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Recent progress in the mathematical analysis of active suspensions

David Gérard-Varet

Progrès récents dans l'analyse mathématique des suspensions actives

Résumé

Cette note s'appuie sur le travail récent [10], dédié à l'analyse du comportement collectif de particules auto-propulsées en suspension dans un écoulement fluide. Le modèle sous-jacent est un système d'EDP de type fluide/cinétique, qui décrit la vitesse du fluide et la distribution des particules en espace et en orientation. L'analyse de stabilité de la distribution isotrope repose sur une étude fine des propriétés de mélange et de diffusion accélérée du système. Mathématiquement, l'intérêt de l'étude vient de la variable d'orientation $p \in \mathbb{S}^2$, qui se substitue à la variable de vitesse $v \in \mathbb{R}^3$ des modèles plus classiques, et est à l'origine de difficultés et phénomènes nouveaux.

Abstract

This note is based on the recent work [10], which analyzes the collective behaviour of suspensions of self-propelled particles in a fluid flow. The underlying model is a coupled Stokes-kinetic system of PDE's, describing the fluid velocity and the distribution of particles in space and orientation. The stability analysis of the isotropic distribution relies on a careful study of the mixing and enhanced diffusion properties of the system. Mathematically, the interest of this study comes from the orientation variable $p \in \mathbb{S}^2$, which substitutes to the usual velocity variable $v \in \mathbb{R}^3$ from more standard models, and is responsible for new phenomena and difficulties.

1. Introduction

The scientific interest for suspensions, defined as large collections of particles immersed in a fluid flow, has more than a century. Since the pioneering work of Einstein about the effective viscosity of dilute suspensions of passive rigid spherical particles [18], numerous studies have improved our understanding of the rheology of suspensions, from dilute to dense [4, 26], from spherical to ellipsoidal, from non-brownian to colloidal [50, 6]. Conversely, the effect of hydrodynamic interaction on the dynamics of the particles, notably their sedimentation, has also been the focus of many studies [3, 37, 39, 46].

Over the last decade, mathematical interest for the derivation and analysis of *effective* or *continuous* models has been renewed, due to recent progress in mean field theory, hydrodynamic limits, or homogenization. See [2, 15, 16, 17, 21, 22, 23, 24, 27, 28, 30, 38, 41] among many. See also the previous contributions [32, 35, 49].

In parallel to the studies of classical passive suspensions, extensive research has been conducted on *active suspensions*, made of particles that have the ability to exert mechanical work, resulting in extra stress on the fluid and self-propulsion. This research is notably motivated by applications

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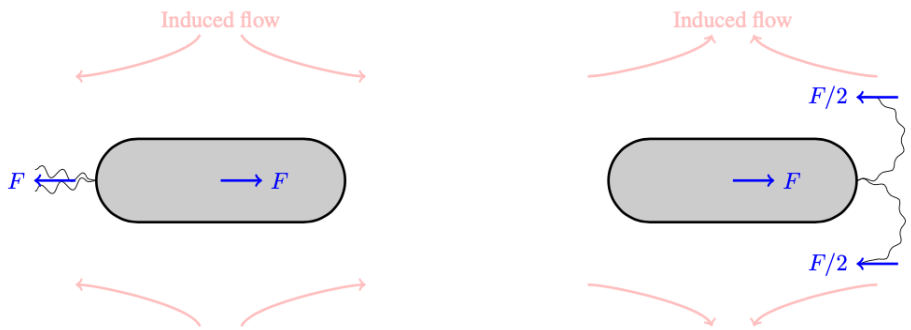

Figure 2.1: Schematic representation of pusher (left) and puller (right) bacteria, taken from [10]. Their effect on the fluid is modelled through combination of point forces (arrows in blue).

to biology (dynamics of bacteria, bioconvection), or to the design of microswimmers for medical purposes. Active suspensions are known to exhibit complex features, including non-newtonian rheology, with possible decrease of the effective viscosity [45, 52]. They may also lead to the emergence of new types of collective dynamics, with chaotic behaviour and/or new patterns. See [47] for a review.

2. The Saintillan–Shelley model

In this note, we focus on one popular fluid/kinetic model, introduced by Saintillan and Shelley in 2008 [48]. It describes a dilute suspension of elongated active particles. Typical example of such particles is bacteria, whose activity is due to the beating of flagellas. We stress that depending on the position of the flagellas, bacteria are distinguished as *pushers* (flagellas at the rear of the body) or *pullers* (flagellas at the front): see Figure 1. The unknowns of the Saintillan–Shelley model are the velocity field $u = u(t,x)$ of the viscous fluid surrounding the particles, and the distribution of particles in orientation and space $\Psi = \Psi(t,x,p)$: for $A \subset \mathbb{R}^3$ and $B \subset \mathbb{S}^2$, $\int_{A\times B} \Psi(t,x,p)\mathrm{d}x\mathrm{d}p$ is the probability of finding at time t a particle with position in A and orientation in B. These unknowns are governed by a set of two equations.

For the fluid, one neglects inertia and considers a Stokes equation:

$$-\Delta_x u + \nabla_x p = \nabla_x \cdot \Sigma, \quad \nabla_x \cdot u = 0, \quad x \in \mathbb{T} := \left(\mathbb{R}/\mathbb{Z}\right)^3, \tag{2.1}$$

where Σ is the additional stress due to the activity of the particles. It is given by

$$\Sigma = \alpha \int_{\mathbb{S}^2} \left(p \otimes p - \frac{I}{3} \right) \psi \, \mathrm{d}p \tag{2.2}$$

with α that can be either negative (for instance for bacteria of pusher type), or negative (for pullers). The idea behind this formula is to consider that the i-th active particle acts on the fluid as a *dipole* that is a sum of opposite point forces located near the center of inertia x_i of the particle, distant from $\varepsilon \ll 1$, and exerted along orientation p_i. For instance, for pushers,

$$F_i \approx (\alpha p_i)\delta_{x_i - \varepsilon p_i} - (\alpha p_i)\delta_{x_i + \varepsilon p_i} \propto \nabla_x \cdot (\alpha p_i \otimes p_i \delta_{x_i}) + \text{higher order term in } \varepsilon$$

See Figure 2.1. Averaging over all particles yields an expression of type (2.2): the extra term $-\alpha \int_{\mathbb{S}^2} \frac{I}{3}\psi \, \mathrm{d}p$ that makes Σ trace-free can always be included, by suitable change of the pressure gradient.

For the dynamics of the particles, a formal mean-field limit (whose justification raises issues, see [29]) gives the following equation on Ψ:

$$\partial_t \Psi + \nabla_x \cdot \big((u(x) + U_0 p)\Psi \big) + \nabla_p \cdot \big((P_{p^\perp}(\gamma S u(x) + A u(x))p)\Psi \big) - \mu \Delta_p \Psi = 0 \quad \text{in } \mathbb{T}\times\mathbb{S}^2 \tag{2.3}$$

where $U_0, \gamma, \mu > 0$, where $P_{p^\perp}$ is the projection tangentially to the sphere at p, and where $Su = \frac{1}{2}(\nabla u + (\nabla u)^t)$ and $Au = \frac{1}{2}(\nabla u - (\nabla u)^t)$ denote the symmetric and skew-symmetric parts of the velocity gradient. In this equation,

- The third diffusive term $-\mu\Delta_p\Psi$, $\mu \ll 1$, corresponds to brownian rotational diffusion.
- The first and second terms correspond to transport, with characteristic equations
$$\dot{x} = u(x) + U_0 p, \quad \dot{p} = \big(P_{p^\perp}(\gamma S u(x) + A u(x))p\big) \times p$$
The formula for $\dot{x}$ reflects the movement of a typical particle under Stokes velocity u and self-propulsion velocity $U_0 p$, where U_0 is the common speed of the particles. The formula for the angular velocity is the one obtained by Jeffery [33] for a single small ellipsoidal particle in a Stokes flow, with γ depending on the geometrical properties of the ellipsoid. The idea behind this formula is that diluteness allows to neglect interactions between particles, so that each particle has an angular velocity as if it was alone in the fluid.

Let us stress that spatial diffusion $-\kappa\Delta_x\Psi$ could be added as well: all results of the next sections would hold uniformly as $\kappa \to 0$. Actually, our mathematical analysis would simplify substantially with a fixed spatial diffusion.

In the case $\alpha > 0$ (pullers), the Saintillan–Shelley model (2.1)–(2.3) has strong similarities with the so-called Doi model for passive colloidal elongated particles [14]. Such Doi model is known to have very good stability properties [9, 43], that will still be true for (2.1)–(2.3), as seen in the next sections. However, in the case $\alpha < 0$ (pushers), the situation is more complex. A main reason for the popularity of the Saintillan–Shelley model is its ability to reproduce numerically a transition phenomenon for suspensions of pushers, in agreement with real laboratory experiments. Roughly, one can observe in numerical computations the existence of a threshold Γ_c such that:

1. For $\Gamma := \frac{\gamma|\alpha|}{U_0} < \Gamma_c$, no coherent collective interaction between the active particles is observed: the distribution of the particles orientations is uniform, which corresponds to the stability of the *incoherent state* $\Psi_{inc} = \frac{1}{4\pi}$.
2. For $\Gamma > \Gamma_c$, the incoherent state loses its stability, and there is emergence of a new collective behaviour. See [42, 48].

A natural but challenging mathematical problem is then to recover this transition picture in a rigorous analytical way. We shall present in the next sections recent results [10] that provide a partial contribution to this problem, namely a sharp linearized analysis of the incoherent regime $\Gamma < \Gamma_c$.

3. Linear analysis: statement of the results

We present the results of [10], dedicated to the linearization of (2.1)–(2.3) around the incoherent state $\Psi_{inc} = \frac{1}{4\pi}$. We consider a Fourier mode in x : $\Psi = \Psi_{inc} + e^{ik'\cdot x}\psi(t, v)$. Once in dimensionless form, the linearized equations reduce to

$$\begin{cases} \partial_t\psi + ip\cdot k\psi - \dfrac{3\Gamma}{4\pi} p\otimes p : Su - \nu\Delta_p\psi = 0 \quad \text{ in } \mathbb{S}^2, \\ u = P_{k^\perp} ik\cdot\Sigma, \\ \Sigma := \varepsilon \displaystyle\int_{\S^2} \left(p\otimes p - \frac{I}{3}\right)\psi \,\mathrm{d}p \\ \psi|_{t=0} = \psi_{init}, \end{cases} \tag{LSS}$$

with $\Gamma := \frac{\gamma|\alpha|}{U_0|k'|}$, $\nu := \frac{\mu}{U_0|k|}$, $\varepsilon := \pm 1$ (+ for pullers, − for pullers), $k := \frac{k'}{|k'|} \in \mathbb{S}^2$.

Note that the second equation in (LSS) is equivalent to the Stokes equation (2.1), with the projection $P_{k^\perp}$ orthogonally to k that corresponds to the divergence-free condition. As u is given by a linear functional of ψ, (LSS) can be seen as a single evolution equation on ψ, made of two parts: a stabilizing advection-diffusion part, with operator $\partial_t + ip\cdot k - \nu\Delta_p$, and a potentially destabilizing *reaction part* $-\frac{3\Gamma}{4\pi} p\otimes p : Su$. This kind of structure is present in many kinetic models, notably in linearized Vlasov like equations. In most settings, variable $p \in \mathbb{S}^2$ is replaced by a velocity variable $v \in \mathbb{R}^d$, and the underlying advection-diffusion equation is

$$\partial_t\psi + iv\cdot k\psi - \nu\Delta_v\psi = 0$$

This simple equation is known to exhibit two types of stabilizing phenomena:

- *Mixing*, inherited from the advection equation $\partial_t \psi_{ad} + iv \cdot k\psi_{ad} = 0$, for which $\psi_{ad}(t,v) = e^{-iv\cdot kt}\psi_{init}(v)$. This transport implies a transfer from low to high frequencies, resulting in decay of ψ_{ad} in weak topology, or equivalently of v-averages

$$\int_{\mathbb{R}^3} \psi_{ad}(t,v)g(v)\mathrm{d}v = \int_{\mathbb{R}^3} e^{-iv\cdot kt}\psi_{init}(v)g(v)\mathrm{d}v = \mathcal{F}(\psi_{init}g)(kt)$$

  Roughly, the decay takes place at a rate governed by the regularity in v of $\psi_{init}g$ (exponential for analytic data, polynomial for Sobolev data).

- Enhanced diffusion, which leads to a decay estimate in strong topology, typically $\|\psi(t,\cdot)\|_{L^2} \lesssim e^{-\delta\nu^{1/3}t}$, for some fixed $\delta > 0$. Such estimate shows exponential decay on typical time $O(\nu^{-1/3})$, much shorter than the usual time scale $O(\nu^{-1})$ of the simple heat equation. Indeed, the transfer from low to high frequencies due to mixing speeds up the diffusion, as diffusion damps more high frequencies, hence the enhanced dissipation.

These phenomena are at the core of various stability results for kinetic equations in the euclidean setting $v \in \mathbb{R}^d$: Landau damping in the context of Vlasov–Poisson equations [36, 40], synchronisation for the kinetic Kuramoto equation [8, 12, 13, 20], stability of Couette flow in fluid mechanics [5, 55].

In the context of (LSS), we show in [10] that for $\Gamma < \Gamma_c$, for some Γ_c determined analytically, Ψ_{inc} is linearly stable. Moreover, linear perturbations ψ solving (LSS) experience mixing and enhanced diffusion, but with strong qualitative differences compared to the euclidean setting. More precisely, we prove the following two results.

Theorem 1 ($\nu = 0$: mixing)**.** *If $\varepsilon = 1$ (pullers), then, for any $\Gamma > 0$, and any smooth ψ_{init}, as $t \to \infty$*

$$|u(t)| = O\Big(\frac{1}{t^2}\Big), \quad \|\psi(t,\cdot)\|_{H^{-1-}} = O\Big(\frac{1}{t}\Big).$$

If $\varepsilon = -1$ (pushers), there exists an absolute constant Γ_c such that

- *for $\Gamma < \Gamma_c$ and any smooth ψ_{init}, the same decay estimates hold.*
- *for $\Gamma > \Gamma_c$, there exists a smooth ψ_{init} such that $|u(t)|$ grows exponentially.*

Theorem 2 ($\nu > 0$: mixing followed by enhanced dissipation)**.** *If $\varepsilon = 1$ or $\varepsilon = -1$ and $\Gamma < \Gamma_c$, there exists M, ν_0, γ such that for all smooth ψ_{init}:*

$$|u(t)| + \frac{1}{t}\|\psi(t,\cdot)\|_{H^{-1-}} \lesssim \min\Big(\frac{|\ln(t)|^M}{t^2}, e^{-\eta\nu^{1/2}t}\Big)$$

uniformly in $\nu \le \nu_0$.

We refer to [10] for more precise statements. We now make several comments on these two theorems.

Remark 3. The decay estimates of Theorem 1 for u and ψ in weak topology are consequence of a mixing phenomenon. We stress that these estimates are optimal : the polynomial decay rate can not be improved, even taking initial data ψ_{init} of infinite regularity. This is in sharp contrast with the usual euclidean situation, and relates to the variable $p \in \mathbb{S}^2$ replacing $v \in \mathbb{R}^d$. We shall come back to this qualitative change in the next section.

Remark 4. In the case of pullers $\varepsilon = 1$, Theorem 1 shows unconditional linear stability of the incoherent state. As already mentioned, this was expected, in view of the similarity between the Saintillan–Shelley model for $\varepsilon = 1$ and the Doi model. In the case $\varepsilon = -1$, the existence of a threshold Γ_c corresponds to a bifurcation phenomenon typical of pushers. Mathematically, this Γ_c is a threshold for the existence of an unstable eigenvalue σ of the linearized operator in (LSS). The associated dispersion relation, parametrized by Γ, $F_\Gamma(\sigma) = 0$ has a root in the unstable half-plane if $\Gamma > \Gamma_c$, none if $\Gamma < \Gamma_c$. This dispersion relation was already present in the seminal paper [48], and studied numerically. See [31] for more. In [10], we compute Γ_c analytically using complex analysis arguments *à la Penrose*, similar to those used in the spectral study of the Vlasov equation [44].

Remark 5. Theorem 2 has two features. First, it shows persistence of mixing in the presence of small rotational diffusion, at the same polynomial rate as in the non-diffusive case (up to a technical logarithmic correction). Second, it shows a phenomenon of enhanced dissipation, resulting in exponential decay for times large than $\nu^{-1/2}$. Let us remark that this timescale $\nu^{-1/2}$ is different from the typical timescale $\nu^{-1/3}$ observed for kinetic equations in euclidean setting. As will be shown in the next section, our proof of persistence of mixing requires several ideas, and notably involves refined dissipation bounds.

Remark 6. Prior to publication of [10], another mathematical analysis of the Saintillan–Shelley model was performed by D. Albritton and L. Ohm in [1]. Regarding mixing in (LSS), [1] contains a weaker version of our Theorem 1, showing for $\Gamma < \Gamma_c$ a L^2 in time stability estimate

$$\int_{\mathbb{R}_+} |u(t)|^2 (1+t)^{3-\epsilon} \mathrm{d}t < \infty$$

Moreover, [1] does not show persistence of mixing for small ν, a result that is central to Theorem 2. On the other hand, when spatial diffusion $-\kappa\Delta_x\psi$ is added, [1] provides interesting nonlinear stability results. When $x \in \mathbb{R}^d$, stability is obtained through linearized Taylor dispersion estimates (no distinction appears here between pullers and pushers). When $x \in \mathbb{T}^d$, stability is obtained either for pullers, or for pushers but in the stringent regime $\Gamma \ll \nu^{1/2}$ instead of $\Gamma < \Gamma_c$. In all these results, the initial datum must be small both compared to ν and κ. A missing step is to prove nonlinear stability in the regime of physical interest $\Gamma < \Gamma_c$, and uniformly in κ. This will be the matter of a forthcoming work [11], based on a non-trivial extension of the linear analysis described in the present note.

4. Linear analysis: proofs

We provide here the key elements of the proofs of Theorem 1 and 2. We pay special attention to the mixing phenomenon and associated polynomial decay.

Step 1. Reduction to a Volterra equation for the velocity.

We first reexpress (LSS) under the abstract form

$$\partial_t \psi = L_\nu \psi + V \cdot u[\psi]$$

where

$$L_\nu = -ik \cdot p + \nu \Delta_p \psi, \quad V(p) = -\frac{3i\Gamma}{4\pi} P_{p^\perp} k$$

and where

$$u[\psi] = i\varepsilon \int_{\mathbb{S}^2} \psi(p)(k \cdot p) P_{k^\perp} p \, \mathrm{d}p$$

is the linear operator giving the velocity field u in terms of ψ. Applying operator $u[\,\cdot\,]$ to both sides of Duhamel's formula:

$$\psi(t) = e^{tL_\nu}\psi_{init} + \int_0^t e^{(t-s)L_\nu} V \cdot u[\psi](s) \, \mathrm{d}s$$

we find a closed equation on $u(t) = u[\psi](t)$, namely:

$$u(t) + \int_0^t K_{\nu,\Gamma}(t-s) u(s) \mathrm{d}s = f_\nu(t) \tag{4.1}$$

where

$$\begin{aligned} K_{\nu,\Gamma}(t) v \cdot \bar{v} &= \frac{3\varepsilon\Gamma}{4\pi} \int_{\mathbb{S}^2} e^{L_\nu t}(k \cdot p)(P_{p^\perp} k \cdot v) \, (P_{k^\perp} p \cdot \bar{v}) \, \mathrm{d}p. \\ f_\nu(t) &= i\varepsilon \int_{\mathbb{S}^2} e^{L_\nu t} \psi_{init}(k \cdot p) P_{k^\perp} p \, \mathrm{d}p \end{aligned} \tag{4.2}$$

Most of the work in [10] consists in the stability analysis of (4.1) and in proving the $O(t^{-2})$ decay of u for Γ in the stable regime. Once the decay of u is obtained, the $O(t^{-1})$ decay of ψ in weak topology is standard. For the analysis of (4.1), two steps are necessary:

(i) Prove the $O(t^{-2})$ decay of K_ν and f_ν, uniformly in ν (up to times $O(\nu^{-1/2})$, after which enhanced dissipation takes over).

(ii) Identify conditions on Γ for this decay to transfer to u.

As Step (i) is the most difficult, we temporarily assume the $O(t^{-2})$ decay of K_ν and f_ν and adress (ii).

Step 2. Pointwise in time decay results for general Volterra equations.

Consider a general Volterra equation

$$u(t) + \int_0^t K(t-s)u(s)\,\mathrm{d}s = f(t) \tag{4.3}$$

where $K(t), f(t)$ are smooth and satisfy $K(t), f(t) = O(t^{-a})$, $a > 1$. In the case of active suspensions, we will show later that this assumption is satisfied with $a = 2$. As $K, f \in L^1(\mathbb{R}_+)$, we can apply the general theory of Volterra equations, see [25], resulting in the following classical theorem

Theorem 7. *If $K \in L^1(\mathbb{R}_+)$ is an integrable matrix kernel, and if its Laplace transform $\mathcal{L}K$ satisfies the spectral condition*

$$\det\big(I + \mathcal{L}K(z)\big) \neq 0 \quad \forall\, \Re z \geq 0 \tag{4.4}$$

then K has a resolvent, that is a matrix kernel $R \in L^1(\mathbb{R}_+)$ satisfying

$$R + K \star R = R + R \star K = R$$

In particular, for all $f \in L^1(\mathbb{R}_+)$, $u = f - R \star f$ is a solution in $L^1(\mathbb{R}_+)$ of (4.3).

Theorem 7 is a weak stability result, meaning that under the stability condition (4.4), the solution u satisfies the weak form of decay $u \in L^1(\mathbb{R}_+)$. This is not sufficient for our purpose. In [10], we go from this qualitative to a quantitative result, and show:

Theorem 8. *If $K(t), f(t) = O(t^{-a})$, $a > 1$, and if (4.4) is satisfied, then $u = O(t^{-a})$.*

See [19] for a similar result.

Proof. We make the change of variables $(\widetilde{u}, \widetilde{f}) := (1+\varepsilon t)^a (u, f)$ where ε will be chosen small. Equation (4.3) turns into

$$\widetilde{u}(t) + \int_0^t k(t,\tau)\widetilde{u}(\tau)\,\mathrm{d}\tau = \widetilde{f}(t)$$

where the non-convolution kernel k is given by

$$k(t,\tau) := \left(\frac{1+\varepsilon t}{1+\varepsilon\tau}\right)^a K(t-\tau)1_{\tau<t}$$

The goal is to prove that under (4.4), $\widetilde{u}$ is bounded if $\widetilde{f}$ is. The key idea is to realize that k belongs to the space

$$\mathcal{K} = \left\{ k, \quad k(t,\tau) = 0 \quad \text{for } \tau \leq t, \quad \|k\| := \sup_t \int_{\mathbb{R}_+} |k(t,\tau)|\,\mathrm{d}\tau < \infty \right\}.$$

This is easily seen to be a Banach algebra for the generalized convolution product

$$k_1 \star k_2(t,\tau) = \int_{\mathbb{R}_+} k_1(t,u)k_2(u,\tau)\,\mathrm{d}u$$

Then, as in any Banach algebra, there is a notion of resolvent: $k \in \mathcal{K}$ has a resolvent $r \in \mathcal{K}$ if $r + r \star k = r + k \star r = k$. Moreover, if r is the resolvent of k, one can easily see that $\widetilde{u}(t) = f(t) - r \star \widetilde{f}(t) = \widetilde{f}(t) - \int_0^t r(t,u)\widetilde{f}(u)\,\mathrm{d}u$ is bounded if $\widetilde{f}$ is.

Finally, as the set of kernels that have a resolvent is open in $\mathcal{B}$, and as k is easily shown to be a small perturbation of $K(t-\tau)1_{t>\tau}$ for ε small, it is enough to prove that $K(t-\tau)1_{t>\tau}$ has a resolvent. But this is easily deduced from Theorem 7, the resolvent being $R(t-\tau)1_{t>\tau}$. □

In the special case of active suspensions, $K = K_{\nu,\Gamma}$ and $f = f_\nu$ are given by (4.2). They will be shown below to be $O(t^{-2})$ so that, from Theorem 8, $u = O(t^{-2})$ under the spectral condition (4.4). We show in [10] that (4.4) is satisfied for all $\Gamma > 0$ for $\varepsilon = 1$, and for all $\Gamma < \Gamma_c$ for some explicit Γ_c for $\varepsilon = -1$. In the case $\varepsilon = 1$, this is deduced from the following simple observation: for any z with $\Re z \geq 0$, for any $v \in \mathbb{C}^3$, defining $\phi = (p \cdot k) P_{k^\perp} p \cdot v$, we have the identity

$$\mathcal{L}K_{\nu,\Gamma}(z)v \cdot v = \frac{3\Gamma}{4\pi}\big((z - L_\nu)^{-1}\phi|\phi\big)_{L^2} \geq 0$$

so that $I + \mathcal{L}K_{\nu,\Gamma}(z)$ is invertible. But in the case $\varepsilon = -1$, $\mathcal{L}K_{\nu,\Gamma}(z)$ has the bad sign, which explains the threshold Γ_c. This threshold can be obtained by arguments à la Penrose [44]. We refer to [10] for all details.

Step 3. Decay of the Volterra kernel: the non-diffusive case.

To complete the proof of Theorems 1 and 2, the remaining major step is to show that the kernel $K_{\nu,\Gamma}$ and the source term f_ν given in (4.2) decay like $1/t^2$. The non-diffusive case $\nu = 0$ corresponding to Theorem 1 is much easier. Indeed, the semigroup of the advection-diffusion operator L_ν reduces to $e^{L_0 t} = e^{-ik\cdot pt}$, so that the formulas for the kernel and the source term are explicit : for instance, one has

$$K_\Gamma(t) = \frac{3\Gamma\varepsilon}{4\pi}\int_{\mathbb{S}^2} e^{-ik\cdot pt}(p \cdot k) P_{p^\perp} k \otimes P_{k^\perp} p \,\mathrm{d}p. \tag{4.5}$$

The keypoint is that K_Γ is given by a Fourier transform *on the sphere*. This explains the difference with the usual euclidean setting $v \in \mathbb{R}^d$. Indeed, in that latter setting, kernels are given by Fourier transforms on $\mathbb{R}^d$, the phase $v \mapsto -ik \cdot v$ has no critical point, so that the decay depends on the regularity of the integrand. But here, the phase $p \mapsto -ik \cdot p$ has a critical point: namely, as the gradient on the sphere is the projection of the euclidean gradient tangentially to the sphere, we find $\nabla_p(-ik \cdot p) = -iP_{p^\perp} k$, which vanishes at the poles $p = \pm k$. Due to this stationary phase, the decay in time is limited, and well-known: one can even have a whole asymptotic expansion in negative powers of t of Fourier transforms of the type (4.5). Interestingly, the first term in such expansion is generically of order $\frac{1}{t}$. But it turns out that in our special case (4.5), the integrand vanishes at the critical points, so that the leading term vanishes, giving decay $O(1/t^2)$ (and in particular integrability of the kernel, which allows to apply the theory of Volterra equations described above).

Step 4. Decay of the Volterra kernel: the diffusive case.

When ν is small but non-zero, we lose the simple explicit formula for the semigroup of the advection-diffusion operator, and the representation of the kernel with a Fourier transform over the sphere. To overcome this issue, we rely on the so-called vector field method, introduced by Klainerman in the context of nonlinear wave equations [34]. In the context of Vlasov equations, see also [51]. Our main inspiration is [7], where the vector field method is applied to mixing in the Vlasov–Poisson–Landau equation. A basic but fundamental observation in [7] is the fact that the vector field $J = \nabla_v + ikt$ commutes to the transport operator $(\partial_t + ik \cdot v)$. In particular, if f is a solution of $\partial_t f + ik \cdot vf = 0$, so is Jf. To deduce some mixing decay estimate for an average of the form $\int_{\mathbb{R}^3} f(t,v)g(v)\,\mathrm{d}v$, we simply write

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} f(t,v)g(v)\,\mathrm{d}v\right| &= \frac{1}{|kt|}\left|\int_{\mathbb{R}^3} Jf(t,v)\,\mathrm{d}v - \int_{\mathbb{R}^3} \nabla_v f(t,v)g(v)\,\mathrm{d}v\right| \\
&= \frac{1}{|kt|}\left|\int_{\mathbb{R}^3} Jf(t,v)\,\mathrm{d}v + \int_{\mathbb{R}^3} f(t,v)\nabla_v \cdot g(v)\,\mathrm{d}v\right| \\
&\lesssim \frac{1}{t}\Big(\|Jf(t,\cdot)\|_{L^1_v} + \|f(t,\cdot)\|_{L^1_v}\Big) \\
&\lesssim \frac{1}{t}\Big(\|Jf(0,\cdot)\|_{L^1_v} + \|f(0,\cdot)\|_{L^1_v}\Big) \\
&\lesssim \frac{1}{t}\|f_{init}\|_{W^{1,1}_v(\mathbb{R}^3)}
\end{aligned}$$

Under smoothness assumptions on f_{init} and g, one can easily iterate this kind of manipulations to obtain $O(t^{-m})$ decay through the control of $J^m f$. As shown in [7], in the euclidean setting, this approach to the derivation of mixing estimates is robust to the introduction of dissipative terms.

In the context of the linearized Saintillan–Shelley model (LSS), the natural analogue of the previous vector field is

$$J = \nabla_p + itP_{p^\perp}k \tag{4.6}$$

But it creates a bad commutator term with the diffusion $-\nu\Delta_p$. Namely, for ψ solving

$$\partial_t\psi + ip\cdot k\psi - \nu\Delta_p\psi = 0 \tag{4.7}$$

the equation for $J\psi$ contains a term of the form $2i\nu t(p\cdot k)\nabla_p\psi$, which is expected to grow like νt^2. As a consequence, $O(1)$ bound on $J\psi$ is not expected over times $T \gg \nu^{-1/3}$. While this threshold $\nu^{-1/3}$ would be enough in the euclidean setting (as enhanced dissipation takes over for larger times), this is not enough in the spherical case, where the natural time scale for enhanced dissipation is $\nu^{-1/2} \gg \nu^{-1/3}$. The same difficulty shows up in the toy model

$$\partial_t f + iky^2 f - \nu\partial_y^2 f = 0,$$

where the term iky^2 has a non-degenerate critical point at $y = 0$, in the same way as the term $ip\cdot k$ in (4.7) has critical points at $p = \pm k$. Now, a key remark is that for this toy model, the vector field:

$$J_{\nu,toy} = \cosh(\sqrt{-2i\nu}t)\partial_y + \frac{2i\sinh(\sqrt{-2i\nu}t)}{\sqrt{-2i\nu}}ky$$

commutes perfectly to $\partial_t + iky^2 - \nu\partial_y^2$. Inspired by this example, one introduces in [10] the viscosity dependent vector field

$$J_\nu = \cosh(\sqrt{-2i\nu}t)\nabla + \frac{i\sinh(\sqrt{-2i\nu}t)}{\sqrt{-2i\nu}}P_{p^\perp}k$$

It behaves similarly to the vector field J defined in (4.6) over times $\lesssim \nu^{-1/2}$. But this time, for ψ a solution of (4.7), the equation for $J_\nu\psi$ contains a term

$$2i\nu t[(p\cdot k) - 1]\nabla_p\psi \tag{4.8}$$

The advantage compared to the previous commutator is that *it vanishes at the critical point* $p = k$. As most difficulties are related to the critical points (slower mixing decay and slower enhanced dissipation), we may expect a tame growth estimate for this new commutator, yielding the desired uniform control of $J_\nu\psi$. To turn this expectation into a solid argument, we need to perform a refined analysis of the enhanced dissipation phenomenon, that leads to better control of quantities vanishing at the critical points. We explain the main elements of this analysis below.

Final step : refined enhanced dissipation estimates and conclusion.

The enhanced dissipation estimate of Theorem 2 results from a sharp long time estimate for solutions ψ of (4.7). The core of this estimate is the so-called hypocoercivity approach described by C. Villani in the monograph [53]. The keypoint is to introduce a good energy functional, which in our case takes the form

$$E[\psi] = \frac{1}{2}\Big[\|\psi\|^2 + a_\nu\|\nabla\psi\|^2 + 2b_\nu\Re\langle i\nabla(p\cdot k)\psi, \nabla\psi\rangle + c_\nu\|\nabla(p\cdot k)\psi\|^2\Big]$$

The coefficients a_ν, b_ν and c_ν are chosen so that this functional is coercive, and so as to derive an estimate of the form

$$\partial_t E[\psi] + D[\psi] \lesssim G \tag{4.9}$$

for $D[\psi]$ some dissipation functional, and G some *good* term (meaning roughly that it is controllable on times $\nu^{-1/2}$). In the original approach of [53], a_ν, b_ν, c_ν are constants. In our setting, we need to refine the energy, by considering as in [54] time dependent weights: namely, we take

$$(a_\nu, b_\nu, c_\nu) = (a_\nu(t), b_\nu(t), c_\nu(t)) = (a\nu t, b\nu t^2, c\nu t^3)$$

for a, b, c well-chosen absolute constants. The associated dissipation functional is given by

$$D[\psi] = \frac{\nu}{2}\|\nabla\psi\|^2 + \frac{a\nu^2 t}{2}\|\nabla\nabla\psi\|^2. + \frac{b\nu t^2}{2}\|\nabla(p\cdot k)\psi\|^2 + \frac{c\nu^2 t^3}{2}\|\nabla(\nabla(p\cdot k)\psi)\|^2$$

In short, once an estimate of type (4.9) is proved, it provides an insight into the decay properties of solutions of (4.7). On one hand, by interpolation, one gets from $D[\psi]$ an extra damping term $\sqrt{\nu}\|\psi\|^2$, responsible for the enhanced dissipation. On the other hand, one can get an improved control for the quantity $\nabla(p\cdot k)\psi$, vanishing at the critical points, due to the term $\frac{b\nu t^2}{2}\|\nabla(p\cdot k)\psi\|^2$ in $D[\psi]$. Roughly, we have

$$\int_0^T \nu t^2 \|\nabla(p\cdot k)\psi\|^2 \lesssim 1$$

up to times T of order $\nu^{-1/2}$. This in turn allows to control the contribution of the commutator term (4.8) in the equation for $J_\nu\psi$, on times of order $\nu^{-1/2}$. Uniform control of $J_\nu\psi$ and application of the vector field method eventually yield the optimal mixing estimates of Theorem 2.

We stress that this schematic picture of the proof hides many technical difficulties. For instance, $J_\nu\psi$ only vanishes at $p=k$, not at the other critical point $p=-k$. This implies to introduce an analogue vector field for the other critical point, and to localize the estimates for $J_\nu\psi$ and its analogue. Also, L^2 estimates for $J_\nu\psi$ are not enough: we need a further estimate for $(J_\nu)^2\psi$, which requires hypocoercive estimates for $J_\nu\psi$ itself. We further need an L^∞ estimate for $J_\nu\psi$, which is derived from maximum principle arguments. All these bounds require a delicate control of the various commutator terms, and we refer to [10] for all details.

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