, f \in L_p, \frac{\partial^l f}{\partial x_n^l} \in L_p \right\}, \quad \|f\|_{W^l_{p,x_n}} = \|f\|_{L_p} + \left\| \frac{\partial^l f}{\partial x_n^l} \right\|_{L_p}.$$

(The existence of $\frac{\partial^j}{\partial x_n^j}(\psi u)$ and the estimate

$$\left\| \frac{\partial^j \psi u}{\partial x_n^j} \right\|_{L_p} \leqq c \,\|u\|_{W^l_{p,x_n}\left(R_{n-1}\times\left(\frac{1}{2},1\right)\right)}, \quad 0 \leqq j \leqq l,$$

follows from the well-known theory for the spaces $W_p^l((a, b))$. But we can approximate $\psi(x_n)\, u(x)$ in the desidered way in $W_{p, x_n}^l(M)$ and so also in $P_{l, \alpha, p}$. So we may assume without loss of generality

$$u \in P_{l, \alpha, p}, \quad u(x) = 0 \quad \text{for} \quad x \in M, \quad 0 < \varkappa < x_n < 1.$$

We set $u(x) = 0$ for $x_n \geqq 1$. For $1 - \varkappa > \delta > 0$ is

$$u_\delta(x) = u(x_1, \ldots, x_{n-1}, x_n + \delta) \in W_{p, x_n}^l(M).$$

It is not hard to show

$$u_\delta(x) \longrightarrow u(x) \quad \text{in} \quad P_{l, \alpha, p} \quad \text{for} \quad \delta \downarrow 0.$$

On the other hand we can approximate $u_\delta \in W_{p, x_n}^l(M)$ in $W_{p, x_n}^l(M)$ (and so also in $P_{l, \alpha, p}$) in the desired way. This completes the proof.

2.2. *Proof of theorem* 2 (a).

1. STEP. First we consider the special case $\alpha_0 = lp$, $\alpha_1 = 0$. For $f \in P_{l, lp, p}$ we want to show

$$(1) \qquad K^p(t, f, P_{l, lp, p}, P_{l, 0, p}) \backsim \backsim \int_M \left[\min(x_n^{lp}, t^p) \left| \frac{\partial^l f}{\partial x_n^l} \right|^p + \min(1, t^p) |f|^p \right] dx.$$

$\backsim$ means that we can estimate the right side of (1) by the left side with help of a positive constant (independent of t) and vice versa. That the right side of (1) is smaller than the left side (with help of a positive constant) is clear. We have to prove the opposite direction by a « good » decomposition of f in $f = f_0 + f_1$, $f_0 \in P_{l, lp, p}$ and $f_1 \in P_{l, 0, p}$. We assume that f is C^∞-function with compact support in $\bar{M}$. For $\frac{1}{2} < t < \infty$ we set $f = f_0$ and $f_1 = 0$. Then follows the desired inequality.

For $0 < t \leqq \frac{1}{2}$ we need a special construction. On the basis of the well-known HARDY-inequality [4]

$$(2) \qquad \int_0^\infty |v(\rho)|^p \rho^\alpha \, d\rho \leqq c \int_0^\infty |v^{(l)}(\rho)|^p \rho^{\alpha + pl} \, d\rho, \quad v(\rho) \in C^\infty([0, \infty)),$$

$\alpha > 1$; $1 < p < \infty$; follows with help of SOBOLEV's inequalities [15]

$$(3)\quad \int_0^1 |w(\varrho)|^p \, d\varrho \leqq c \left(\int_0^1 |w^{(l)}(\varrho)|^p \, \varrho^{pl} \, d\varrho + \int_{2-\frac{1}{l}}^1 |w(\varrho)|^p \, d\varrho \right); \quad w(\varrho) \in C^\infty([0,1]).$$

Approximation shows that (3) is true for $w \in W_p^l((0,1))$. We return to the case $0 < t \leqq \frac{1}{2}$ and set $x' = (x_1, \dots, x_{n-1})$, and $x = (x', x_n)$. We choose

$$f_1(x) = \begin{cases} f(x) & \text{for} \quad t^{1/l} \leqq x_n < 1 \\ \sum_{j=0}^{l-1} \frac{(x_n - t^{1/l})^j}{j!} \frac{\partial^j f}{\partial x_n^j}(x', t^{1/l}) & \text{for} \quad 0 < x_n < t^{1/l} \end{cases}$$

$f_0(x) = f(x) - f_1(x)$. It holds $f_1 \in W^l_{p,xn}(M)$. With help of (3) follows

$$(4)\quad K^p(t,f) \leqq c\,(\|f_0\|^p_{P_{l,lp,p}} + t^p \|f_1\|^p_{P_{l,0,p}})$$

$$\leqq c' \int_M \left[\min(x_n^{lp}, t^p) \left| \frac{\partial^l f}{\partial x_n^l} \right|^p + t^p |f|^p \right] dx + c' t^p \sum_{j=0}^{l-1} \int_{R_{n-1}} \left| \frac{\partial^j f}{\partial x_n^j}(x', t^{1/l}) \right|^p dx'.$$

Using SOBOLEV's embedding theorems for the intervall $(0,1)$ (see [12] or [15]) we find

$$(5)\quad \sum_{j=0}^{l-1} \int_{R_{n-1}} \left| \frac{\partial^j f}{\partial x_n^j}(x', t^{1/l}) \right|^p dx' \leqq c \int_{t^{1/l}}^{(1-2^{-1/l})+t^{1/l}} \int_{R_{n-1}} \left(\left| \frac{\partial^l f}{\partial x_n^l} \right|^p + |f|^p \right) dx$$

(4) and (5) lead to the desired inequality. This completes the proof of (1) for C^∞-functions with compact support in $\bar{M}$. $K(t, f, P_{l,lp,p}, P_{l,0,p})$ and the $\frac{1}{p}$ — power of the right side of (1) are equivalent norms in $P_{l,lp,p}$. Now the proof of (1) for $f \in P_{l,lp,p}$ follows from 2.1.

2. STEP. We prove theorem 2 (a) for $\alpha_0 = lp$ snd $\alpha_1 = 0$. (1) shows

$$\|f\|^p_{(P_{l,lp,p}, P_{l,0,p})_{\theta,p}} \sim \int_0^\infty t^{-\theta p} \int_M \left[\min(x_n^{lp}, t^p) \left| \frac{\partial^l f}{\partial x_n^l} \right|^p + \min(1, t^p) |f|^p \right] dx \frac{dt}{t}$$

We compute the right side in a completely elementary way. It is equal to

$$c \int_M |f|^p \, dx + c' \int_M x_n^{lp(1-\theta)} \left| \frac{\partial^l f}{\partial x_n^l} \right|^p dx$$

by suitable choice of the positive constants c and c'. But this proves theorem 2 (a) for $\alpha_0 = lp$ and $\alpha_1 = 0$.

3. STEP. The full proof of theorem 2 (a) follows now from the reiteration-theorem of interpolation theory [9] and the special case of the second step.

2.3. *Proof of theorem* 2 (*b*). Let $u(t) \in C^\infty([0,1])$, α a number with $0 \leqq \alpha p' < 1$, where p' is determined by $\frac{1}{p} + \frac{1}{p'} = 1$, and $t' \in [0,1]$ with

$$u(t') = \int_0^1 u(t) \, dt.$$

(We assume without loss of generality that $u(t)$ is real, so that t' exists). Then is

$$|u(0)|^p = \left| -\int_0^{t'} \frac{du}{dt} \, dt + u(t') \right|^p \leqq c \int_0^1 \left(t^{\alpha p} \left| \frac{du}{dt} \right|^p + |u|^p \right) dt.$$

Using HARDY's inequality (3) and SOBOLEV's inequalities [15] we find

$$|u(0)|^p \leqq c \int_0^1 \left(t^\sigma \left| \frac{d^k u}{dt^k} \right|^p + |u|^p \right) dt, \tag{6}$$

σ is an arbitrary number with

$$0 \leqq \sigma < \frac{p}{p'} + (k-1)p = kp - 1.$$

If $f(x)$ a C^∞-function with compact support in $\bar{M}$ then follows form (6)

$$\left\| \frac{\partial^j f}{\partial x_n^j}(x', 0) \right\|_{L_p(R_{n-1})} \leqq c \, \|f\|_{P_{l,\alpha,p}}$$

with

$$0 \leqq \alpha < (k-j)\,p - 1.$$

This leads to theorem 2 (b) for C^∞-functions. The full proof follows from the density property 2.1.

2.4. *Proof of theorem* 2 (c).

1. STEP. A trivial consequence of theorem 2 (b) is

$$(7) \qquad \overset{\circ}{P}_{l,\alpha,p} \subset \left\{ f \,|\, f \in P_{l,\alpha,p}\,,\ \frac{\partial^j f}{\partial x_n^j}\Big|_{x_n=0} = 0 \text{ for } 0 \leqq j \leqq l - \left[\frac{\alpha+1}{p}\right] - 1 \right\}.$$

2. STEP. For the proof of the opposite direction we start with a remark. Let f be a function of the right side of (7). 2.1 shows that we find C^∞-functions φ_k; $k = 1, 2, \ldots$; with compact support in $\bar{M}$ with

$$\varphi_k \to f \text{ in } P_{l,\alpha,p} \quad \text{for} \quad k \to \infty.$$

Then follows from theorem 2 (b)

$$\left\| f - \left(\varphi_k - \sum_{j=0}^{l-\left[\frac{\alpha+1}{p}\right]-1} \frac{x_n^j}{j!} \frac{\partial^j \varphi_k(x', 0)}{\partial x_n^j} \right) \right\|_{P_{l,\alpha p,p}}$$

$$\leqq \| f - \varphi_k \|_{P_{l,\alpha p,p}} + c \sum_{j=0}^{l-\left[\frac{\alpha+1}{p}\right]-1} \left\| \frac{\partial^j \varphi_k(x', 0)}{\partial x_n^j} \right\|_{L_p(R_{n-1})} \to 0$$

for $k \to \infty$. But this shows that it is sufficient to approximate C^∞-functions f with compact support in $\bar{M}$ and with

$$(8) \qquad \frac{\partial^j f}{\partial x_n^j}(x', 0) = 0\,;\ j = 0, \ldots, l - \left[\frac{\alpha+1}{p}\right] - 1\,;$$

by C^∞-functions with compact support in $M+$.

3. STEP. We prove theorem 2 (c) for the non-singular cases $\alpha \neq$ integer $-\frac{1}{p}$. In this case we can use standard estimate technique. We use a set of functions $\chi_\lambda(t)$; $0 < \lambda < \frac{1}{2}$; with

$$\chi_\lambda(t) \in C^\infty([0, 1])\,;\quad \chi_\lambda(t) = 1 \quad \text{for} \quad 0 \leqq t \leqq \lambda,$$

$$\chi_\lambda(t) = 0 \quad \text{for} \quad 2\lambda \leqq t \leqq 1, \quad |\chi_\lambda^{(k)}(t)| \leqq c\,\lambda^{-k}\,,$$

(c independent of λ). Let f be a C^∞-function with compact support in $\bar{M}$ and with (8). For the proof of theorem 2 (c) it is sufficient to show

$$(1 - \chi_\lambda) f \to f \quad \text{in} \quad P_{l,\alpha,p} \quad \text{for} \quad \lambda \downarrow 0. \tag{9}$$

For this it is sufficient to show

$$\int_M x_n^\alpha \left| \frac{\partial^l}{\partial x_n^l} (\chi_\lambda f) \right|^p dx \to 0 \quad \text{for} \quad \lambda \downarrow 0. \tag{10}$$

We set $m = l - \left[\frac{\alpha + 1}{p}\right]$. Now it is

$$\int_M x_n^\alpha \left| \frac{\partial^l}{\partial x_n^l} (\chi_\lambda f) \right|^p dx \leqq c \sum_{j=0}^{l} \int_0^{2\lambda} \int_{R_{n-1}} x_n^\varkappa \lambda^{-(l-j)p} \left| \frac{\partial^j f}{\partial x_n^j} \right|^p dx' \, dx_n$$

$$\leqq c' \int \sum_{j=m}^{l} x_n^\alpha \lambda^{-(l-j)p} \, dx_n + c' \sum_{j=0}^{m-1} \int_0^{2\lambda} x_n^\alpha \lambda^{-(l-j)p} \, x_n^{(m-j)p} \, dx_n$$

$$\leqq c'' \int_0^{2\lambda} \lambda^{-(l-m)p} \, x_n^\alpha \, dx_n \leqq c''' \, \lambda^{\alpha+1-(l-m)p} \to 0$$

for $\lambda \downarrow 0$ because $\alpha + 1 - (l - m)p > 0$.

4. Step. We prove theorem 2 (c) for the singular cases $\alpha = kp - 1$; $k = 1, 2, \ldots, l$. The estimate of the last step does not work because $\alpha + 1 - (l - m)p = 0$. We generalize the estimate technique developed in [16]. Let f be a C^∞-function with compact support in $\bar{M}$ and with (8). Now we have

$$m = 1 - \left[\frac{\alpha + 1}{p}\right] = l - k.$$

We write $f(x)$ in the form

$$f(x) = x_n^{l-k} \frac{1}{(l-k)!} \frac{\partial^{l-k}}{\partial x_n^{l-k}} (x', 0) + g(x) = x_n^{l-k} \, h(x') + g(x). \tag{11}$$

For the function $g(x)$ the estimate of the last step works (there we can replace m by $m + 1$). So we can assume (without loss of generality) $g(x) \equiv 0$

in (11), $h(x') \in D(R_{n-1})$. For given $\varepsilon > 0$ and $\beta < 0$ we set

$$\varphi_\varepsilon(x', x_n) = (-\varepsilon \log x_n)^\beta x_n^{l-k} h(x'); \quad 0 < x_n < e^{-1/\varepsilon}, \ x' \in R_{n-1}.$$

It holds

$$\frac{\partial^j \varphi_\varepsilon}{\partial x_n^j}(x) = \varepsilon^\beta (-\log x_n)^\beta h(x') \frac{\partial^j}{\partial x_n^j} x_n^{l-k} + \varepsilon^\beta h(x') x_n^{l-k-j} \sum_{s=1}^{j} b_{j,s} (-\log x_n)^{\beta-s}; \quad 0 \leqq j \leqq l \tag{12}$$

$b_{j,s}$ are constants. Especially we hawe

$$\frac{\partial^j \varphi_\varepsilon}{\partial x_n^j}(x', \varepsilon^{-1/\varepsilon}) = h(x') \frac{\partial^j}{\partial x_n^j} x_n^{l-k} \Big|_{x_n = e^{-1/\varepsilon}} + h(x') \, o(e^{-1/\varepsilon(l-k-j)}) \tag{13}$$

for $0 \leqq j \leqq l-1$. $o(\varkappa(\varepsilon))$ is LANDAU's symbol in the sense $\varepsilon \downarrow 0$. We extend the function $\varphi_\varepsilon(x)$ into the x_n-intervall $[e^{-1/\varepsilon}, 2e^{-1/\varepsilon}]$ by the polynom in x_n

$$P_\varepsilon(x', x_n) = \tag{14}$$

$$= h(x')[x_n^{l-k} + (x_n - 2e^{-1/\varepsilon})(a_1(x_n - e^{-1/\varepsilon}) + \ldots + a_{l-1}(x - e^{-l/\varepsilon})^{l-1}]$$

in such a way that

$$\frac{\partial^j}{\partial x_n^j} \varphi_\varepsilon(x', e^{-1/\varepsilon}) = \frac{\partial^j P_\varepsilon}{\partial x_n^j}(x', e^{-1/\varepsilon}) \tag{15a}$$

and

$$\frac{\partial^j P_\varepsilon}{\partial x_n^j}(x', 2e^{-1/\varepsilon}) = h(x') \frac{\partial^j}{\partial x_n^j} x_n^{l-k} \Big|_{x_n = 2e^{-1/\varepsilon}} \tag{15b}$$

for $j = 0, \ldots, l-1$ hold. We determine the coefficients a_j by induction. With holp of (13) and the definition of $\varphi_\varepsilon(x)$ we find

$$a_j = o(e^{1/\varepsilon(k+j)}). \tag{16}$$

We set

$$\psi_\varepsilon(x', x_n) = \begin{cases} \varphi_\varepsilon(x', x_n) & \text{for} \quad 0 < x_n \leqq e^{-1/\varepsilon} \\ P_\varepsilon(x', x_n) & \text{for} \quad e^{-1/\varepsilon} < x_n \leqq 2e^{1/\varepsilon} \\ f(x) & \text{for} \quad 2e^{-1/\varepsilon} < x_n < 1 \end{cases} \tag{17}$$

Then holds $\psi_\varepsilon(x) \in W^l_{p, x_n}(R_{n-1} \times (\delta, 1))$ for all $\delta > 0$. We want to show $\psi_\varepsilon(x) \in$

$\in P_{l,\alpha,p}$ and

$$\|f(x) - \psi_\varepsilon(x)\|_{P_{l,\alpha,p}} \to 0 \quad \text{for} \quad \varepsilon \downarrow 0. \tag{18}$$

For this purpose are the following three estimates are sufficient.

(a) From (12) with $j = 1$ follows

$$\int_0^{e^{-1/\varepsilon}} x_n^{kp-1} \left| \frac{\partial^l \varphi_\varepsilon}{\partial x_n^l}(x) \right|^p dx_n \leqq c\, \varepsilon^{\beta p} \int_0^{e^{-1/\varepsilon}} \frac{1}{x_n} (-\log x_n)^{p(\beta-1)}\, dx_n = c\, \frac{\varepsilon^{p-1}}{p(1-\beta)-1}$$

(b) From (16) follows

$$\int_{e^{-1/\varepsilon}}^{2e^{-1/\varepsilon}} |P_\varepsilon(x)|^p\, dx_n = o(1) + \varepsilon^{-\frac{p}{\varepsilon} l} \sum_{j=1}^{l-1} o\left(e^{\frac{p}{\varepsilon}(k+j)}\right) e^{-\frac{p}{\varepsilon} j} = o(1).$$

(c) From (16) follows

$$\int_{e^{-1/\varepsilon}}^{2e^{-1/\varepsilon}} x_n^{kp-1} \left| \frac{\partial^l P_\varepsilon}{\partial x_n^l} \right|^p dx_n = e^{-\frac{1}{\varepsilon}(kp-1)} \sum_{j=1}^{l-1} o\left(e^{\frac{p}{\varepsilon}(k+j)}\right) e^{-\frac{p}{\varepsilon} j - \frac{1}{\varepsilon}} = o(1).$$

This proves (18). Now we have to show the possibility of approximation of $\psi_\varepsilon(x)$ for a fixed ε by C^∞-functions with compact support in M^+ in the space $P_{l,\alpha,p}$. For this purpose we choose a number ρ with $0 < \rho < 1$ and determine a polynom in x_n

$$Q_{\varepsilon,\varrho}(x) = h(x')\, x_n^{l-k+1} \left(b_0 + b_1 (x_n - e^{-\frac{1}{\varrho\varepsilon}}) + \ldots + b_{l-1} (x_n - e^{-\frac{1}{\varrho\varepsilon}})^{l-1}\right),$$

in such a way that

$$\frac{\partial^j Q_{\varepsilon,\varrho}}{\partial x_n^j}(x', e^{-\frac{1}{\varrho\varepsilon}}) = \frac{\partial^j \varphi_\varepsilon}{\partial x_n^j}(x', e^{-\frac{1}{\varrho\varepsilon}}); \quad j = 0, \ldots, l-1,$$

holds. We compute the coefficients by induction. With help of (12) we find

$$b_j = 0\left(\rho^{-\beta}\, e^{\frac{j+1}{\varrho\varepsilon}}\right); \qquad j = 0, \ldots, l-1. \tag{19}$$

By this $0(\varkappa(\rho))$ is to understand in the sense $\rho \downarrow 0$, (ε is fixed). Now we

construct

$$\eta_{\varepsilon,\varrho}(x) = \begin{cases} Q_{\varepsilon,\varrho}(x) & \text{for} \quad 0 < x_n < e^{-\frac{1}{\varrho\varepsilon}} \\ \psi_\varepsilon(x) & \text{for} \quad e^{-\frac{1}{\varrho\varepsilon}} \leqq x_n < 1. \end{cases} \tag{20}$$

$\eta_{\varepsilon,\varrho}(x) \in P_{l,\alpha,p}$. We want to show

$$\| \eta_{\varepsilon,\varrho} - \psi_\varepsilon \|_{P_{l,\alpha,p}} \to 0 \quad \text{for} \quad \varrho \downarrow 0, \quad (\varepsilon \text{ fixed}). \tag{21}$$

We know $\psi_\varepsilon \in P_{l,\alpha,p}$. Using (19) the relation (21) follows from

$$\int_0^{e^{-\frac{1}{\varrho\varepsilon}}} | Q_{\varepsilon,\varrho}(x) |^p \, dx_n = o(1)$$

and

$$\int_0^{e^{-\frac{1}{\varrho\varepsilon}}} x_n^{kp-1} \left| \frac{\partial^l Q_{\varepsilon,\varrho}}{\partial x_n^l}(x) \right|^p dx_n = o(1).$$

(18) and (21) show that the functions $\eta_{\varepsilon,\varrho}(x)$ approximate $f(x)$ in $P_{l,\alpha,p}$. But for the functions $\eta_{\varepsilon,\varrho}(x)$ the estimate technique of the third step works (we may replace m by $m+1$). It follows that we can approximate $f(x)$ by functions $\chi_\lambda(x_n)\,\eta_{\varepsilon,\varrho}(x)$, $2\lambda < e^{-\frac{1}{\varrho\varepsilon}}$, in the sense of the third step. But $\chi_\lambda(x_n)\,\eta_{\varepsilon,\varrho}(x) \in W^l_{p,x_n}(M)$ and vanishes near the plane $\{x \mid x_n = 0\}$. Such a function we can approximate in $W^l_{p,x_n}(M)$ (and so also in $P_{l,\alpha,p}$) by C^∞-functions with compact support in M^+. This completes the proof.

2.5. *An embedding theorem.* We go over to the proof of theorem 1. We start with an embedding theorem and define $W^s_{p,x_n}(M)$ for $s = [s] + \{s\}$, $[s]$ integer, $0 < \{s\} < 1$.

$$W^s_{p,x_n}(M) = \Big\{ f \mid f \in D'(M),\ \|f\|_{W^s_{p,x_n}} = \|f\|_{L_p} + \Big(\int_{R_{n-1}} \int_{[0,1]\times[0,1]} \frac{\left| \frac{\partial^{[s]} f}{\partial x_n^{[s]}}(x',t) - \frac{\partial^{[s]}}{\partial x_n^{[s]}}(x',\tau) \right|^p}{|t-\tau|^{1+\{s\}p}} \, dt\, d\tau\, dx' \Big)^{\frac{1}{p}} < \infty \Big\}.$$

From the well-known fact (see [9] or [10])

$$(L_p((0,1)),\ W_p^k((0,1)))_{\theta,p} = W_p^{\theta k}(0,1)),$$

$$k \text{ integer},\quad \theta k \neq \text{integer};\quad 0<\theta<1,\quad 1<p<\infty;$$

and the structure of the interpolationfunctional $K(t,u)$ follows

$$(L_p(M),\ W_{p,x_n}^k(M))_{\theta,p} = W_{p,x_n}^{\theta k}(M), \tag{22}$$

$$k \text{ integer},\quad \theta k \neq \text{integer}.$$

Indeed, for a C^∞-function u with compact support in $\bar{M}$ is

$$K^p(t,u,L_p(M),\ W_{p,x_n}^k(M)) \sim$$

$$\inf_{\substack{u=u_0+u_1\\ u_1\in W_{p,x_n}^k(M)}} \int_M \left(|u_0|^p + t^p |u_1|^p + t^p \left|\frac{\partial^k u_1}{\partial x_n^k}\right|^p \right) dx$$

$$= \int_{R_{n-1}} \inf_{\substack{u(x',x_n)=u_0(x',x_n)+u_1(x',x_n)\\ u_1(x',x_n)\in W_p^k((0,1))}} \int_0^1 \Big(|u_0(x',x_n)|^p + t^p |u_1(x',x_n)|^p +$$

$$+ t^p \left|\frac{\partial^k u_1(x',x_n)}{\partial x_n^k}\right|^p \Big) dx_n\, dx' \sim \int_{R_{n-1}} K^p(t, u(x',\cdot), L_p((0,1)),\ W_p^k((0,1)))\, dx'.$$

Approximation shows that the first and the last expression in this relation are equivalent also for $u \in L_p(M)$. From this follows (22).

Now we want to prove

$$P_{l,p(l-\varkappa),p} \subset W_{p,x_n}^\varkappa(M);\quad 0 \leqq \varkappa \leqq l,\quad 1<p<\infty. \tag{23}$$

If $\varkappa$ an integer this follows from the inequality (3) and the smoothness property 2.1. If $\varkappa \neq$ integer the result is a consequence of theorem 2 (a) and (22).

2.6. *The spaces* $\widetilde{W}_{p,x_n}^\varkappa(M)$. The completion of all C^∞-functions with compact support in M^+ in the space $W_{p,x_n}^\varkappa(M)$ we denote with $\widetilde{W}_{p,x_n}^\varkappa(M)$.

We want to show: $f \in W^{\varkappa}_{p,x_n}(M)$ belongs to $\widetilde{W}^{\varkappa}_{p,x_n}(M)$ iff

$$\frac{\partial^j f}{\partial x_n^j}(x', 0) = 0 \quad \text{for} \quad j = 0, \dots, \left[\varkappa - \frac{1}{p}\right]^-. \tag{24}$$

$\left(\text{If } \varkappa \leqq \frac{1}{p} \text{ this means } W^{\varkappa}_{p,x_n}(M) = \widetilde{W}^{\varkappa}_{p,x_n}(M)\right)$.

1. STEP. It is well known that for a function $f \in W^{\varkappa}_{p,x_n}(M)$ the operators $\frac{\partial^j f}{\partial x_n^j}$ have boundary values, $j = 0, \dots, \left[\varkappa - \frac{1}{p}\right]^-$, and

$$\sum_{j=0}^{\left[\varkappa-\frac{1}{p}\right]^-} \left\| \frac{\partial^j f}{\partial x_n^j}(x', 0) \right\|_{L_p} \leqq c \, \| f \|_{W^{\varkappa}_{p,x_n}(M)} \tag{25}$$

holds. Indeed, this relation follows from one-dimensional embedding theorem

$$\sum_{j=0}^{\left[\varkappa-\frac{1}{p}\right]^-} \left| \frac{\partial^j f}{\partial x_n^j}(x', 0) \right|^p \leqq c \, \| f(x', \cdot) \|^p_{W^{\varkappa}_p((0,1))},$$

[12], and an integration over R_{n-1}. This proves that the conditions (24) are necessary.

2. STEP. We assume $f \in W^{\varkappa}_{p,x_n}(M)$ and (24) holds. In the same way as in the second step of 2.4 we approximate f in $W^{\varkappa}_{p,x_n}(M)$ by C^∞-functions with compact support in $\bar{M}$ for which (24) also holds. (We used that the C^∞-functions with compact support in $\bar{M}$ are dense in $W^{\varkappa}_{p,x_n}(M)$). So we assume without loss of generality that f is a C^∞-function with compact support in $\bar{M}$ and (24) holds. Now it is easy to see

$$\left[\varkappa - \frac{1}{p}\right]^- = l - \left[\frac{\alpha + 1}{p}\right] - 1, \tag{26}$$

$$l \text{ integer}, \quad l > \varkappa, \quad \alpha = p\,(l - \varkappa).$$

Then follows from theorem 2 (c) that we can approximate f in $P_{l,\alpha,p}$ by C^∞-functions with compact support in M^+. Now (23) and (26) show that the same is true in $W^{\varkappa}_{p,x_n}(M)$. Hence (24) is also sufficient.

2.7. *Proof of theorem* 1 *for the W-spaces.*

1. STEP. Let $f \in \overset{\circ}{W}{}_p^s(\Omega)$. With help of the usual method of local coordinates and the embedding theorems [12], p. 291, follows

$$\frac{\partial^j f}{\partial \nu^j}\Big|_{\partial\Omega} = 0, \quad j = 0, \dots, \left[s - \frac{1}{p}\right]^- . \tag{27}$$

2. STEP. Let be $f \in W_p^s(\Omega)$ and (27) holds. (27) is equivalent to

$$D^\gamma f|_{\partial\Omega} = 0 \quad \text{for} \quad |\gamma| \leqq \left[s - \frac{1}{p}\right]^- .$$

We use again the method of local coordinates and the last relation. It follows that we can restrict the considerations to the case:

$$f \in W_p^s(M), \quad \operatorname{supp} f \subset \left\{x \,|\, |x| > \frac{1}{2}, \ 0 < x_n < 1\right\}$$

$$\frac{\partial^j f}{\partial x_n^j}(x', 0) = 0 \quad \text{for} \quad 0 \leqq j \leqq \left[s - \frac{1}{p}\right]^- . \tag{28}$$

With help of similar arguments as above we may assume that f is a C^∞-function with compact support in $\overline{M}$. From the interpolation theory for W-spaces and also from the theory of equivalent norms [5] follows for $\{s\} > 0$

$$\|u\|_{W_p^s} \backsim \|u\|_{L_p} \tag{29}$$

$$+ \sum_{j=1}^{n-1} \left[\int_0^\infty t^{-\{s\}p} \left\| \frac{\partial^{[s]} u}{\partial x_j^{[s]}}(x_1, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_n) - \frac{\partial^{[s]} u}{\partial x_j^{[s]}}(x) \right\|^p_{L_p(M)} \frac{dt}{t} \right]^{\frac{1}{p}}$$

$$+ \left[\int_0^1 t^{-\{s\}p} \left\| \frac{\partial^{[s]} u}{\partial x_n^{[s]}}(x', x_n + t) - \frac{\partial^{[s]} u}{\partial x_n^{[s]}}(x) \right\|^p_{L_p(R_{n-1}\times(0,\,1-t))} \frac{dt}{t} \right]^{\frac{1}{p}} .$$

The first and the last term together are equivalent to $\|u\|_{W_{p,x_n}^s}$ from 2.5. For $\{s\} = 0$ we have

$$\|u\|_{W_p^s} = \|u\|_{L_p} + \sum_{j=1}^n \left\| \frac{\partial^s u}{\partial x_j^s} \right\|_{L_p} .$$

Now we approximate f in the sense of 2.6 by C^∞-functions φ_k with compact support in M^+ in the space $W^s_{p,x_n}(M)$. But the approximation-method developed in the third and the fourth step of 2.4 and the explicit expression for the norm in $W^s_p(M)$ show

$$\|f - \varphi_k\|_{W^s_p} \to 0 \quad \text{for} \quad k \to \infty.$$

For the method in the third step this is clear. For the method in the fourth step also, when we take into consideration that we need only the case $x_n^{l-k} \cdot h(x')$. Further we may assume

$$\operatorname{supp} \varphi_k \subset \left\{x \mid |x| < \frac{1}{2},\ x_n > 0\right\}.$$

This completes the proof.

2.8. *Proof of theorem* 1 *for the H-spaces.*

1. STEP. Let $f \in \overset{\circ}{H}{}^s_p(\Omega)$. With help of embedding theorems [12] p. 420, and local coordinates, follows again (27).

2. STEP. We consider the case $1 < p \leqq 2$. Then we have

$$W^s_p \subset H^s_p, \quad \|f\|_{H^s_p} \leqq c\,\|f\|_{W^s_p}.$$

Let $f \in H^s_p$ and (27) holds. With the same method as above we show that we can assume without loss of generality $f \in C^\infty(\bar{\Omega})$. But then we approximate f in $W^s_p(\Omega)$ by functions from $D(\Omega)$. The last estimate shows that this is also an approximation in H^s_p. This proves the theorem 1 for H^s_p with $1 < p \leqq 2$.

3. STEP. The case $2 < p < \infty$. Let $f \in H^s_p(\Omega)$ and (27) holds. We may again assume $f \in C^\infty(\bar{\Omega})$. If $s \neq \text{integer} + \dfrac{1}{p}$ we can approximate f in $W^{s+\varepsilon}_p(\Omega)$ for $0 < \varepsilon \leqq \left[s - \dfrac{1}{p}\right] + 1 - \left(s - \dfrac{1}{p}\right)$ by $D(\Omega)$-functions. The possibility of approximation in H^s_p follows from

$$W^{s+\varepsilon}_p(\Omega) \subset H^s_p(\Omega).$$

If $s = \text{integer} + \dfrac{1}{p}$ we use again local coordinates and restrict the consi-

derations to the case

$$f \text{ is } C^\infty\text{-function with supp } f \subset \left\{x \,|\, |x| < \frac{1}{2},\ 0 < x_n < 1\right\},$$

$$\frac{\partial^j f}{\partial x_n^j}(x', 0) = 0 \quad \text{for} \quad 0 \leqq j \leqq \left[s - \frac{1}{p}\right]^-.$$

The considerations to the begin of the fourth step of 2.4 show that we can restrict our attention to the case

$$f(x', x_n) = x_n^{s-\frac{1}{p}} h(x'),\ h(x') \in D(R_{n-1}). \tag{30}$$

Now we use an one-dimensional embedding theorem. It is

$$L_p((0,1)) \supset W_2^\varrho((0,1)), \quad W_p^l((0,1)) \supset W_2^{\varrho+l}((0,1)), \tag{31}$$

with $\varrho = \frac{1}{2} - \frac{1}{p}$ [12]; $l = 1, 2, \ldots$. With help of (23) (one-dimensional case) follows

$$L_p((0,1)) \supset P_{l+1,\,2(l+1-\varrho),\,2}, \quad W_p^l((0,1)) \supset P_{l+1;\,2(1-\varrho),\,2}.$$

Theorem 2 (a) and the interpolation theory for W-spaces lead to

$$B_{p,2}^s((0,1)) \supset P_{l+1,\,2(l-s+1-\varrho),\,2} \qquad 0 < s < l. \tag{32}$$

$B_{p,2}^s$ are BESOV-spaces. For definition and interpolation theorems see [3, 12, 17]. (See also formula (33)). With $s = \text{integer} + \frac{1}{p}$ and $\varrho = \frac{1}{2} - \frac{1}{p}$ follows

$$l + 1 - \left[\frac{2(l - s + 1 - \varrho) + 1}{2}\right] - 1 = s - \frac{1}{p} - 1 = \left[s - \frac{1}{p}\right]^-.$$

Theorem 2 (c) shows that we can approximate $x_n^{s-\frac{1}{p}}$ in the space $P_{l+1,\,2(l-s+1-\varrho),\,2}$ in the desired way (one-dimensional case). But the relation (32) proves that we can approximate $x_n^{s-\frac{1}{p}}$ also in the space $B_{p,2}^s((0,1))$ in the desired way. Now we go over to the space $B_{p,2}^s(M)$.

From the interpolation theory for BESOV-spaces follows

$$(33)\qquad \| u \|_{B^s_{p,2}(M)} \sim \| u \|_{L_p(M)} +$$

$$\sum_{j=1}^{n-1} \left[\int_0^\infty t^{-2\{s\}} \left\| \frac{\partial^{[s]} u}{\partial x_j^{[s]}} (x_1, \ldots, x_{j-1}, x_j + t, x_{j+1}, \ldots, x_n) - \frac{\partial^{[s]} u}{\partial x_j^{[s]}} (x) \right\|^2_{L_p(M)} \frac{dt}{t} \right]^{\frac{1}{2}}$$

$$+ \left[\int_0^1 t^{-2\{s\}} \left\| \frac{\partial^{[s]} u}{\partial x_n^{[s]}} (x', x_n + t) - \frac{\partial^{[s]} u}{\partial x_n^{[s]}} (x) \right\|^2_{L_p(R_{n-1} \times (0,\, 1-t))} \frac{dt}{t} \right]^{\frac{1}{2}}.$$

In our case is $\{s\} = \frac{1}{p}$. In the one dimensional case the first and the last term of the right side together are a norm in $B^s_{p,2}((0,1))$. This formula is similar to formula (29). A repetition of the consideration after formula (29) leads to: Each function of type (30) we can approximate in $B^s_{p,2}(M)$ by C^∞-functions with compact support in M^+. It holds

$$B^s_{p,2}(M) \subset H^s_p(M),$$

[12]. From this follows the possibility of approximation in the desired way in $H^s_p(M)$. This proves the theorem.

2.2. *Proof of theorem* 3 (a). Let $s \neq \text{integer} + \frac{1}{p}$. Then the theorem follows immediately from

$$W^{s+\varepsilon}_p(\Omega) \subset B^s_{p,q}(\Omega) \subset W^{s-\varepsilon}_p(\Omega), \qquad \varepsilon > 0,$$

theorem 1 and similar arguments as above. Let $s = \text{integer} + \frac{1}{p}$. That for a function $f \in \overset{\circ}{B}{}^s_{p,q}$ the conditions (27) hold follows again from the embedding theorems [12], p. 291. Now we assume $f \in B^s_{pq}(\Omega)$ and (27). Similar considerations as above and

$$B^s_{p,q_1}(\Omega) \subset B^s_{p,q_2}(\Omega) \quad \text{for} \quad 1 \leqq q_1 \leqq q_2 \leqq \infty$$

show that we can restrict our attention to $B^s_{p,q}(\Omega)$ with $1 < q < p$. Now we generalize (31) and (32) and find

$$L_p((0,1)) \supset W^g_q((0,1)), \qquad W^l_p((0,1)) \supset W^{l+\varrho}_q((0,1))$$

with $\varrho = \frac{1}{q} - \frac{1}{p}$ [12]; $l = 1, 2, \ldots$;

$$B^s_{pq}((0,1)) \supset P_{l+1,\, q(l+1-\varrho-s),\, q}, \quad 0 < s < 1,$$

(one-dimensional case). Further we have again

$$l + 1 - \left[\frac{q(l+1-\varrho-s)+1}{q}\right] - 1 = \left[s - \frac{1}{p}\right]^- .$$

The rest is only a repetition of the arguments of the third step of 2.8. (The norm of $B^s_{pq}(M)$ we can write in the form (33) after replacing the number 2 by q).

2.10. *Proof of theorem* 3 (b). Again we can restrict our attention to the singular case $s = \text{integer} + \frac{1}{p}$. Let $f \in B^s_{p,1}(\Omega)$ and

$$\left.\frac{\partial^j f}{\partial \nu^j}\right|_{\partial\Omega} = 0, \quad j = 0, \ldots, \left[s - \frac{1}{p}\right] = s - \frac{1}{p}. \tag{34}$$

From

$$W_p^{s+\varepsilon}(\Omega) \subset B^s_{p,1}(\Omega), \quad 0 < \varepsilon < 1,$$

and theorem 1 follows $f \in \overset{\circ}{B}{}^s_{p,1}(\Omega)$. We have to show that (34) holds for a function $f \in \overset{\circ}{B}{}^s_{p,1}(\Omega)$. For this purpose we prove a special one dimensional embedding theorem. Let $C^\varkappa([0,1])$ the HÖLDER-space,

$$C^\varkappa([0,1]) = \{f \mid f^{(j)} \in C([0,1]) \quad \text{for} \quad j = 0, \ldots, \varkappa\}, \quad \varkappa \text{ integer},$$

$$C^\varkappa([0,1]) = \left\{f \mid f^{(j)} \in C([0,1]) \quad \text{for} \quad j = 0, \ldots, [\varkappa];\ \sup_{\substack{x \neq y \\ x,\, y \in (0,1)}} \frac{|f^{([\varkappa])}(x) - f^{([\varkappa])}(y)|}{|x - y|^{\{\varkappa\}}} < \infty\right\},$$

$\varkappa \neq$ integer, with the usual norms. Is $0 < \varepsilon < 1$, and l an integer; $l = 1, 2, \ldots$; then holds (one-dimensional)

$$C^s \supset W_p^{s+\frac{1}{p}}, \quad C^{s+l} \supset W_p^{l+s+\frac{1}{p}},$$

[12]. Interpolation leads to

$$C^{s+\theta l} \supset (C^s,\ C^{s+l})_{\theta,1} \supset (W_p^{s+\frac{1}{p}},\ W_p^{l-s+\frac{1}{p}})_{\theta,1} = B^{s+\frac{1}{p}+\theta l}_{p,1}.$$

This proves

$$C^{s-\frac{1}{p}} \supset B^s_{p,1} \tag{35}$$

for $s > \frac{1}{p}$. We need the relation (35) also for $s = \frac{1}{p}$. For this purpose we introduce the subspaces

$$\widetilde{B}^s_{p,1} = \{f \mid f \in B^s_{p,1},\ f(0) = 0\}, \quad s > \frac{1}{p},$$

and

$$\widetilde{C}^\sigma = \{f \mid f \in C^\sigma,\ f(0) = 0\}, \quad \sigma > 0.$$

(In the same way we define $\widetilde{W}^s_p$). It is easy to see that the operator $Af = f'$ leads to an isomorphic map from $\widetilde{C}^l$ ($\widetilde{W}^l_p$) onto C^{l-1} (W^{l-1}_p), where l is an integer; $l = 1, 2, \ldots$.

By interpolation follows that A is also an isomorphic map from $\widetilde{C}^\varkappa$ onto $C^{\varkappa-1}$ and from $\widetilde{B}^\varkappa_{p,1}$ onto $B^{\varkappa-1}_{p,1}$ for $\varkappa > 1$. Now we can prove (35) for the limit case $s = \frac{1}{p}$. For $g \in B^{\frac{1}{p}}_{p,1}$ is

$$\| g \|_{B^{\frac{1}{p}}_{p,1}} \leqq c \| A^{-1} g \|_{B_{p,1}^{1+\frac{1}{p}}} \leqq c' \| A^{-1} g \|_{C^1} \leqq c'' \| g \|_{C^0}$$

c, c', c'' positive numbers. This proves (35) for $s = \frac{1}{p}$. We consider a C^∞-function with compact support in $\bar{M}$. With help of (35) and the explicit norm of $B^s_{p,1}(M)$ (we have only to replace in (33) the number 2 by 1) follows

$$\sum_{j=0}^{s-\frac{1}{p}} \left\| \frac{\partial^j f}{\partial x_n^j}(x', 0) \right\|_{L_p(R_{n-1})} \leqq c \left\| \| f(x', \cdot) \|_{B^s_{p,1}((0,1))} \right\|_{L_p(R_{n-1})}$$

$$\leqq c' \left\| \int_0^1 t^{-\{s\}} \left\| \frac{\partial^{[s]} f}{\partial x_n^{[s]}}(x', x_n + t) - \frac{\partial^{[s]} f}{\partial x_n^{[s]}}(x', x_n) \right\|_{L_p((0,1-t))} \frac{dt}{t} + \right.$$

$$\left. + \| f(x', \cdot) \|_{L_p((0,1))} \right\|_{L_p(R_{n-1})}$$

$$\leqq c'' \int_0^1 t^{-\{s\}} \left\| \frac{\partial^{[s]} f}{\partial x_n^{[s]}}(x', x_n + t) - \frac{\partial^{[s]} f}{\partial x_n^{[s]}}(x) \right\|_{L_p(R_{n-1} \times (0,\, 1-t))} \frac{dt}{t} + c'' \|f\|_{L_p(M)}$$

$$\leqq c''' \|f\|_{B^s_{p,1}(M)} .$$

From this relation with help of the method of local coordinates follows (34) for a function $f \in \overset{\circ}{B}{}^s_{p,1}(\Omega)$. This proves the theorem.

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