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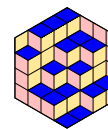


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Kazhdan–Lusztig cells of $\mathbf{a}$ -value 2 in $\mathbf{a}(2)$ -finite Coxeter systems

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ABSTRACT A Coxeter group is said to be $\mathbf{a}(2)$ -finite if it has finitely many elements of $\mathbf{a}$ -value (in the sense of Lusztig) equal to 2. In this paper, we give explicit combinatorial descriptions of the left, right, and two-sided Kazhdan–Lusztig cells of $\mathbf{a}$ -value 2 in an irreducible $\mathbf{a}(2)$ -finite Coxeter group. In particular, we introduce elements we call *stubs* to parameterize the one-sided cells and we characterize the one-sided cells via both star operations and weak Bruhat orders. We also compute the cardinalities of all the one-sided and two-sided cells of $\mathbf{a}$ -value 2 in irreducible $\mathbf{a}(2)$ -finite Coxeter groups.

1. INTRODUCTION

Let (W, S) be an arbitrary Coxeter system and let H be the associated Hecke algebra. In the landmark paper [22], Kazhdan and Lusztig introduced the Kazhdan–Lusztig basis of H and used it to partition the Coxeter group W into left, right and two-sided cells. Each two-sided cell is a union of left cells as well as a union of right cells. These Kazhdan–Lusztig cells have connections with numerous objects from representation theory, such as primitive ideals of universal enveloping algebras of Lie algebras [3], characters of reductive groups over finite fields [26], and unipotent conjugacy classes of reductive groups [29]. Consequently, the determination of cells has been an important problem in representation theory. The goal of this paper is to parameterize the one-sided cells of $\mathbf{a}$ -value 2 in irreducible $\mathbf{a}(2)$ -finite Coxeter systems, as well as to count all one-sided and two-sided cells of $\mathbf{a}$ -value 2 in such Coxeter systems.

Before explaining the meaning of the “ $\mathbf{a}$ -value” and “ $\mathbf{a}(2)$ -finite”, let us briefly summarize for context some known results about Kazhdan–Lusztig cells. For Weyl groups of types A , B , D and affine A , the cells may be obtained via various combinatorial models involving Young tableaux, the Robinson–Schensted correspondence, and their suitable generalizations; see [22, 2] for type A , [3, 13, 6] for types B and D , and [32, 10] for type affine A . Cells of some other specific Coxeter systems (especially those with small ranks or complete Coxeter diagrams) have also been computed, often on a case-by-case basis via carefully designed algorithms. For example, the paper [36] treats cells in the Coxeter group F_4 , [33] treats $\tilde{C}_4$, [1] treats H_3 and H_4 , [9] and [14] treat E_8 ; the paper [20] studies affine Weyl groups of rank 2, and the papers [37, 38] deal with Coxeter systems with complete Coxeter diagrams. For a more comprehensive summary of the cell literature, see [7, §26].

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Given that most of the above references focus on Coxeter systems of particular types, it is perhaps natural to wonder whether one can systematically study Kazhdan–Lusztig cells of an arbitrary Coxeter system (W, S) . The **a**-function, a certain function $\mathbf{a} : W \rightarrow \mathbb{Z}_{\geq 0}$, offers an important tool in this study. The **a**-function is due to Lusztig, who first defined it for Weyl and affine Weyl groups in [27, 28] and then extended the definition to general Coxeter groups in [30]. Lusztig showed that under a certain boundedness condition on (W, S) , the **a**-function takes a constant value on every two-sided cell of W and hence on every left or right cell. Lusztig conjectures that the boundedness condition is satisfied by every Coxeter system (see [30, §13.4]), and we shall assume that the conjecture holds throughout the paper (as is common in the literature). Under this assumption, it makes sense to speak of the **a**-value of a cell, and the set $W_n := \{w \in W : \mathbf{a}(w) = n\}$ is a union of two-sided cells for all $n \in \mathbb{Z}_{\geq 0}$. In principle, one may also hope to organize and study cells by **a**-values. Indeed, cells of **a**-values 0 and 1 in a general Coxeter system (W, S) are well understood. For example, if we assume (W, S) is irreducible (see Remark 2.1), then both W_0 and W_1 contain a single two-sided cell, we have $W_0 = \{1\}$, and W_1 consists of all non-identity elements in W with a unique reduced word. Moreover, Coxeter systems where W_1 is finite have been classified, and the cardinality of W_1 is known for such systems. For more details on the cell W_1 , see Proposition 2.4, [25], [21], [7, Chapter 13] and [39].

Motivated by the desire to understand elements and cells of **a**-value 2, we considered the set $W_2 = \{w \in W : \mathbf{a}(w) = 2\}$ in [19] and classified all Coxeter systems for which W_2 is finite. We called these systems **a**(2)-finite, and our classification states that an irreducible Coxeter system is **a**(2)-finite if and only if it has a complete Coxeter diagram or is of Coxeter types $A_n, B_n, \tilde{C}_n, E_{q,r}, F_n, H_n$ and I_n . The Coxeter groups of type $E_{q,r}$ encompass all Weyl groups of type D, E and affine E ; see Remark 2.5. We will recall the classification as Part (4) of Proposition 2.4 in subsection 2.2.

In this paper, we determine and count the left, right and two-sided cells within W_2 for all **a**(2)-finite Coxeter systems. Our results center around the notion of *stubs*. To define them, recall that every pair of noncommuting generators $I = \{i, j\} \subseteq S$ of a Coxeter group W gives rise to four partially defined functions on W , namely, the left upper, left lower, right upper and right lower star operations (see subsection 2.4). These operations generalize the star operations introduced in [22]. We define a *left stub* in W to be an element that admits no right lower star operation with respect to any pair of noncommuting generators; similarly, we define a *right stub* to be an element admitting no left lower star operation. We equip the set of left stubs in W_2 with two equivalence relations, one called *slide equivalence* and the other called *simple slide equivalence* (Definition 3.24), and we will show that stubs and these two relations “control everything” about the cells of W_2 in **a**(2)-finite Coxeter systems.

To be more precise, denote the set of left stubs in W_2 by $\mathcal{S}(W)$, denote slide equivalence by $\approx$, and denote simple slide equivalence by $\sim$. Call a two-sided cell a *2-cell*, a left or right cell a *1-cell*, and call the intersection of a left cell with a right cell in the same 2-cell a *0-cell*. For each stub $w \in \mathcal{S}(W)$, define the *right upper star closure* to be the set R_w of all elements that can be obtained from w by a sequence of right upper star operations. Then we will describe the 1-cells and 2-cells in terms of stubs as follows:

THEOREM 1.1. *Let (W, S) be any irreducible **a**(2)-finite Coxeter system.*

- (1) Stubs parameterize 1-cells (Theorem 3.15 & Theorem 3.17):
There is a bijection from $\mathcal{S}(W)$ to the set of right cells in W_2 that sends each stub $w \in \mathcal{S}(W)$ to its right upper star closure R_w . Moreover, the cell R_w coincides with the set of the elements of W_2 that are stronger than w in

the right weak Bruhat order. (The last fact may be viewed as an analog of Proposition 2.4.(3).)

Similar results hold for left cells. In fact, each left cell in W_2 is of the form $R_w^{-1} = \{x^{-1} : x \in R_w\}$ for some $w \in \mathcal{S}(W)$.

- (2) Slide equivalence determines 2-cells (Theorem 3.31):

Let $\mathcal{C}$ be the set of $\approx$ -classes in $\mathcal{S}(W)$ and let $\mathcal{E}$ be the set of 2-cells in W_2 . Then there is a bijection $\Phi : \mathcal{C} \rightarrow \mathcal{E}$ given by $\Phi(C) = \cup_{w \in C} R_w$ for all $C \in \mathcal{C}$. If (W, S) is not of type $E_{1,r}$ for any $r \geq 1$, then $\mathcal{S}(W)$ contains a single $\sim$ -class and W_2 is itself a 2-cell. If (W, S) is of type $E_{1,r} = D_{r+3}$ (see Remark 2.5), then we can explicitly describe the sets $\mathcal{C}$ and $\mathcal{E}$ as well. In particular, there are three 2-cells in W_2 if $r = 1$ and two 2-cells in W_2 if $r > 1$.

We will also count all left, right and two-sided cells in W_2 . As explained in Corollary 2.6, these enumeration problems can be reduced to the enumeration of 0-cells by general theory. This motivates us to examine 0-cells, which are in fact also interesting in their own right as we will soon explain. By Theorem 1.1.(1), the 0-cells in W_2 in an $\mathbf{a}(2)$ -finite Coxeter system are all of the form $I(x, y) = R_x \cap (R_y)^{-1}$ where $x, y \in \mathcal{S}(W)$. We have the following results on 0-cells and on the enumeration of all cells in W_2 . The relations $\approx$ and $\sim$ play crucial roles in these results.

THEOREM 1.2. *Maintain the setting and notation of Theorem 1.1.*

- (1) Elements of W_2 decompose nicely (Theorem 4.3 & Corollary 4.4):
Each element in W_2 can be canonically decomposed into an “ $\mathbf{a}(2)$ -triple” (Definition 4.1). The decomposition gives rise to a bijection between the set of $\mathbf{a}(2)$ -triples and W_2 , and we can use it to characterize each 0-cell $I(x, y)$ where $x, y \in \mathcal{S}(W)$ via a set of “core elements compatible with x and y ” (Definition 4.1).
- (2) Different 0-cells can be related (Remark 4.13 & Proposition 4.12):
For all x, y, x', y' , we can use Part (1) and the relation $\approx$ to obtain the 0-cells $I(x, y)$ and $I(x', y')$ from each other whenever they are in the same 2-cell. In addition, we have $|I(x, y)| = |I(x', y')|$ whenever $x \sim x'$ and $y \sim y'$.
- (3) We can count all 0-cells (Theorem 4.17 & Proposition 4.18):
We can compute one particular 0-cell $I(x, y)$ of each 2-cell in W_2 and then use Part (2) to compute and count all 0-cells in W_2 .
- (4) We can count all 1-cells and 2-cells (Theorem 4.20):
We can use Part (3) to compute the cardinalities of all left, right and two-sided cells in W_2 . The cardinalities are given by Table 9.

The results from Theorems 1.1 and 1.2 have applications for distinguished involutions and cell modules. Here, distinguished involutions are certain special involutions in Coxeter groups, and each 1-cell contains a unique distinguished involution (see [30, Chapter 14]). The canonical decompositions mentioned in Theorem 1.1 lead to a natural description of all distinguished involutions in W_2 , as we will explain in Remark 4.8. Cell modules are modules of Hecke algebras associated to Kazhdan–Lusztig cells. In the simply-laced $\mathbf{a}(2)$ -finite Coxeter types, namely types A_n and $E_{q,r}$, we will show that each 0-cell contains exactly one element. In a forthcoming paper, we will exploit this fact to interpret the left cell module afforded by L in terms of the reflection representation of the Coxeter group. Doing so realizes the cell module without needing input from the Kazhdan–Lusztig basis used to define it.

Our results are also helpful for understanding the J -rings and their categorical analogs attached to Coxeter systems. Introduced by Lusztig [28, 30], the J -ring of a Coxeter system decomposes into a direct sum of subrings corresponding to the two-sided cells of the group, and the structures of these subrings can sometimes be deduced

from knowledge of 0-cells. For example, since the 0-cells in W_2 are all singletons in types A_n and $E_{q,r}$ as we just mentioned, the subrings of the J -ring corresponding to the two-sided cells in W_2 must be matrix rings by general theory. We hope to discuss similar subrings of J -rings in other $\mathbf{a}(2)$ -finite Coxeter types in future work. In [24, 23, 31], Mazorchuk and his coauthors studied 2-representations of categorical analogs of the J -ring and of its subring corresponding to the cell W_1 . Facts about W_1 , including the classification of Coxeter groups where the cell is finite and combinatorial descriptions of the 1-cells in W_1 , proved useful in these studies. We hope our investigation of the cells in W_2 would be useful in a similar way.

Let us comment on the key ingredients and methods used in our study of cells in W_2 . First, we recall our observation from [19] that elements in W_2 are fully commutative in the sense of Stembridge [35]. Each fully commutative element has a Cartier–Foata canonical form and an associated poset called a heap, both of which are essential to our analysis of elements in W_2 . In particular, antichains in heaps and the widths of heaps (see subsection 2.3) play important roles in our proofs. A result of Ernst on fully commutative elements in the Coxeter groups $\tilde{C}_{n-1}$ is also key to our determination of those particular 0-cells mentioned in Theorem 1.2.(3); see Lemma 5.11 and subsection 5.2.

Next, we remark that the determination and enumeration of cells in W_2 can often be done in multiple ways. For example, it is possible to use the Robinson–Schensted correspondence or Temperley–Lieb diagrams to find and count cells of W_2 in type A . Various generalized Temperley–Lieb diagrams also exist in types $B, \tilde{C}, D, E$ and H (see, for example, [11, 15]) and can be indirectly applied to study cells of W_2 , although in type F such diagrams have not been defined to our knowledge. After obtaining the stub characterizations of 1-cells in each $\mathbf{a}(2)$ -finite Coxeter system, one can also count the cells with brute-force computation by considering Cartier–Foata forms or heaps. Finally, since we are interested in the case where W_2 is a finite set, we were able to automate the computation and counting of cells with a computer, even in cases where W is infinite. We do not elaborate on these alternative methods, however, since our approach via stubs seems the most general, conceptual and convenient overall.

Last but not least, we should mention that some of our results may seem reminiscent of those in the paper [12] by Fan, where the author studied certain cells defined via the so-called monomial bases of the generalized Temperley–Lieb algebras of certain Coxeter systems. These Coxeter systems do not include those of type $\tilde{C}_n$ or $E_{q,r}$ (for general values of q, r) that are studied in this paper. We also note that, as pointed out in [12, §4.1] and explained in [8, 17, 18], monomial cells are different from Kazhdan–Lusztig cells in general, even for general finite Coxeter groups. For example, we will see in Examples 2.8 and 4.14 that the left Kazhdan–Lusztig cells of $\mathbf{a}$ -value 2 in the Coxeter group of type B_4 have sizes 8 and 10, but the left monomial cells of $\mathbf{a}$ -value 2 in the same group have sizes 2, 4 and 6 by [12, §7.1]. In view of the above facts, we do not attempt to use the results of [12] in any form in this paper.

The rest of the paper is organized as follows. In section 2 we recall the necessary background for this paper on Coxeter systems, Hecke algebras, fully commutative elements and star operations. We also recall the classification of $\mathbf{a}(2)$ -finite Coxeter systems in subsection 2.2. In section 3, we define stubs, describe them, and explain how stubs parameterize 1-cells in W_2 . We also introduce the equivalence relations $\approx, \sim$ on stubs and use the former relation to find the two-sided cells of W_2 in subsection 3.3. Section 4 investigates 0-cells in detail and explains how to deduce the cardinalities of all cells in W_2 via the 0-cells. The starting point of the deduction is Theorem 4.17, which is stated without proof in subsection 4.3. The entire section 5 is dedicated to the

proof of this theorem. We prepare a set of technical lemmas on heaps in subsection 5.1 and use them to complete the proof in subsection 5.2.

2. BACKGROUND

We review some preliminary facts on Coxeter groups and Hecke algebras relevant to the paper, including the definition of Kazhdan–Lusztig cells, the definition of the $\mathbf{a}$ -function, and the classification of $\mathbf{a}(2)$ -finite Coxeter systems. We then recall the notions of fully commutative elements and generalized star operations, both of which will be essential to the study of Kazhdan–Lusztig cells of $\mathbf{a}$ -value 2.

2.1. COXETER SYSTEMS. Throughout the paper, (W, S) stands for a Coxeter system with a finite generating set S , Coxeter group W and Coxeter matrix $M = [m(s, t)]_{s, t \in S}$. Thus, we have $m(s, s) = 1$ for all $s \in S$, $m(s, t) = m(t, s) \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ for all distinct generators $s, t \in S$, and W is the group generated by S subject to the relations $\{(ss')^{m(s, s')} = 1, s, s' \in S, m(s, s') < \infty\}$. It is well known that st has order $m(s, t)$ for all distinct $s, t \in S$; in particular, s and t commute if and only if $m(s, t) = 2$.

The *Coxeter diagram* of (W, S) is the undirected diagram G on vertex set S where two vertices s, t are connected by an edge $\{s, t\}$ if and only if $m(s, t) \geq 3$. It follows that two distinct generators in S commute in W if and only if they are not adjacent in G . For each edge $\{s, t\}$ in G , we think of $m(s, t)$ as the *weight* of the edge, call the edge *simple* if $m(s, t) = 3$, and call the edge *heavy* otherwise. When drawing G , we label all heavy edges by their weights, so that the defining data of (W, S) can be fully recovered from the drawing. The system (W, S) is *irreducible* if the Coxeter diagram G is connected and *reducible* otherwise.

REMARK 2.1. To simplify statements, we shall assume all Coxeter systems to be irreducible for the rest of the paper. The assumption is well justified as far as the $\mathbf{a}$ -function is concerned, because the $\mathbf{a}$ -function behaves additively across connected components of the Coxeter graph. As a consequence, Kazhdan–Lusztig cells of a particular $\mathbf{a}$ -value in a reducible Coxeter system can be easily obtained via cells of the same or lower $\mathbf{a}$ -values in suitable irreducible Coxeter systems. Since we are only interested in cells of $\mathbf{a}$ -value 2, and cells of $\mathbf{a}$ -values 1 or 0 in irreducible Coxeter systems are well understood as explained in the introduction, it suffices to consider only irreducible Coxeter systems as we study cells of $\mathbf{a}$ -value 2 in this paper; see also [19, Theorem 1.3].

For every subset J of S , the subgroup $W_J := \langle s : s \in J \rangle$ of W is called the *parabolic subgroup generated by J* . Note that if for another Coxeter system (W', S') there is a bijection $f : S' \rightarrow J$ with $m(s, t) = m(f(s), f(t))$ for all $s, t \in S'$ (in other words, if the Coxeter diagram of (W', S') is isomorphic to the subgraph of the Coxeter diagram of (W, S) induced by J), then this bijection naturally extends to a group isomorphism from W' to W_J . For example, in the notation of Proposition 2.4.(4) the Coxeter group A_{n-1} naturally embeds into B_n for all $n \geq 2$ in the sense that A_{n-1} is isomorphic to the parabolic subgroup generated by the set $J = \{2, \dots, n\}$.

Let S^* be the free monoid on S , viewed as the set of words on the alphabet S . To distinguish words in S^* from elements of W , we denote each word with an underline. For example, the words $\underline{u} = st, \underline{v} = ts \in S^*$ represent the same element $w = st = ts$ in W if s, t are commuting distinct generators in S . For each $w \in W$, the words $\underline{w} \in S^*$ that express w with a minimum number of letters are called the *reduced words* of w ; that minimum number is called the *length* of w and written $l(w)$. More generally, we call a factorization $w = w_1 w_2 \dots w_k$ in w *reduced* if $l(w) = \sum_{i=1}^k l(w_i)$. In this

paper, the notation $w = w_1 \cdot w_2 \cdots w_k$ will always indicate that the factorization $w = w_1 w_2 \cdots w_k$ is reduced. Whenever we have $z = x \cdot y$ for elements $x, y, z \in W$, we write $x \leq^R z$ and $y \leq^L z$. The relations $\leq^R$ and $\leq^L$ define two partial orders called the *right weak Bruhat order* and *left weak Bruhat order* on W , respectively.

For each $w \in W$, we define a *left descent* of w to be a generator $s \in S$ such that $l(sw) < l(w)$, and we denote the set of left descents of w by $\mathcal{L}(w)$. Similarly, a *right descent* of w is a generator $s \in S$ such that $l(ws) < l(w)$, and we denote the set of right descents of w by $\mathcal{R}(w)$. Descents can be described via reduced words: it is well known that for all $s \in S$ and $w \in W$, we have $s \in \mathcal{L}(w)$ if and only if some reduced word of w begins with the letter s , and $s \in \mathcal{R}(w)$ if and only if some reduced word of w ends with the letter s .

We end the subsection by recalling the Matsumoto–Tits Theorem. Define a *braid relation* to be a relation of the form $sts \cdots = tst \cdots$ where $s, t \in S, 2 \leq m(s, t) < \infty$, and both sides contain $m(s, t)$ factors. Then the theorem states that for all $w \in W$, every pair of reduced words of w can be obtained from each other by applying a finite sequence of braid relations. It follows that every two reduced words of an element w contain the same set of letters. We call the set the *support* of w and denote it by $\text{Supp}(w)$.

2.2. CELLS AND THE $\mathbf{a}$ -FUNCTION. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and let H be the *Hecke algebra* of a Coxeter system (W, S) . We recall that H is a unital, associative $\mathcal{A}$ -algebra given by a certain presentation and that H has an $\mathcal{A}$ -linear basis $\{C_w : w \in W\}$ called the *Kazhdan–Lusztig basis*; see [39, §2.2]. The Kazhdan–Lusztig basis gives rise to Kazhdan–Lusztig cells and the $\mathbf{a}$ -function as follows.

Let $x, y \in W$. For all $z \in W$, let $D_z : H \rightarrow \mathcal{A}$ be the unique linear map such that $D_z(C_w) = \delta_{z,w}$ for all $w \in W$, where δ is the Kronecker delta symbol. Write $x \prec_L y$ or $x \prec_R y$ if $D_x(C_s C_y) \neq 0$ or $D_x(C_y C_s) \neq 0$ for some $s \in S$, respectively, and write $x \prec_{LR} y$ if either $x \prec_L y$ or $x \prec_R y$. The preorders $\leq_L, \leq_R, \leq_{LR}$ defined as the reflexive, transitive closures of the relations $\prec_L, \prec_R, \prec_{LR}$ naturally generate equivalence relations $\sim_L, \sim_R, \sim_{LR}$, and we call the resulting equivalence classes the *left, right and two-sided Kazhdan–Lusztig cells* of W , respectively. For example, we have that $x \sim_L y$, i.e. x and y are in the same left Kazhdan–Lusztig cell, if and only if $x \leq_L y$ and $y \leq_L x$, which happens if and only if there are sequences $x = w_0, w_1, \dots, w_n = y$ and $y = z_0, z_1, \dots, z_k = x$ where $n, k \geq 0$ and $w_i \prec_L w_{i+1}, z_j \prec_L z_{j+1}$ for all $0 \leq i < n, 0 \leq j < k$. For brevity, from now on we will refer to Kazhdan–Lusztig cells simply as “cells”, and we call both left and right cells *one-sided cells*. By the definition of $\prec_{LR}$, each two-sided cell of W is a union of left cells, as well as a union of right cells.

To define the $\mathbf{a}$ -function from the basis $\{C_w : w \in W\}$, consider the structure constants $h_{x,y,z} \in \mathcal{A}(x, y, z \in W)$ such that

$$(1) \quad C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$$

for all $x, y \in W$. By §15.2 of [30], for each $z \in W$ there exists a unique integer $\mathbf{a}(z) \geq 0$ such that

- (a) $h_{x,y,z} \in v^{\mathbf{a}(z)} \mathbb{Z}[v^{-1}]$ for all $x, y \in W$;
- (b) $h_{x,y,z} \notin v^{\mathbf{a}(z)-1} \mathbb{Z}[v^{-1}]$ for some $x, y \in W$.

The assignment $z \mapsto \mathbf{a}(z)$ defines the function $\mathbf{a} : W \rightarrow \mathbb{Z}_{\geq 0}$.

In the sequel, we will make extensive use of the following standard results on cells and the $\mathbf{a}$ -function. We note that the proofs of Parts (5)–(9) of the proposition either directly or indirectly rely on the use of Lusztig’s boundedness conjecture [30, §13.4].

The conjecture states that for the standard basis $\{T_w : w \in W\}$ of the Hecke algebra (see [39, §2.2]) and the structure constants $f_{x,y,z}$ such that $T_x T_y = \sum_{z \in W} f_{x,y,z} T_z$ for all $x, y \in W$, there is a nonnegative integer N such that $v^{-N} f_{x,y,z}$ is an element of the ring $\mathbb{Z}[v^{-1}]$ for all $x, y, z \in W$. In the following proposition, Parts (5)–(9) contain statements from conjectures P9–P12 and P14 in [30, §14], and the proofs of these five conjectures in turn rely on Lusztig’s boundedness conjecture (despite the fact that conjectures P1–P6 from [30, §14] are known to hold in the setting of our current paper, where the Hecke algebra has “equal parameters”, without reliance on the boundedness conjecture); see also [30, §15.1], [7, §14.3] and [7, Theorem 15.2.5].

PROPOSITION 2.2. *Let (W, S) be a Coxeter system and let $x, y \in W$.*

- (1) *If $x \leq^L y$ then $y \leq_L x$; if $x \leq^R y$ then $y \leq_R x$.*
- (2) *If x is a product of k pairwise commuting generators in S , then $\mathbf{a}(x) = k$.*
- (3) *We have $\mathcal{L}(x) = \mathcal{L}(y)$ if $x \sim_R y$, and $\mathcal{R}(x) = \mathcal{R}(y)$ if $x \sim_L y$;*
- (4) *For each left cell L in W , the set $L^{-1} = \{z^{-1} : z \in L\}$ is a right cell; for each right cell R in W , the set $R^{-1} = \{z^{-1} : z \in R\}$ is a left cell.*
- (5) *If $x \leq_{LR} y$, then we have $\mathbf{a}(x) \geq \mathbf{a}(y)$. In particular, if $x \sim_{LR} y$, then we have $\mathbf{a}(x) = \mathbf{a}(y)$. Moreover, if $x \leq_L y$ but $x \not\sim_L y$, or if $x \leq_R y$ but $x \not\sim_R y$, then we have $\mathbf{a}(x) > \mathbf{a}(y)$.*
- (6) *We have $z \sim_{LR} z^{-1}$ for all $z \in W$.*
- (7) *Let $J \subseteq S$ and consider the corresponding parabolic subgroup W_J of W . Let $\mathbf{a}_J$ denote the $\mathbf{a}$ -function associated to the Coxeter system (W_J, J) . If $x \in W_J$, then the $\mathbf{a}$ -value $\mathbf{a}_J(x)$ computed in the system (W_J, J) equals the value $\mathbf{a}(x)$ computed in the system (W, S) .*
- (8) *Let L_1, L_2 be two left cells in the same two-sided cell. Then the intersection $L_1 \cap (L_2^{-1})$ is nonempty.*
- (9) *If $x \sim_{LR} y$, then W contains an element z such that $z \sim_L x$ and $z \sim_R y$.*

Proof. Parts (1)–(7) are due to Lusztig and can be found in [30]: Parts (1)–(2) follow immediately from Theorem 6.6 and Proposition 2.6; Parts (3)–(4) are proved in Section 8; the statements in Parts (5)–(7) appear in Sections 14 as conjectures P9–P12 and P14. Part (8) is Lemma 2.4 of [4]. (The lemma relies on the construction of the so-called J -ring of (W, S) from [30, §18], which in turn depends on Lusztig’s boundedness conjecture.) To see Part (9), note that if $x \sim_{LR} y$ then $x \sim_{LR} y^{-1}$ by Part (6); therefore the left cell of x intersects the inverse of the left cell of y^{-1} nontrivially by Part (8). This inverse is precisely the right cell of y by Part (4), so any element z in the intersection satisfies $z \sim_L x$ and $z \sim_R y$. $\square$

DEFINITION 2.3. *For each Coxeter system (W, S) and each integer $n \geq 0$, we let $W_n = \{w \in W : \mathbf{a}(w) = n\}$, and we say that (W, S) is $\mathbf{a}(n)$ -finite if W_n is finite.*

By Proposition 2.2.(5), for each integer $n \geq 0$ the set W_n defined above is a union of two-sided cells, of left cells, and of right cells. We are interested in the cells in W_2 within $\mathbf{a}(2)$ -finite Coxeter systems. Our motivation partly comes from the following facts about W_0 and W_1 .

PROPOSITION 2.4. *Let (W, S) be a Coxeter system with Coxeter diagram G . (Recall from Remark 2.1 that we assume (W, S) is irreducible.)*

- (1) *We have $W_0 = \{1\}$. In particular, W_0 is finite and simultaneously the unique left, right and two-sided cell of $\mathbf{a}$ -value 0.*
- (2) *We have $W_1 = \{w \in W : w \neq 1 \text{ and } w \text{ has a unique reduced word}\}$. The set W_1 is finite (i.e. the system (W, S) is $\mathbf{a}(1)$ -finite) if and only if G is acyclic, contains no edge with infinite weight, and contains at most one heavy edge with finite weight.*

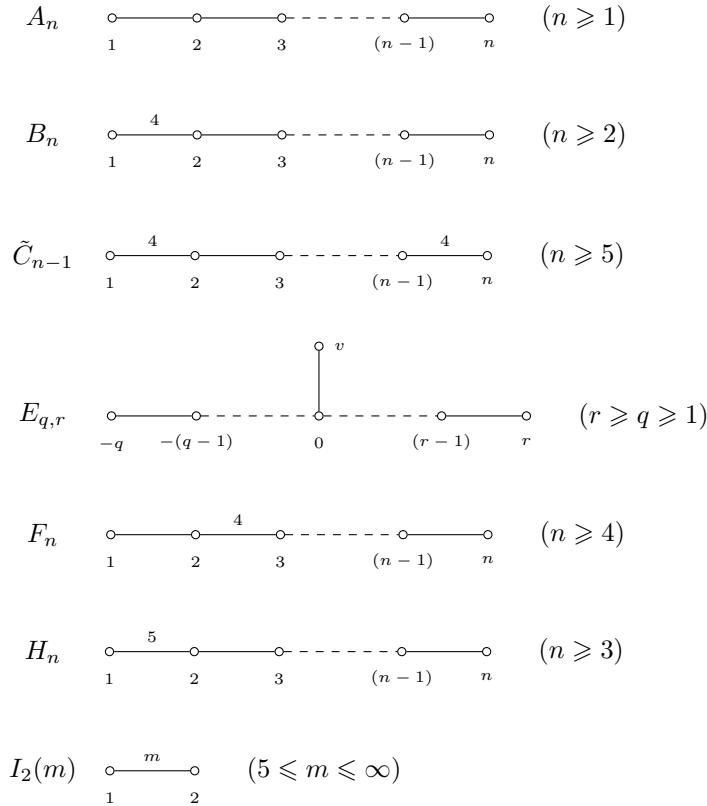


FIGURE 1. Irreducible $\mathbf{a}(2)$ -finite Coxeter groups with acyclic diagrams.

- (3) The set W_1 is itself a two-sided cell and hence the unique two-sided cell of $\mathbf{a}$ -value 1 in W_1 . The left cells in W_1 are in bijection with S and given by the sets

$$L_s := \{z \in W_1 : s \leq^L z\}$$

while the right cells in W_1 are in bijection with S and given by the sets

$$R_s := \{z \in W_1 : s \leq^R z\}$$

where $s \in S$.

- (4) If G contains a cycle, then W is $\mathbf{a}(2)$ -finite if and only if G is complete. If G is acyclic, then W is $\mathbf{a}(2)$ -finite if and only if G is one of the graphs in Figure 1, where n equals the number of vertices in the Coxeter diagram and there are q and r vertices strictly to the left and the right of the trivalent vertex in $E_{q,r}$.

Proof. Part (1) is proved in [30, §13]. Let

$$C = \{w \in W : w \neq 1 \text{ and } w \text{ has a unique reduced word}\}.$$

Then the first assertion in Part (2), the assertion that $W_1 = C$, is proved in [39, Corollary 3.1]. Given the equality $W_1 = C$, the second assertion in Part (2) follows immediately from [25, Proposition 3.8(h)]. Part (3) follows from [25, Proposition 3.8(c)] and Proposition 2.2(4). Part (4) is the main result of [19]. $\square$

REMARK 2.5 (Labeling conventions). For each $\mathbf{a}(2)$ -finite Coxeter system (W, S) , we will adhere to the labeling of the set S from Figure 1 for the rest of the paper. For example, the generators in B_n will always be labeled $1, 2, \dots, n$, with $\{1, 2\}$ forming

	$L(1)$	$L(2)$	$L(3)$	$L(4)$	$L(5)$	$L(6)$
$R(1)$	2	2	2	2	1	1
$R(2)$	2	2	2	2	1	1
$R(3)$	2	2	2	2	1	1
$R(4)$	2	2	2	2	1	1
$R(5)$	1	1	1	1	2	2
$R(6)$	1	1	1	1	2	2

TABLE 1. Sizes of 0-cells of $\mathbf{a}$ -value 2 in type B_4

the unique heavy edge, and generators in $E_{q,r}$ will always be labeled by the numbers $-q, -q+1, \dots, 0, \dots, r$ and the letter v under the assumption that $r \geq q$, which causes no loss of generality. Note that in the cases where $q = 1$, $(q, r) = (2, 2)$, $(q, r) = (2, 3)$, $(q, r) = (2, 4)$, $(q, r) = (3, 3)$ and $(q, r) = (2, 5)$, the Coxeter group of type $E_{q,r}$ coincides with the Weyl or affine Weyl group of type $D_{r+3}, E_6, E_7, E_8, \tilde{E}_7$ and $\tilde{E}_8$, respectively; the group F_5 coincides with the affine Weyl group of type $\tilde{F}_4$.

As we study cells within the set W_2 for $\mathbf{a}(2)$ -finite Coxeter groups, Part (4) of Proposition 2.2 allows us to consider only right cells because each left cell L is the setwise inverse of a right cell $R := L^{-1}$. Furthermore, Part (6) of the same proposition implies that the set R is not only a right cell but also a right cell in the same two-sided cell as L . The following corollary, which we will use to count cells in W_2 , is now immediate.

COROLLARY 2.6. *Let E be a two-sided cell in a Coxeter group W . Let $\mathcal{LC}$ and $\mathcal{RC}$ be the collections of left cells and right cells in E , respectively. Then $\mathcal{LC} = \{R^{-1} : R \in \mathcal{RC}\}$. In particular, for each $R \in \mathcal{RC}$ we have*

$$R = \bigsqcup_{L \in \mathcal{LC}} R \cap L = \bigsqcup_{R' \in \mathcal{RC}} R \cap (R')^{-1}$$

and hence

$$|R| = \sum_{L \in \mathcal{LC}} |R \cap L| = \sum_{R' \in \mathcal{RC}} |R \cap (R')^{-1}|.$$

The above corollary suggests that intersections of left and right cells in a two-sided cell are important. This motivates the following definition.

DEFINITION 2.7. *We define a zero-sided cell in a Coxeter system (W, S) to be a set of the form $R \cap L$ where R is a right cell and L is a left cell in the same two-sided cell as R . For brevity, we also refer to two-sided, one-sided and zero-sided cells of W as 2-cells, 1-cells and 0-cells.*

EXAMPLE 2.8. Let (W, S) be the Coxeter system of type B_4 , which is $\mathbf{a}(2)$ -finite by Proposition 2.2.(10). Later we will see that W_2 is the unique two-sided cell of $\mathbf{a}$ -value 2 in W ; moreover, the cell W_2 can be partitioned into 6 right cells as well as into 6 left cells. Under a certain labeling of the right and left cells respectively as $R(i)$ and $L(j)$ for $1 \leq i, j \leq 6$, the sizes of the 0-cells in W_2 are given by Table 1, where the entry in the $R(i)$ -row, $L(j)$ -column equals the size of the 0-cell $R(i) \cap L(j)$. By Corollary 2.6, we may count each 1-cell $R(i)$ by summing the entries in the corresponding row, and then we may count the 2-cell W_2 by summing all entries in the table. It follows that $|R(i)| = 10$ if $1 \leq i \leq 4$, that $|R(i)| = 8$ if $5 \leq i \leq 6$, and that $|W_2| = 56$.

REMARK 2.9 (Symmetry). As is evident in Propositions 2.2 and 2.4, there is a large amount of “left-right symmetry” in the properties of Coxeter group elements and

Kazhdan–Lusztig cells. The rest of the paper contains many more pairs of “left-right symmetric” notions and assertions, such as the two statements in Proposition 2.18. For such pairs, we will often formulate one notion or assertion carefully and leave the formulation of the other to the reader (as we actually already did in Corollary 2.6 and Example 2.8), or prove one of the assertions and invoke symmetry as justification for the other. The reader may assume that the obvious symmetry always works as expected.

2.3. FULLY COMMUTATIVE ELEMENTS. Recall that a braid relation in a Coxeter system is a relation of the form $st \cdots = ts \cdots$ where $s, t \in S, 2 \leq m(s, t) < \infty$ and both sides contain $m(s, t)$ factors. We call the relation a *commutation relation* if $m(s, t) = 2$ and call each side of the relation a *long braid* if $m(s, t) \geq 3$. An element $w \in W$ is called *fully commutative*, or *FC*, if every pair of reduced words of w can be connected by a finite sequence of commutation relations. Proposition 2.1 of [35] gives a well-known “word criterion for full commutativity”: we have w if FC if and only if no reduced word of w contains a long braid as a subword. Here and henceforth, by a subword of a word $s_1 \dots s_q$ we always mean a contiguous subword, i.e. a word of the form $s_i s_{i+1} \dots s_{j-1} s_j$ for some $1 \leq i \leq j \leq q$.

We denote the set of all FC elements in W by $\text{FC}(W)$. We shall study cells of $\mathbf{a}$ -value 2 via FC elements because of the following fact:

PROPOSITION 2.10 ([19, Proposition 3.9]). *Let $w \in W$. If $\mathbf{a}(w) = 2$, then w is FC.*

We review two important tools for studying FC elements. The first is a canonical form called the *Cartier–Foata form*. For an FC element w , this refers to the reduced word $w = w_p \cdots w_2 \cdot w_1$ satisfying the following properties:

- (1) For all $1 \leq i \leq p$, the element w_i is a product of pairwise commuting generators in S ;
- (2) For all $1 < i \leq p$, every generator $t \in \text{Supp}(w_i)$ fails to commute with some generator $s \in \text{Supp}(w_{i-1})$.

The factors $w_1, w_2, w_3, \dots$ can be obtained inductively as the product of the right descents of the elements $x_1 = w, x_2 = x_1 w_1, x_3 = x_2 w_2, \dots$, respectively, and the factorization $w = w_p \cdots w_1$ is unique up to the re-ordering of the generators appearing in w_i for each i ; see [16]. We call each w_i the *i -th layer* of w . When the generators in S are labeled by integers, we shall insist that within each layer we order the generators in increasing order from left to right. For example, in the group $W = A_5$ the element $w = 1532 \in \text{FC}(W)$ has Cartier–Foata form $w = w_2 w_1$ where $w_1 = 25$ and $w_2 = 13$. For more on the Cartier–Foata form, see [16].

Our second tool for studying an FC element w is a labeled poset called a heap. To define it, we first define the *heap of a word* $\underline{w} = s_1 \dots s_q \in S^*$ to be the labeled poset $H(\underline{w}) := ([q], \preceq)$ where the underlying set is $[q] = \{1, 2, \dots, q\}$, the partial order $\preceq$ is the reflexive, transitive closure of the relation $\prec$ defined by

$$i \prec j \quad \text{if } i < j \text{ and } m(s_i, s_j) \neq 2,$$

and the label of the element i is s_i for each $i \in [q]$. The heaps of two words related by a commutation relation are isomorphic as labeled posets in the sense that there exists a poset isomorphism $f : H(\underline{w}) \rightarrow H(\underline{w}')$ such that $f(i)$ and i have the same label for all $i \in H(\underline{w})$; see [35, §2.2]. Thus, it makes sense to define the *heap of an element* $w \in W$, up to isomorphism, to be the heap of any reduced word of w ; we denote this heap by $H(w)$. The following “heap criterion for full commutativity” is a well-known analog of the word criterion mentioned earlier.

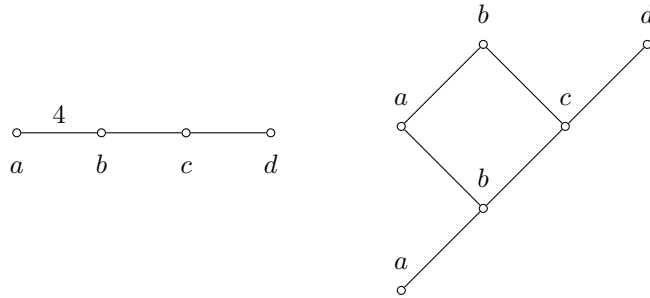


FIGURE 2. Lattice embedding of a heap.

PROPOSITION 2.11 ([35, Proposition 3.3]). *Let $\underline{w} = s_1 s_2 \cdots s_q \in S^*$. Then $\underline{w}$ is the reduced word of a fully commutative element in W if and only if the heap $H(\underline{w})$ satisfies the following conditions:*

- (1) *there is no covering relation $i \prec j$ such that $s_i = s_j$;*
- (2) *there is no convex chain $i_1 \prec i_2 \prec \cdots \prec i_m$ in $H(\underline{w})$ with $s_{i_1} = s_{i_3} = \cdots = s_{i_m}$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $s, t \in S$ and $m = m(s, t) \geq 3$.*

REMARK 2.12. Let $w \in \text{FC}(W)$. By the definition of $H(w)$, the left and right descents of w are exactly the labels of the minimal and maximal elements in $H(w)$, respectively. In particular, the support of the first layer in the Cartier–Foata form of w coincides with the set of the labels of the maximal elements in $H(w)$, because they both equal $\mathcal{R}(w)$.

REMARK 2.13. By the definition of a heap, for every chain of coverings $i_1 \prec i_2 \prec \cdots \prec i_k$ in the heap of an FC element $w = s_1 \cdots s_q$, the generators s_{i_j} and $s_{i_{j+1}}$ must be distinct and adjacent in the Coxeter diagram. In other words, the sequence of generators $s_{i_1}, \dots, s_{i_k}$ forms a *walk* from s_{i_1} to s_{i_k} on the Coxeter diagram in the usual graph theoretical sense. We also recall, for later use, that such a walk has *length* $(k - 1)$ and that a *path* from a vertex v to a vertex u on a graph is a walk from v to u of minimal length. On an acyclic and connected graph, there is a unique path from u to v for any two vertices u, v .

There is an intuitive way to visualize the heap of a word $\underline{w} = s_1 \cdots s_q \in S^*$ in the lattice $S \times \mathbb{Z}_{\geq 0}$. Here, we think of the elements of S as *columns* and the elements of $\mathbb{Z}_{\geq 0}$ as *levels*, and we say two columns s, t *commute* if they commute as generators, i.e. if $m(s, t) = 2$. To embed $H(\underline{w})$ into the lattice, we read the letters in $\underline{w}$ from left to right, and for the j -th letter we drop a vertex p_j in the column s_j to the lowest level possible subject to the condition that p_j should fall above every vertex p_i that has been dropped into a column that does not commute with s_j , i.e. above every vertex p_i for which $i \prec j$. In addition, if a column s does not commute with s_j and contains at least one vertex p_i with $i < j$, i.e. if $i \prec j$, then we draw an edge to connect p_j to the highest such vertex, i.e. to the vertex p_i in column s with i maximal. It follows that the vertices $p_1, \dots, p_q$ and the edges of the form $\{p_i, p_j\}$ where $i \prec j$ form exactly the Hasse diagram of the heap $H(\underline{w})$.

When drawing the embedding of $H(\underline{w})$, it is customary to not draw the columns or levels and to label each point p_i simply by s_i . This ensures that the isomorphic heaps arising from the reduced words of w are given by the same graph, the Hasse diagram of the heap $H(w)$. For example, in Figure 2 the picture on the right shows the heap $H(w)$ of the element $w = \text{abcadb}$ in the Coxeter group whose Coxeter diagram is drawn on the left.

Heaps can help compute $\mathbf{a}$ -values. Let $n(w)$ be the *width* of the poset $H(w)$, i.e. let

$$n(w) = \max\{|A| : A \text{ is an antichain in } H(w)\}.$$

Then $n(w)$ bounds and may equal $\mathbf{a}(w)$ (see also Remark 3.10):

PROPOSITION 2.14 ([34]). *Let $w \in \text{FC}(W)$. Then*

- (1) *we have $\mathbf{a}(w) \geq n(w)$;*
- (2) *we have $\mathbf{a}(w) = n(w)$ if W is a Weyl or affine Weyl group.*

Proof. The equality in (2) is the main result, Theorem 3.1, of [34]. The inequality in (1) essentially follows the results in §1.3 of the same paper, but let us give a short proof in our notation: the definition of heaps implies that w admits a reduced factorization of the form $w = x \cdot y \cdot z$ where y is a product of $n(w)$ pairwise commuting generators, whence Parts (1), (2) and (5) of Proposition 2.2 imply that $\mathbf{a}(w) \geq \mathbf{a}(y) = n(w)$. $\square$

COROLLARY 2.15. *Suppose that $\mathbf{a}(w) = 2$ (so w is FC by Proposition 2.10).*

- (1) *An antichain in $H(w)$ has at most two elements, and every antichain in $H(w)$ with two elements is maximal.*
- (2) *Every layer in the Cartier–Foata form of w has at most two generators.*
- (3) *We have $|\mathcal{R}(w)| \leq 2$.*

Proof. We must have $n(w) \leq 2$ by Proposition 2.14.(1). This implies (1). By the definition of heaps, each layer in the Cartier–Foata form of w corresponds to an antichain that has as many elements as the support of the layer, so (1) implies (2). Finally, the set $\mathcal{R}(w)$ equals the support of the first layer of w by Remark 2.12; therefore (3) follows from (2). $\square$

Antichains of heaps appear frequently in this paper. We highlight one property of maximal antichains now. Recall that an *ideal* in a poset P is a set $\mathcal{I}$ such that if $y \in \mathcal{I}$ and $x < y$ for $x, y \in P$ then $x \in \mathcal{I}$; dually, a *filter* in P is a set $\mathcal{F}$ such that if $y \in \mathcal{F}$ and $x > y$ for $x, y \in P$ then $x \in \mathcal{F}$. Any set $A \subseteq P$ *generates* an ideal

$$\mathcal{I}_A := \{i \in H : i \leq j \text{ for some } j \in A\}$$

and a filter

$$\mathcal{F}_A := \{i \in H : i \geq j \text{ for some } j \in A\}.$$

If A is an antichain, then any element in the set $P \setminus (\mathcal{I}_A \cup \mathcal{F}_A)$, if it exists, can be adjoined to A to form a larger antichain. This implies the following:

LEMMA 2.16. *If A is a maximal antichain in P , then $P = \mathcal{I}_A \cup \mathcal{F}_A$.*

2.4. STAR OPERATIONS. In this subsection we discuss star operations on W . We emphasize the so-called right star operations and omit details on the analogous left star operations in the spirit of Remark 2.9.

A right star operation is a function from W to W partially defined with respect to a pair of noncommuting generators $I = \{s, t\} \subseteq S$. Let $m = m(s, t)$ and consider the parabolic subgroup $W_I = \langle s, t \rangle$. Then each element w in W admits a unique reduced factorization $w = w^I \cdot w_I$ where $\mathcal{R}(w^I) \cap I = \emptyset$ and $w_I \in W_I$. This factorization is the *left coset decomposition* of w with respect to I ; see [5, §2.4]. Depending on the value of w_I , one of the following mutually exclusive conditions must hold; the sequences in (2) and (3) are both infinite if $m = \infty$.

- (1) $w_I = 1$, or w_I is the longest element $sts \dots$ of length $m(s, t)$ in W_I ;
- (2) w is one of the $(m - 1)$ elements

$$x_1 := w^I \cdot s, \quad x_2 := w^I \cdot st, \quad x_3 := w^I \cdot sts, \quad \dots$$

where $\mathcal{L}(w_I) = \{s\}$ and $l(w_I) < m$;

(3) w is one of the $(m - 1)$ elements

$$y_1 := w^I \cdot t, \quad y_2 := w^I \cdot ts, \quad y_3 := w^I \cdot tst, \quad \dots$$

where $\mathcal{L}(w_I) = \{t\}$ and $l(w_I) < m$.

By definition, we may apply a *right upper star operation with respect to I* on w if and only if w is an element of the form x_i or y_i for some $1 \leq i \leq m - 2$; in these cases, the result of the operation is denoted by w^* and defined to be x_{i+1} or y_{i+1} , respectively. Similarly, a *right lower star operation with respect to I* is defined on w precisely when w is an element of the form x_i or y_i for some $2 \leq i \leq m - 1$, whence the result of the operation is the element $w_* := x_{i-1}$ or $w_* := y_{i-1}$, respectively. Left star operations are defined similarly, and we denote the result of applying a left upper or lower star operation on w by *w or ${}_{}w$, respectively.

When $m = m(s, t) = 3$, the (generalized) star operations defined above recover the original star operations introduced by Kazhdan and Lusztig in [22]. In this case at most one of w^* and w_* is defined depending on the value of w_I ; therefore we may unambiguously speak of *the right star operation with respect to I* , without indicating if the operation is upper or lower. More precisely, both w^* and w_* are undefined if $w_I \in \{1, sts = tst\}$, w^* is defined while w_* is not if $w_I \in \{s, t\}$, and w_* is defined while w^* is not if $w_I \in \{st, ts\}$. In the last two cases, we denote whichever one of w^* and w_* is defined by $w*$, placing the star sign in the middle. We call the operation $w \mapsto w*$ a *simple star operation*; it is an involution in the sense that $(w*)^* = w$ whenever $w*$ is defined. Similarly, the pair $I = \{s, t\}$ gives rise to a partially defined involution $w \mapsto {}^*w$ on W that we call the *simple left star operation with respect to I* .

EXAMPLE 2.17. Let (W, S) be a Coxeter system with $S = \{a, b, c\}$, $m(a, b) = 3$, $m(b, c) = 4$ and $m(a, c) = 2$. Let $I = \{a, b\}$, $J = \{b, c\}$, and let $w = abcab$. Then with respect to I , the coset decompositions of w are given by

$$w = w_I \cdot {}^Iw = aba \cdot cb, \quad w = w^I \cdot w_I = abc \cdot ab,$$

so $w_* = abc \cdot a$ while w^* , ${}_{}w$, *w are not defined. Moreover, since $m(a, b) = 3$, we have $w* = w_* = abca$ while *w is undefined. With respect to J , we have

$$w = w_J \cdot {}^Jw = b \cdot abcb, \quad w = w^J \cdot w_J = ba \cdot bcb,$$

so ${}^*w = cb \cdot abcb$, $w_* = ba \cdot bc$, while ${}_{}w$ and w^* are not defined.

Star operations are intimately related to Kazhdan–Lusztig cells. First, left and right cells are closed under left and right star operations, respectively:

PROPOSITION 2.18 ([19, Proposition 3.3]). *Let (W, S) be a Coxeter system. Then the following holds with respect to every pair of noncommuting generators $\{s, t\}$ in S :*

- (1) *If $w \in W$ and ${}_{}w$ is defined, then $w \sim_L {}^*w$;*
- (2) *If $w \in W$ and w_* is defined, then $w \sim_R w_*$.*

Second, simple star operations preserve cell equivalence in the following way:

PROPOSITION 2.19 ([22, Corollary 4.3]). *Let (W, S) be a Coxeter system, let $y, w \in W$, and suppose $m(s, t) = 3$ for some generators $s, t \in S$. Then the following hold with respect to the pair $\{s, t\}$:*

- (1) *if $y \sim_L w$ and $y*$, $w*$ are defined, then $y* \sim_L w*$;*
- (2) *if $y \sim_R w$ and ${}_{}y$, ${}_{}w$ are defined, then ${}_{}y \sim_R {}^*w$.*

Later, we will often combine Propositions 2.18 and 2.19 with Proposition 2.2 to show two elements are in the same cell or in distinct cells. Note that Proposition 2.18 implies that star operations preserve $\mathbf{a}$ -values:

COROLLARY 2.20. *Let $x, y \in W$. If y is obtained from x via any star operation, then $\mathbf{a}(y) = \mathbf{a}(x)$.*

Proof. This is immediate from Proposition 2.18 and Proposition 2.2.(5). $\square$

Our discussion of star operations so far applies to arbitrary elements of W , rather than only FC elements. On the other hand, if an element $w \in W$ is FC, then the heap of w provides a simple characterization of what star operations can be applied to w . We explain why this is true for right lower star operations below. Let $I = \{s, t\}$ be a pair of noncommuting generators in S as before. By definition, the element w_* with respect to I is defined and given by $w_* = ws$ if and only if in the coset decomposition $w = w^I \cdot w_I$ we have $2 \leq l(w_I) < m(s, t)$ and the reduced word of w_I is of the form $w = \dots ts$. The last condition holds if and only if $s \in \mathcal{R}(w)$ and $t \in \mathcal{R}(ws)$, in which case w has a reduced word ending in s and removal of that s results in a reduced word of w_* . Remark 2.12 now implies the following result:

PROPOSITION 2.21. *Let $w \in W$ be FC and suppose $3 \leq m(s, t)$ for some $s, t \in S$. Then w admits a right lower star operation with respect to $\{s, t\}$ which results in ws if and only if the conditions below all hold:*

- (1) *the heap $H(w)$ contains a maximal element i labeled by s ;*
- (2) *the element i covers an element $j \in H(w)$ labeled by t ;*
- (3) *the element i is the unique maximal element covering j in $H(w)$.*

Note that Condition (3) is equivalent to the condition that j becomes maximal upon removal of i from $H(w)$. The proposition allows us to tell what lower star operations applies to an FC element from a glance at its heap. For example, given the element $w = abcadb$ and the embedding of $H(w)$ considered in Figure 2, we note that the two highest vertices in the figure are the only maximal elements in $H(w)$ and that only the removal of top vertex labeled by b results in a new maximal vertex, namely, the higher of the two vertices labeled by a . It follows that only one right lower star operation is defined on w , namely, the operation with respect to $\{a, b\}$ that removes the right descent $b \in \mathcal{R}(w)$.

3. KAZHDAN–LUSZTIG CELLS VIA STUBS

Recall the following definition of stubs from the introduction:

DEFINITION 3.1. *Let (W, S) be a Coxeter system and let $w \in W$. We call w a left stub if no right lower star operation can be applied to w . Similarly, we call w a right stub if w admits no left lower star operation. We denote the set of left stubs of $\mathbf{a}$ -value 2 in W by $\mathcal{S}(W)$, and denote the set of right stubs of $\mathbf{a}$ -value 2 in W by $\mathcal{S}'(W)$.*

In this section, we assume that (W, S) is an irreducible $\mathbf{a}(2)$ -finite Coxeter system, describe the stubs in $\mathcal{S}(W)$, and use stubs to find the left, right and two-sided cells in W_2 . Our main results on cells are Theorems 3.7, 3.15, 3.17 and 3.31. As we study 2-cells in subsection 3.3, we also introduce two equivalence relations on $\mathcal{S}(W)$ that will play a crucial role as we study 0-cells in section 4.

We note that while some results in the section apply to arbitrary Coxeter systems, stubs seem to control cells in W_2 in the nicest ways only for $\mathbf{a}(2)$ -finite systems; see Remark 3.19. We also note that since W_2 certainly contains no cell when it is empty, we shall often assume that (W, S) is *nontrivially* $\mathbf{a}(2)$ -finite in the sense that W_2 is finite but nonempty. By Proposition 2.4.(4) and the results of [19], the nontrivially $\mathbf{a}(2)$ -finite systems are exactly those of types A_n ($n \geq 3$), B_n ($n \geq 3$), $\tilde{C}_{n-1}$ ($n \geq 5$), $E_{q,r}$ ($r \geq q \geq 1$), F_n ($n \geq 4$) and H_n ($n \geq 3$). Our description of stubs and cells will therefore often be given type by type.

We will often use the prefix “ $\mathbf{a}(2)$ -” as an adjective to indicate an object has $\mathbf{a}$ -value 2. For example, the set $\mathcal{S}(W)$ is exactly the set of left $\mathbf{a}(2)$ -stubs in W , and an $\mathbf{a}(2)$ -cell means a cell of $\mathbf{a}$ -value 2.

3.1. DESCRIPTION OF STUBS. We study $\mathbf{a}(2)$ -stubs in this subsection. As $\mathbf{a}(2)$ -elements are FC, we start with the following observation:

PROPOSITION 3.2. *Let (W, S) be an arbitrary Coxeter system and let w be an FC element in W . Let $w = w_p \dots w_2 w_1$ be the Cartier–Foata form of w . Then w is a left stub if and only if every generator $t \in \text{Supp}(w_2)$ fails to commute with at least two generators $s, s' \in \text{Supp}(w_1)$.*

REMARK 3.3. Note that the above condition on t is vacuously true when $w = w_1$, i.e. when w is a product of commuting generators.

Proof of Proposition 3.2. Let $t \in \text{Supp}(w_2)$ and $s \in \text{Supp}(w_1)$. Then $s \neq t$ by Proposition 2.11.(1), so t is covered by s in $H(w)$ if and only if $m(s, t) \geq 3$. The proposition then follows from Remark 2.12 and Proposition 2.21. $\square$

The proposition shows that whether an FC element is a left stub is determined “locally”, by only the first two layers of its Cartier–Foata form.

COROLLARY 3.4. *Let (W, S) be an arbitrary Coxeter system. Let G be the Coxeter diagram of (W, S) . Then an $\mathbf{a}(2)$ -element $w \in W$ is a left stub if and only if in the Cartier–Foata form $w = w_p \dots w_2 w_1$ of w , we have*

- (1) $w_1 = ss'$ for two commuting generators s, s' in S ;
- (2) Every generator $t \in \text{Supp}(w_2)$ is adjacent to both s and s' in G .

Proof. By Proposition 3.2, it suffices to show that $\text{Supp}(w_1)$ contains exactly two generators. By Corollary 2.15, it further suffices to show that $\text{Supp}(w_1)$ is not empty or a singleton. This follows from Proposition 2.4: if w_1 is empty, then $w = 1$ and $\mathbf{a}(w) = \mathbf{a}(1_W) = 0$; if $\text{Supp}(w_1)$ contains only one generator s , then w_2 must be empty by Proposition 3.2, which forces $w = w_1 = s$ and $\mathbf{a}(w) = \mathbf{a}(s) = 1$. $\square$

For nontrivially $\mathbf{a}(2)$ -finite systems, Corollary 3.4 turns out to impose strong restrictions on the first two layers of $\mathbf{a}(2)$ -stubs. The following observation imposes further restrictions on the deeper layers.

LEMMA 3.5. *Let (W, S) be an arbitrary Coxeter system. Let G be the Coxeter diagram of (W, S) . Let $w \in \text{FC}(W)$, let $w = w_p \dots w_1$ be the Cartier–Foata form of w , and let $2 < i \leq p$.*

- (1) Every element $t \in \text{Supp}(w_i)$ is adjacent in G to some element $s \in \text{Supp}(w_{i-1})$.
- (2) If $s, t \in S$ are generators connected by a simple edge in G such that s appears in w_{i-2} and $w_{i-1} = t$, then s does not appear in w_i .

Proof. Part (1) simply paraphrases Condition (2) from the characterization of the Cartier–Foata form in subsection 2.3. To prove (2), note that if s appears in w_i then we may commute the s in w_{i-2} to the left and the s in w_i to the right, if necessary, until they are both next to the t in w_{i-1} . This results in a reduced factorization $w = x \cdot sts \cdot y$ of w , contradicting the word criterion for full commutativity. $\square$

The following example shows in detail how Conditions (1)–(2) from Corollary 3.4 and Lemma 3.5 strongly restrict the Cartier–Foata forms of left $\mathbf{a}(2)$ -stubs in a nontrivially $\mathbf{a}(2)$ -finite Coxeter system. Roughly speaking, the restrictions are strong because the Coxeter diagram of such a system always contains few pairs of vertices that share a neighbor as well as few heavy edges.

1 3 ○ ○	2 4 ○ ○	1 4 ○ ○
1 3 ○ ○ 2	2 4 ○ ○ 3	
1 3 ○ ○ 2 1		

TABLE 2. The six stubs of B_4

EXAMPLE 3.6. We find all left $\mathbf{a}(2)$ -stubs in the Coxeter system (W, S) of type B_n in this example. Table 2 contains the heaps of the six stubs in $\mathcal{S}(B_4)$, which will serve as an example.

Let $w \in \mathcal{S}(W)$ and let $w = w_p \dots w_2 w_1$ be the Cartier–Foata form of w . By Condition (1) of Corollary 3.4, we must have either $w_1 = ij$ for some $1 \leq i, j \leq n$ where $j > i + 2$ or $w_1 = i(i + 2)$ for some $1 \leq i \leq n - 1$. In the former case—which holds for only the stub in the third column in Table 2 if $n = 4$ —the second layer of w cannot exist by Condition (2) of Corollary 3.4 since i, j share no neighbor in G . In the latter case—which corresponds to the first two columns of Table 2—either w_2 does not exist and $w = w_1 = i(i + 2)$ (first row), or w_2 exists and is given by $w_2 = (i + 1)$, the only common neighbor of i and $(i + 2)$ in G (second row). In the latter subcase, if $i > 1$ then we have $m(i + 1, i) = m(i + 1, i + 2) = 3$, hence Lemma 3.5 rules out the possibility of a third layer in w and forces $w = w_2 w_1 = (i + 1) \cdot i(i + 2)$; if $i = 1$, then Lemma 3.5 implies that we have either $w = w_2 w_1 = 2 \cdot (13)$ or $w = w_3 w_2 w_1 = 1 \cdot 2 \cdot (13)$. In particular, if w contains at least three layers then we have to have $w_1 = 13, w_2 = 2$ and $w_3 = 1$, whence w cannot have a fourth layer w_4 : an element $t \in \text{Supp}(w_2)$ has to be adjacent to the only generator 1 in w_3 and thus has to be $t = 2$, so if w contains a fourth layer then $w = \dots 2 \cdot 1 \cdot 2 \cdot (13) = \dots (2121) \cdot 3$, contradicting the fact that w is FC. The stub $1 \cdot 2 \cdot (13)$ is shown in the third row and first column in Table 2.

In summary, we can find all left $\mathbf{a}(2)$ -stubs in B_n , and among them there is a unique stub with more than two layers, namely $1 \cdot 2 \cdot (13)$. The problem of finding left $\mathbf{a}(2)$ -stubs in the Coxeter system A_n is similar but easier.

We classify the left $\mathbf{a}(2)$ -stubs of all nontrivially $\mathbf{a}(2)$ -finite Coxeter systems in the following theorem. In the theorem, we express every element in its Cartier–Foata form and use $\cdot$ to separate the different layers, and the symbol $\sqcup$ denotes disjoint union. We will draw the heaps of typical stubs immediately after the proof of the theorem. The classification lays the foundation for the subsequent study of $\mathbf{a}(2)$ -cells.

THEOREM 3.7. *Let (W, S) be an irreducible and nontrivially $\mathbf{a}(2)$ -finite Coxeter system. Suppose that (W, S) is of type X , and let G be the Coxeter diagram of (W, S) , labeled in the convention explained in Remark 2.5. Let $\mathcal{S}(W)$ be the set of all left stubs of $\mathbf{a}$ -value 2 in W . Then $\mathcal{S}(W)$ can be described as follows.*

- (1) *If $X = A_n$ for some $n \geq 3$, then $\mathcal{S}(W) = \mathcal{S}_1 \sqcup \mathcal{S}_2$ where*

$$\mathcal{S}_1 = \{x_{ij} := ij \mid 1 \leq i, j \leq n, j > i + 1\}$$

and

$$\mathcal{S}_2 = \{y_i := i \cdot (i-1)(i+1) \mid 1 < i < n\}.$$

- (2) If $X = B_n$ for some $n \geq 3$, then $\mathcal{S}(W) = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \mathcal{S}_3$ where $\mathcal{S}_1, \mathcal{S}_2$ are the same (see Remark 3.8.(1)) as in (1) and

$$\mathcal{S}_3 = \{z_1 := 1 \cdot 2 \cdot 13\}.$$

- (3) If $X = \tilde{C}_{n-1}$ for some $n \geq 5$, then $\mathcal{S}(W) = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \mathcal{S}_3 \sqcup \mathcal{S}_4$ where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are the same as in (2) and

$$\mathcal{S}_4 = \{z_n := n \cdot (n-1) \cdot (n-2)n\}.$$

- (4) If $X = E_{q,r}$ for some $r \geq q \geq 1$, then $\mathcal{S}(W) = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \mathcal{T}_2 \sqcup \mathcal{S}_3$ for the sets

$$\mathcal{S}_1 = \{x_{st} \mid s, t \in S, m(s, t) = 2\},$$

$$\mathcal{S}_2 = \{y_i := i \cdot (i-1)(i+1) \mid -q < i < r\},$$

$$\mathcal{T}_2 = \{y'_0 := 0 \cdot (-1)v\} \sqcup \{y''_0 := 0 \cdot 1v\},$$

and $\mathcal{S}_3 = \{z_s : s \in S, s \neq 0\}$ where

$$z_s = \begin{cases} s \cdot (s-1) \cdot \dots \cdot 0 \cdot (-1)v & \text{if } s \neq v \text{ and } 1 \leq s \leq r, \\ s \cdot (s+1) \cdot \dots \cdot 0 \cdot (1)v & \text{if } s \neq v \text{ and } -q \leq s \leq -1, \\ v \cdot 0 \cdot (-1)1 & \text{if } s = v. \end{cases}$$

- (5) If $X = F_n$ for some $n \geq 4$, then $\mathcal{S}(W) = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \mathcal{S}_3$ where $\mathcal{S}_1, \mathcal{S}_2$ are the same as in (1) and

$$\mathcal{S}_3 = \{z_1 := 1 \cdot 2 \cdot 3 \cdot 24, z_2 := 2 \cdot 3 \cdot 24\} \sqcup \{z_i \mid 3 \leq i \leq n\},$$

where $z_i = i \cdot (i-1) \cdot \dots \cdot 3 \cdot 2 \cdot 13$ for each $3 \leq i \leq n$.

- (6) If $X = H_n$ for some $n \geq 3$, then $\mathcal{S}(W) = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \mathcal{S}_3 \sqcup \mathcal{S}_4$ where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are the same as in (2) and

$$\mathcal{S}_4 = \{z_i := i \cdot (i-1) \cdot \dots \cdot 1 \cdot 2 \cdot 13 \mid 2 \leq i \leq n\}.$$

- (7) In each of the above, we listed each stub of $\mathbf{a}$ -value 2 exactly once. (For example, two $x_{ij}, x_{i'j'}$ from $\mathcal{S}_1$ are equal only if $i = i', j = j'$.) The cardinality of $\mathcal{S}(W)$ in W is given by

$$|\mathcal{S}(W)| = \begin{cases} \binom{n}{2} - 1 & \text{if } X = A_n \ (n \geq 3); \\ \binom{n}{2} & \text{if } X = B_n \ (n \geq 3); \\ \binom{n}{2} + 1 & \text{if } X = \tilde{C}_{n-1} \ (n \geq 5); \\ \binom{n+1}{2} - 1 & \text{if } X = E_{q,r} \ (q, r \geq 1, n = q + r + 2 = |S|); \\ \binom{n+1}{2} - 1 & \text{if } X = F_n \ (n \geq 4); \\ \binom{n+1}{2} - 1 & \text{if } X = H_n \ (n \geq 3). \end{cases}$$

REMARK 3.8. (1) In Theorem 3.7, by saying that a certain set $\mathcal{S}_i$ in some part is the same as the set $\mathcal{S}_i$ given from an earlier part, we mean that the former set consists of the elements given by the same reduced words as those of the elements in the latter set.

- (2) By symmetry, an element $w \in W$ is right stub if and only if w^{-1} is a left stub, so $\mathcal{S}'(W) = \{w^{-1} : w \in \mathcal{S}(W)\}$ for each Coxeter system (W, S) appearing in Theorem 3.7.

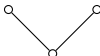
short stubs: 	medium stubs: 
--	--

TABLE 3. The short and medium stubs of $\mathcal{S}(W)$

Proof of Theorem 3.7. Part (7) follows from Parts (1)–(6) by inspection (to see that every element in $\mathcal{S}(W)$ is indeed listed only once) and by easy counting arguments, so it suffices to show that in each of Cases (1)–(6), an element $w \in W$ is a left $\mathbf{a}(2)$ -stub if and only if $w \in \mathcal{S}(W)$. The “only if” implication follows from analysis of the layers of left $\mathbf{a}(2)$ -stubs: Corollary 3.4 implies that the first two layers of such a stub w must satisfy Conditions (1)–(2) given in the corollary, whence Lemma 3.5 forces w to be an element in $\mathcal{S}(W)$ by arguments similar to those we used for type B_n in Example 3.6. The “if” implication, i.e. the fact that every element $w \in \mathcal{S}(W)$ is a left $\mathbf{a}(2)$ -stub, can be proven as follows: first, draw the heap of the specified reduced word of w we gave (such as $x_{13} = 13 \in A_n$) and use Proposition 2.11 to see that w is indeed FC; second, use Proposition 2.21 to see that w is a stub; finally, note that we can transform w to its first layer w_1 by a sequence of left lower star operations by the analog of Proposition 2.21 for left star operations, and then invoke Corollary 2.20 to conclude that $\mathbf{a}(w) = \mathbf{a}(w_1) = 2$. The proof is complete. $\square$

Let us describe and draw the heaps of stubs described in Theorem 3.7. Call a stub $w \in \mathcal{S}(W)$ *short* if $l(w) = 2$, *medium* if $l(w) = 3$, and *long* if $l(w) > 3$. Then the short, medium, and long stubs in $\mathcal{S}(W)$ are precisely those labeled by “ x ”, “ y ” (including y'_0 and y''_0 in type $E_{q,r}$) and “ z ” in Theorem 3.7; they are also precisely those with one, two, and more than two layers, respectively. The heaps of the short and medium stubs take the forms shown in Table 3. Long stubs exist in all $\mathbf{a}(2)$ -finite Coxeter systems except A_n . In type $B, \tilde{C}, F$ and H , every long stub involves a “turn” at the second layer, that is, there is always a heavy edge $\{s, t\}$ in the Coxeter diagram such that s appears in the first and third layers of the stub while t appears in the second layer. The stub $1213 \in B_4$ pictured in Table 2 is such an example. Table 4 contains more typical such stubs. Finally, in type $E_{q,r}$, each long stub w with Cartier–Foata form $w = w_p \dots w_2 w_1$ satisfies the conditions $\text{Supp}(w_2) = \{0\}$, $\text{Supp}(w_1) = \{s, t\}$, $\text{Supp}(w_2) = \{u\}$ where $\{s, t, u\} = \{-1, 1, v\}$; the heaps of the long stubs are shown in Table 5.

We record a few features of the stubs in Theorem 3.7 for future use:

LEMMA 3.9. *Let (W, S) be an arbitrary $\mathbf{a}(2)$ -finite Coxeter group from Theorem 3.7 and keep the notation of the theorem. Then the stubs in $\mathcal{S}(W)$ have the following properties.*

- (1) *The left descent set of each short stub $x_{tt'}$, medium stub y_t (including y'_0, y''_0 in type $E_{q,r}$) and long stub z_t equals $\{t, t'\}, \{t\}$ and $\{t\}$, respectively. In particular, a stub $w \in \mathcal{S}(W)$ has two left descents if and only if it is a short stub, and in this case we can recover w from $\mathcal{L}(w)$: if $\mathcal{L}(w) = \{t, t'\}$ then $w = x_{tt'}$.*
- (2) *Let $w \in \mathcal{S}(W)$ be a medium or long stub and let $w = w_p \dots w_1$ be its Cartier–Foata factorization. Then $l(w_1) = 2$ and $l(w_j) = 1$ for all $2 \leq j \leq p$, so we may pick a reduced word $\underline{w} = s_1 \dots s_k(s_{k+1}s_{k+2})$ of w such that $w_1 = s_{k+1}s_{k+2}$, $w_2 = s_k$, $w_3 = s_{k-1}$, etc.*
- (3) *Let w and $\underline{w}$ be as in (2). Then in the heap $H(\underline{w})$, every element $i < k + 1$ is comparable both to $k + 1$ and to $k + 2$ via the convex chains of coverings $i \prec i + 1 \prec \dots k \prec k + 1$ and $i \prec i + 1 \prec \dots k \prec k + 2$. In particular, the*

$z_n \in \tilde{C}_{n-1}$: 	$z_1 \in F_n$:
$z_i \in F_n$: (3 ≤ i ≤ n) 	$z_i \in H_n$: (1 ≤ i ≤ n)

TABLE 4. Typical long stubs of $\mathcal{S}(W)$ in $B_n, \tilde{C}_{n-1}, F_n$ and H_n

z_i : ($i \in \mathbb{Z}_{<0}$) 	z_v : 	z_i : ($i \in \mathbb{Z}_{>0}$)
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TABLE 5. The long stubs of $\mathcal{S}(E_{q,r})$

- element k is both the only element covered by $k + 1$ and the only element covered by $k + 2$ in $H(\underline{w})$.
- (4) Let w and $\underline{w}$ be as in (2). Then we may start with w and successively apply left lower star operations with respect to the noncommuting pairs of generators $\{s_1, s_2\}, \{s_2, s_3\}, \dots, \{s_k, s_{k+1}\}$ to obtain the sequence $w, s_2 s_3 \dots s_k (s_{k+1} s_{k+2}), s_3 \dots s_k (s_{k+1} s_{k+2}), \dots, s_k (s_{k+1} s_{k+2}), w_1$. All elements in this sequence lie in $\mathcal{S}(W)$ and share the same left cell (and hence the same right descents) as w . In particular, w is in the same left cell and has the same right descents as its first layer $w_1 = s_{k+1} s_{k+2}$, which is itself a short stub.

Proof. All the claims are readily confirmed by inspection of Theorem 3.7 and the pictures in Tables 3–5. □

REMARK 3.10. The question of whether the n -value (as defined in the paragraph before Proposition 2.14) and **a**-value of an FC element always agree in a general Coxeter system is open. The results of this subsection allow us to enhance Proposition 2.14.(1) and give a partial answer to this question in the following way:

PROPOSITION 3.11. *Let (W, S) be an arbitrary Coxeter system and let $w \in \text{FC}(W)$.*

- (1) We have $\mathbf{a}(w) = 0$ if and only if $n(w) = 0$.
- (2) We have $\mathbf{a}(w) = 1$ if and only if $n(w) = 1$.
- (3) If (W, S) is $\mathbf{a}(2)$ -finite, then $\mathbf{a}(w) = 2$ if and only if $n(w) = 2$.

Proof. (1) By Proposition 2.4.(1), the equations $\mathbf{a}(w) = 0$ and $n(w) = 0$ both hold exactly when w is the identity element and has an empty reduced word, so they are equivalent.

(2) If $\mathbf{a}(w) = 1$, then $n(w) \neq 0$ by Part (1) and $n(w) \leq 1$ by Proposition 2.14.(1), and therefore $n(w) = 1$. Conversely, if $n(w) = 1$ then the heap $H(w)$ is a chain, so no reduced word of w contains two consecutive generators that commute. The definition of FC elements then implies that w has a unique reduced word, so $\mathbf{a}(w) = 1$ by Proposition 2.4.(2). It follows that $\mathbf{a}(w) = 1$ if and only if $n(w) = 1$.

(3) Suppose (W, S) is $\mathbf{a}(2)$ -finite. If $\mathbf{a}(w) = 2$, then $n(w) \notin \{0, 1\}$ by Parts (1) and (2) and $n(w) \leq 2$ by Proposition 2.14.(1), and therefore $n(w) = 2$. Now suppose that $n(w) = 2$. We prove that $\mathbf{a}(w) = 2$ by induction on the length $l(w)$ of w . We have $l(w) = |H(w)| \geq n(w) = 2$, so in the base case we have $l(w) = 2$. Since $n(w) = 2$, we must have $w = st$ for two commuting generators $s, t \in S$ in this case; thus $\mathbf{a}(w) = 2$ by Proposition 2.2.(2), as desired. If $l(w) > 2$, then w either admits or does not admit a right lower star operation. In the former case, if a right lower star operation takes w to some element w' then we have $\mathbf{a}(w) = \mathbf{a}(w') = n(w') = n(w) = 2$, where the first equality holds by Corollary 2.20, the second equality holds by induction, and the third equality holds because star operations do not change n -values of FC elements (see the proof of Proposition 3.15 in [19]). In the latter case, the element w is a left stub, so Corollary 3.4 implies that the first two layers of w satisfy Conditions (1)–(2) from the corollary. As pointed out in the proof of Theorem 3.7, this forces w to be an element in the set $\mathcal{S}(W)$ given in the theorem, and therefore w indeed has $\mathbf{a}$ -value 2. It follows by induction that $\mathbf{a}(w) = 2$. We have proved that $\mathbf{a}(w) = 2$ if and only if $n(w) = 2$. $\square$

3.2. STUBS PARAMETERIZE 1-CELLS. Let (W, S) be a nontrivially $\mathbf{a}(2)$ -finite Coxeter system throughout this subsection. We characterize 1-cells in W_2 via stubs in two ways, first in terms of star operations in Theorem 3.15 and then in terms of reduced words and the weak Bruhat order in Theorem 3.17. To start, we show that distinct left stubs lie in distinct right cells in the following two results.

LEMMA 3.12. *Let u, w be two distinct left $\mathbf{a}(2)$ -stubs in W . Then we either have $\mathcal{L}(u) \neq \mathcal{L}(w)$ or have $\mathcal{L}(u) = \mathcal{L}(w) = \{s\}$ for some $s \in S$. Moreover, in the latter case the elements $u' := su$ and $w' := sw$ are also stubs, and we have $\mathcal{L}(u') \neq \mathcal{L}(w')$.*

Proof. We use Lemma 3.9. Suppose that $\mathcal{L}(u) = \mathcal{L}(w)$. Since $u \neq w$ by assumption and left stubs with two left descents can be recovered from the descent by Part (1) of the lemma, we must have $\mathcal{L}(u) = \mathcal{L}(w) = \{s\}$ for some $s \in S$. This proves the first claim. For the second claim, note that in the notation of Theorem 3.7, we can have $\mathcal{L}(u) = \mathcal{L}(w) = \{s\}$ for distinct stubs u, w only in the following cases where (W, S) is of type $E_{q,r}, F_n$ or H_n ; outside these types, the claim holds vacuously.

(i) **Case 1:** (W, S) is of type $E_{q,r}$, and we have either one of two subcases:

(a) $s = 0$ and u, w are two y -stubs from the set $\{y_0, y'_0, y''_0\}$ where

$$y_0 = 0 \cdot x_{(-1)1}, \quad y'_0 = 0 \cdot x_{(-1)v}, \quad y''_0 = 0 \cdot x_{1v};$$

(b) $s = i$ for some nonzero integer i with $-q < i < r$, and

$$\{u, w\} = \{y_i = i \cdot x_{(i-1)(i+1)}, z_i = i \cdot z'\}$$

where z' is a long stub.

- (ii) **Case 2:** (W, S) is of type F_n , and there is some $1 < i < n$ such that

$$\{u, w\} = \{y_i = i \cdot x_{(i-1)(i+1)}, z_i = i \cdot z'\}$$

where z' is a long stub.

- (iii) **Case 3:** (W, S) is of type H_n , and there is some $1 < i < n$ such that

$$\{u, w\} = \{y_i = i \cdot x_{(i-1)(i+1)}, z_i = i \cdot z'\}$$

where z' is a long stub.

We have $\mathcal{L}(u') \neq \mathcal{L}(w')$ for the stubs $u' = su$ and $w' = sw$ in Case 1(a) by inspection, and the same is true in all the other cases by Lemma 3.9.(1) because one of u', w' is a short stub while the other is not. $\square$

PROPOSITION 3.13. *Let u, w be distinct left $\mathbf{a}(2)$ -stubs of W . Then $u \not\sim_R w$.*

Proof. Keep the notation from Theorem 3.7, Lemma 3.12, and the proof of the lemma. We have $u \not\sim_R w$ if $\mathcal{L}(u) \neq \mathcal{L}(w)$ by Proposition 2.2.(3), so Lemma 3.12 implies that it suffices to treat Cases 1–3 from its proof, where $\mathcal{L}(u) = \mathcal{L}(w) = \{s\}$ for some $s \in S$. We may show $u \not\sim_R w$ by finding a generator $t \in S$ such that $m(s, t) = 3$ and $\mathcal{L}(*u) \neq \mathcal{L}(*w)$, where $*$ denotes the simple left star operation with respect to $\{s, t\}$: the fact that $\mathcal{L}(*u) \neq \mathcal{L}(*w)$ implies that $*u \not\sim_R *w$, so $u \not\sim_R w$ by Proposition 2.19.(2). We explain how to find such a generator t below.

In Case 1(a) of the proof of Lemma 3.12, we may take t to be the unique generator appearing in both the stubs $u' = su$ and $w' = sw$. For example, if $u = y_0$ and $w = y'_0$, then $u' = x_{(-1)1}$, $w' = x_{(-1)v}$ and we may take $t = -1$, whence $*u = u'$, $*w = w'$ and $\mathcal{L}(*u) \neq \mathcal{L}(*w)$. In Cases 1(b), 2 and 3, we note that s has a numerical label i and that $i - 1, i + 1$ are also generators in S . By our labeling of generators (Remark 2.5), it follows that $m(i, i + 1) = 3$ except when (W, S) is of Coxeter type F_n , $i = 2$ and $\{u, w\} = \{y_2 = 2(13), z_2 = 2 \cdot 3 \cdot (24)\}$. In this exceptional case we pick $t = i - 1 = 1$, so that $m(s, t) = 3$ and $\{*u, *w\} = \{13 = x_{13}, 1 \cdot 2 \cdot 3 \cdot (24) = z_1\}$ with respect to $\{s, t\}$. In all other cases we may pick $t = i + 1$, whence $m(s, t) = 3$ and $\{*u, *w\} = \{*y_i, *z_i\} = \{x_{(i-1)(i+1)}, z_{i+1}\}$ with respect to $\{s, t\}$. Here, the fact that $*z_i = z_{i+1}$ in the last set can be easily seen from heaps in Tables 4 and 5, with the star operation on z_i being a lower one if and only if (W, S) is of type $E_{q,r}$ and $-q < i < 0$. In all cases, one of $*u$ and $*w$ is an short stub and the other is a long stub, so $\mathcal{L}(*u) \neq \mathcal{L}(*w)$ by Lemma 3.9.(1), as desired. $\square$

We are ready to describe cells in terms of stubs and star operations.

DEFINITION 3.14. *For each element $w \in W$, we define the right upper star closure of w to be the set of all elements $y \in W$ for which there exists a sequence $z_1 = w, z_2, \dots, z_q = y$ such that z_{i+1} can be obtained from z_i via a right upper star operation for all $1 \leq i \leq q - 1$; we denote the set by R_w . Similarly, for each $w \in W$ we define its left upper star closure to be the set L_w containing all elements that can be obtained from w via a sequence of left upper star operations.*

THEOREM 3.15. *Let W be a nontrivially $\mathbf{a}(2)$ -finite Coxeter group, let W_2 be the set of $w \in W$ with $\mathbf{a}(w) = 2$, and let $\mathcal{S}(W)$ be the set of left $\mathbf{a}(2)$ -stubs in W . Then the set R_w forms a right Kazhdan–Lusztig cell for every $w \in \mathcal{S}(W)$. Moreover, we have $R_w \cap R_{w'} = \emptyset$ for distinct $w, w' \in \mathcal{S}(W)$, and $W_2 = \sqcup_{w \in \mathcal{S}(W)} R_w$. In particular, the number of right cells in W_2 equals the cardinality of $\mathcal{S}(W)$ given in Part (7) of Theorem 3.7.*

Proof. Let $R(w)$ be the right Kazhdan–Lusztig cell containing w for each $w \in W$. Then we have $R_w \subseteq R(w)$ by Proposition 2.18. On the other hand, by the definition of stubs we have $W_2 = \cup_{w \in \mathcal{S}(W)} R_w$. It follows that to prove the theorem it suffices

to show that $w \not\sim_R w'$ whenever w, w' are distinct elements in $\mathcal{S}(W)$. This holds by Proposition 3.13. $\square$

Using the fact that the set R_w is a right cell for each $w \in \mathcal{S}(W)$, we now work towards a second description of the cells, in terms of reduced words.

DEFINITION 3.16. *Let $w \in W$. We define a left stub decomposition of w to be a reduced factorization of the form $w = x \cdot z$ where x is a left stub and $z \in W$. Similarly, we define a right stub decomposition of w to be a reduced factorization of the form $w = z \cdot y$ where y is a right stub.*

THEOREM 3.17. *Let $x \in \mathcal{S}(W)$ and let w be an element of $\mathbf{a}$ -value 2 in W . Let $R(x)$ be the right cell containing x (so that $R(x) = R_x$ by Theorem 3.15).*

- (1) *We have $w \in R_x$ if and only if w has a left stub decomposition of the form $w = x \cdot z$.*
- (2) *We have $w \in R(x)$ if and only if $x \leq^R w$, i.e. we have*

$$R(x) = \{z \in W_2 : x \leq^R z\}.$$

- (3) *The element w has a unique left stub decomposition in the sense that the stub x' and the element z' in the decomposition $w = x' \cdot z'$ are both unique.*

Note that Part (2) of the theorem, which characterizes the right cell in W_2 in terms of the right weak Bruhat order, may be viewed as an analog of Proposition 2.4.(3). To prove the theorem, we will use the following lemma.

LEMMA 3.18. *Let (W, S) be an arbitrary Coxeter system and let $w \in \text{FC}(W)$.*

- (1) *Suppose that w has a reduced word of the form*

$$\underline{w} = s_1 \dots s_k (s_{k+1} s_{k+2} \dots s_{k+j}) s_{k+j+1} \dots s_q$$

where the set $A := \{k+1, k+2, \dots, k+j\}$ forms a maximal antichain in the heap $H(w)$. Let $x = s_1 \dots s_k (s_{k+1} \dots s_{k+j})$ and $y = (s_{k+1} \dots s_{k+j}) s_{k+j+1} \dots s_q$. Then we have $H(x) = \mathcal{I}_A$ and $H(y) = \mathcal{F}_A$ as sets.

- (2) *In the above setting, we have $\mathcal{L}(w) = \mathcal{L}(x)$ and $\mathcal{R}(w) = \mathcal{R}(y)$.*
- (3) *Let $x \in \mathcal{S}(W)$ and let w be an element of $\mathbf{a}$ -value 2 with a left stub decomposition of the form $w = x \cdot z$. Then $\mathcal{L}(w) = \mathcal{L}(x)$.*

Proof. (1) Since A is a maximal antichain in $H(w)$, we have $H(\underline{w}) = \mathcal{I}_A \cup \mathcal{F}_A$ by Lemma 2.16. For any $k+j+1 \leq i \leq q$, we have $i \notin \mathcal{I}_A$ by the definition of heaps; therefore $i \in \mathcal{F}_A$. Similarly, we have $i \in \mathcal{I}_A$ for all $k+1 \leq i \leq q$. It follows that $H(x) = \mathcal{I}_A$ and $H(y) = \mathcal{F}_A$.

(2) Part (1) implies that the set of minimal elements of $H(x)$ and $H(w)$ coincide, so $\mathcal{L}(w) = \mathcal{L}(x)$ by Remark 2.12. Similarly we have $\mathcal{R}(w) = \mathcal{R}(y)$.

(3) By Lemma 3.9.(2), there is a reduced word $\underline{x} = s_1 \dots s_k (s_{k+1} s_{k+2})$ of x where $(s_{k+1} s_{k+2})$ equals the first layer of x . Let $s_{k+3} \dots s_q$ be a reduced word of z , so that $\underline{w} := s_1 \dots s_k (s_{k+1} s_{k+2}) \cdot s_{k+3} \dots s_q$ is a reduced word of w . The set $A := \{k+1, k+2\}$ forms a maximal antichain in the heap $H(\underline{w})$ by Corollary 2.15.(1), and therefore $\mathcal{L}(w) = \mathcal{L}(x)$ by Parts (1) and (2). $\square$

Proof of Theorem 3.17. Part (1) implies Part (2) by the definition of $\leq^R$ and Proposition 2.2.(5). Part (3) also follows from Part (1): if $w = x'' \cdot z''$ is another left stub decomposition of w then $w \in R(x')$ and $w \in R(x'')$, and therefore $x \sim_R w \sim_R x''$; this forces $x' = x''$, and hence $z' = z''$, by Proposition 3.13.

It remains to prove Part (1). The “only if” implication follows from the definition of R_x and upper star operations. To prove the “if” implication, suppose that w has a

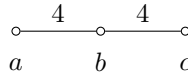


FIGURE 3. The Coxeter system of type $\tilde{C}_2$

stub factorization of the form $w = x \cdot z$, and let $y \in \mathcal{S}(W)$ be the unique stub such that $w \in R_y$. The unique existence of such a stub is guaranteed by Theorem 3.15, and we need to prove that $x = y$.

Since $w \in R_y$, we have $w \sim_R y$ and hence $\mathcal{L}(y) = \mathcal{L}(w)$ by Proposition 2.2.(3). On the other hand, since $w = x \cdot z$, we have $\mathcal{L}(x) = \mathcal{L}(w)$ by Lemma 3.18. It follows that $\mathcal{L}(x) = \mathcal{L}(y)$, where $1 \leq |\mathcal{L}(x)| = |\mathcal{L}(y)| \leq 2$ by the analog of Corollary 2.15.(3) for left descents. If $|\mathcal{L}(x)| = |\mathcal{L}(y)| = 2$, then $x = y$ by Lemma 3.9.(1). If $|\mathcal{L}(x)| = |\mathcal{L}(y)| = 1$, say with $\mathcal{L}(x) = \mathcal{L}(y) = \{s\}$, then $l(x), l(y) > 2$ and the elements $x' := sx$ and $y' := sy$ are stubs with $\mathcal{L}(x') \neq \mathcal{L}(y')$ by Lemma 3.9.(4). The element $w' := sw$ has stub decompositions of the form $w' = x' \cdot z'$ and $w' = y' \cdot z''$, which forces $\mathcal{L}(x') = \mathcal{L}(w') = \mathcal{L}(y')$ by Lemma 3.18. Since $\mathcal{L}(x) = \mathcal{L}(y)$ and $\mathcal{L}(x') = \mathcal{L}(y')$, Lemma 3.12 implies that $x = y$. The proof is complete. $\square$

REMARK 3.19. The assumption that (W, S) be $\mathbf{a}(2)$ -finite is crucial for the validity of most results in the subsection. For example, consider the affine Weyl system $\tilde{C}_2$, whose Coxeter diagram is shown in Figure 3. The system is not $\mathbf{a}(2)$ -finite, and the elements $x = acb$ and $y = acbac$ in W are both left $\mathbf{a}(2)$ -stubs by Corollary 3.4. Note that $\mathcal{L}(x) = \mathcal{L}(y) = \{a, c\}$, that $\mathbf{a}(x) = n(x) = \mathbf{a}(y) = n(y) = 2$ by Proposition 2.14.(2), that $x \leq^R y$, and that consequently $x \sim_R y$ by Proposition 2.2.(5). It is then easy to check that the conclusions of Lemma 3.12, Proposition 3.13, Theorem 3.15 and Theorem 3.17 all fail for x and y . On the other hand, the assertions in Lemma 3.18 do hold for all Coxeter systems, as indicated in the statement of the lemma.

As usual, Theorem 3.15 and Theorem 3.17 have obvious left-handed analogs which hold by symmetry. First, for every right $\mathbf{a}(2)$ -stub x' , the left upper star closure $L_{x'}$ of x' must be identical with the left cell that contains x' ; moreover, we have $W_2 = \sqcup_{x' \in \mathcal{S}'(W)} L_{x'}$ where $\mathcal{S}'(W)$ is the set of all right stubs in W_2 . Second, for each right stub $x' \in \mathcal{S}'(W)$, the left cell $L_{x'}$ consists precisely of all $\mathbf{a}(2)$ -elements w that admit a right stub decomposition of the form $w = z \cdot x'$. It is also possible to describe left cells in W_2 via left stubs: for every $w \in \mathcal{S}(W)$, Remark 3.8.(1) and Proposition 2.2.(4) imply that

$$(2) \quad L_{w^{-1}} = R_w^{-1}$$

where $R_w^{-1} = \{x^{-1} : x \in R_w\}$, because both sets can be described as the left cell containing w^{-1} . We can now label the 0-cells in W_2 with pairs of left stubs:

DEFINITION 3.20. For all $x, y \in \mathcal{S}(W)$, we define $I(x, y) := R_x \cap R_y^{-1} = R_x \cap L_{y^{-1}}$ and define $N(x, y) := |I(x, y)|$.

REMARK 3.21. By Equation (2), for all $x, y \in \mathcal{S}(W)$, the 0-cell $I(y, x) = R_y \cap R_x^{-1}$ is in bijection with the 0-cell $I(x, y)$ via the inversion map $I(x, y) \rightarrow I(y, x), w \mapsto w^{-1}$. In particular, we have $N(x, y) = N(y, x)$.

The following proposition, where the last part follows from the first two parts, is immediate from Corollary 2.6.

PROPOSITION 3.22. Let $\mathcal{S}(W)$ be the set of left $\mathbf{a}(2)$ -stubs of W , and let E be a two-sided $\mathbf{a}(2)$ -cell of W . Set $\mathcal{T} = E \cap \mathcal{S}(W)$. Then

- (1) For each $x \in \mathcal{T}$, the right cell $R_x \subseteq E$ equals the disjoint union

$$R_x = \bigsqcup_{y \in \mathcal{T}} I(x, y).$$

Consequently, we have

$$|R_x| = \sum_{y \in \mathcal{T}} N(x, y).$$

- (2) The decomposition of the two-sided cell E into right cells is given by

$$E = \bigsqcup_{x \in \mathcal{T}} R_x.$$

Consequently, we have

$$|E| = \sum_{x \in \mathcal{T}} |R_x|.$$

- (3) The two-sided cell E equals the disjoint union

$$E = \bigsqcup_{x, y \in \mathcal{T}} I(x, y).$$

Consequently, we have

$$|E| = \sum_{x, y \in \mathcal{T}} N(x, y).$$

Note that in light of the proposition, to count the cells in W_2 it suffices to understand the 0-cells $I(x, y)$. We will study these 0-cells in section 4.

3.3. SLIDE EQUIVALENCE AND 2-CELLS. In this subsection, we introduce two equivalence relations on $\mathcal{S}(W)$ and use one of them to determine how the right cells R_w ($w \in \mathcal{S}(W)$) coalesce into 2-cells in W_2 . Recalling that the first layer of every stub $w \in \mathcal{S}(W)$ is itself a short stub by Lemma 3.9.(4), we define the relations in two steps, as follows.

DEFINITION 3.23. We define a slide to be a transformation taking a short stub $st \in \mathcal{S}(W)$ to a short stub $su \in \mathcal{S}(W)$ where $m(t, u) \geq 3$, i.e., where $\{t, u\}$ forms an edge in the Coxeter diagram. We say the move is along the edge $\{t, u\}$ and call it a simple slide if the edge $\{t, u\}$ is simple, i.e. if $m(t, u) = 3$.

DEFINITION 3.24. Let $w, w' \in \mathcal{S}(W)$ and let x, x' be their respective first layers.

- (1) We say that w and w' are slide equivalent, and write $w \approx w'$, if x and x' can be related by a (possibly empty) sequence of slides.
- (2) We say that w and w' are simple slide equivalent, and write $w \sim w'$, if x and x' can be related by a (possibly empty) sequence of simple slides.

It is clear that $\approx$ and $\sim$ are equivalence relations, and that $\sim$ refines $\approx$.

REMARK 3.25. Let w, w' and x, x' be as in Definition 3.24.

- (1) Short stubs equal their first layers, so $w \approx x$, that is, every stub is slide equivalent to a short stub. It also follows that every slide equivalence class C in $\mathcal{S}(W)$ can be recovered from the set Z_C of short stubs it contains, namely, as the set of stubs whose first layer lies in Z_C . The above facts also hold for the relation $\sim$. For these reasons, we will often first focus on short stubs when studying $\approx$ and $\sim$.

- (2) If $w \approx w'$ then $w \approx x \approx x' \approx w'$, where we can connect w to x and connect x' to w' with left star operations by Lemma 3.9.(4). Since x and x' can be related by slides by Definition 3.24, it follows that the relation $\approx$ is generated by left star operations and slides. Similarly, the relation $\sim$ is generated by left star operations and simple slides.

For the rest of this subsection we will focus on the relation $\approx$ and its connection with 2-cells. The relation $\sim$ will be studied further in section 4 in connection with 0-cells; see Proposition 4.12 and subsection 4.3.

Slide equivalent stubs lie in the same 2-cells:

LEMMA 3.26. *Let $w, w' \in \mathcal{S}(W)$. If $w \approx w'$, then we have $w \sim_{LR} w'$.*

Proof. Slides can be achieved by star operations: for any two short stubs $x = st$ and $x' = su$ where $m(t, u) \geq 3$, we have $*(x*) = *(stu) = x'$ where the two star operations are performed with respect to $\{t, u\}$. It follows from Remark 3.25.(2) that if $w \approx w'$, then w and w' can be related by left and right star operations, hence $w \sim_{LR} w'$ by Proposition 2.18, $\square$

Lemma 3.26 suggests that to find the 2-cells of W_2 it is helpful to find the equivalence classes of $\approx$. We do so in Lemma 3.27, Lemma 3.28 and Proposition 3.30 below. The lemmas focus on short stubs, as justified in Remark 3.25.(1). For convenience, we will say a subset of $\mathcal{S}(W)$ is *slide connected* if its elements are pairwise slide equivalent.

LEMMA 3.27. *Let G be the Coxeter diagram of an arbitrary Coxeter system (W, S) . Suppose a subset S' of S induces a linear subgraph in G in the sense that we can label the elements of S' as $1, 2, \dots, k$ in a way such that $m(i, j) \geq 3$ if and only if $|j - i| = 1$ for all $1 \leq i, j \leq k$. Let X be the set of short stubs in $\mathcal{S}(W)$ whose supports lie in S' .*

- (1) *We have $X = \{ij : 1 \leq i, j \leq n, |i - j| > 1\}$.*
- (2) *The set X is slide connected.*

Proof. Part (1) follows from definitions. Part (2) holds since the elements of X can be arranged into the array

$$\begin{array}{ccccccc} 13 & 14 & 15 & & \dots & & 1n \\ & 24 & 25 & & \dots & & 2n \\ & & & & \dots & & \dots \\ & & & & (n-3)(n-1) & & (n-3)n \\ & & & & & & (n-2)n \end{array}$$

where every two adjacent entries in a row or in a column are short stubs related by a slide along an edge of the form $\{i, i + 1\}$ in G . $\square$

LEMMA 3.28. *Let (W, S) be a Coxeter system of type $E_{q,r}$ where $r \geq q \geq 1$. Consider the short stubs $x_1 := (-1)v$, $x_2 := 1v$ and $x_3 := (-1)1$ in $\mathcal{S}(W)$, and let Z be the set of all short stubs in $\mathcal{S}(W)$.*

- (1) *If $q = r = 1$, then none of x_1, x_2, x_3 admits any slides.*
- (2) *If $q = 1$ and $r > 1$, then x_1 admits no slides, we have $x_2 \approx x_3$, and the set $Z \setminus \{x_1\}$ is slide connected.*
- (3) *If $r \geq q > 1$, then Z is slide connected.*

Proof. We start with some observations, the key one being that if $r > 1$, then we have $x_2 \approx x_3$ via the sequence of slides (see Figure 4)

$$(3) \quad x_2 = 1v = v1 \rightarrow v2 \rightarrow 02 \rightarrow (-1)2 \rightarrow (-1)1 = x_3.$$

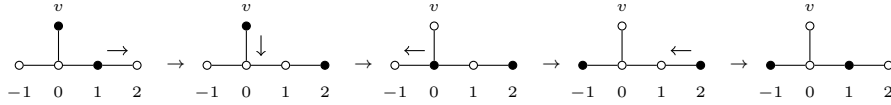


FIGURE 4. Sliding from x to z when $r > 1$

Similarly, we have $x_1 \approx x_3$ whenever $q > 1$. We also consider the partition of the set $Z' := Z \setminus \{x_1\}$ into the sets

$$X = \{x \in Z \setminus \{x_1\} : v \in \text{Supp}(x)\}, Y = \{x \in Z \setminus \{x_1\} : v \notin \text{Supp}(x)\}$$

where we have $x_2 = 1v \in X$ and $x_3 = (-1)1 \in Y$. The set X is slide connected because its elements can be listed in the sequence $1v, 2v, \dots, rv$ where every two adjacent stubs are related by a slide. The set Y is slide connected by an application of Lemma 3.27 with $S' = S \setminus \{v\}$.

To prove the lemma, first note that if $q = 1$ then $x_1 = (-1)v$ admits no slide because both the generators -1 and v are adjacent to and only to 0 in the Coxeter graph, so the claim about x_1 from Part (2) holds. Part (1) holds by similar arguments. Next, suppose that $r > 1$. Then $x_2 \approx x_3$ as we observed, and the set $Z' = X \sqcup Y$ is slide connected because $x_2 \in X, x_3 \in Y$, and X, Y are slide connected. This completes the proof of Part (2). Finally, if in addition we have $q > 1$ then $x_3 \approx x_1$. Since the set $Z' = Z \setminus \{x_1\}$ is slide connected and contains x_3 , it follows that Z is slide connected, so Part (3) holds. $\square$

EXAMPLE 3.29. The sequence of slides in (3) is depicted in Figure 4, where in each copy of the Coxeter diagram we have omitted the vertices $3, 4, \dots, r$, filled the generators in the support of the short stub, and indicated the slide to be performed with arrows. Note that the slides rely on the existence of the vertex 2 , and hence on the assumption that $r > 1$, in a crucial way: sliding from x_2 to x_3 is impossible when $r = 1$ but becomes possible when $r > 1$ because the edge $\{1, 2\}$ permits more slides.

We can now describe the slide equivalence classes of $\mathcal{S}(W)$.

PROPOSITION 3.30. *Let (W, S) be an $\mathbf{a}(2)$ -finite Coxeter system.*

- (1) *Suppose (W, S) is of type A_n ($n \geq 3$), B_n ($n \geq 3$), $\tilde{C}_{n-1}$ ($n \geq 5$), F_n ($n \geq 4$), H_n ($n \geq 3$) or of type $E_{q,r}$ where $r \geq q > 1$. Then $\mathcal{S}(W)$ contains a single slide equivalence class.*
- (2) *Suppose (W, S) is of type $E_{q,r}$ where $r \geq q = 1$. Let $x_1 = (-1)v, x_2 = 1v, x_3 = (-1)1$ and let $C_i = \{w \in \mathcal{S}(W) : w \approx x_i\}$ for $1 \leq i \leq 3$.*
 - (a) *If $r = 1$, then $\mathcal{S}(W)$ consists of three slide equivalence classes. The classes are exactly C_1, C_2, C_3 , and we have*

$$C_1 = \{(-1)v, 0(-1)v, 10(-1)v\},$$

$$C_2 = \{1v, 01v, (-1)0v\},$$

$$C_3 = \{(-1)1, 0(-1)1, v0(-1)1\}.$$

- (b) *If $r > 1$, then $C_2 = C_3$ and the set $\mathcal{S}(W)$ consists of two slide equivalence classes. The classes are exactly C_1, C_2 , and we have*

$$C_1 = \{(-1)v, 0(-1)v, 10(-1)v, 210(-1)v, \dots, r \dots 210(-1)v\}.$$

Proof. (1) By Remark 3.25.(1) it suffices to show that the set of all short stubs in $\mathcal{S}(W)$ is slide connected. This holds in type $E_{q,r}$ when $r \geq q > 1$ by Lemma 3.28.(3) and holds in types $A_n, B_n, \tilde{C}_{n-1}, F_n, H_n$ by applications of Lemma 3.27.(2) with $S' = S$.

(2) If $r = 1$, then the stubs x_1, x_2, x_3 are all the short stubs in $\mathcal{S}(W)$, and they lie in three distinct slide equivalence classes by Lemma 3.28.(1). Thus, Remark 3.25.(1) implies that C_1, C_2, C_3 are the slide equivalence classes in $\mathcal{S}(W)$ and are given by

$$C_i = \{w \in \mathcal{S}(W) : \text{the first layer of } w \text{ equals } x_i\}$$

for each i . The equations in Part (a) then follow from the above equality by direct computation. Part (b) can be proved using Lemma 3.28.(2) in a similar way, with $C_2 = C_3$ since $x_2 \approx x_3$ when $r > 1$. $\square$

We are ready to find the 2-cells in W_2 in all $\mathbf{a}(2)$ -finite Coxeter systems:

THEOREM 3.31. *Let (W, S) be an irreducible nontrivially $\mathbf{a}(2)$ -finite Coxeter system.*

- (1) *Suppose (W, S) is of type A_n ($n \geq 3$), B_n ($n \geq 3$), $\tilde{C}_{n-1}$ ($n \geq 5$), F_n ($n \geq 4$), H_n ($n \geq 3$) or of type $E_{q,r}$ where $r \geq q > 1$. Then W_2 contains a single 2-cell.*
- (2) *Suppose (W, S) is of type $E_{q,r}$ where $r \geq q = 1$. Let C_1, C_2, C_3 be as in Proposition 3.30.(2) and let $E_i = \sqcup_{w \in C_i} R_w$ for all $1 \leq i \leq 3$. If $r = 1$, then W_2 contains three 2-cells and they are exactly E_1, E_2 and E_3 . If $r > 1$, then W_2 contains two 2-cells and they are exactly E_1 and E_2 .*
- (3) *Let $\mathcal{C}$ be the set of slide equivalence classes in $\mathcal{S}(W)$ and let $\mathcal{E}$ be the set of 2-cells in W_2 . Then there is a bijection $\Phi : \mathcal{C} \rightarrow \mathcal{E}$ given by $\Phi(C) = \cup_{w \in C} R_w$ for all $C \in \mathcal{C}$.*

Proof. Part (1) follows directly from Theorem 3.15, Lemma 3.26 and Proposition 3.30.(1). To prove Part (2), first note that E_i is a subset of a 2-cell by Theorem 3.15 and Lemma 3.26 for all $1 \leq i \leq 3$. Also note that by Proposition 3.30.(2), we have $W_2 = E_1 \sqcup E_2 \sqcup E_3$ if $r = 1$ and $W_2 = E_1 \sqcup E_2$ if $r > 1$. These facts, together with the symmetry in the Coxeter diagram of $E_{1,1}$, imply that to show the 2-cells of W_2 in type $E_{1,r}$ ($r \geq 1$) are as claimed it remains to prove that E_1 and E_2 lie in different 2-cells for all $r \geq 1$. We do so by showing below that the stubs $x := (-1)v \in E_1$ and $y := 1v \in E_2$ are not in the same 2-cell.

Suppose $x \sim_{LR} y$. Then Part (9) of Proposition 2.2 implies that some element $z \in W$ satisfies $z \sim_L x$ and $z \sim_R y$. Parts (3) and (5) of the same proposition then imply that $\mathcal{R}(z) = \mathcal{R}(x) = \{-1, v\}$, $\mathcal{L}(z) = \mathcal{L}(y) = \{1, v\}$ and $\mathbf{a}(z) = \mathbf{a}(x) = 2$. Let $z = w_p \dots w_2 w_1$ be the Cartier–Foata form of z . Recall that w_1 must equal the product of the right descents of z and that every generator in $\text{Supp}(w_2)$ must fail to commute with some generator in $\text{Supp}(w_1)$. It follows that $w_1 = (-1)v$ and $w_2 = 0$. Corollary 3.4 then implies that $z \in \mathcal{S}(W)$, whence the fact that $\mathcal{L}(z) = \{1, v\}$ forces $z = 1v$ by Lemma 3.9.(1). This contradicts the fact that $z = w_p \dots w_3 \cdot w_2 w_1 = w_p \dots w_3 \cdot (01v)$ has length at least 3, so $x \not\sim_{LR} y$, and the proof of Part (2) is complete.

Part (3) follows immediately from comparison of Parts (1)–(2) of the theorem with Proposition 3.30. $\square$

As mentioned in the introduction, the partition of some Weyl and affine Weyl groups such as A_n and B_n into cells are known in the literature. Cells in the finite Coxeter groups of types H_3 and H_4 have also been determined using computational methods in [1]. In this sense, the descriptions of the Kazhdan–Lusztig cells of W_2 from Theorems 3.15, 3.17 and 3.31 are not new for those groups. On the other hand, we note that these three theorems offer convenient combinatorial descriptions of the cells of W_2 , and it does so in uniform way for all $\mathbf{a}(2)$ -finite Coxeter groups. For the Coxeter systems of types F_n where $n \geq 6$, H_n where $n \geq 5$, $\tilde{C}_{n-1}$ for general values of n , and $E_{q,r}$ for general values of q, r , the descriptions of cells from these theorems are new to our knowledge.

4. ENUMERATION VIA 0-CELLS

We maintain the setting and notation in section 3, with (W, S) being an irreducible nontrivially $\mathbf{a}(2)$ -finite Coxeter system. In this section we compute the cardinalities of the 1-cells and 2-cells of W_2 found in subsections 3.2 and 3.3. By Proposition 3.22, to do so it suffices to understand the 0-cells $I(x, y)$ and their cardinalities $N(x, y)$ in W_2 where $x, y \in \mathcal{S}(W)$, so the study of 0-cells occupies almost the entire section. We will interpret each 0-cell as a certain set of *core elements* in subsection 4.1, explain how different 0-cells relate to each other in subsection 4.2, and exploit such relationships to count all 0-cells, 1-cells and 2-cells of W_2 in subsection 4.3.

4.1. ANATOMY OF $\mathbf{a}(2)$ -ELEMENTS. In this subsection we establish a canonical decomposition that illuminates the structure of both the elements and the 0-cells in W_2 . The decomposition uses the following notions:

DEFINITION 4.1. (1) We call an element $w \in W_2$ a *core element*, or simply a *core*, if $|\mathcal{L}(w)| = |\mathcal{R}(w)| = 2$.

- (2) We say that an ordered pair (w, w') of FC elements in W are *descent compatible* if the sets $\mathcal{R}(w) = \mathcal{L}(w')$ and $|\mathcal{R}(w)| = |\mathcal{L}(w')| = 2$. When this is the case, Remark 2.12 guarantees the existence of reduced words

$$\underline{w} = s_1 \dots s_k(s_{k+1}s_{k+2}), \quad \underline{w'} = (s_{k+1}s_{k+2})s_{k+3} \dots s_{k'}$$

of w and w' , respectively, such that $\mathcal{R}(w) = \mathcal{L}(w') = \{s_{k+1}, s_{k+2}\}$, and we define the *glued product* of w and w' to be the element of W expressed by the word

$$\underline{w} * \underline{w'} := s_1 \dots (s_{k+1}s_{k+2}) \dots s_{k'}.$$

We denote the *glued product* by $w * w'$.

- (3) Let $x \in \mathcal{S}(W), y' \in \mathcal{S}'(W)$ and let $w \in W$ be a core. We say w is *compatible* with both x and y if both the pairs (x, w) and (w, y') are descent compatible. We denote the set of cores compatible with both x and y' by $\text{Cor}(x, y')$.
- (4) We define an $\mathbf{a}(2)$ -triple in W to be a triple (x, w, y') of elements in W_2 such that $x \in \mathcal{S}(W), y' \in \mathcal{S}'(W)$, and $w \in \text{Cor}(x, y')$. We denote the set of $\mathbf{a}(2)$ -triples in W by $\text{Tri}(W)$.

REMARK 4.2. The element $w * w'$ in Part (2) of Definition 4.1 is well-defined, independent of the choice of the reduced words $\underline{w}$ and $\underline{w'}$. Indeed, in the notation of the definition, we have $w * w' = ws_{k+1}s_{k+2}w'$. It is worth noting, however, that the element $w * w'$ need not be FC when w and w' are FC. For example, in the Coxeter system of type A_4 the glued product of the FC elements $w = 124, w' = 241$ equals $1241 = (121)4$, which is not FC.

We describe the canonical decomposition of $\mathbf{a}(2)$ -elements below.

THEOREM 4.3. Let (W, S) be an irreducible nontrivially $\mathbf{a}(2)$ -finite Coxeter system. Then there is a bijection $g : \text{Tri}(W) \rightarrow W_2$ given by

$$g(x, w, y') = (x * w) * y'$$

for all $(x, w, y') \in \text{Tri}(W)$. Furthermore, for all $x \in \mathcal{S}(W)$ and $y' \in \mathcal{S}'(W)$, the map g restricts to a bijection $g_{xy'} : \{x\} \times \text{Cor}(x, y') \times \{y'\} \rightarrow R_x \cap L_{y'}$.

The theorem immediately relates 0-cells to sets of cores:

COROLLARY 4.4. Let (W, S) be an irreducible nontrivially $\mathbf{a}(2)$ -finite Coxeter system. Then for all $x, y \in \mathcal{S}(W)$, there is a bijection $f_{xy} : \text{Cor}(x, y^{-1}) \rightarrow I(x, y)$ given by

$$f_{xy}(w) = (x * w) * y^{-1}$$

for every $w \in \text{Cor}(x, y^{-1})$. In particular, we have $N(x, y) = |\text{Cor}(x, y^{-1})|$.

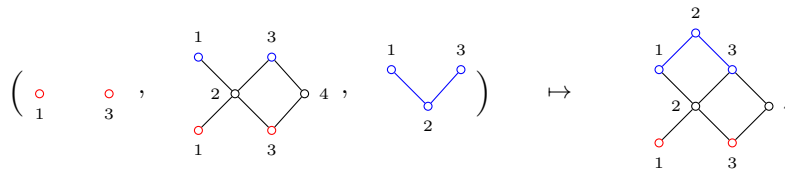


FIGURE 5. The glued product of an $\mathbf{a}(2)$ -triple

EXAMPLE 4.5. Suppose (W, S) is of type B_4 . Consider the left stubs $x = 13, y = 213$ and the core element $w = 132413 \in \text{Cor}(x, y')$ compatible with both x and $y' := y^{-1}$. The canonical bijection g of Theorem 4.3 sends the $\mathbf{a}(2)$ -triple (x, w, y) to the element $z := 1324132 \in I(x, y)$, as shown in Figure 5. In consequence, we have $f_{xy}(w) = z$ for the induced bijection $f_{xy} : \text{Cor}(x, y^{-1}) \rightarrow I(x, y)$ of Corollary 4.4.

We now begin the proof of Theorem 4.3. An important subtlety about the theorem concerns the definition of the map g : in order for the supposed output $(x * w) * y'$ to be defined, we need $x * w$ to be an FC element that is descent compatible with y , yet as we explained in Remark 4.2 the glued product of two FC elements need not be FC in general. This subtlety is addressed in the following proposition and reflected in its rather technical proof, where we frequently have to invoke certain features of $\mathbf{a}(2)$ -stubs in $\mathbf{a}(2)$ -finite Coxeter systems highlighted in Lemma 3.9.

PROPOSITION 4.6. *Let $x \in \mathcal{S}(W)$ and let $w \in W_2$. Suppose that the pair (x, w) is descent compatible, and let $\underline{x}, \underline{w}, \underline{x} * \underline{w}$ and $x * w$ be as in Part (2) of Definition 4.1. Then $\underline{x} * \underline{w}$ is a reduced word, the element $x * w$ is FC, and $\mathbf{a}(x * w) = 2$. Moreover, we have $\mathcal{R}(x * w) = \mathcal{L}(y)$, so that $x * w$ is descent compatible with y .*

Proof. Let $\underline{x} = s_1 \dots s_k(s_{k+1}s_{k+2})$ and $\underline{w} = (s_{k+1}s_{k+2})s_{k+3} \dots s_{k'}$ be reduced words of x and w , respectively, with $\mathcal{R}(x) = \mathcal{L}(w) = \{s_{k+1}, s_{k+2}\}$. Let $\underline{z} = \underline{x} * \underline{w} = x_1 \dots s_{k'}$. We first use Proposition 2.11 to show that $\underline{z}$ is a reduced word of an FC element, and hence $x * w$ is FC, by examining the heap $H(\underline{z}) = \{1, \dots, k'\}$. To do so, we view the heaps $H(\underline{x}) = \{1, \dots, k+2\}$ and $H(\underline{w}) = \{k+1, \dots, k'\}$ of $\underline{x}$ and $\underline{w}$ naturally as sub-posets of $H(\underline{z})$. Let $A = \{k+1, k+2\}$. Recall from Lemma 3.9.(3) that for every element $i \in H(\underline{x})$, we have both $i \preceq k+1$ and $i \preceq k+2$ in $H(\underline{x})$. On the other hand, we claim that every element $j \in H(\underline{w})$ satisfies either $k+1 \leq j$ or $k+2 \leq j$ in $H(\underline{w})$. To see this, note that since $\mathcal{L}(w) = \{s_{k+1}, s_{k+2}\}$, the set A consists of minimal elements in $H(w)$, so it is an antichain in $H(\underline{w})$ by Remark 2.12. Since $\mathbf{a}(w) = 2$, this antichain is maximal by Corollary 2.15. Since A is a maximal antichain consisting of all minimal elements in $H(\underline{w})$, the filter generated by A in $H(\underline{w})$ must equal $H(\underline{w})$ by Lemma 2.16. Our claim follows.

By Proposition 2.11, to prove that $\underline{z}$ is the reduced word of an FC element it suffices to show that $H(\underline{z})$ contains no covering relation $i \prec j$ with $s_i = s_j$ or convex chains $C = (i_1 \prec i_2 \prec \dots \prec i_m)$ where the elements' labels alternate in two noncommuting generators $s, t \in S$ and $m = m(s, t)$. Since x and w are both FC, such covering relations or convex chains cannot exist in either $H(\underline{x}) = \{1, \dots, k+2\}$ or in $H(\underline{w}) = \{k+1, \dots, k'\}$, so if they exist in $H(\underline{z})$, then they must involve indices from both $\underline{x}$ and $\underline{w}$, that is, we must have either (a) a covering relation $i \prec j$ where $i < k+1, j > k+2$, and $s_i = s_j$ or (b) a convex chain of the form C where $i_1 < k+1, i_m > k+2$. Write $i = i_1$ and $j = i_m$ in Case (b). Then in both Cases (a) and (b), the last paragraph shows that in $H(\underline{z})$ the index i is smaller than both elements of A in the poset order and the index j is larger than at least one element of A . It follows that that $i \preceq k+1 \preceq j$ or $i \preceq k+2 \preceq j$, so j cannot cover i and Case (a) cannot occur.

It also follows that in Case (b) the convex chain $i_1, \dots, i_m$ contains three consecutive elements a, b, c such that $a < k + 1, b \in \{k + 1, k + 2\}, c > k + 2$, and $s_a = s_c$. By Lemma 3.9.(3), the element a covered by b must be k . Let $t = s_a = s_k$. Then $s_c = t$ and t commutes with neither s_{k+1} nor s_{k+2} by Corollary 3.4.(2). But then we must have both $a \preceq k + 1 \preceq c$ and $a \preceq k + 2 \preceq c$ in $H(\underline{z})$, contradicting the convexity of the chain $i_1, \dots, i_m$. We may now conclude that $x * w$ is a FC element with reduced word $\underline{x} * \underline{w}$.

Next, we show that $\mathbf{a}(x * w) = 2$ by induction on the length $l(x) = k + 2 \geq 2$ of the stub x . In the base case, we have $k = 2$ and $x * w = w$, and therefore $\mathbf{a}(x * w) = \mathbf{a}(w) = 2$. If $k + 2 > 2$, then the element $x' := s_1 x = s_2 \dots s_{k+1} s_{k+2}$ is also a stub and satisfies $\mathcal{R}(x') = \mathcal{R}(x)$ by Lemma 3.9.(4), so $x' * w$ is defined and has $\mathbf{a}$ -value 2 by induction. The element 1 is the unique minimal element in the heap $H(\underline{x})$, so it is also the unique minimal element in $H(\underline{z})$ since we argued that $k + 1 \preceq j$ or $k + 2 \preceq j$ in $H(\underline{z})$ for all $j \in H(\underline{z}) \setminus H(\underline{x})$. Also, in the heap $H(\underline{x})$ we have the covering relation $1 \prec 2$ by Lemma 3.9.(3). It then follows from the left-sided analog of Proposition 2.21 that $x * w$ admits a left lower star operation with respect to $\{s_1, s_2\}$ that takes it to the element $s_2 s_3 \dots s'_k = x' * w$; therefore $\mathbf{a}(x * w) = \mathbf{a}(x' * w) = 2$ by Corollary 2.20, as desired.

It remains to show that $\mathcal{R}(x * w) = \mathcal{R}(w)$. Since $\mathbf{a}(x * w) = 2$, the antichain $A = \{k + 1, k + 2\}$ in $H(\underline{z}) = H(x * w)$ must be maximal by Corollary 2.15. Since $x * w$ is FC, it follows from Definition 4.1.(2) and Lemma 3.18.(2) that $\mathcal{R}(x * w) = \mathcal{R}(w) = \mathcal{L}(y)$, so $x * w$ is descent compatible with y . $\square$

Besides Proposition 4.6, we prepare one more ingredient for the proof of Theorem 4.3, namely how suitable glued products relate to stub decompositions and hence 1-cells.

PROPOSITION 4.7. (1) *For every stub $x \in \mathcal{S}(W)$ and every element $w \in W_2$ such that the pair (x, w) is descent compatible, the element $z := x * w$ is in the right cell of x .*
 (2) *For every element $z \in W_2$, there exists a unique way to write z as a glued product $z = x * w$ where $x \in \mathcal{S}(W)$ and w is an $\mathbf{a}(2)$ -element such that the pair (x, w) is compatible.*

Proof. (1) We have $\mathbf{a}(z) = 2$ by Proposition 4.6, and writing $\underline{x}, \underline{w}$ and $\underline{z} = \underline{x} * \underline{w}$ as in Proposition 4.6 yields a left stub decomposition $z = x \cdot w'$ where $w' = s_{k+3} \dots s_{k'}$. It follows that $z \sim_R x$ by Theorem 3.17.(1).

(2) We can find the candidates for the elements x and w as follows. Let $x \in \mathcal{S}(W)$ be the unique stub such that $z \in R_x$, so that by Theorem 3.17 we have $z = x \cdot w'$ for some element w' . Let $\underline{w}' = s_{k+3} s_{k+4} \dots s'_{k'}$ be a reduced word of w' , and let w be the element expressed by the word $\underline{w} := (s_{k+1} s_{k+2}) s_{k+3} \dots s_{k'} = s_{k+1} \dots s_{k'}$.

We now check that $\mathbf{a}(w) = 2$ and $z = x * w$. To do so, note that since the factorization $z = x \cdot w' = s_1 \dots s_{k'}$ is reduced, the word $\underline{w}$ is reduced and $w \leq_L z$. In particular, we have $\mathbf{a}(w) \leq \mathbf{a}(z) = 2$ by Parts (1) and (5) of Proposition 2.2. Meanwhile, the elements s_{k+1} and s_{k+2} are left descents of w , so the elements $(k + 1)$ and $(k + 2)$ form an antichain in the heap $H(w) = \{k + 1, k + 2, \dots, k'\}$ and $\mathbf{a}(w) \geq n(w) \geq 2$. It follows that $\mathbf{a}(w) = 2$ and $\mathcal{L}(w) = \{s_{k+1}, s_{k+2}\}$. It further follows that $z = x * w$, as desired.

To prove the uniqueness claim, suppose $z = x_0 * w_0$ is another factorization where $x_0 \in \mathcal{S}(W)$, $\mathbf{a}(w_0) = 2$ and (x_0, w_0) is descent compatible. Then $x_0 \sim_R z$ by Part (1), so $x_0 = x$ by Theorem 3.15. Remark 4.2 now implies that $x s_{k+1} s_{k+2} w = z = x_0 s_{k+1} s_{k+2} w_0$ where $\{s_{k+1}, s_{k+2}\} = \mathcal{R}(x) = \mathcal{R}(x_0)$, so $w = w_0$. The uniqueness claim follows. $\square$

We are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let $(x, w, y') \in \text{Tri}(W)$ and let $z = g(x, w, y')$. By Proposition 4.6, the pair $(x * w, y')$ is descent compatible since $x * w$ is an $\mathbf{a}(2)$ -element with $\mathcal{R}(x * w) = \mathcal{R}(w) = \mathcal{L}(y')$. A second application of the proposition then shows that $(x * w) * y'$ has $\mathbf{a}$ -value 2, so the proposed formula for g does define a map from $\text{Tri}(W)$ to W_2 . Note that by symmetry, we have $z = (x * w) * y' = x * (w * y')$, and therefore $z \in R_x$ by Proposition 4.7.(1). Similarly we have $z \in L_{y'}$, and therefore g restricts to a map $g_{xy'} : \{x\} \times \text{Cor}(x, y') \times \{y'\} \rightarrow R_x \cap L_{y'}$ as claimed.

It remains to prove that g is bijective. The surjectivity and injectivity of g follow respectively from the existence and uniqueness claims in Proposition 4.7.(2). More precisely, for every $z \in W_2$, the proposition guarantees a unique factorization $z = x * w'$ with $x \in \mathcal{S}(W)$, $w' \in W_2$ and (x, w') descent compatible, and the obvious analog of the proposition, involving right stubs and left cells, guarantees a unique factorization $w' = w * y'$ with $w \in W_2$, $y' \in \mathcal{S}'(W)$ and (w, y') descent compatible; it follows that (x, w, y') is the unique triple in $\text{Tri}(W)$ such that $g(x, w, y') = z$. $\square$

REMARK 4.8 (Distinguished involutions). The tools developed in this subsection are well-suited for describing the distinguished involutions in W_2 . Distinguished involutions are certain important involutions in Coxeter groups, and it is known that every right cell R contains a unique distinguished involution, which is a member of the 0-cell $R \cap R^{-1}$. For each $\mathbf{a}(2)$ -finite Coxeter system (W, S) and each stub $x \in \mathcal{S}(W)$, we claim that the distinguished involution in the cell R_x is precisely the glued product $d := x * x^{-1}$. This can be proved by showing d is the unique involution in R_x for which a certain number $\gamma_{x, x^{-1}, d}$ is nonzero. Here, the number $\gamma_{x, x^{-1}, d}$ can be obtained by directly computing the product $C_x C_{x^{-1}}$ in the Hecke algebra of (W, S) (see [30, §14]), but we omit the details. Instead, we note that d is clearly an involution and we have $d \in R_x$ by Proposition 4.7.(1). Consequently we have $d^{-1} = d \in R_x$, so that $d \in R_x^{-1}$ and $d \in R_x \cap R_x^{-1} = R_x \cap L_{x^{-1}} = I(x, x)$. In particular, when the 0-cell $I(x, x)$ is a singleton (which is sometimes the case, see Proposition 4.18), we can conclude that d is the distinguished involution in R_x without any computation.

4.2. RELATING 0-CELLS. In this subsection we use the relations $\sim$ and $\approx$ introduced in Definition 3.23 to connect different 0-cells in W_2 . Our results explain how two 0-cells $I(x, y)$ and $I(x', y')$ relate to each other for $x, y, x', y' \in \mathcal{S}(W)$ when $x \approx x', y \approx y'$ or when $x \sim x', y \sim y'$. Since $\approx$ and $\sim$ are generated by left star operations and suitable slides, we first study how these generating transformations affect 0-cells. The effect of the left star operations can be understood via the canonical decomposition of $\mathbf{a}(2)$ -elements described in subsection 4.1:

LEMMA 4.9. *Let $x, x', y \in \mathcal{S}(W)$, suppose that x and x' can be related by left star operations, and let*

$$f_{xy} : \text{Cor}(x, y^{-1}) \rightarrow I(x, y), \quad f_{x'y} : \text{Cor}(x', y^{-1}) \rightarrow I(x', y)$$

be the bijections defined in Corollary 4.4. Then the sets $\text{Cor}(x, y^{-1})$ and $\text{Cor}(x', y^{-1})$ are equal and the map

$$\varphi_{xx'} := f_{x'y} \circ f_{xy}^{-1} : I(x, y) \rightarrow I(x', y)$$

is a bijection. In particular, we have $N(x, y) = N(x', y)$.

Proof. Since x and x' can be related by left star operations, we have $\mathcal{R}(x) = \mathcal{R}(x')$ by Propositions 2.18 and 2.2.(3). By Definition 4.1, for any two stubs $z, z' \in \mathcal{S}(W)$ the set $\text{Cor}(z, z')$ depends only on the descent sets $\mathcal{R}(z)$ and $\mathcal{L}(z')$, so $\text{Cor}(x, y) = \text{Cor}(x', y)$. The remaining claims follow immediately. $\square$

REMARK 4.10. The bijection $\varphi_{xx'}$ from Lemma 4.9 has two simple descriptions. First, in the notation of Theorem 4.3 and Corollary 4.4, the elements of $I(x, y)$ and $I(x', y)$ are precisely the glued products $g(x, w, y) = x * w * y$ and $g(x', w, y) = x' * w * y$ where $w \in \text{Cor}(x, y) = \text{Cor}(x', y)$, so for each element in $I(x, y)$ the map $\varphi_{xx'}$ simply replaces the left stub x with x' in the canonical decomposition of the element into $\mathbf{a}(2)$ -triples. Second, the above description implies that $\varphi_{xx'}(z) = x'x^{-1}z$ by Remark 4.2; that is, applying $\varphi_{xx'}$ amounts to left multiplication by $x'x^{-1}$ in the group W .

Next, we study how slides on short stubs affect relevant 0-cells.

LEMMA 4.11. *Let (W, S) be an arbitrary Coxeter system. Suppose $s, t, u \in S$ are generators such that $m(s, t) = m(t, u) = 2$ and $m(t, u) \geq 3$, and let $J = \{t, u\}$.*

- (1) *Let $w \in \text{FC}(W)$ and suppose that $\mathcal{L}(w) = \{s, t\}$. Then w admits at least one left star operation σ with respect to J , and for any such operation we have $\{s, u\} \subseteq \mathcal{L}(\sigma(w))$.*
- (2) *Now suppose that (W, S) is $\mathbf{a}(2)$ -finite and consider the short stubs $x = st$, $x' = su$ in $\mathcal{S}(W)$. For each subset A of W , let*

$$\Sigma(A) = \{\sigma(w) : w \in A, \sigma \text{ is a left star operation with respect to } J\}.$$

Then we have $\Sigma(R_x) = R_{x'}$ and $\Sigma(R_{x'}) = R_x$. For every $y \in \mathcal{S}(W)$, we also have $\Sigma(I(x, y)) = I(x', y)$ and $\Sigma(I(x', y)) = I(x, y)$.

- (3) *Let (W, S) , x, x' and y be as in Part (2), and suppose that $m(t, u) = 3$. Then $N(x', y) = N(x, y)$.*

Proof. (1) Suppose $\mathcal{L}(w) = \{s, t\}$ and consider the right coset decomposition $w = w_J \cdot {}^J w$ of w relative to J . Since $w \in \text{FC}(W)$, we have $l(w_I) < m(s, t)$ by the word criterion for full commutativity (subsection 2.3). Since $t \in \mathcal{L}(w)$, we have $l(w_I) \geq 1$. It follows that w admits at least one left star operation with respect to I . By the definition of star operations, any such operation must introduce u as a new left descent and keep s as a descent, hence we have $\{s, u\} \subseteq \mathcal{L}(\sigma(w))$.

(2) Let $w \in R_x$ and let L be a left cell in W_2 . Then $\mathcal{L}(w) = \mathcal{L}(x) = \{s, t\}$ by Proposition 2.2.(3), so Part (1) guarantees that w admits at least one left star operation, say σ , with respect to J . Let $z \in \mathcal{S}(W)$ be the unique stub such that $\sigma(w) \in R_z$. Then $\{s, u\} \subseteq \mathcal{L}(\sigma(w)) = \mathcal{L}(z)$ by Part (1) and Proposition 2.2.(3). This forces $\mathcal{L}(z) = \{s, u\}$ by Corollary 2.15, which in turn forces $z = su = x'$ by Lemma 3.9.(1). It follows that $\sigma(w) \in R_{x'}$, and therefore $\Sigma(R_x) \subseteq R_{x'}$. A similar argument shows that $\Sigma(R_{x'}) \subseteq R_x$. Note that since left cells are closed under left star operations by Proposition 2.18, it further follows that $\Sigma(R_x \cap L) \subseteq R_{x'} \cap L$ and $\Sigma(R_{x'} \cap L) \subseteq R_x \cap L$.

The operation σ can be reversed by another left star operation σ' with respect to J , with σ' being a lower operation if σ is upper and vice versa; therefore $w = \sigma' \circ \sigma(w) \in \Sigma(\Sigma(R_x))$. It follows that $R_x \subseteq \Sigma(\Sigma(R_x))$. The same argument shows that $R_x \cap L \subseteq \Sigma(\Sigma(R_x \cap L))$.

We have shown that $\Sigma(R_x) \subseteq R_{x'}$ and $R_x \subseteq \Sigma(\Sigma(R_x))$, so $R_x \subseteq \Sigma(\Sigma(R_x)) \subseteq \Sigma(R_{x'})$. Since $\Sigma(R_{x'}) \subseteq R_x$, it follows that $\Sigma(R_x) = R_{x'}$. Similarly we may conclude that $\Sigma(R_{x'}) = R_x$, $\Sigma(R_x \cap L) = R_{x'} \cap L$ and $\Sigma(R_{x'} \cap L) = R_x \cap L$ from the previous two paragraphs. The last two equalities imply the desired claims on $I(x, y)$ and $I(x', y)$ when we set $L = R_y^{-1}$, and the proof of (2) is complete.

(3) Let $w \in R_x$. Since $m(t, u) = 3$, the element w admits at most one left star operation relative to J , namely, the simple left star operation $*$. On the other hand w admits at least one left star operation relative to $J = \{t, u\}$ by Part (1). It follows that the set $\Sigma(R_x)$ defined in Part (2) equals the set $*(R_x) := \{*w : w \in R_x\}$. Being an involution, the map $*$ is injective on the set of elements it is defined on, so we

have $|R'_x| = |\Sigma(R_x)| = |*(R_x)| = |R_x|$, where the first equality holds by Part (2). Similarly we have $|I(x', y)| = |\Sigma(I(x, y))| = |*(I(x, y))| = |I(x, y)|$, i.e. we have $N(x', y) = N(x, y)$. $\square$

Lemma 4.9 and Lemma 4.11.(3) lead to a numerical invariance that will be very useful for counting 1-cells and 2-cells in W_2 (see Proposition 4.15):

PROPOSITION 4.12. *Let $x, y, x', y' \in \mathcal{S}(W)$ and suppose $x' \sim x, y' \sim y$. Then $N(x', y') = N(x, y)$.*

Proof. The relation $\sim$ is generated by left star operations and simple slides by Remark 3.25.(2), we have $N(x', y) = N(x, y)$ when x' and x are related by left star operations by Lemma 4.9, and we have $N(x', y) = N(x, y)$ if x' and x are short stubs related by a simple slide by Lemma 4.11.(3), hence $N(x', y) = N(x, y)$. Similarly we have $N(y', x') = N(y, x')$. Remark 3.21 now implies that $N(x', y') = N(y', x') = N(y, x') = N(x', y) = N(x, y)$, as desired. $\square$

REMARK 4.13. In the same way that Lemma 4.11.(3) is a reflection of Lemma 4.11.(2) in the special case where the slide involved is simple, Proposition 4.12 is a numerical reflection of the more general fact that in an $\mathbf{a}(2)$ -finite Coxeter system (W, S) , we can always deduce all 0-cells in the same 2-cell of W_2 from each other. More precisely, if the system (W, S) is not of type $E_{1,r}$ for some integer $r \geq 1$, then W_2 is itself a single 2-cell by Theorem 3.31, and we can deduce $I(x', y')$ from $I(x, y)$ for all stubs $x, y, x', y' \in \mathcal{S}(W)$ as follows:

- (1.a) If x and x' are short, then we may obtain x' from x by a sequence of slides because all short stubs in $\mathcal{S}(W)$ are slide equivalent by the proof of Theorem 3.31. By Lemma 4.11.(2), this implies that we may start from $I(x, y)$ and apply a corresponding sequence of left star operations setwise to obtain $I(x', y)$.
- (1.b) More generally, let x_1 and x'_1 be the first layers of x and x' , respectively. Then x and x' can be related to x_1 and x'_1 by left star operations by Lemma 3.9.(4), respectively, so by Lemma 4.9 and Remark 4.10 we may obtain $I(x_1, y)$ from $I(x, y)$, and $I(x'_1, y)$ from $I(x'_1, y)$, via suitable left multiplications. Since x_1 and x'_1 are short stubs, we can obtain $I(x'_1, y)$ from $I(x_1, y)$ as explained in (1.a), so we can obtain $I(x', y)$ from $I(x, y)$.
- (2) By symmetry, the suitable counterparts of Lemma 4.9, Remark 4.10, and Lemma 4.11.(2) allow us to obtain $I(x', y')$ from $I(x', y)$ via right star operations and right multiplications. It follows that we can obtain $I(x', y')$ from $I(x, y)$.

When (W, S) is of type $E_{1,r}$ where $r \geq 1$, we may obtain different 0-cells from each other in each 2-cells E_i (in the notation of Theorem 3.31) by the same reasoning because the short stubs in E_i are always pairwise slide equivalent by Lemma 3.28.

EXAMPLE 4.14. The Coxeter system (W, S) of type B_4 contains six $\mathbf{a}(2)$ -stubs, namely $x_1 := 1 \cdot 2 \cdot 13, x_2 := 2 \cdot 13, x_3 = 13, x_4 = 14, x_5 = 24$ and $x_6 = 3 \cdot 24$. The six stubs are all slide equivalent but fall into two simple slide equivalence classes, with $x_1 \sim x_2 \sim x_3 \sim x_4$ and $x_5 \sim x_6$. Let $R(i) = R_{x_i}, L(i) = R(i)^{-1}, I(i, j) = R(i) \cap L(j)$ and $N(i, j) = |I(i, j)|$ for all $1 \leq i, j \leq 6$. Then the 0-cells in W_2 are given in Table 6, where x_i labels the i -th row and i -th column and where $I(i, j)$ appears in the i -th row and j -column for all $1 \leq i, j \leq 6$.

We may use the procedures described earlier to determine the rows of the table from each other, i.e. to obtain an entry from another entry in the same column. For example, since x_3 and x_4 differ by a sliding move along the edge $\{3, 4\}$, we may obtain the third and fourth row from each other via left star operations with respect to $\{3, 4\}$; since

	1 · 2 · 13	2 · 13	13	14	24	3 · 24
1 · 2 · 13	{121321, 1213241321}	{12132, 121324132}	{1213, 12132413}	{12134, 1213241}	{121324}	{1213243}
2 · 13	{21321, 213241321}	{2132, 21324132}	{213, 2132413}	{2134, 213241}	{21324}	{213243}
13	{1321, 13241321}	{132, 1324132}	{13, 132413}	{134, 13241}	{1324}	{13243}
14	{41321, 1241321}	{4132, 124132}	{413, 12413}	{14, 1241}	{124}	{1243}
24	{241321}	{24132}	{2413}	{214}	{24, 2124}	{243, 21243}
3 · 24	{3241321}	{324132}	{32413}	{3214}	{324, 32124}	{3243, 321243}

TABLE 6. Zero-sided cells of **a**-value 2 in type B_4

x_5 and x_6 are related to each other by left star operations and $x_5x_6^{-1} = x_6x_5^{-1} = 3$, we may obtain the fifth and sixth row from each other using left multiplication by the generator 3. Similarly, we can determine the columns of the table from each other via right star operations and right multiplications, so we can recover the entire table from any single entry.

We note that taking the cardinality $N(i, j)$ of each listed 0-cell $I(x, y)$ in Table 6 recovers Table 1 from Example 2.8. Given that $x_1 \sim x_2 \sim x_3 \sim x_4$ and $x_5 \sim x_6$, both these tables conform to Proposition 4.12. The tables also demonstrate the fact that the conclusion $N(x', y') = N(x, y)$ of Proposition 4.12 no longer holds when we weaken the assumption $x \sim x', y \sim y$ to the assumption $x \approx x', y \approx y'$. In this sense, the theorem cannot be strengthened.

4.3. PREPARATION FOR CELL ENUMERATION. Proposition 4.12 brings the relation $\sim$ on $\mathcal{S}(W)$ to the forefront of the enumerations of 1-cells and 2-cells in W_2 . To be more precise, suppose $w_1, \dots, w_d$ is a complete, irredundant list of representatives of the $\sim$ -classes in $\mathcal{S}(W)$, let n_i be the size of the $\sim$ -class of w_i , let $N_i = |R_{w_i}|$ and let $I_{ij} = I(w_i, w_j)$ and $N_{ij} = N(w_i, w_j)$ for all $1 \leq i, j \leq d$. Then by Proposition 4.12 and Proposition 3.22, we can count all cells in W_2 via the following results:

PROPOSITION 4.15. *Maintain the notation of Theorem 3.31.*

- (1) *For all $x, y \in \mathcal{S}(W)$, if $x \not\sim_{LR} y$ then $N(x, y) = 0$.*
- (2) *For all $x, y \in \mathcal{S}(W)$, we have $N(x, y) = N_{ij}$ where i and j are the unique integers such that $x \sim w_i$ and $y \sim w_j$.*
- (3) *For all $x \in \mathcal{S}(W)$, we have*

$$|R_x| = \sum_{j=1}^d n_j N_{ij} = |R_{w_i}| = N_i$$

where i is the unique integer such that $x \sim w_i$.

- (4) *Suppose (W, S) is not of type E_{1r} for any $r \geq 1$. Then the unique two-sided cell in W_2 , i.e. the set W_2 itself, has cardinality*

$$|W_2| = \sum_{1 \leq i, j \leq d} n_i n_j N_{ij} = \sum_{1 \leq i \leq d} n_i N_i.$$

- (5) *Suppose (W, S) is of type E_{1r} for some $r \geq 1$, and let $1 \leq i, j \leq d$. Then we have $N_{ij} = 0$ if $i \neq j$. Moreover, each two-sided cell E_i in W_2 has cardinality*

$$|E_i| = |\sqcup_{y \sim w_i} R_y| = \sum_{y \sim w_i} |R_y| = n_i N_i = n_i \cdot \sum_{j=1}^d n_j N_{ij} = n_i^2 N_{ii}.$$

- (6) *We have $N_{ij} = N_{ji}$ for all $1 \leq i, j \leq d$.*

X	representatives and cardinalities of $\approx$ -classes
$A_n, n \geq 3$	$\{13 : \beta_n - 1\}$
B_3	$\{13 : 3\}$
$B_n, n > 3$	$\{13 : n, \quad 24 : \beta_{n-1} - 1\}$
$\tilde{C}_{n-1}, n \geq 5$	$\{13 : n - 1, \quad 24 : \beta_{n-2} - 1, \quad (n - 2)n : n - 1, \quad 1n : 1\}$
$E_{1,1} = D_4$	$\{(-1)v : 3, 1v : 3, \quad (-1)1 : 3\}$
$E_{q,r}, r > q = 1$	$\{(-1)v : 2 + r = n - 1, \quad 1v : \beta_n\}$
$E_{q,r}, r \geq q \geq 2$	$\{(-1)1 : \beta_{q+r+3} - 1 = \beta_{n+1} - 1\}$
F_4	$\{13 : 9\}$
$F_n, n > 4$	$\{13 : 3n - 3, \quad 35 : \beta_{n-2} - 1\}$
H_3	$\{13 : 5\}$
$H_n, n > 3$	$\{13 : 2n - 1, \quad 24 : \beta_{n-1} - 1\}$

TABLE 7. The slide equivalence classes of $\mathcal{S}(W)$, where $\beta_n = \binom{n}{2}$ for each integer n

Proof. Let $x, y \in \mathcal{S}(W)$. If $x \not\sim_{LR} y$, then $x \not\sim_{LR} y^{-1}$ by Proposition 2.2.(6); therefore $R_x \cap L_{y^{-1}} = \emptyset$ and $N(x, y) = 0$. This proves Part (1). Part (2) follows from Proposition 4.12. To prove (3), note that by Proposition 3.22.(1) we have $|R_x| = \sum_{y \in \mathcal{T}} N(x, y)$ where $\mathcal{T} = E \cap \mathcal{S}(W)$ and E is the 2-cell containing x . By Part (1), we may enlarge the set $\mathcal{T}$ to obtain $|R_x| = \sum_{y \in \mathcal{S}(W)} N(x, y)$. Part (3) then follows from Part (2). Part (4) follows directly from Proposition 3.22.(3) and Part (3). To see Part (5), first note that if $i \neq j$ then $w_i \not\sim_{LR} w_j$ by Theorem 3.31, so $N_{ij} = 0$ by Part (1). This implies the last equality in the displayed equation; the other equalities hold by the definition of E_i and Part (3). Finally, Part (6) follows from Remark 3.21. $\square$

We now present the data w_i, n_i, N_{ij} defined earlier in this subsection, starting with a description of the equivalence classes of $\sim$ in terms of $\{w_1 : n_1, w_2 : n_2, \dots, w_d : n_d\}$.

PROPOSITION 4.16. *Let (W, S) be an irreducible nontrivially $\mathbf{a}(2)$ -finite Coxeter system of type X , and let $\mathcal{S}(W)$ be the set of left $\mathbf{a}(2)$ -stubs in W , denoted in the same way as in Theorem 3.7. Let $\beta_n = \binom{n}{2}$ for all $n \geq 2$. Then the equivalence classes of $\sim$ are described by Table 7.*

Proof. Let G be the Coxeter diagram of (W, S) . Recall that the number $|\mathcal{S}(W)|$ is given in Theorem 3.7.(7), which we will use without comment from now on.

When (W, S) is of type A_n ($n \geq 3$) or $E_{q,r}$ ($r \geq q \geq 1$), all edges of G are simple, so the relations $\approx$ and $\sim$ coincide. It follows from Proposition 3.30 and Theorem 3.7.(7) that Table 7 gives the correct information in these types. In particular, in type $E_{1,r}$ where $r > 1$, counting the set C_1 in Proposition 3.30.(2).(b) yields $n_1 = |C_1| = r + 2 = n - 1$, and therefore $n_2 = |C_2| = |\mathcal{S}(W)| - |C_1| = (\beta_{n+1} - 1) - (n - 1) = \beta_n$.

For the Coxeter systems of types B_n ($n \geq 3$), $\tilde{C}_{n-1}$ ($n \geq 5$), F_n ($n \geq 4$) and H_n ($n \geq 3$), we sketch the key ideas for obtaining the set $\{w_1 : n_1, \dots, w_d : n_d\}$. Let Z be the set of all short stubs of $\mathcal{S}(W)$. By Remark 3.25.(1), we may choose all the class representatives $w_1, \dots, w_d$ from Z , and to understand the $\sim$ -classes of $\mathcal{S}(W)$ it suffices to understand how the short stubs in Z fall into different $\sim$ -classes. To this end, note that removing the heavy edge or heavy edges in G results in $r = 2$ or $r = 3$ linear subgraphs of G whose edges are simple and whose vertex sets partition S : these are the sets $S_1 = \{1\}, S_2 = \{2, \dots, n - 1\}, S_3 = \{n\}$ in type $\tilde{C}_{n-1}$, the sets $S_1 = \{1, 2\}, S_2 = \{3, \dots, n\}$ in type F_n ($n \geq 4$), and the sets $S_1 = \{1\}, S_2 = \{2, \dots, n\}$

otherwise. Associate a tuple $\text{dist}(z) := (\text{Supp}(z) \cap S_i)_{1 \leq i \leq r}$ to each stub $z \in Z$ to record how its support is distributed among these subsets, and note that by the definition of simple slides, we have $z \sim z'$ if and only if $\text{dist}(z) = \text{dist}(z')$ for all $z, z' \in Z$. Observe that in every Coxeter type we have included in the list $w_1, \dots, w_d$ one short stub corresponding to every possible distribution. It follows that $w_1, \dots, w_d$ is a complete, irredundant list of class representatives of $\sim$ in $\mathcal{S}(W)$.

To obtain the sizes $n_1, \dots, n_d$ of the equivalence classes, start with the class represented by the short stub whose support lies entirely in the set S_2 specified earlier. This stub is always w_2 in our list of representatives. If $|S_2| = k$ then the size of its $\sim$ -class in $\mathcal{S}(W)$ is $n_2 = \beta_k - 1$ because S_2 generates a parabolic subgroup of W isomorphic to A_k and $\mathcal{S}(A_k) = \beta_k - 1$. Three $\sim$ -classes remain in type $\tilde{C}_{n-1}$ and one class remains otherwise. In the latter case we can obtain cardinality n_1 of the remaining class as $n_1 = |\mathcal{S}(W)| - n_2$. In the former case, we note that the element $w_4 = 1n$ admits no slides and is the only stub with first layer $1n$ by Corollary 3.4, so $\{1n\}$ is a $\sim$ -class and $n_4 = 1$. By the symmetry in G , it further follows that $n_1 = n_3 = (|\mathcal{S}(W)| - n_2 - n_4)/2 = (n - 1)$. Alternatively, in both cases we may list the elements of the remaining equivalence classes and count them without difficulty with the help of Corollary 3.4 and Theorem 3.7, but we omit the details. $\square$

Our next result describes selected 0-cells and will lead to the cardinalities N_{ij} for all $1 \leq i, j \leq d$. We postpone its proof to section 5, where we will refer to the cells given in the theorem *representative 0-cells*.

THEOREM 4.17. *Let (W, S) be a nontrivially $\mathbf{a}(2)$ -finite Coxeter system of type X . Then the following set equalities hold in W .*

- (1) *If $X = A_n$ where $n \geq 3$, then $I(13, 13) = \{13\}$.*
- (2) *If $X = B_3$, then $I(13, 13) = \{13\}$.*
- (3) *If $X = B_n$ where $n > 3$, then $I(24, 24) = \{24, 2124\}$.*
- (4) *If $X = \tilde{C}_{n-1}$ where $n \geq 5$, then $I(24, 24) = \{24, 2124, 2 \cdot z, 212 \cdot z\}$ where $z = 45 \cdots (n-1)n(n-1) \cdots 54$.*
- (5) *If $X = E_{q,r}$ where $r \geq q \geq 1$, then for all $x \in \{(-1)v, 1v, (-1)1\}$ we have $I(x, x) = \{x\}$.*
- (6) *If $X = F_4$, then $I(24, 24) = \{24\}$.*
- (7) *If $X = F_n$ where $n > 4$, then $I(24, 24) = \{24, 243524\}$.*
- (8) *If $X = H_3$, then $I(13, 13) = \{13\}$.*
- (9) *If $X = H_n$ where $n > 3$, then $I(24, 24) = \{24, 2124\}$.*

PROPOSITION 4.18. *Maintain the notation of Proposition 4.16. Then the numbers N_{ij} where $1 \leq i \leq j \leq d$ are given by Table 8.*

Proof. We sketch the proof. In Theorem 4.17 we have described $I_{ij} = I(w_i, w_j)$ for some particular i, j . Taking the cardinality of this set shows that the number N_{ij} given in Table 8 is correct. For every other pair (k, l) with $1 \leq k \leq l \leq d$, we can obtain I_{kl} from I_{ij} as explained in subsection 4.2 and check that that $N_{kl} = |I_{kl}|$ agrees with the value given by Table 8. $\square$

REMARK 4.19. In the above proof we mentioned computing N_{kl} by computing the 0-cell I_{kl} and taking its cardinality, but the computation of N_{kl} can often be simplified. Since we are not interested in the set $I_{kl} = I(w_k, w_l)$ per se, we do not have to compute I_{kl} from the set I_{ij} . Instead, it is often convenient to compute a 0-cell $I'_{kl} := I(a, b)$ from a 0-cell $I'_{ij} := I(c, d)$ where a, b, c, d are short stubs such that $a \sim w_k, b \sim w_l, c \sim w_i, d \sim w_j$ and such that we can relate c to a , and d to b , with a small number of slides. Here the set I'_{ij} has N_{ij} elements by Proposition 4.12 and N_{ij} is at most 4 by Table 8, so in practice it is not difficult to obtain I_{ij} by

X	Values of N_{ij} ($1 \leq i < j \leq d$)
$A_n, n \geq 3$	$N_{11} = 1$
B_3	$N_{11} = 1$
$B_n, n > 3$	$N_{11} = N_{22} = 2, N_{12} = 1$
$\tilde{C}_n, n \geq 5$	$N_{11} = N_{22} = N_{33} = N_{44} = 4,$ $N_{12} = N_{23} = N_{14} = N_{34} = 2, N_{13} = N_{24} = 1$
$E_{q,r}, r > q = 1$	$N_{ii} = 1$ for all $1 \leq i \leq d$, $N_{ij} = 0$ whenever $i \neq j$
$E_{q,r}, r > q \geq 2$	$N_{11} = 1$
F_4	$N_{11} = 1$
$F_n, n > 4$	$N_{11} = N_{22} = 2, N_{12} = 1$
H_3	$N_{11} = 1$
$H_n, n > 3$	$N_{ij} = 2$ for all $1 \leq i \leq j \leq 2$.

TABLE 8. Sizes of 0-cells of **a**-value 2

finding N_{ij} elements that are obviously in it. The point of using the stubs a, b, c, d is that if we select them carefully then we may obtain I'_{kl} from I'_{ij} using a smaller number of setwise star operations than we need for obtaining I_{kl} from I_{ij} . For example, in the group $\tilde{C}_{n-1}$ we have $w_1 = 13, w_2 = 24, w_4 = 1n$ by Proposition 4.16 and $N_{22} = |I_{22}| = 4$ by Theorem 4.17.(4), and to compute N_{14} it is convenient to use the stubs $a := 14 \sim w_1, b := 1n \sim w_4, c := 24 \sim w_2, d := 2(n-1) \sim w_2$ as follows: the set $I'_{22} = I(c, d) = I(24, 2(n-1))$ should contain $N_{22} = 4$ elements, so once we note that the set

$$I' := \{2 \cdot z', 2 \cdot z' \cdot n(n-1), 212 \cdot z', 212 \cdot z' \cdot n(n-1) \mid z' = 45 \cdots (n-1)\}$$

is contained in I'_{22} we can conclude that $I'_{22} = I'$. Since a can be obtained from c with one slide along the edge $\{1, 2\}$, applying the corresponding setwise left star operations to $I'_{22} = I(c, d)$ yields

$$I(a, d) = \{12 \cdot z', 12 \cdot z' \cdot n(n-1) \mid z' = 45 \cdots (n-1)\}.$$

Similarly, since b can be obtained from d with two slides, along the edges $\{1, 2\}$ and $\{n-1, n\}$, applying the two corresponding setwise right star operations in succession to $I(a, d)$ yields

$$I'_{14} = I(a, b) = \{1 \cdot z'', 121 \cdot z'' \mid z'' = 45 \cdots n\},$$

so $N_{14} = 2$. Note that we only need three setwise star operations, even if n is large. In contrast, if we computed I_{14} from I_{22} by working with $w_1 = 13, w_2 = 24$ and $w_4 = (n-1)n$ directly, then we need many more setwise star operations to obtain I_{14} when n is large.

4.4. SIZES OF **a**(2)-CELLS. We are ready to compute the sizes of all left, right, and two-sided Kazhdan–Lusztig cells of **a**-value 2 in all **a**(2)-finite Coxeter groups:

THEOREM 4.20. *Let (W, S) be an irreducible nontrivially **a**(2)-finite Coxeter system of type X and maintain the notation of subsection 4.3. In particular, let $\beta_n = \binom{n}{2}$ for all $n \geq 2$. Then the sizes of 1-cells and 2-cells in W_2 are given by Table 9. When W_2 is a single two-sided cell of size N , we simply write N instead of $|W_2| = N$ in the corresponding box in the table.*

Proof. We know the numbers n_i and N_{ij} for all $1 \leq i, j \leq d$ by Propositions 4.16, 4.18 and 4.15.(6), so the sizes of all 1-cells and 2-cells can be calculated by directly applying the formulas from Proposition 4.15.(3)–(5). The calculations are straightforward and

X	sizes of 1-cells	sizes of 2-cells
$A_n, n \geq 3$	$N_1 = \beta_n - 1$	$(\beta_n - 1)^2$
B_3	$N_1 = 3$	9
$B_n, n > 3$	$N_1 = \beta_{n+1}, N_2 = n^2 - 2n$	$n^4/2 - 2n^3 + 7n^2/2$
$\tilde{C}_{n-1}, n \geq 5$	$N_1 = N_3 = n^2 + 1,$ $N_2 = 2n^2 - 6n + 5,$ $N_4 = n^2/2 + 3n/2 + 2$	$n^4 - 6n^3 + 20n^2 - 21n + 10$
$E_{11} = D_4$	$N_1 = N_2 = N_3 = 3$	$ C_1 = C_2 = C_3 = 9$
$E_{q,r}, r > q = 1$	$N_1 = n - 1,$ $N_2 = \beta_n$	$ C_1 = (n - 1)^2,$ $ C_2 = \beta_n^2$
$E_{q,r}, r \geq q \geq 2$	$N_1 = \beta_{n+1} - 1$	$N_1 = (\beta_{n+1} - 1)^2$
F_4	$N_1 = 9$	81
$F_n, n > 4$	$N_1 = n^2/2 + 7n/2 - 4,$ $N_2 = (n - 1)^2$	$n^4/2 - 2n^3 + 33n^2/2 - 29n + 14$
H_3	$N_1 = 5$	25
$H_n, n > 3$	$N_1 = N_2 = 2(\beta_{n+1} - 1)$	$2(\beta_{n+1} - 1)^2$

TABLE 9. Sizes of 1-cells and 2-cells of **a**-value 2, where $\beta_n = \binom{n}{2}$ for each integer n

	$n - 1$	$\beta_{n-2} - 1$	$n - 1$	1
$n - 1$	4	2	1	2
$\beta_{n-2} - 1$	2	4	2	1
$n - 1$	1	2	4	2
1	2	1	2	4

TABLE 10.

similar for all **a**(2)-finite Coxeter types, so instead of carrying out the computations in detail for all types, we only illustrate below how to count the cells in $\tilde{C}_{n-1}$ ($n \geq 5$), where the number d of $\sim$ -classes in $\mathcal{S}(W)$ is the largest and the computations are the most complex. Recall from Proposition 4.16 that $\mathcal{S}(W)$ contains $d = 4$ slide equivalence classes and their sizes are $n_1 = n - 1, n_2 = \beta_{n-2} - 1, n_3 = n - 1$ and $n_4 = 1$. Consider the following $d \times d$ table, where the rows and columns are labeled by n_i and the entry in the i -th row, j -th column is the number N_{ij} from Table 8.

By Proposition 4.15.(2), for each $1 \leq i \leq d$ we can compute N_i as a weighted sum of the i -th row in the above table, with the column labels as weights. For example, we have

$$N_1 = \sum_{j=1}^d n_j N_{1j} = 4 \cdot (n - 1) + 2 \cdot (\beta_{n-2} - 1) + 1 \cdot (n - 1) + 2 \cdot 1 = n^2 + 1.$$

Similar computations show that $N_2 = 2n^2 - 6n + 5, N_3 = n^2 + 1$ and $N_4 = n^2/2 + 3n/2 + 2$. Having computed $N_1, \dots, N_d$, we can then compute the size of the 2-cell W_2 as the weighted sum $\sum_{i=1}^d n_i N_i$ by Proposition 4.15.(4). Doing so yields $|W_2| = n^4 - 6n^3 + 20n^2 - 21n + 10$. $\square$

5. REPRESENTATIVE 0-CELLS

We assume (W, S) is a nontrivially $\mathbf{a}(2)$ -finite Coxeter system and prove Theorem 4.17 in this section. We complete the proof in subsection 5.2 after preparing a series of technical definitions and lemmas in subsection 5.1.

5.1. LEMMAS ON HEAPS. Let G denote the Coxeter diagram of (W, S) . Then G has no cycles; therefore for any two vertices u, v in G there is a unique path from u to v (see Remark 2.13). For pairwise distinct vertices u, v, t , we say that u and v *lie on different sides of* a vertex t if the path connecting u to v passes through t . We call two vertices *neighbors* of each other if they are adjacent in G , define the *degree* of a vertex v in G to be the number of its neighbors, and call a vertex an *end vertex* if it has degree 1. For each element $w \in \text{FC}(W)$ and any two elements i, j in the heap $H(w)$, we denote the label of i by $e(i)$, write $(i, j) := \{h \in H(w) : i \not\leq h \not\leq j\}$, and write $[i, j] := \{h \in H(w) : i \leq h \leq j\}$.

The following two lemmas allow us to deduce the existence of additional elements and structure in $H(w)$ from certain local configurations. The vertex whose existence is asserted in the lemmas allows us to expand the lattice embedding of the heap vertically and horizontally, which is why we name the lemmas vertical and horizontal completion.

LEMMA 5.1 (Vertical Completion). *Let a be an end vertex in G and let b be its only neighbor. Suppose $b \in \mathcal{L}(w)$ and $e(i) = a$ for an element $i \in H(w)$. Then $H(w)$ contains an element that is covered by i and has label b .*

Proof. Since $b \in \mathcal{L}(w)$, some minimal element $j \in H(w)$ has label b by Remark 2.12. The elements i and j are comparable since a and b are adjacent in G . Moreover, we must have $j \leq i$ since j is minimal, so there is a chain $j = j_1 \prec \dots \prec j_{p-1} \prec j_p = i$ of coverings connecting j to i in $H(w)$. Since b is the only neighbor of a , we must have $e(j_{p-1}) = b$. It follows that j_{p-1} has the desired properties. $\square$

LEMMA 5.2 (Horizontal Completion). *Let a be an end vertex in G and let b be its only neighbor. Suppose that $C = (j \prec i \prec j')$ is a chain of coverings in $H(w)$ with $e(i) = a$ and $e(j) = e(j') = b$. Then i, j, j' account for all elements in $[j, j']$ that are labeled by a or b . Moreover, if C is not convex, then j and j' are connected by a chain $D = (j = j_1 \prec j_2 \prec \dots \prec j_q = j')$ of coverings where $q \geq 3$. In this case, at least one element in D is labeled by a neighbor of b in G that is distinct from a .*

Proof. An element in (j, j') with label b would be comparable to i and thus contradict the fact that i covers j and j' covers i , so j and j' are the only elements with label b in $[j, j']$. Since b is the only neighbor of a , it further follows that (j, j') contains no element with label a other than i , because otherwise such an element would cover or be covered by i , violating Condition (1) of Proposition 2.11. We have proved the first claim.

If C is not convex, then j and j' must both be comparable to some element $k \in (j, j')$ that is not in C . Concatenating chains of coverings $j \prec \dots \prec k$ and $k \prec \dots \prec j'$ connecting j to k and k to j' yields a chain D in $[j, j']$ that connects j to j' and has at least three elements, as desired. The chain D corresponds to a walk of length at least 3 from b to b in G by Remark 2.13. Such a walk must involve a neighbor of b , and such a neighbor cannot be a by the first claim since D lies in the interval $[j, j']$. The proof is complete. $\square$

The next result proves that certain subsets of heaps are antichains.

LEMMA 5.3. *Let x, y, z be elements of S such that x and z lie on different sides of y . Suppose that $H(w)$ contains two elements i and k labeled by x and z , respectively.*

Then the set $\{i, k\}$ is an antichain in $H(w)$ whenever one of the following conditions holds.

- (1) The heap $H(w)$ contains two elements j, j' labeled by y such that $j \prec i \prec j'$ and $k \in (j, j')$.
- (2) The heap $H(w)$ contains an element j labeled by y and the sets $\{i, j\}$ and $\{j, k\}$ are antichains in $H(w)$.

Proof. We prove the lemma by contradiction. Suppose i and k are comparable and assume, without loss of generality, that $i \preceq k$ in $H(w)$. Then by Remark 2.13, there is a chain of coverings $i = i_1 \prec i_2 \prec \cdots \prec i_l = k$ in $H(w)$ whose elements' labels form a walk on G from x to z . The walk must pass through y , so $e(i_p) = y$ for some $1 \leq p \leq l$. Elements sharing a label are comparable in a heap, so i_p is comparable to j and j' under Condition (1) and is comparable to j under Condition (2).

Suppose Condition (1) holds. Then the interval (j, j') contains no element with label y by Lemma 5.2, so we have either $i_p \preceq j$ or $j' \preceq i_p$ in $H(w)$. But in these cases we would have $i \preceq i_p \preceq j$ or $j' \preceq i_p \preceq k$, respectively, contradicting the fact that k and i lie in (j, j') .

Under Condition (2), we have either $i_p \prec j$ or $j \prec i_p$, which would imply that $i \preceq i_p \preceq j$ or $j \preceq i_p \preceq k$ and violate the assumption that $\{i, j\}$ and $\{j, k\}$ are antichains. $\square$

Lemma 5.3 will be used in tandem with Lemma 5.2 in subsection 5.2. It also helps establish our next proposition, Proposition 5.7, which concerns the following notions:

DEFINITION 5.4. We define a special quadruple in S to be an ordered tuple (a, b, c, d) of four distinct elements in S with the following properties.

- (1) We have $m(a, c) = m(a, d) = m(b, d) = 2$, $m(a, b) \geq 3$, $m(b, c) \geq 3$ and $m(c, d) = 3$. In particular, the generators a, b, c, d induce a subgraph of the form $a-b-c-d$ in G if we ignore edge weights.
- (2) The vertices a and c have degrees 1 and 2 in G , respectively. In other words, the vertex b is the only neighbor of a , and the vertices b, d are the only neighbors of c .

DEFINITION 5.5. Let (a, b, c, d) be a special quadruple in S . We call an antichain $\{i, j\}$ of size 2 in $H(w)$ a trapped antichain relative to (a, b, c, d) if the following conditions hold.

- (1) Neither i nor j is maximal or minimal in $H(w)$.
- (2) The label of i is a and the label of j is c .

EXAMPLE 5.6. In each Coxeter system (W, S) of type $\tilde{C}_{n-1}$ where $n \geq 5$, the tuple $(1, 2, 3, 4)$ is a special quadruple. In the heap of the FC element $w = 2413524 \in W$, the elements with labels 1 and 3 form a trapped antichain relative to $(1, 2, 3, 4)$. Note that $n(w) = 3$, hence $\mathbf{a}(w) = n(w) > 2$ by Proposition 2.14. The next proposition shows that the fact that $\mathbf{a}(w) \neq 2$ is to be expected.

PROPOSITION 5.7. Let (W, S) be a nontrivially $\mathbf{a}(2)$ -finite Coxeter system, let $w \in W_2$, and suppose $|\mathcal{L}(w)| = |\mathcal{R}(w)| = 2$. Then the heap $H(w)$ cannot contain any trapped antichain relative to a special quadruple in S .

Proof. Suppose that $H(w)$ contains a trapped antichain $A = \{i, j\}$ relative to a special quadruple (a, b, c, d) in S , with $e(i) = a$ and $e(j) = c$. We will derive a contradiction. To start, note that A is a maximal antichain in $H(w)$ by Corollary 2.15. By Lemma 2.16 and Definition 4.1.(2), it follows that w can be written as a product $w = x * y$ such that $H(x)$ and $H(y)$ coincide with the ideal $\mathcal{I}_A$ and filter $\mathcal{F}_A$ when

viewed naturally as subsets of $H(w)$. Lemma 3.18 then implies that $\mathcal{L}(x) = \mathcal{L}(w)$, so x has two left descents. On the other hand, Condition (1) of Definition 5.5 implies that $\mathcal{I}_A$ properly contains A , so $x \neq ac$ and $l(x) > 2$. It follows that $x \notin \mathcal{S}(W)$, because no stub in $\mathcal{S}(W)$ with length at least 3 has two left descents by Lemma 3.9.(1).

Since i is not minimal, it must cover an element $g \in H(w)$. The label of g must be a neighbor of $e(i) = a$, which has to be b by Condition (2) of Definition 5.4. Since b is also adjacent to c in G , the element g is comparable to j as well. If $j \preceq g$ then $j \preceq g \prec i$ and A would not be antichain, so $g \preceq j$. It follows that neither the set $H(x) \setminus \{i\}$ nor the set $H(x) \setminus \{j\}$ contains a maximal element labeled by b ; therefore x admits no right lower star operation with respect to $\{a, b\}$ or $\{c, b\}$ by Proposition 2.21. On the other hand, since x is not a stub it must admit a right lower star operation with respect to some pair $\{s, t\}$ of noncommuting generators, and one of s, t has to come from the set $\mathcal{R}(x) = \{a, c\}$. Condition (2) of Definition 5.4 now forces $\{s, t\} = \{c, d\}$. It follows that j covers an element $h \in H(x) \subseteq H(w)$ labeled by d . A similar argument shows that $H(w)$ contains an element h' that lies in $\mathcal{F}_A$, covers j , and has label d .

Since $m(c, d) = 3$ by Definition 5.4, the chain $h \preceq j \preceq h'$ cannot be convex by Proposition 2.11, so Lemma 5.2 implies that h and h' are connected by a chain of coverings containing an element k labeled by some neighbor e of d distinct from c . The vertices a, b, c, d, e induce a subgraph of G of the form $a-b-c-d-e$, so we may now apply Lemma 5.3 as follows: first, since c, e lie on different sides of d , Part (1) of the lemma implies that $\{j, k\}$ forms an antichain; next, since a, e lie on different sides of c and $\{i, j\}, \{j, k\}$ are both antichains, Part (2) of the lemma implies that $\{i, k\}$ forms an antichain. It follows that $\{i, j, k\}$ is an antichain in $H(w)$. Proposition 2.14 then implies that $\mathbf{a}(w) \geq n(w) \geq 3$, contradicting the assumption that $\mathbf{a}(w) = 2$. $\square$

5.2. PROOF OF THEOREM 4.17. Let (W, S) be a nontrivially $\mathbf{a}(2)$ -finite Coxeter system, and recall that each part of Theorem 4.17 claims that if (W, S) has a certain type X , then a representative 0-cell $I(x, y) := I(x, y) \subseteq W$ where $x, y \in \mathcal{S}(W)$ equals a certain set. Denoting this set by $K(x, y)$, we first note that $K(x, y) \subseteq I(x, y)$ regardless of what X is:

PROPOSITION 5.8. *We have $K(x, y) \subseteq I(x, y)$ for every Coxeter system (W, S) in Theorem 4.17.*

Proof. Let $w \in K(x, y)$. Then w has reduced factorizations $w = x \cdot w'$ and $w = w'' \cdot y^{-1}$ by inspection, so we can conclude that $w \in I(x, y)$ by Theorem 3.17 once we can show $\mathbf{a}(w) = 2$. If w is a product st of two commuting generators $s, t \in S$, then $\mathbf{a}(w) = 2$ by Proposition 2.2.(2). Otherwise, we observe that w can be obtained from such a product $w = 24$ (i.e. the product $w = s_2 s_4$) via a sequence of right upper star operations; therefore $\mathbf{a}(w) = \mathbf{a}(24) = 2$ by Corollary 2.20. Specifically, the element $w = 243524$ in Part (7) can be built from 24 by applying right upper star operations with respect to $\{4, 5\}, \{2, 3\}, \{2, 3\}$ and $\{4, 5\}$ successively, and a suitable sequence of star operations taking 24 to w is straightforward to find in all other cases. $\square$

To prove Theorem 4.17 it remains to prove that $I(x, y) \subseteq K(x, y)$ for each group W in the theorem. We will do so by treating one Coxeter type at a time below, and the following notions will be used frequently. For every element $w \in \text{FC}(W)$ and any generator $p \in S$, we denote the number of elements labeled by p in the heap by $o_w(p)$, where we drop the subscript w if there is no danger of confusion. We define a p -interval in $H(w)$ to be an interval of the form $[j, j'] \subseteq H(w)$ where j, j' are consecutive elements labeled by p in the sense that $j \preceq j', j \neq j'$, and j, j' are the only elements with label p in the interval $[j, j']$. Finally, when we wish to emphasize the ambient group W that a 0-cell $I(x, y)$ lies in, we will write $I_W(x, y)$ for $I(x, y)$.

The following facts will also be used frequently without further comment. First, if x, y are short stubs, then every element $w \in I(x, y)$ has left descent set $\mathcal{L}(w) = \mathcal{L}(x) = \text{Supp}(x)$ and right descent set $\mathcal{R}(w) = \mathcal{R}(y^{-1}) = \mathcal{R}(y) = \text{Supp}(y)$ by Lemma 3.9.(1). Second, recall from Remark 2.12 that the left and right descents of an FC elements are precisely the labels of the minimal and maximal elements of its heap, respectively. Third, consider the situation where an $\mathbf{a}(2)$ -finite Coxeter system (W, S) is isomorphic to a parabolic subgroup W'_J of another $\mathbf{a}(2)$ -finite Coxeter system (W', S') for some subset J of S' via a group isomorphism $\phi : W \rightarrow W'_J$ induced by a bijection $f : S \rightarrow J$ with the property that $m(f(s), f(t)) = m(s, t)$ for all $s, t \in S$. (By “induced” we mean the restriction of ϕ to S coincides with f .) Then the next lemma holds for any two stubs $x, y \in \mathcal{S}(W)$, where $f(Z) = \{f(z) : z \in Z\}$ for every subset Z of a Coxeter group and any map f whose domain contains Z . The lemma allows us to deduce 0-cells of $\mathbf{a}$ -value 2 in W from such cells in W' by “parabolic restriction.”

LEMMA 5.9. *In the above setting, let $x' = \phi(x), y' = \phi(y)$, denote the 0-cell $I_W(x, y)$ in W by I_1 , denote the 0-cell $I_{W'_J}(x', y')$ in W'_J by I_2 , and denote the 0-cell $I_{W'}(x', y')$ in W' by I_3 . Then we have $I_2 = \phi(I_1)$ and $I_2 = \{z \in I_3 : \text{Supp}(z) \subseteq J\}$. In particular, we have*

$$I_W(x, y) = \{\phi^{-1}(z) : z \in I_{W'}(x', y'), \text{Supp}(z) \subseteq J\}.$$

Proof. We first sketch why $I_2 = \phi(I_1)$: the fact that ϕ is induced by the bijection $f : S \rightarrow J$ implies that ϕ is in fact an isomorphism of Coxeter systems, so that ϕ in turn induces an isomorphism from the Hecke algebra of (W, S) to the Hecke algebra of (W'_J, J) that sends the Kazhdan–Lusztig basis element C_w to $C_{\phi(w)}$ for all $w \in W$. It follows that $I_2 = \phi(I_1)$.

Next, let $\mathbf{a}_{S'}$ and $\mathbf{a}_J$ denote the $\mathbf{a}$ -functions associated to the Coxeter system (W', S') and (W'_J, J) , respectively. By Proposition 2.2.(7), the system (W'_J, J) is $\mathbf{a}(2)$ -finite since (W', S') is, so Theorem 3.17 and its analog for left cells imply that

$$I_2 = \{z \in W'_J : \mathbf{a}_J(z) = 2, x \leq^R z, y \leq^L z\}$$

and

$$I_3 = \{z \in W' : \mathbf{a}_{S'}(z) = 2, x \leq^R z, y \leq^L z\}.$$

An element $z \in W'$ is in the parabolic subgroup $z \in W'_J$ if and only if $\text{Supp}(w) \subseteq J$, and for such an element we have $\mathbf{a}_{S'}(z) = \mathbf{a}_J(z)$ by Proposition 2.2.(8). It follows that $I_2 = \{z \in I_3 : \text{Supp}(z) \subseteq J\}$. $\square$

5.2.1. *Type $\tilde{C}_{n-1}$.* Suppose $X = \tilde{C}_{n-1}$ ($n \geq 5$). We complete the proof of Theorem 4.17.(4) by proving the following:

PROPOSITION 5.10. *If $X = \tilde{C}_{n-1}$ ($n \geq 5$), then we have*

$$I(24, 24) \subseteq \{24, 2124, 2 \cdot z, 212 \cdot z\}$$

where $z = 45 \dots (n-1)n(n-1) \dots 54$.

To prove the proposition, let

$$z_p = p(p-1) \dots 212 \dots (p-1)p,$$

$$z'_p = p(p+1) \dots (n-1)n(n-1) \dots (p+1)p$$

for all $1 < p < n$. For example, the elements 212 and z from the proposition equal z_2 and z'_4 , respectively. We will take advantage of the following result of Ernst:

LEMMA 5.11 ([11]). *Let $n \geq 4$ and let $w \in \text{FC}(\tilde{C}_{n-1})$. Let $1 < p < n$ and suppose $[j, j']$ is a p -interval in $H(w)$ for two elements $j \preceq j'$ in $H(w)$.*

- (1) If the interval $[j, j']$ contains no element labeled by $(p+1)$, then either $\mathbf{a}(w) = 1$ or w has a reduced factorization $w = u \cdot z_p \cdot u'$ where $\text{Supp}(u) \cup \text{Supp}(u') \subseteq \{p+1, p+2, \dots, n\}$.
- (2) If the interval $[j, j']$ contains no element labeled by $(p-1)$, then either $\mathbf{a}(w) = 1$ or w has a reduced factorization $w = u \cdot z'_p \cdot u'$ where $\text{Supp}(u) \cup \text{Supp}(u') \subseteq \{1, 2, \dots, p-1\}$.

Proof. Part (1) is a restatement of Lemma 3.6 of [11]. (We note that each element “of Type I” in [11] has a unique reduced word by [11, §3.1] and thus has $\mathbf{a}$ -value 1 by Proposition 2.4.(2).) Part (2) follows from Part (1) by the symmetry in the Coxeter diagram of $\tilde{C}_{n-1}$. $\square$

Proof of Proposition 5.10. Let $w \in I(24, 24)$. Then 2 labels both a maximal and a minimal element in $H(w)$, as does 4. It follows that $o(2), o(4) \geq 1$, and that if $3 \in \text{Supp}(w)$ then every element labeled by 3 in $H(w)$ has to lie in a 2-interval. Note that every 2-interval in $H(w)$ must contain an element labeled by 1: otherwise, Lemma 5.11.(2) implies that $w = u \cdot z'_2 \cdot u'$ where 4 labels no maximal element in z'_2 and does not appear in u or u' , so 4 labels no maximal element in $H(w)$, a contradiction. On the other hand, since $(1, 2, 3, 4)$ is a special quadruple in S , no 2-interval in $H(w)$ can contain both an element labeled by 1 and an element labeled by 3 by Proposition 5.7. It follows that $3 \notin \text{Supp}(w)$. For $i \in \{2, 4\}$, let a_i and b_i stand for the minimal and maximal element in $H(w)$ labeled by i , respectively. Since $3 \notin \text{Supp}(w)$, Remark 2.13 implies that no two elements in $H(w)$ labeled by two generators in S that lie on different sides of 3 are comparable. Since every element in $H(w)$ must be comparable to a maximal and to a minimal element, it follows that $H(w)$ equals the disjoint union of the two convex intervals $I_2 := [a_2, b_2]$ and $I_4 := [a_4, b_4]$ as sets, where each element in I_2 and I_4 carries a label smaller and larger than 3, respectively. It follows that $w = x \cdot y$ for FC elements x, y such that $H(x) = I_1$ and $H(y) = I_2$. Applying Lemma 5.11.(1) with $p = 2$ and Lemma 5.11.(2) with $p = 4$ reveals that $x \in \{2, 212\}$ and $y \in \{4, z'_4 = z\}$, respectively, so we have $w \in \{24, 2124, 2z, 212z\}$. We conclude that $I(24, 24) \subseteq \{24, 2124, 2z, 212z\}$. $\square$

5.2.2. *Type B_n .* We complete the proof of Theorem 4.17.(2)–(3) by deducing the following proposition from Proposition 5.10.

PROPOSITION 5.12. *If $X = B_n$ ($n \geq 3$), then $I(13, 13) \subseteq \{13\}$ if $n = 3$ and $I(24, 24) \subseteq \{24, 2124\}$ if $n > 3$.*

Proof. Let $k = n + 2 \geq 5$, consider the Coxeter system of type (W', S') of type $\tilde{C}_{k-1}$, and let $J = \{1, 2, \dots, n\} \subseteq S'$. The group B_n is naturally isomorphic to parabolic subgroup W'_J of W' via the isomorphism $\phi : W \rightarrow W'_J$ with $\phi(i) = i$ for each generator $i \in S$. Thus, Lemma 5.9 implies that if $n > 3$ then $I(24, 24) = I_W(24, 24) \subseteq \{\phi^{-1}(w') : w' \in I_{W'}(24, 24), \text{Supp}(w') \subseteq J\}$, where the last set equals $\{\phi^{-1}(24), \phi^{-1}(2124)\} = \{24, 2124\}$. It follows that $I(24, 24) \subseteq \{24, 2124\}$. Similarly, if $n = 3$, then using the methods from Remark 4.13 we can see that $I_{W'}(13, 13) = \{13, 134543, 132413, 13245413\}$, whence Lemma 5.9 implies that $I(13, 13) \subseteq \{13\}$ because $4 \notin J$. $\square$

5.2.3. *Type A_n .* We complete the proof of Theorem 4.17.(1) by deducing the following proposition from Proposition 5.12.

PROPOSITION 5.13. *If $X = A_n$ ($n \geq 3$), then $I(13, 13) \subseteq \{13\}$.*

Proof. We use parabolic restriction as in the proof of Proposition 5.12. More precisely, let (W, S) and (W', S') be the Coxeter systems of type A_n and B_{n+1} , respectively, let

$J = \{2, 3, \dots, n+1\}$, and consider the isomorphism $\phi : W \rightarrow W'_J$ with $\phi(i) = i+1$ for all $i \in S$. We have $I_{W'}(24, 24) \subseteq \{24, 2124\}$ by Proposition 5.12 and $I(13, 13) = I_W(13, 13) \subseteq \{\phi^{-1}(z) : z \in I_{W'}(24, 24), \text{Supp}(z) \in J\}$ by Lemma 5.9; therefore $I(13, 13) \subseteq \{\phi^{-1}(24)\} = \{13\}$. $\square$

5.2.4. *Type $E_{q,r}$.* We complete the proof of Theorem 4.17.(5) by deducing the following result from Theorem 4.17.(1):

PROPOSITION 5.14. *If $X = E_{q,r}$ where $r \geq q \geq 1$, then $I(x, x) \subseteq \{x\}$ for all $x \in \{(-1)v, 1v, (-1)1\}$.*

Proof. Let $x \in \{(-1)v, 1v, (-1)1\}$. To prove the proposition we may assume that $r \geq q > 1$ in the Coxeter system W of type $E_{q,r}$: any system (W', S') of type $E_{1,r}$ lies in W as a parabolic subgroup, and Lemma 5.9 implies that if $I_W(x, x) \subseteq \{x\}$ then $I_{W'}(x, x) \subseteq \{x\}$. Under the assumption that $r \geq q > 1$, it further suffices to show that $I((-1)1, (-1)1) \subseteq \{(-1)1\}$: since $\mathcal{S}(W)$ contains a single $\sim$ -class by Proposition 3.30.(1), if $I((-1)1, (-1)1) \subseteq \{(-1)1\}$ then all 0-cells have size 1 in W_2 by Proposition 4.12, which in turn implies that the 0-cell $I(y, y)$ has to contain only the element y that is obviously in it for all short stubs $y \in \mathcal{S}(W)$.

Let $x = (-1)1$ and let $w \in I(x, x)$. Suppose $v \in \text{Supp}(w)$, so that some element i in the heap $H(w)$ has label v . Since $\mathcal{R}(w) = \mathcal{L}(w) = \{-1, 1\}$, the element i is neither minimal nor maximal in $H(w)$, so it must cover some element j and be covered by some element j' in $H(w)$ by Lemma 5.1. The labels of j and j' must both be 0, the only neighbor of v in S . Since $m(v, 0) = 3$, the chain $j \prec i \prec j'$ cannot be convex by Proposition 2.11, so Lemma 5.2 implies that j and j' are connected by a chain in $H(w)$ that contains an element k labeled by either -1 or 1 . Lemma 5.3 then implies that the set $\{i, k\}$ forms a trapped antichain relative to either the special quadruple $(v, 0, -1, -2)$ or the special quadruple $(v, 0, 1, 2)$. This contradicts Proposition 5.7, so $v \notin \text{Supp}(w)$. But then w lies in the parabolic subgroup of W of type A_{q+r+1} generated by the set $\{-q, \dots, -1, 0, 1, \dots, r\}$. It follows from Theorem 4.17.(1), Lemma 5.9 and Proposition 4.12 that $I(x, x)$ is the singleton $\{x\}$. The proof is complete. $\square$

5.2.5. *Type F_n .* We complete the proof of Theorem 4.17.(6)–(7) by deducing the following proposition from Proposition 5.12.

PROPOSITION 5.15. *If $X = F_n$ ($n \geq 3$), then $I(24, 24) \subseteq \{24\}$ if $n = 4$ and $I(24, 24) \subseteq \{24, 243524\}$ if $n > 4$.*

Proof. Consider the parabolic group W_J of W generated by the set $J = \{2, 3, \dots, n\}$. It is a Coxeter group of type B_{n-1} , so $I_{W_J}(24, 24) = \{24\}$ if $n-1 = 3$ and $I_{W_J}(35, 35) = \{35, 3235\}$ if $n-1 > 3$ by Theorem 4.17.(2)–(3). In the latter case, computing the 0-cell $I_{W_J}(24, 24)$ from $I_{W_J}(35, 35)$ using the methods of Remark 4.13 gives $I_{W_J}(24, 24) = \{24, 243524\}$.

Let $w \in I(24, 24)$. We claim that $1 \notin \text{Supp}(w)$, so that w lies in the parabolic subgroup of type B_{n-1} generated by the set $J = \{2, 3, \dots, n\}$. It follows that $I(24, 24) = I_W(24, 24) \subseteq I_{W_J}(24, 24)$, so $I(24, 24) \subseteq \{24\}$ if $n = 4$ and $I(24, 24) \subseteq \{24, 243524\}$ if $n > 4$ by the last paragraph. To prove the claim, suppose that $1 \in \text{Supp}(w)$, so that some element i in the heap $H(w)$ has label 1. This leads to a contradiction to Proposition 5.7 in the same way the element i does in the proof of Proposition 5.14: Lemmas 5.1, 5.2 and 5.3 force i to be part of an antichain $\{i, k\}$ contained in a 2-interval $[j, j'] \subseteq H(w)$ where $j \prec i \prec j'$ and $e(k) = 3$, and this antichain is a trapped antichain relative to the special quadruple $(1, 2, 3, 4)$. It follows that $1 \notin \text{Supp}(w)$, as claimed. $\square$

5.2.6. *Type H_n .* We complete the proof of Theorem 4.17.(8)–(9) by deducing the following proposition from Proposition 5.12.

PROPOSITION 5.16. *If $X = H_n$ ($n \geq 3$), then $I(13, 13) \subseteq \{13\}$ if $n = 3$ and $I(24, 24) \subseteq \{24, 2124\}$ if $n > 3$.*

Proof. If we can show $I(24, 24) = \{24, 2124\}$ whenever $n > 3$, then we can use Remark 4.13 to obtain $I(13, 13) = \{13, 132413\}$ when $n = 4$. Viewing H_3 as the parabolic subgroup of H_4 generated by $\{1, 2, 3\}$ naturally, we can then use Lemma 5.9 to deduce that $I(13, 13) = \{13\}$ in H_3 . Thus, it suffices to prove that $I(24, 24) \subseteq \{24, 2124\}$ in H_n whenever $n > 3$.

Assume $n > 3$ and let $w \in I(24, 24)$. Let $\underline{w}$ be a reduced word of w . We claim that the heap $H(w) = H(\underline{w})$ does not contain a convex chain of the form $i \prec j \prec i' \prec j'$ where $e(i) = e(i') = 1, e(j) = e(j') = 2$ or where $e(i) = e(i') = 2, e(j) = e(j') = 1$. By Proposition 2.11, the claim implies that $\underline{w}$ is the reduced word of an element w' in the Coxeter group of type B_n . Moreover, we have $\mathbf{a}(w') = n(w') = n(w) = 2$ by Proposition 3.11.(3), whence Theorem 3.17.(3) implies that $w' \in I_{B_n}(24, 24) = \{24, 2124\}$ by reduced word considerations. It follows that $w \in \{24, 2124\}$ in H_n , so that $I(24, 24) \subseteq \{24, 2124\}$ in H_n , as desired.

It remains to prove the claim, which we do by contradiction. Suppose that $e(i) = e(i') = 1$ and $e(j) = e(j') = 2$ in the chain C . Since $1 \notin \mathcal{L}(w)$, the element i must cover an element h with label 2 in $H(w)$ by Lemma 5.1. Let $C = \{h, i, j, i', j'\}$ and let $k \in [h, j']$. Applying Lemma 5.2 to the chains $h \prec i \prec j$ and $j \prec i' \prec j'$, we note that if $e(k) \in \{1, 2\}$ then $k \in C$. Also observe that $e(k) \neq 3$, for otherwise $\{i, k\}$ is a trapped antichain in $H(w)$ with relative to the special quadruple $(1, 2, 3, 4)$, contradicting Proposition 5.7. Finally, we cannot have $e(k) > 3$ either, because otherwise $e(k)$ and 2 lie on different sides of the vertex 3 in G , so h must be connected to k by a chain in $[h, j']$ passing through an element labeled by 3, contradicting the last observation. It follows that $[h, j'] = C$ as sets. But then the chain $h \prec i \prec j \prec i' \prec j'$ is convex, which contradicts the fact that $w \in \text{FC}(H_n)$ by Proposition 2.11. A similar contradiction can be derived if $e(i) = e(i') = 2$ and $e(j) = e(j') = 1$, and the proof is complete. $\square$

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