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HARMONIC SYNTHESIS FOR SUBGROUPS (*)

by Carl HERZ

0. Introduction.

Let G be a locally compact group and $L_p(G)$ the complex Lebesgue space with respect to the left invariant Haar measure. The bounded linear operators on $L_p(G)$ which commute with right translation form a Banach algebra $CONV_p(G)$ in the operator norm $|||\ |||_p$. We denote by $PM_p(G)$ the smallest ultraweakly closed subspace on $CONV_p(G)$ containing the left translations. We have

$$PM_p(G) = CONV_p(G)$$

whenever $p = 2$ or G is amenable ; in any case it will be more convenient to work with PM_p.

Let H be a closed subgroup of G. It is not too hard to prove

THEOREM A. – *There is a canonical isometric inclusion of Banach algebra* $PM_p(H) \to PM_p(G)$.

There is an obvious notion of "support" for a convolution operator. In terms of this we have

THEOREM B. – *If* H *is amenable or normal then the image of the canonical inclusion* $PM_p(H) \to PM_p(G)$ *consists of all elements of* $PM_p(G)$ *with support in* H.

The conclusion of Theorem B amounts to the statement that H is a set of spectral synthesis in G.

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Theorems A and B are immediate corollaries of statements about certain spaces $A_p(G)$ of continuous functions vanishing at ∞ on G whose conjugate Banach spaces may be identified with $PM_p(G)$. The first of these is

THEOREM 1. – *Restriction of functions gives an extremal epimorphism of Banach spaces* $A_p(G) \to A_p(H)$.

There are really two assertions in Theorem 1, and it is worthwhile distinguishing them.

THEOREM 1a. – *Restriction of functions gives a morphism (linear contraction) of Banach spaces* $A_p(G) \to A_p(H)$.

THEOREM 1b. – *Given* $h \in A_p(H)$ *and* $\varepsilon > 0$ *there exists* $g \in A_p(G)$ *with* $\|g\| < \|h\| + \varepsilon$ *such that g restricted to* H *is h. (In case* $p = 2$ *one may take* $\|g\| = \|h\|$).

The point is that Theorem 1b is purely existential ; in general, given h we know very little about g. In particular we are led to formulate

CONDITION (C) *on* G, H, *and p : In the formulation of Theorem 1b, if h has compact support then g may be chosen to have compact support.*

We have previously [9] outlined the proof of Theorem 1. We repeat the proof here by way of getting to

PROPOSITION 1. – *Condition* (C) *holds in each of the following situations*

i) H *is an amenable group* ;

ii) H *is a normal subgroup or, more generally, the normalizer of* H *is open in* G.

iii) *There exists a closed subgroup* N *of* G *such that*

a) HN *is open in* G,

b) H *is contained in the normaliser of* N,

c) $H \cap N = \{1\}$,

d) N *contains an* H*-invariant compact subset of positive measure with respect to the Haar measure of* N.

The class of amenable groups includes the compact groups and the solvable groups. Thus situation i) covers many cases. In situation iii) there exists a fixed $\varphi \in A_p(N)$ with $\|\varphi\| = 1 = \varphi(1)$ such that $h \to g$ defined by

$$g(x) = 0 \quad \text{for} \quad x \notin HN$$

$$g(yn) = h(y)\varphi(n) \quad \text{for} \quad y \in H\ ,\ n \in N\ ,$$

gives a Banach space retraction $A_p(H) \to A_p(G)$. Here we have much more than is required for condition (C) ; nevertheless it is not clear that hypothesis d) can be eliminated. Proposition 1 can be extended somewhat at the expense of complicating the statement, but here are three examples in which condition (C) remains in doubt.

Example 1. – $H = SL_2(\mathbf{Z})$, $G = SL_2(\mathbf{R})$

Example 2. – $H = SL_2(\mathbf{Z})$, $G = HN$ with $N = \mathbf{R}^2$ and the usual action of $SL_2(\mathbf{Z})$ on $\mathbf{R}^2$.

Example 3. – $H = SL_2(\mathbf{R})$, $G = HN$ with $N = \mathbf{R}^2$ and the usual action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2$.

We say that a closed subset $E \subset G$ is a set of *spectral synthesis* for $A_p(G)$ if for each $f \in A_p(G)$ which vanishes on E and each $\varepsilon > 0$ there exists $\varphi \in A_p(G)$ such that supp φ is a compact disjoint from E and $\|f - \varphi\| < \varepsilon$. We say that E is a set of *local* spectral synthesis if the conditions in the definition of spectral synthesis obtain under the additional assumption that f has compact support.

The spaces $A_p(G)$ are actually Banach algebras under pointwise multiplication of functions ; the Gelfand spectrum of these algebras may be identified with G. Thus "spectral synthesis" has its ordinary meaning for regular, commutative, semi-simple Banach algebras ; but the algebra structure plays no role in the proof of the next.

THEOREM 2. – *A closed subgroup* H *is a set of local spectral synthesis for* $A_p(G)$.

To say that the group G is amenable is equivalent to saying that the algebra $A_p(G)$ has approximate identities of compact support. In this case it is immediate that local spectral synthesis implies spectral synthesis. When G is not amenable this implication is doubtful. In fact we have

PROPOSITION 2. – *The closed subgroup* H *is a set of spectral synthesis for* $A_p(G)$ *iff condition* (C) *holds.*

Theorem B is simply the dual version of Proposition 2 combined with Proposition 1 i) and ii).

We had previously [10] proved Theorem 2 under the strong extra hypothesis that H was normal. At the same time as the present work was done Dunkl and Ramirez [4] proved Theorem B for G compact eliminating our normality hypothesis.

In case $p = 2$ it is known [5] that for $h \in A_2(H)$ one can choose u, $v \in L_2(H)$ with $h = v * \check{u}$ and $\|h\| = \|u\|_2 \, \|v\|_2$. If h is positive-definite then one can take $v = \overline{u}$; in particular $h(1) = \|h\|$ which is characteristic for positive-definite elements of A_2. Thus the proof of Theorem 1 gives

ADDENDUM TO THEOREM 1. – *If* h *is a positive-definite function in* $A_2(H)$ *then there exists a positive-definite function* $g \in A_2(G)$ *whose restriction to* H *is* h.

The above statement is known to be false in general, see [5 ; p. 204], if the hypothesis $h \in A_2(H)$ is dropped, even if H is a commutative normal subgroup.

The restriction to $p = 2$ yields no simplification in the proofs of Theorems 1 and 2. Indeed, for Theorem 2 there is no advantage in imposing any additional hypothesis.

After giving the basic definitions in § 1 we quickly dispose of Theorem 2 in § 2. Next, in § 3, we prove that $A_p(G)$ is a regular, tauberian Banach algebra of functions on G, a statement which (by Theorem 3 in § 3) includes the fact that G is the Gelfand space of maximal ideals of $A_p(G)$. This is a known result, – Eymard's proof for the case $p = 2$ [5, Th. 3.34], carries over, – but the procedure is simpler and, we think, pinpoints what has to be proved.

The discussion in § 3 of regular, tauberian algebras of functions on locally compact Hausdorff spaces treats spectral synthesis in the appropriate abstract context. The main concern is the passage from local to global spectral synthesis, and Proposition 2 is proved in this context. The passage can always be carried out in algebras with bounded approximate identities, but this remains true if it is only assumed that the quotient algebra of restrictions to the set in question

has bounded approximate identities. This is the abstract version of Proposition 1 i). (If one assumes that G is amenable the Proposition 1 is banal ; the point is that we only suppose the amenability of H). In passing we prove an abstract result, Theorem 4, which shows that for any amenable group G, given a compact subset K and $\varepsilon > 0$ there exists $k \in A_p(G)$ with $k = 1$ on K and $\| k \| < 1 + \varepsilon$. We believe([1]) that this is the first proof of that fact, even for G commutative and $p = 2$, which does not use structure theory.

The rest of the paper is mainly concerned with the proof of Proposition 1. In § 4 we deal with reduction steps ; we get open subgroups and quotients by compact normal subgroups out of the way, and this allows one to assume that G satisfies the second axiom of countability. Case iii) of Proposition 1 is quite easy and we dispose of it in § 5. We give a new proof of Theorem 1 in § 6 ; this is somewhat complicated but the difficulties seem unavoidable. The proof of Theorem 1 yields a constructive proof of case i) of Proposition 1. We do not repeat the technicalities in § 7 where case ii) is handled.

The basic facts about convolution operators are given in § 8. In particular we show that a pseudomeasure, i.e. a linear functional on A_p, corresponds to a convolution operator on L_p in such a way that the support, defined by duality with the algebra A_p, is equal to the support of the convolution operator. Once one has this Theorems A and B are immediate corollaries of what has been proved previously.

Amenable groups are characterized by the property that their A_p algebras have approximate identities of bound 1. This fact is used throughout, but since we cannot find a convenient reference we give an outline in § 9. The full proof that $A_p(G)$ has bounded approximate identities implies G amenable is very long. What we need here is the converse, and the fact that Reiter's property (P_p) implies that A_p has approximate identities of bound 1 is completely proved here by a short argument.

([1]) *Added in proof.* At the time of writing, [15] had not yet appeared ; it is first, and the two proofs are substantially the same.

1. Preliminaries.

Let G be a locally compact group. For $1 \leqslant p < \infty$ we write $L_p(G)$ for the Lebesgue space with respect to the left-invariant Haar measure on G. We denote by $\mathcal{K}(G)$ the vector space of continuous complex-valued functions of compact support on G. One may view $L_p(G)$ as the completion of $\mathcal{K}(G)$ for the norm

$$\|u\|_p = \left\{ \int_G |u(x)|^p \, dx \right\}^{1/p}.$$

The completion of $\mathcal{K}(G)$ for the supremum norm $\| \ \|_\infty$, is denoted $C_0(G)$. Left-translation by elements of G gives a Banach space automorphism of each $L_p(G)$; for $\sigma \in G$, $\lambda_p(\sigma) : L_p(G) \to L_p(G)$ is defined by $\lambda_p(\sigma)\, u(x) = u(\sigma^{-1}x)$. There are also automorphisms given by right translations ; for $\sigma \in G$, $\rho_p(\sigma) : L_p(G) \to L_p(G)$ is defined by $\rho_p(\sigma)\, u(x) = u(x\sigma)\Delta^{1/p}(\sigma)$ where Δ designates the modular function of G. To simplify notation we shall suppress the index p when the context is obvious.

If we form the Banach space tensor product $L_p(G) \otimes L_{p'}(G)$ where $1/p + 1/p' = 1$, then for $1 < p < \infty$ we have a Banach space morphism $L_p(G) \otimes L_{p'}(G) \to C_0(G)$ defined by $u \otimes v \mapsto f$ where $f(\sigma) = \langle \lambda(\sigma)u \, , \, v \rangle$. The coimage of this morphism is denoted by $P : L_p(G) \otimes L_{p'}(G) \to A_p(G)$; thus $A_p(G)$ consists of certain continuous functions vanishing at ∞ on G with the norm being the quotient norm from the tensor product. If $k, l \in \mathcal{K}(G)$ their convolution is defined by $k * l(\sigma) = \int k(x)\, l(x^{-1}\sigma)\, dx$. Thus we may write

$$P(u \otimes v) = v * \check{u} \qquad \text{where} \qquad \check{u}(x) = u(x^{-1}).$$

A tensor $t \in L_p(G) \otimes L_{p'}(G)$ can be expressed, non-uniquely, as a sum $t = \sum_{n=1}^{\infty} u_n \otimes v_n$ where $\{u_n\} \subset L_p(G)$,

$$\{u_n\} \subset L_{p'}(G), \text{ and } \Sigma \, \|u_n\|_p \, \|v_n\|_{p'} < \infty \ ;$$

the tensor product norm is $\|t\| = \inf \Sigma \, \|u_n\|_p \, \|v_n\|_{p'}$ where the infimum is taken over all possible representations. For the sake of concreteness we remark that the tensor t is determined by the corresponding locally summable function, i.e. for $w \in \mathcal{K}(G \times G)$ the

quantity $\int w\, d\mu_t = \int_{G\times G} w(x,y)\, t(x,y)\, dx\, dy$ can be well-defined and the corresponding Radon measure μ_t is zero iff t is the zero-tensor. The support of t is defined by $\operatorname{supp} t = \operatorname{supp} \mu_t$.

It follows from the above that given $f \in A_p(G)$ and $\varepsilon > 0$ we can write f as an absolutely and uniformly convergent sum $f = \Sigma\, v_n * \check{u}_n$ where $\Sigma\, \|u_n\|_p\, \|v_n\|_{p'} < \|f\| + \varepsilon$, and we can take $\{u_n\}, \{v_n\} \subset \mathfrak{K}(G)$ if so desired. It is obvious that the elements of compact support are dense in $A_p(G)$ and that left and right translations give strongly continuous automorphisms of $A_p(G)$.

Without further ado the reader may proceed to the proof of Theorem 2. In the matter of harmonic synthesis complications arise in passing from local to global results. The space $A_2(G)$ is the Fourier Algebra which can be described conveniently by Fourier series when G is a compact group or by the Banach algebra isomorphism with the convolution algebra $L_1(\hat{G})$ given by the Fourier transform when G is commutative and $\hat{G}$ is the character group. Compact and commutative groups belong to the more general class of amenable groups. If G is amenable, then, as we shall indicate later, given $f \in A_p(G)$ of compact support and $\varepsilon > 0$ there exists $t \in L_p(G) \otimes L_{p'}(G)$ of compact support such that $\|t\| < \|f\| + \varepsilon$ and $Pt = f$. This fact alone suffices for the passage to global harmonic synthesis for amenable subgroups. In general one can say that when the group is amenable there are no real difficulties at infinity and otherwise there are.

A key idea in the study of A_p is to introduce the Lebesgue spaces of Banach-valued functions $L_p(G\,;\,B)$. If B is a Banach space and B' its dual then we get a morphism (linear contraction)

$$L_p(G\,;\,B) \otimes L_{p'}(G\,;\,B') \to L_p(G) \otimes L_{p'}(G)$$

such that $u\,\alpha \otimes v\,\beta \mapsto \langle \alpha, \beta \rangle\, u \otimes v$ where $u \in L_p(G)$, $v \in L_{p'}(G)$, $\alpha \in B$, $\beta \in B'$, precisely in the case that B is a p-space (see [11]). In this article we shall only need the case in which B is another L_p space ; there is then no difficulty in establishing the existence of the required morphism provided one works abstractly and doesn't get confused by irrelevant measure-theoretic issues which might appear to arise if one writes everything in terms of point functions. Now as long as B is a non-trivial p-space we can just as well define $A_p(G)$ by the extremal epimorphism $P : L_p(G\,;\,B) \otimes L_{p'}(G\,;\,B') \to A_p(G)$ given by $u\,\alpha \otimes v\,\beta \mapsto f$ where $f(\sigma) = \langle \lambda(\sigma)\, u, v \rangle \langle \alpha, \beta \rangle$.

As a particular instance of the above suppose that $B = L_p(G)$. If $g \in A_p(G)$ has the form $g = v * \check{u}$ and $\varphi \in A_p(G)$ has the form $\varphi = \beta * \check{\alpha}$ then if we define $U \in L_p(G\ ;\ B)$, $V \in L_{p'}(G\ ;\ B')$ by $U(x) = u(x)\ \lambda(x^{-1})\ \alpha$, $V(x) = v(x)\ \lambda(x^{-1})\ \beta$, i.e. as "function" on $G \times G$ we have $U(x\ ,\ \xi) = u(x)\ \alpha(x\xi)$, $V(x\ ,\ \xi) = v(x)\ \beta(x\xi)$ then for $f = P(U \otimes V)$ we get $f = g\varphi$. More generally, if $s \in L_p(G) \otimes L_{p'}(G)$ and $t \in B \otimes B'$ then for $w \in L_p(G\ ;\ B) \otimes L_{p'}(G\ ;\ B')$ defined by $w(x\ ,\ y) = s(x\ ,\ y)\ (\lambda_1(x^{-1}) \otimes \lambda_2(y^{-1})\ t)$, where λ_1 is left-translation on B and λ_2 is left-translation on B', we have $Pw = (Ps)\ (Pt)$. This proves that $A_p(G)$ is a Banach algebra ; it was given in [8, Th. 1] for the case $p = 2$, but no change is required for other values of p. The whole point is that contraction of tensors gives a morphism $L_p(G\ ;\ B) \otimes L_{p'}(G\ ;\ B') \to L_p(G) \otimes L_{p'}(G)$ so that we can extend P ; this may be false if B is not a p-space, e.g. if $p = 2$ and G is a non-trivial group we must have B a Hilbert space. Actually the proof gives a little more ; namely, if $s \in L_p(G) \circ L_{p'}(G)$ and $\varphi \in A_p(G)$ then for $t(x\ ,\ y) = \varphi(yx^{-1})\ s(x\ ,\ y)$ we have $t \in L_p(G) \otimes L_{p'}(G)$ with $\|t\| \leqslant \|\varphi\|\ \|s\|$.

2. Proof of Theorem 2.

Let H be a given closed subset of G. Write J_H for the subset of $A_p(G)$ consisting of the elements whose supports are compacts disjoint from H. Fix $f \in A_p(G)$ having compact support. We shall describe a procedure for estimating the distance from f to J_H.

For $0 < \varepsilon < \|f\|_\infty$ put $W_\varepsilon = \{x \in G : \|\rho(x)f - f\| \leqslant \varepsilon\}$; each W_ε is a compact neighborhood of the identity. Let V be an open set such that $1 \in V \subset W_\varepsilon$. Put $v = f$ on HV, $v = 0$ elsewhere. Suppose $u \in L_p^+(G)$, $\int u(x)\,dx = 1$ and supp $u \subset V$. Consider the function $\varphi = (f - u) * \check{u}$. Observe that $\varphi \in A_p(G)$ since $f - v \in L_{p'}(G)$ and $u \in L_p(G)$. Now $\varphi(\sigma) = \int (f - v)\ (\sigma x)\, u(x)\,dx$ which is zero whenever σ supp $u \subset HV$; it follows that supp φ is a compact disjoint from H. We have

$$f - \varphi = (f - f * \check{u}) + (v * \check{u})\,.$$

Now $\|f - f * \check{u}\| \leqslant \int \|f - \rho(x)f\|\, u(x)\, dx \leqslant \varepsilon$ since supp $u \subset W_\varepsilon$. Also $\|\nu * \check{u}\| \leqslant \|u\|_p \ \|\nu\|_{p'}$, but the conditions on u entail the fact that the greatest lower bound of the possible $\|u\|_p$ is exactly $|V|^{-1/p'}$. Thus we have

$$\operatorname{dist}(f, J_H) \leqslant \varepsilon + |V|^{-1/p'} \left\{ \int_{HV} |f(x)|^{p'}\, dx \right\}^{1/p'}.$$

If H consists of a single point x then dist $(f, J_H) = |f(x)|$, and we have already proved spectral synthesis.

Henceforth we assume that f vanishes on H. For $0 < \delta < \|f\|_\infty$ put $\Omega_\delta = \{x \in G : \|\rho(x)f - f\|_\infty < \delta\}$. Suppose that $HV \subset H\Omega_\delta$; it then follows that $|f| \leqslant \delta$ on HV, and we get the new estimate

$$\operatorname{dist}(f, J_H) \leqslant \varepsilon + \delta |V|^{-1/p'} |F_V|^{1/p'}$$

where $$F_V = \{x \in HV : f(x) \neq 0\}.$$

The above estimate is useful when smoothness assumptions on f are tied in with the nature of H. In one extreme, if G is an n-dimensional Lie group and $p' \geqslant n$ then we get dist $(f, J_H) = 0$ for all H on which f vanishes provided f satisfies a Lipschitz condition of order n/p'. The argument here is that for V a small cubical neighborhood of the identity we shall have $\delta^{p'} |V|^{-1}$ bounded while $|F_V| \to 0$ as $V \to 1$. The opposite extreme is the case in which no supplementary hypotheses on f are imposed, but we take H to be a subgroup. We need an elementary fact.

LEMMA 1. — *Let* H *be a closed subgroup of* G, K *a compact subset of* H, *and* W *a compact neighborhood of the identity in* G. *Then there exists a constant* $c = c$ (K , W) *such that for any neighborhood* Ω *of the identity in* G *we can find an open set* V *such that*

i) $1 \in V \subset W$

ii) $HV \subset H\Omega$

iii) $|KV| \leqslant c\,|V|$

Given the Lemma, we finish the proof of Theorem 2 this way. Put $K = [(\text{supp } f)\ W_\varepsilon^{-1}] \cap H$; then $F_V \subset KV$ for all $V \subset W_\varepsilon$. Now apply the Lemma with $W = W_\varepsilon$ and $\Omega = \Omega_\delta$. We get :

$$\text{dist}\,(f, J_H) \leqslant \varepsilon + \delta\, c^{1/p'}(\varepsilon)$$

where $c(\varepsilon)$ is independent of δ. Thus $\text{dist}\,(f, J_H) \leqslant \varepsilon$.

Proof of Lemma 1. – There exists an open neighborhood W' of the identity in G and a compact neighborhood U of the identity in H such that $UW' \subset W$. Given the compact subset K of H, the set KU is also compact ; so there exist $y_1, \ldots, y_c \in H$ such that $KU \subset \bigcup_{i=1}^{c} y_i U$. Given Ω an open subset of G containing the identity, put $V = (UW') \cap (H\Omega)$. Then V is open, $1 \in V \subset \Omega$, and $HV \subset H\Omega$. Now

$$KV \subset (KUW') \cap (H\Omega) \subset (U y_i\, UW') \cap H\Omega = \bigcup y_i\, V.$$

Thus KV is contained in c left-translates of V ; so $|KV| \leqslant c\,|V|$.

3. Regular tauberian algebras of functions.

The Banach algebra properties of A_p can conveniently be studied in a more general context. Suppose G is an arbitrary locally compact Hausdorff space. A Banach algebra of functions on G is a Banach algebra A whose elements may be identified with complex-valued continuous functions on G with the algebraic operations on A corresponding to pointwise addition and multiplication of functions ; this is equivalent to saying that there is a Banach algebra monomorphism $A \to C(G)$ where C (G) is the algebra of bounded continuous functions on G in the supremum norm.

DEFINITION. – *The Banach algebra* A *is a regular tauberian algebra of functions on* G *if three conditions hold*

(R) *Given a compact subset* $K \subset G$ *and a closed subset* F *disjoint from* K *there exists* $f \in A$ *such that* $f = 1$ *on* K *and* $f = 0$ *on* F.

(T) *The elements of compact support are dense in* A.

(G) *If* M *is a continuous multiplicative linear functional on* A *whose support is a single point* $\{x\} \subset G$ *then* $M = \delta_x$, i.e. $\langle f, M \rangle = f(x)$ *for all* $f \in A$.

If A is an algebra of functions satisfying the regularity condition (R) then there exist partitions of unity in A over each compact in G. We may define the *support* of a linear functional $T \in A'$ as the subset of G characterized by : $x \notin \text{supp } T$ iff there exists a neighborhood U of x such that $\langle f, T \rangle = 0$ for all $f \in A$ with $\text{supp } f \subset U$. It follows from the existence of suitable partitions of unity that supp T is the smallest closed subset $E \subset G$ such that $T \perp J_E$ where J_E is the set of $f \in A$ whose support is a compact disjoint from E. The tauberian condition (T) is equivalent to the statement supp $T = \emptyset$ iff $T = 0$.

Condition (G) is not in general a consequence of (R) and (T), although in many cases it is easily checked. For example, if G is a $C^{(\infty)}$-manifold and the infinitely differentiable functions of compact support are dense in A then (G) is valid since the only distribution whose support is a single point $\{x\}$ and which is multiplicative is δ_x. The situation is even simpler when G is totally disconnected and the elements of A which are locally constant form a dense subset.

When G is a locally compact group it is immediate from the definition that the elements of compact support are dense in $A_p(G)$. Given a compact K and a closed set F disjoint from K there exists a compact neighborhood U of the identity such that $(KUU^{-1}) \cap F = \emptyset$. Define $u = |U|^{-1}$ on U, $u = 0$ elsewhere and $v = 1$ on KU, $v = 0$ elsewhere ; then $u \in L_p(G)$, $v \in L_{p'}(G)$, and $k = v * \check{u} \in A_p(G)$ with $k = 1$ on K and $k = 0$ on F. Hence once one knowns that $A_p(G)$ is a Banach algebra it is clear that $A_p(G)$ is a Banach algebra of functions on G satisfying (R) and (T). The simplest way to verify that condition (G) holds is to use Theorem 2 in the case $H = \{1\}$. This amounts to the statement that if $f \in A_p(G)$ has compact support and $f(x) = 0$ then $f \in \overline{J}_x$. From this it follows that any linear functional $T \in A_{p'}(G)$ with $\text{supp } T \subset \{x\}$ is necessarily of the form $T = c\delta_x$ for some constant c. In summary we have

PROPOSITION 3. – *For any locally compact group* G, *the algebras* $A_p(G)$, $1 < p < \infty$, *are regular, tauberian algebras of functions on* G.

In the rest of this section we use only that $A_p(G)$ is a regular tauberian algebra of functions on G. When $A = A_p(G)$ and H is a closed subgroup, then the A(H) introduced below coincides with $A_p(H)$; this is the assertion of Theorem 1.

Whenever A is a Banach algebra of functions on G which separates the points of G there is a canonical homeomorphism of G *into* the Gelfand spectrum of A.

THEOREM 3. – *If* A *is a regular tauberian algebra of functions on a locally compact Hausdorff space* G *then* G *may be identified with the Gelfand space of regular maximal ideals of* A.

Proof. – Since A is an algebra, the dual space A' is an A-module : for $k \in A$ and $T \in A'$ the element $kT \in A'$ is defined by

$$\langle f , kT \rangle = \langle fk , T \rangle.$$

When A is an algebra of functions on G satisfying (R) we have supp $(kT) \subset (\text{supp } k) \cap (\text{supp } T)$. Now suppose A also satisfies (T) and let $M \in A'$ be a non-trivial multiplicative linear functional ; then supp $M \neq \phi$. Given $x \in \text{supp } M$ and an arbitrary open neighborhood U of x there must exist $f \in A$ with compact support such that $\langle f , M \rangle \neq 0$ and supp $f \subset U$. Choose $k \in A$ with $k = 1$ on supp f and $k = 0$ outside U. Since M is multiplicative we have

$$\langle f , M \rangle = \langle fk , M \rangle = \langle f , M \rangle \langle k , M \rangle ;$$

so $\langle k , M \rangle = 1$ and hence $kM = M$. Therefore supp $M \subset U$, but since U was an arbitrary open set containing x it follows that supp $M = \{x\}$. Condition (G) is now invoked to conclude that $M = \delta_x$.

Given a closed set $E \subset G$ put $\ker_A(E)$ for the closed ideal of A consisting of the elements which vanish on E. Then $\ker_A(E)$ is an algebra of functions on $G \setminus E$ for which conditions (R) and (G) obviously hold. If the tauberian condition (T) also holds then E is said to be a set of *spectral synthesis* for A. Equivalently, to say that E is a set of spectral synthesis for A is to say that given $f \in A$ with $f = 0$ on E and given $\varepsilon > 0$ there exists $\varphi \in A$ such that supp φ is a compact disjoint from E and $\|f - \varphi\| < \varepsilon$. The closed set E is a set of *local* spectral synthesis if the previous condition holds under the additional hypothesis that f has compact support.

We write A(E) for the Banach algebra $A/\ker_A(E)$. It is obvious that A(E) is a regular, tauberian algebra of functions on E. The question of spectral synthesis for E is related to the behaviour of extrapolations of elements of A(E) by elements of A. More precisely we have

PROPOSITION 4. – *A closed subset* E *of* G *is a set of spectral synthesis for* A *iff the following condition holds : given* $g \in A$ *with compact support and* $\varepsilon > 0$ *there exists* $h \in A$ *with compact support such that* $h = g$ *on a neighborhood of* E *and* $\|h\| < \|g\|_E + \varepsilon$ *where* $\|g\|_E$ *is the norm in* A(E) *of the restriction of* g *to* E.

Proof of necessity. – Given $g \in A$ *and* $\varepsilon > 0$ there exists $\psi \in A$ such that $g - \psi = 0$ on E and $\|\psi\| < \|g\|_E + \varepsilon/2$. If E is of synthesis then there exists $\varphi \in A$ such that supp φ is a compact disjoint from E and $\|(g - \psi) - \varphi\| < \varepsilon/2$. Put $h = g - \varphi$; then $h = g$ on a neighborhood of E and h has compact support if g has compact support. For the norm we have $\|h\| \leqslant \|g - \psi - \varphi\| + \|\psi\| < \|g\|_E + \varepsilon$.

Proof of sufficiency. – We may weaken the condition to

$$\|h\| < c\,\|g\|_E + \varepsilon$$

where c is a constant depending only on A and E. Given $f \in A$ with $f = 0$ on E and $\varepsilon > 0$ choose $g \in A$ with compact support such that $\|f - g\| < \varepsilon/3c$. Pick h with compact support equal to g on a neighborhood of E such that $\|h\| < c\,\|g\|_E + \varepsilon/3$. We have

$$\|g\|_E \leqslant \|f - g\|$$

since $f = 0$ on E. Putting $\varphi = g - h$ we have that supp φ is a compact disjoint from E and $\|f - \varphi\| < \varepsilon$.

Let $\mathrm{Res}_E : A \to A(E)$ be the quotient morphism given by restriction of functions. We know that given $h \in A(E)$ and $\varepsilon > 0$ there exists $g \in A$ such that $\mathrm{Res}_E g = h$ and $\|g\| < \|h\| + \varepsilon$. If we seek to control supports we are led to formulate

CONDITION (C). – *Given* $h \in A(E)$ *with compact support and* $\varepsilon > 0$ *there exists* $g \in A$ *with compact support such that* $\mathrm{Res}_E g = h$ *and* $\|g\| < \|h\| + \varepsilon$.

The abstract version of Proposition 2 is

PROPOSITION 2*. – *Let* E *be a set of local spectral synthesis for* A. *Then* E *is a set of spectral synthesis iff condition (C) holds.*

Proof of necessity. – Let $h \in A(E)$ have compact support. Given $\varepsilon > 0$ choose $g_1 \in A$ such that $\mathrm{Res}_E g_1 = h$ and $\|g_1\| < \|h\| + \varepsilon/3$.

Pick $g_2 \in A$ with compact support such that $\|g_1 - g_2\| < \varepsilon/3$. By Proposition 4 there exists $g_3 \in A$ having compact support such that

$$\mathrm{Res}_E g_3 = \mathrm{Res}_E(g_1 - g_2) = h - \mathrm{Res}_E g_2$$

and

$$\|g_3\| < \|g_1 - g_2\|_E + \varepsilon/3 < 2\varepsilon/3.$$

Put $g = g_2 + g_3$; then g has compact support, $\mathrm{Res}_E g = h$, and $\|g\| < \|h\| + \varepsilon$.

Proof of sufficiency. – Suppose $f \in A$ and $f = 0$ on E. Given $\varepsilon > 0$ choose $g_1 \in A$ with compact support such that $\|f - g_1\| < \varepsilon/4$. Then $\mathrm{Res}_E g_1$ has compact support and $\|\mathrm{Res}_E g_1\| = \|f - g_1\|_E < \varepsilon/4$. By Condition (C) there exists $g_2 \in A$ with compact support such that $\mathrm{Res}_E g_2 = \mathrm{Res}_E g_1$ and $\|g_2\| < \varepsilon/2$. Now $g_1 - g_2$ vanishes on E and has compact support ; since E is assumed to be of local synthesis there exists $\varphi \in A$ such that $\mathrm{supp}\,\varphi$ is a compact disjoint from E and $\|g_1 - g_2 - \varphi\| < \varepsilon/4$. It follows that $\|f - \varphi\| < \varepsilon$.

When G is a discrete space, any algebra of functions satisfying (R) and (T) necessarily satisfies (G), and moreover all subsets are of local synthesis. Nevertheless, Mirkil [13] has shown that spectral synthesis may fail for regular tauberian algebras of functions on a discrete space. Thus Condition (C) need not hold in general.

We can make a hypothesis on A(E) which ensures that Condition (C) holds without making reference to the original algebra A. For our purpose it is convenient to bring in approximate identities.

A Banach algebra $\mathfrak{A}$ is said to have an *approximate identity* of bound b if for each $\varepsilon > 0$ and each finite subset $\{h_1, \ldots, h_n\} \subset \mathfrak{A}$ there exists $u \in \mathfrak{A}$ with $\|u\| \leqslant b$ such that $\|h_i - uh_i\| < \varepsilon$ for $i = 1, \ldots, n$. We recall

COHEN'S FACTORIZATION THEOREM [1]. *Let $\mathfrak{A}$ be a Banach algebra with approximate identities of bound b. Given $h \in \mathfrak{A}$ and $\varepsilon > 0$ there exists $u \in \mathfrak{A}$ with $\|u\| \leqslant b$ and $g \in \overline{\mathfrak{A}h}$, the closure of $\mathfrak{A}h$, such that $h = ug$ and $\|h - g\| < \varepsilon$.*

A simple but valuable corollary is

THEOREM 4. – *Let $\mathfrak{A}$ be a regular tauberian algebra of functions on a locally compact Hausdorff space* H, *and suppose that $\mathfrak{A}$ has*

approximate identifies of bound b. Then for each compact $K \subset H$ *and each* $\varepsilon > 0$ *there exists* $k \in \mathfrak{A}$ *with compact support such that* $k = 1$ *on* K *and* $\|k\| < b + \varepsilon$.

Proof. – There exists $l \in \mathfrak{A}$ such that $l = 1$ on K. Given $\delta > 0$, apply Cohen's Factorization Theorem to get $l = ul_0$ where $\|u\| \leqslant b$ and $\|l - l_0\| < \delta$. Put $k_0 = u - u(l - l_0)$; then $k_0 = 1$ on K and $\|k_0\| < b(1 + \delta)$. Take $k_1 \in \mathfrak{A}$ with compact support such that $\|k_0 - k_1\| < \delta$. Define k by the norm-convergent sum

$$k = k_1 \sum_{n=0}^{\infty} (k_0 - k_1)^n.$$

Then k has compact support and $k = 1$ on K ; further we have $\|k\| \leqslant b(1 + 2\delta)(1 - \delta)^{-1} < b + \varepsilon$ for properly chosen δ.

For H a locally compact group and $1 < p < \infty$ the algebras $A_p(H)$ have bounded approximate identities, in fact of bound 1, iff the group H is amenable. Thus an abstract version of Proposition 1 i) is

PROPOSITION 1*. – *Let* A *be a regular tauberian algebra of functions on* G *and* H *a closed subset of* G. *If* A(H) *has bounded approximate identities then Condition* (C) *holds.*

Proof. – Let $h \in A(H)$ with compact support and $\varepsilon > 0$ be given. Suppose A(H) has approximate identities of bound b. Choose $g_0 \in A$ such that $\mathrm{Res}_H g_0 = h$ and $\|g_0\| < \|h\| + \varepsilon/6$; then pick $g_1 \in A$ with compact support such that $\|g_0 - g_1\| < \varepsilon/6b$. Put $v = h - \mathrm{Res}_H g_1$. Then there exists $u \in A$ such that $\mathrm{Res}_H u = v$ and $\|u\| \leqslant 2\|v\| < \varepsilon/3b$. Now v has compact support, and the argument used to prove Theorem 4 shows that there exists $k \in A$ with compact support such that $k = 1$ on supp v and $\|k\| < 2b$. The function $g = g_1 + kv$ has the properties required in Condition (C).

COUNTEREXAMPLE – Let $G = [-1, 1]$ and let $\mathcal{D}$ be the infinitely differentiable functions on G. Given $f \in \mathcal{D}$ put $M_n(f) = \sup |f^{(n)}(x)|$ for $|x| \leqslant (n + 1)^{-1}$, and put $\|f\| = \sum_{n=0}^{\infty} M_n(f)/n!$. Let

$$\mathcal{D}_0 = \{f \in \mathcal{D} : \|f\| < \infty\};$$

Leibnitz' rule shows that if $f, g \in \mathcal{D}_0$ then $\|fg\| \leqslant \|f\| \, \|g\|$. It follows that the completion of $\mathcal{D}_0$ for the norm gives a Banach algebra A of functions on G. Condition (T) is trivially satisfied, and (R) is easily verified. Now for any complex number z with $|z| \leqslant 1$ we get an element $T_z \in A'$ by putting $\langle f, T_z \rangle = \sum_{n=0}^{\infty} f^{(n)}(0) z^n/n!$. It is easy to check that T_z is a multiplicative linear functional with supp $T_z = \{0\}$.

4. Reduction steps.

The situation in which H is an open subgroup of G is especially simple and deserves special mention.

PROPOSITION 5. – *If* H *is an open subgroup of* G *then* $A_p(H)$ *may be identified with the subalgebra of* $A_p(G)$ *consisting of the functions which vanish outside* H. *Moreover* $f \to \chi f$ *where* χ *is the indicator function of* H *gives a Banach algebra retraction* $A_p(G) \to A_p(H)$.

Proof. – Suppose $u, v \in \mathcal{K}(G)$. Then we may write $u = \Sigma \rho(\sigma_i) u_i$, $v = \Sigma \rho(\sigma_i) v_i$ (finite sum) where supp u_i, supp $v_i \subset H$ and $H\sigma_i \neq H\sigma_j$ if $i \neq j$. We have $\|u\|_p^p = \Sigma \|u_i\|_p^p$ and $\|v\|_p^p = \Sigma \|v_i\|_p^p$. Now

$$\chi(v * \check{u}) = \Sigma \, v_i * \check{u}_i \in A_p(H),$$

and the Holder inequality gives the A_p-norm estimate

$$\|\chi(v * \check{u})\| \leqslant \Sigma \, \|u_i\|_p \, \|v_i\|_{p'} \leqslant \|u\|_p \, \|v\|_p .$$

Compact normal subgroups can be factored out with a very simple effect on A_p.

PROPOSITION 6. – *If* K *is a compact normal subgroup of* G *then* $A_p(G/K)$ *may be identified with the subalgebra of* $A_p(G)$ *consisting of the functions which a periodic with respect to* K. *Moreover* $f \to M_K f$ *where*

$$M_K f(\sigma) = \int_K f(\sigma\kappa) \, d\kappa$$

gives a Banach space retraction $A_p(G) \to A_p(G/K)$.

Proof. – We may regard $L_p(G/K)$ as the subspace of $L_p(G)$ consisting of the periodic functions, M_K again gives a retraction, and

$$M_K(v * \check{u}) = (M_K v) * (M_K \check{u}) .$$

An invariant pseudometric on a group G is a function $\omega : G \to [0, \infty)$ with the properties : i) $\omega(1) = 0$, ii) $\omega(x) = \omega(x^{-1})$, iii) $\omega(xy) \leqslant \omega(x) + \omega(y)$. The set $\omega^{-1}(0)$ is a subgroup ; we say that ω is *normal* if $\omega^{-1}(0)$ is a normal subgroup. Given a pair (ω , ω') of invariant pseudometrics we say that ω' is *stronger* than ω if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in G$, $\omega'(x) < \delta$ implies $\omega(x) < \varepsilon$.

LEMMA 2. – *Let* G *be a σ-compact group. For each continuous invariant pseudometric* ω *on* G *there exists a continuous invariant pseudometric* ω' *which is normal and stronger than* ω.

Proof. – Let $\{K_n\}$ be a countable collection of compact subsets of G such that $\cup K_n = G$. For each n put $\omega_n(x) = \sup_{y \in K_n} \omega(yxy^{-1})$. Then each ω_n is a continuous invariant pseudometric. The function $\omega' = \Sigma\, 2^{-n}\, \omega_n(1 + \omega_n)^{-1}$ fulfills the requirements.

We use Lemma 2 to reduce everything to the situation in which the group G is separable and metrizable. To illustrate the application we prove

LEMMA 3. – *If Theorem 1 is valid when* G *is a separable, metrizable, locally compact group then it is valid for all locally compact groups.*

Proof (For Theorem 1a). – Suppose $g \in A_p(G)$ is a given non-trivial element. Then g vanishes outside a σ-compact open subgroup G_0 of G. Let g_0 be the restriction of g to G_0 ; so $g_0 \in A_p(G_0)$ by Proposition 5. Define ω on G_0 by $\omega(x) = \| \lambda(x) g_0 - g_0 \|_\infty$; then ω is a continuous invariant pseudometric. Let ω' be normal and stronger than ω. Put $K = \{x \in G_0 : \omega'(x) = 0\}$; then K is a compact normal subgroup of G_0, – it is normal since ω' is normal and compact since $K \subset \omega^{-1}(0)$ and g_0 vanishes at ∞ which implies that $\lim_{x \to \infty} \omega(x)$ exists and equals $\| g_0 \|_\infty$. Put $G_1 = G_0/K$; write $\pi : G_0 \to G_1$ for the projection. By Proposition 6 we have $g_0 = g_1 \circ \pi$ where

$g_1 \in A_p(G_1)$. Now G_1 is metrizable, (ω' induces a metric) and σ-compact. Theorem 1a applied to G_1 and H_1 where $H_1 = \pi(H \cap G_0)$ gives the fact that $h_1 \in A_p(H_1)$ and $\|h_1\| \leqslant \|g_1\| \leqslant \|g\|$ where h_1 is the restriction of g_1 to H_1. The function h defined on H by $h(y) = h_1(\pi y)$ for $y \in H \cap G_0$ and $h(y) = 0$ otherwise meets the requirements.

Proof (For Theorem 1b). – Given $h \in A_p(H)$ let $g' \in C_0(G)$ be an extension of h as a continuous function vanishing at ∞. Assuming that g' is non-trivial, we form G_0, ω, and K as above but this time in terms of g'. Let h_0 be the restriction of h to $H \cap G_0$; then h_0 is periodic with respect to $H \cap K$, and so $h_0 = h_1 \circ \pi$ where $h_1 \in A_p(H_1)$ and $\|h_1\| \leqslant \|h\|$. Etc.

5. Strong semi-direct products.

Consider the situation in which $G = HN$ where H is a closed subgroup, N is a closed normal subgroup, and $H \cap N = \{1\}$. If in addition N contains a compact set K of positive measure (with respect to the Haar measure of N) which is invariant under conjugation by elements of H we shall say that G is the strong semi-direct product of H by N. The extra hypothesis rules out many semi-direct products of interest in analysis, but it is essential for what follows here. We put

$$A_p^H(N) = \{\varphi \in A_p(N) : \varphi(y^{-1} n y) = \varphi(n) \quad \text{for all } y \in H, n \in N\}.$$

If the semi-direct product is not strong then $A_p^H(N) = \{0\}$. Otherwise, if K is a compact, H-invariant set of positive measure and κ its indicator function then $\varphi = |K|^{-1} \kappa * \check{\kappa} \in A_p^H(N)$, $\|\varphi\| = 1$, and $\hat{\varphi}(1) = 1$, also φ has compact support.

PROPOSITION 7. – *Let* $G = HN$ *be a strong semi-direct product. Then* $h \otimes \varphi \to g$, *where* $g(yn) = h(y)\, \varphi(n)$, *gives a Banach algebra monomorphism* $A_p(H) \otimes A_p^H(N) \to A_p(G)$.

Proof. – In the case of a semi-direct product we have a canonical morphism $L_p(H) \otimes L_p(N) \to L_p(G)$ given by $u \otimes v \mapsto U$ where $U(yn) = u(y)\, v(n)$. This induces a morphism

$$(L_p(H) \otimes L_{p'}(H)) \otimes (L_p(N) \otimes L_{p'}(N)) \to L_p(G) \otimes L_{p'}(G)$$

which we write as $s \otimes t \mapsto S = st$. Now suppose $g = P_G S$ where $P_G : L_p(G) \otimes L_{p'}(G) \to A_p(G)$ is the canonical morphism defining A_p. Then $g(\sigma) = \int S(\sigma^{-1}x, x)\, dx$; if $\sigma = \tau\nu$ where $\tau \in H$ and $\nu \in N$ this becomes

$$g(\tau\nu) = \int_H \int_N s(\tau^{-1}y, y)\, t(y^{-1}\tau\nu^{-1}\tau^{-1}yn, n)\, dn\, dy,$$

since the Haar measure on G is the product of the Haar measures of H and N. The integration over N yields

$$g(\tau\nu) = \int_H s(\tau^{-1}y, y)\, \varphi(y^{-1}\tau\nu\tau^{-1}y)\, dy,$$

where $\varphi = P_N t$. If φ is invariant under conjugation of its argument by elements of H, i.e. if $\varphi \in A_p^H(N)$, then we get

$$g(\tau\nu) = \int_H s(\tau^{-1}y, y)\, dy\ \varphi(\nu) = h(\tau)\, \varphi(\nu)$$

where $h = P_H s$.

Proposition 1, iii) is an immediate corollary of Proposition 7 and Proposition 5.

6. Proof of Theorem 1.

Let λ_G denote left-translation in the G-variable on $L_p(G \times H)$ and λ_H left-translation in the H-variable. We get two extremal epimorphisms

$$P_G : L_p(G \times H) \otimes L_{p'}(G \times H) \to A_p(G)$$

$$P_H : L_p(G \times H) \otimes L_{p'}(G \times H) \to A_p(H)$$

given by $P_G(u \otimes v) = g$, $P_H(u \otimes v) = h$ where

$$g(\sigma) = \langle \lambda_G(\sigma)\, u, v \rangle, \quad h(\tau) = \langle \lambda_H(\tau)\, u, v \rangle.$$

Suppose there is an automorphism $\Theta : L_p(G \times H) \to L_p(G \times H)$ such that $\Theta\lambda_H(\tau) = \lambda_G(\tau)\Theta$ for all $\tau \in H$. Put

$$\overline{\Theta} : L_{p'}(G \times H) \to L_{p'}(G \times H)$$

for the automorphism $\overline{\Theta} = \Theta'^{-1}$, the transposed inverse. Then we have

$$\langle \lambda_H(\tau)\, u\,,\, v \rangle = \langle \lambda_G(\tau)\ \Theta u\,,\, \overline{\Theta} v \rangle\,, \quad \text{all} \quad \tau \in H.$$

It follows that $\mathrm{Res}_H \circ P_G \circ (\Theta \circledast \overline{\Theta}) = P_H$ where Res_H is the restriction of functions from G to H. Since P_G and P_H are extremal epimorphisms, Theorem 1 is proved.

It remains only to exhibit a suitable automorphism Θ. We do this in the case that the locally compact group G is separable metric. In this situation there is a Borel measurable map $\theta : G \to H$ such that $\theta(\tau x) = \tau\, \theta(x)$ for all $x \in G$ and $\tau \in H$. (We shall say more about θ in a moment). In terms of point functions we put

$$\Theta u(x\,,\, y) = u\,[y^{-1}\ (\theta x)^{-1} x\,,\, (\theta x) y]\,.$$

It is immediate that $\Theta \lambda_H(\tau) = \lambda_G(\tau)\, \Theta$, but some readers may not feel comfortable with the change of variables required to show that Θ is an automorphism of $L_p(G \times H)$. To make matters more obvious we write $\Theta = \Theta_1 \circ F$ where F and Θ_1 are the automorphisms described below. There is always a canonical isomorphims

$$L_p(G \times H) \simeq L_p(H\ ;\ L_p(G))$$

where the latter space may be viewed as the completion of $\mathcal{K}(H\ ;\ L_p(G))$, the space of continuous functions of compact support $u : H \to L_p(G)$, with respect to the L_p-norm for the left-invariant Haar measure on H. Since λ_G operates continuously as an automorphism of $L_p(G)$ we have that $F\ u(y) = \lambda_G(y)\, u(y)$ defines an element $F u \in \mathcal{K}(H\ ;\ L_p(G))$ whenever $u \in \mathcal{K}(H\ ;\ L_p(G))$. The inverse of $F : \mathcal{K}(H\ ;\ L_p(G)) \to \mathcal{K}(H\ ;\ L_p(G))$ is given by

$$F^{-1}\, w(y) = \lambda_G(y^{-1})\, w(y).$$

Since F is obviously isometric in the L_p-norm it extends uniquely to an automorphism of $L_p(H\ ;\ L_p(G))$.

To describe Θ_1 we start with the canonical isomorphism $L_p(G \times H) \simeq L_p(G\ ;\ L_p(H))$ and view the right-hand side as the completion of $\mathcal{K}(G\ ;\ L_p(H))$. Given a continuous map $\varphi : G \to H$ we get an automorphism Φ of $\mathcal{K}(G\ ;\ L_p(H))$ given by

$$\Phi u(x) = \lambda_H[(\varphi x)^{-1}]\, u(x).$$

Up to now everything is just like what went on before, but at this point we shall use the assumption that H is a metric space. Suppose $\{\varphi_n\}$ is a sequence of continuous maps which converge in measure on supp u where $u \in \mathcal{K}(G\ ;\ L_p(H))$; then $\Phi_n u$ also converges in measure on supp u. The argument is that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\lambda(y_1)\ u(x) - \lambda(y_2)\ u(x)| < \varepsilon$ for all x wherever $d(y_1\ , y_2) < \delta$ where $|\ |$ is the norm in $L_p(H)$ and d is the distance on H. Thus $|\Phi_m u(x) - \Phi_n u(x)| < \varepsilon$ for all $x \in E^{\delta}_{m,n} \cap \text{supp } u$ where $E^{\delta}_{m,n} = \{x\ :\ d(\varphi_m(x)\ ,\ \varphi_n(x)) \geqslant \delta\}$. The assumption is that

$$\mu\,(E^{\delta}_{m,n} \cap \text{supp } u) \to 0$$

as $m\ ,\ n \to \infty$ on G. Since $|\Theta_n u(x)| \leqslant |u(x)|$ everywhere, it follows by dominated convergence that $\Phi_n u$ converges in the norm of $L_p(G\ ;\ L_p(H))$. What we have proved is that if $\{\varphi_n\}$ is a sequence of continuous maps $\varphi_n\ :\ G \to H$ which converge in measure on each compact to a map $\theta\ :\ G \to H$ then $\Phi_n u$ converges in the L_p-norm for each $u \in \mathcal{K}(G\ ;\ L_p(H))$. The limit may be identified with $\Theta_1 u$ where $\Theta_1 u(x) = \lambda_H[(\theta x)^{-1}]\, u(x)$. It is routine that Θ_1 extends to an automorphism of $L_p(G\ ;\ L_p(H))$. Since G is a σ-compact, locally compact space and μ is a Radon measure, any Borel map from G to a metric space is the limit in measure of a sequence of continuous maps.

The existence of the map $\theta\ :\ G \to H$ for arbitrary separable metric groups G is obtained this way. Let $\Gamma = H \backslash G$ be the space of right cosets and $G \xrightarrow{\pi} \Gamma$ the projection. According to [12, Lemma 1.1] there exists a Borel set F in G such that $G = HF\ ,\ H \cap F = \{1\}$, and, moreover, $(\pi^{-1}\, \pi K) \cap F$ is pre-compact in G whenever K is a compact subset of G. In effect, this says there is a Borel cross section $\psi\ :\ \Gamma \to G$ such that $\pi \circ \psi = \text{id}_\Gamma$ and ψ maps compacts into pre-compacts. We define θ by $\theta(x) = x[\psi \circ \pi(x)]^{-1}$. It follows that in addition to the properties already used, θ maps compacts into sets whose closure is compact. This implies that if $u \in L_p(G \times H)$ has compact support then the same is true of Θu.

The preceding remarks raise some questions related to Condition (C). There is an obvious notion of support for a tensor

$$t \in L_p(G) \otimes L_{p'}(G)\ ;$$

the definition may be formulated by saying that supp t is the closed of $G \times G$ characterized by the property that $(x_0, y_0) \notin \text{supp } t$ iff there exists a neighborhood W of (x_0, y_0) such that

$$\int_{G\times G} w(x, y)\, t(x, y)\, dx\, dy = 0$$

for all $w \in \mathcal{K}(G \times G)$ with supp $w \subset W$. One can show that if $f = Pt$ then supp f is contained in the closure of the set

$$\{z \in G : z = yx^{-1},\ (x, y) \in \text{supp } t\}.$$

This prompts

DEFINITION. – *An element* $f \in A_p(G)$ *is formally of compact support if for each* $\varepsilon > 0$ *there exists* $t \in L_p(G) \circledast L_{p'}(G)$ *such that* $\|t\| < \|f\| + \varepsilon$, *supp t is compact, and* $f = Pt$.

QUESTION. – *For a given group* G *is it true that each* $f \in A_p(G)$ *with compact support is formally of compact support ?*

For amenable groups the answer to the Question is Yes. Given any compact $K \subset G$, where G is amenable, and $\varepsilon > 0$ one can construct $k \in A_p(G)$ such that $k = 1$ on K and $k = Ps$ where $s \in L_p(G) \circledast L_{p'}(G)$ has compact support with $\|s\| < 1 + \varepsilon$. This is a strengthening of what we obtained from Theorem 4. Now if $f \in A_p(G)$ and supp $f \subset K$, – so $f = fk$, – then $f = Pt$ where $t(x, y) = f(yx^{-1})\, s(x, y)$. Clearly supp $t \subset$ supp s, and we know from § 1 that $\|t\| \leqslant \|f\|\, \|s\| \leqslant (1 + \varepsilon)\, \|f\|$.

The point of the above is that if $h \in A_p(H)$ is formally of compact support then given $\varepsilon > 0$ we can find $g \in A_p(G)$ with $\text{Res}_H g = h$ and $\|g\| < \|h\| + \varepsilon$ such that g is formally of compact support. To do this fix some $s \in L_p(G) \circledast L_{p'}(G)$ such that

$$\|s\| = 1 = \int_{G'} s(x, x)\, dx$$

and supp s is compact. Choose $t \in L_p(H) \circledast L_{p'}(H)$ with compact support such that $h = Pt$ and $\|t\| < 1 + \varepsilon$. Define

$$T \in L_p(G \times H) \circledast L_{p'}(G \times H)$$

by $T = st$; we use the morphism $L_p(G) \otimes L_p(H) \to L_p(G \times H)$ to get a morphism

$$[L_p(G) \otimes L_{p'}(G)] \otimes [L_p(H) \otimes L_{p'}(H)] \to L_p(G \times H) \otimes L_{p'}(G \times H)$$

which we designate by $s \otimes t \to st$. Then $\|T\| < \|h\| + \varepsilon$ and supp t is compact. Put $S = (\Theta \times \overline{\Theta})\, T$; then supp S is also compact. If we put $g = P_G S$ all the requirements are met.

The above gives a constructive proof of Proposition 1 i) once one has found for each compact $K \subset H$ and $\varepsilon > 0$ an element $k \in A_p(H)$ which is formally of compact support such that $k = 1$ on K and $\|k\| < 1 + \varepsilon$. One way to do this is to find a compact set $U \subset G$ such that $|KU| < (1 + \varepsilon)\ |U|$ and put $k = v * \check{u}$ where $u(y) = |U|^{-1}$ for $y \in K$, $= 0$ for $y \notin K$, and v is the indicator function of KU. Actually, the procedure of § 3 is simpler. We know from Theorem 4 that for H amenable there exists $k_1 \in A_p(H)$ with $k_1 = 1$ on K and $\|k_1\| < 1 + \varepsilon/2$. Take k_0 formally compact support such that $\|k_1 - k_0\| < \varepsilon/3$; then $k = k_0 \sum_{n=0}^{\infty} (k_1 - k_0)^n$ is formally of compact support, $\|k\| < 1 + \varepsilon$, and $k = 1$ on K.

7. Normal subgroups.

We suppose here that G is a locally compact group satisfying the second axiom of countability and that H is a closed subgroup of G. Let $\Gamma = G/H$ be the space of left cosets. If one forms Lebesgue spaces with respect to the quasi-invariant measure on Γ he obtains non-canonical isomorphisms of $L_p(\Gamma\ ;\ L_p(H))$ with $L_p(G)$. These yield morphism $L_p(\Gamma) \otimes L_p(H) \to L_p(G)$, and if one proceeds as in § 5 he obtains non-canonical morphism

$$(L_p(\Gamma) \otimes L_{p'}(\Gamma)) \otimes A_p(H) \to A_p(G).$$

The morphisms depend on the choice of Borel cross-section $\psi : \Gamma \to G$. More explicitly, let $\pi : G \to \Gamma$ be the projection homomorphism. The action of G on Γ is defined by putting $\xi^\sigma = \pi(\sigma^{-1}x)$ where $\sigma \in G$ and $x \in G$ is any element such that $\pi(x) = \xi$. The quasi-invariant measure on Γ satisfies

$$\int_\Gamma w(\xi^\sigma)\, m(\sigma, \xi)\, d\xi = \int_\Gamma w(\xi)\, d\xi$$

for all $w \in \mathcal{K}(\Gamma)$ where $m : G \times \Gamma \to \mathbf{R}^+$ is a certain continuous function whose exact nature does not concern us here. Given a Borel mapping $\psi : \Gamma \to G$ such that $\pi \circ \psi = id_\Gamma$, we define a Borel mapping $\vartheta : G \times \Gamma \to H$ by

$$\sigma\psi(\xi^\sigma) = \psi(\xi)\, \vartheta(\sigma, \xi)\ ;\ \sigma \in G\ ,\ \xi \in \Gamma.$$

Given $s \in L_p(\Gamma) \circledast L_{p'}(\Gamma)$ and $h \in A_p(G)$, the function

$$g(\sigma) = \int_\Gamma s(\xi^\sigma, \xi)\, m^{1/p}(\sigma, \xi)\, h[\vartheta(\sigma, \xi)]\, d\xi$$

belongs to $A_p(G)$ and $\|g\| \leqslant \|s\|\ \|h\|$. In general, there is no close relation between $\mathrm{Res}_H g$ and h.

When H is a normal subgroup considerable simplifications occur. If $G = \Gamma H$ is a semi-direct product we are reduced to a retracing of § 5. Here, however, the subgroup H is the normal subgroup, and a different approach is needed for which the assumption that ψ is a homomorphism is irrelevant.

PROPOSITION 8. – *Suppose that* H *is a normal subgroup of* G. *Given an open subset* Ω *of* G *and* $\varepsilon > 0$ *then for each* $h \in A_p(H)$ *with supp* h *a compact subset of* $\Omega \cap H$ *there exists* $g \in A_p(G)$ *with* $\|g\| \leqslant \|h\|$ *such that supp* g *is a compact subset of* Ω *and*

$$\|\mathrm{Res}_H g - h\| < \varepsilon.$$

Proof. – We first observe that G has a strongly continuous representation as automorphisms of $A_p(H)$ according to $x \mapsto \alpha(x)$ where $\alpha(x)\, h(y) = h(x^{-1}yx)$. Hence there is a compact neighborhood U of the identity in G such that $U(\mathrm{supp}\, h)\, U^{-1} \subset \Omega$ and $\|\alpha(x)\, h - h\| < \varepsilon$ for all $x \in U$. Put $K = \pi U$ and let $\psi : \Gamma \to G$ be a Borel cross-section such that $\psi K \subset U$. Take $s \in L_p(\Gamma) \circledast L_{p'}(\Gamma)$ given by $s = |K|^{-1}\, \kappa^{1/p} \circledast \kappa^{1/p'}$ where κ is the indicator function of K. Then for the function g defined on G by

$$g(\sigma) = \int_\Gamma s(\xi^\sigma, \xi)\, h[\vartheta(\sigma, \xi)]\, d\xi$$

we have $g \in A_p(G)$ and $\|g\| \leqslant \|h\|$. If

$$\xi^{\sigma} \in K \ , \ \xi \in K, \quad \text{and} \quad \vartheta(\sigma \ , \xi) \in \operatorname{supp} h,$$

then $\sigma \in U \, (\operatorname{supp} h) U^{-1}$; hence $\operatorname{supp} g \subset \Omega$. It remains only to observe that for $\tau \in H$ we get

$$g(\tau) = |K|^{-1} \int_K \alpha(\psi(\xi)) \, h(\tau) \, d\xi.$$

Therefore $\| \operatorname{Res}_H g - h \| < \varepsilon$.

As an immediate consequence we get Proposition 1 ii) in a more precise form.

PROPOSITION 1 ii). – *Suppose that* H *is a normal subgroup of* G. *Given an element of compact support* $h \in A_p(H)$, $\varepsilon > 0$, *and an open subset* Ω *of* G *with* $\operatorname{supp} h \subset \Omega \cap H$, *there exists* $g \in A_p(G)$ *with* $\|g\| < \|h\| + \varepsilon$ *such that* $\operatorname{Res}_H g = h$ *and* $\operatorname{supp} g \subset \overline{\Omega}$.

Proof. – We define inductively sequences $\{h_n\}$ and $\{g_n\}$ with

$$h_n \in A_p(H) \ , \ \operatorname{supp} h_n \subset \Omega \cap H \ , \ g_n \in A_p(G) \ , \ \|g_n\| \leqslant \|h_n\|,$$

and $\operatorname{supp} g_n \subset \Omega$ as follows : put $h_1 = h$; assuming that h_n has been given choose g_n according to Proposition 8 so that

$$\| \operatorname{Res}_H g_n - h_n \| < \varepsilon \, 2^{-n} \ ;$$

put $h_{n+1} = h_n - \operatorname{Res}_H g_n$. Then $g = \Sigma g_n$ meets the requirements.

8. Convolution operators.

Given a Banach space B, let END(B) designate the Banach algebra of bounded linear operators on B. If B is reflexive then END(B) is naturally isomorphic to the dual space of $B \otimes B'$ under the pairing $\langle T \, , u \otimes v \rangle = \langle Tu \, , v \rangle$. If B is a module for the group G then the dual space of $B \otimes_G B'$ is $\mathrm{END}^G(B)$, the operators which commute with the operators giving the G-module structure. As a particular example, let G be a locally compact group, $B = L_p(G)$, and take the G-module structure on B as that coming from the right translations. We write $\mathrm{CONV}_p(G)$ for $\mathrm{END}^G(L_p(G))$; this is the algebra of (left-) convolution operators.

It is obvious that the canonical morphism

$$P : L_p(G) \otimes L_{p'}(G) \to A_p(G)$$

factors through $L_p(G) \otimes_G L_{p'}(G) \to A_p(G)$. This last, as we shall see, is an isomorphism for $p = 2$ for all G, and the same is true for all p when G is amenable. It is immediate that we have an isometric inclusion $P' : A_p'(G) \to \mathrm{END}(L_p(G))$, the image being exactly the smallest ultraweakly closed subspace containing the left translations $\lambda(\sigma)$, $\sigma \in G$. The ultraweak topology on END(B) is the weak topology from the pairing with $B \otimes B'$. Thus $A_p'(G)$ can be identified with a subspace of $\mathrm{CONV}_p(G)$. We view the identification in this fashion. Suppose μ is a bounded complex Radon measure on G. Then $\lambda(\mu) \in \mathrm{CONV}_p(G)$ can be defined by $\lambda(\mu)u = \int [\lambda(\sigma)u]\, d\mu(\sigma)$ for $u \in L_p(G)$ where we have a Bochner integral with $\sigma \mapsto \lambda(\sigma)u$ regarded as an element of $C_0(G ; L_p(G))$. Another notation for $\lambda(\mu)u$ is $\mu * u$. In particular, if $k, u \in \mathcal{K}(G)$ then

$$\lambda(k)u = \int [\lambda(x)u]\, k(x)\, dx$$

coincides with the previously defined $k * u$. The measures are weak*-dense in $A_p'(G)$; we call the elements of $A_p'(G)$ *pseudomeasures*, notation : $A_p'(G) = PM_p(G)$, and we shall write $PM_p(G) \xrightarrow{\lambda} \mathrm{CONV}_p(G)$ for the isometric inclusion. We shall write

$$T * u = \lambda(T)u \quad \text{for} \quad T \in PM_p(G), u \in L_p(G),$$

and $|||T|||_p$ will be used for norm of $T \in PM_p(G)$ as well as for the norm of $T \in \mathrm{CONV}_p(G)$. The completion of $L_1(G)$ for the norm $|||\ |||_p$ gives the space $PF_p(G)$ of *pseudofunctions.*

The space $PM_p(G)$, being the dual of the Banach algebra $A_p(G)$, is canonically an $A_p(G)$-module. In § 1 we showed that $L_p(G) \otimes L_{p'}(G)$ was an $A_p(G)$-module, and, by duality, the same is true on $\mathrm{CONV}_p(G)$. In fact $\lambda : PM_p(G) \to \mathrm{CONV}_p(G)$ is a morphism of Banach $A_p(G)$-modules.

Suppose $k \in L_p(G)$ and $u \in \mathcal{K}(G)$. Then we can define $k * u \in L_p(G)$ by $k * u = \int [\rho_p(x)k]\, u(x^{-1})\Delta^{-1/p}(x)\, dx$ and be consistent with previous notation. Since elements of $\mathrm{CONV}_p(G)$ commute with right-translations we have

$T(k * u) = (Tk) * u$ for $T \in CONV_p(G)$, $k \in L_p(G)$, $u \in \mathcal{K}(G)$.

Now suppose $k \in L_1 \cap L_p(G)$; we see that $(Tk) * u = T\lambda(k)u$, and this extends the definition of $(Tk) * u$ to all $u \in L_p(G)$.

The support of an element $u \in L_p(G)$ is defined by saying $x \notin \operatorname{supp} u$ iff there exists a neighborhood V of x such that for all $v \in \mathcal{K}(G)$ with $\operatorname{supp} v \subset V$ we have $\langle u, v \rangle = 0$. The support of a convolution operator $T \in CONV_p(G)$ is defined by saying that $x \notin \operatorname{supp} T$ iff there exists a neighborhood U of the identity such that $x \notin \operatorname{supp}(Tu)$ for all $u \in \mathcal{K}(G)$ with $\operatorname{supp} u \subset U$.

PROPOSITION 9. – *Suppose* $T \in CONV_p(G)$ *is a convolution operator with compact support. Then for each neighborhood* Ω *of supp* T, *the operator* T *is in the ultrastrong closure of the set of operators* $\lambda(w)$ *where* $w \in \mathcal{K}(G)$, $\operatorname{supp} w \subset \Omega$, *and* $|||w|||_p \leqslant |||T|||_p$.

Proof. – Let U be a neighborhood of the identity such that $(\operatorname{supp} T)\, U \subset \Omega$. Take $k \in \mathcal{K}^+(G)$ with $\operatorname{supp} k \subset U$ and $\int k(x)\,dx = 1$. Put $w = Tk$; then $\operatorname{supp} w$ is a compact subset of Ω and, since $w \in L_p$, we also have $w \in L_1$. We have previously established that

$$T\lambda(k)u = T(k * u) = w * u = \lambda(w)u$$

for all $u \in L_p(G)$. Since the identity operator is the ultrastrong limit of operators of the form $\lambda(k)$ it follows that T is an ultrastrong limit of the $\lambda(w)$. Here $|||w|||_p \leqslant |||T|||_p \; \|k\|_1 \leqslant |||T|||_p$. We have constructed $w \in L_1(G)$, but it is easy to modify this to get $w \in \mathcal{K}(G)$.

A basic question in harmonic analysis is whether the conclusion of Proposition 9 persists when the convolution operator is no longer assumed to have compact support. I gave the following at the Warwick Summer Institute 1968 (see [6]) which generalizes earlier results of Figà-Talamanca [7] for compact or commutative groups.

THEOREM 5. – *Let* G *be an amenable group. Given* $T \in CONV_p(G)$ *and* Ω *any open set containing supp* T, *the operator* T *is in the ultrastrong closure of the set of operators* $\lambda(w)$ *where* $w \in \mathcal{K}(G)$, *supp* $w \subset \Omega$, *and* $|||w|||_p \leqslant |||T|||_p$. *In particular,* $\lambda : PM_p(G) \to CONV_p(G)$ *is an isomorphism.*

Proof. – Let $\mathfrak{A}$ be the set of $f \in A_p(G)$ with $\|f\| \leqslant 1$ and supp f compact. By Theorem 4, for each compact $K \subset G$ and each $\varepsilon > 0$ there exists $f \in (1+\varepsilon)\mathfrak{A}$ with $f = 1$ on K ; hence for each $t \in L_p(G) \circledast L_{p'}(G)$ with compact support we have $t = ft$ for some $f \in (1+\varepsilon)\mathfrak{A}$. It follows that every $t \in L_p(G) \circledast L_{p'}(G)$ is in the norm closure of $\mathfrak{A} \cdot t$, and therefore each $T \in \mathrm{END}(L_p(G))$ is in the ultrastrong closure of $\mathfrak{A} \cdot T$. When $T \in \mathrm{CONV}_p(G)$ and $f \in \mathfrak{A}$ then, as will be seen in Proposition 10 below, supp $(f \cdot T)$ is a compact subset of Ω while $|||f \cdot T|||_p \leqslant \|f\| \, |||T|||_p \leqslant |||T|||_p$. The conclusion now follows from Proposition 9.

It is not at all obvious that Theorem 5 remains valid for non-amenable groups. For discrete groups G the conclusion of Theorem 5 implies that every subset $E \subset G$ is a set of narrow spectral synthesis, i.e. each $T \in \mathrm{PM}_p(E)$, the pseudomeasures with support in E, is the weak*-limit of finite measures μ with supp $\mu \subset E$ and $|||\mu|||_p \leqslant |||T|||_p$. When $p = 2$ a weaker statement is available.

THEOREM 5'. – *For an arbitrary locally compact group* G *each* $T \in \mathrm{PM}_2(G)$ *is in the ultrastrong closure of the set of operators* $\lambda(w)$ *where* $w \in \mathcal{K}(G)$ *and* $|||w|||_2 \leqslant |||T|||_2$. *In particular*

$$\lambda : \mathrm{PM}_2(G) \to \mathrm{CONV}_2(G)$$

is an isomorphism.

Theorem 5' is well-known. The proof is this. What we call $\mathrm{PF}_2(G)$, namely the norm closure of the $\lambda(w)$ with $w \in \mathcal{K}(G)$ is the regular C*-algebra of G. Its bicommutator in END $(L_2(G))$ is $\mathrm{CONV}_2(G)$; see [2 ; Ch. I, § 5], but the proof works for any p. Where the Hilbert space techniques first come in is that $\mathrm{CONV}_2(G)$, being the von Neumann algebra generated by $\mathrm{PF}_2(G)$, is the ultrastrong closure of $\mathrm{PF}_2(G)$, see [2 ; Ch. 1, § 3]. Now by the Kaplansky Density Theorem (loc. cit.) the unit ball of $\mathrm{CONV}_2(G)$ is the ultrastrong closure of the $\lambda(w)$ with $w \in \mathcal{K}(G)$ and $|||w|||_2 \leqslant 1$.

It remains to give some more explicit information about the support of a convolution operator.

PROPOSITION 10. – *If* $f \in A_p(G)$ *and* $T \in \mathrm{CONV}_p(G)$ *then*

$$\{x \in \mathrm{supp}\ T : f(x) \neq 0\} \subset \mathrm{supp}\ (f \cdot T) \subset (\mathrm{supp}\ f) \cap (\mathrm{supp}\ T).$$

Proof. – Suppose $1 \notin \operatorname{supp}(f \cdot T)$ and $f(1) \neq 0$. Then there exists a compact neighborhood U of 1 such that

$$\langle (f \cdot T)\alpha , \beta \rangle = \langle f \cdot T , \alpha \otimes \beta \rangle = 0$$

for all $\alpha , \beta \in \mathcal{K}(G)$ with $\operatorname{supp}\alpha$, $\operatorname{supp}\beta \subset U$, and, at the same time, $|f| > 0$ on UU^{-1}. Since $A_p(G)$ is a regular Banach algebra with Gelfand spectrum G (Theorem 2), there exists $g \in A_p(G)$ such that $gf = 1$ on UU^{-1}. Now suppose $u , v \in \mathcal{K}(G)$ with $\operatorname{supp} u$, $\operatorname{supp} v \subset U$. Then $\langle Tu , v \rangle = \langle T , u \otimes v \rangle = \langle f \cdot T , g \cdot (u \otimes v) \rangle = \Sigma \langle f \cdot T , \alpha_n \otimes \beta_n \rangle$ where $g \cdot (u \otimes v) = \Sigma \alpha_n \otimes \beta_n$ with

$$\{\alpha_n\} , \{\beta_n\} \subset \mathcal{K}(G) \quad \text{and} \quad \Sigma \|\alpha_n\|_p \ \|\beta_n\|_{p'} < \infty .$$

Since $g(yx^{-1})\ u(x)\ v(y) = \Sigma\, \alpha_n(x)\, \beta_n(y)$ we can assume $\operatorname{supp}\alpha_n$, $\operatorname{supp}\beta_n \subset U$, and hence $\langle f \cdot T, \alpha_n \otimes \beta_n \rangle = 0$ for each n. Thus

$$\langle Tu , v \rangle = 0 \quad \text{and} \quad 1 \notin \operatorname{supp} T.$$

We have proved that if $f(1) \neq 0$ and $1 \in \operatorname{supp} T$ then $1 \in \operatorname{supp}(f \cdot T)$. Conversely, suppose $1 \in \operatorname{supp}(f \cdot T)$. If we take $g \in A_p(G)$ with compact support such that $g(1) = 1$ then $1 \in \operatorname{supp}(gf \cdot T)$. If f were 0 on a neighborhood U of 1 we could take $\operatorname{supp} g \subset U$ which would give $gf = 0$, a contradiction. Hence we have $1 \in \operatorname{supp} f$, and it remains to prove that $1 \in \operatorname{supp} T$. Suppose $1 \notin \operatorname{supp} T$ and let U be a neighborhood of 1 such that U^2U^{-1} does not meet $\operatorname{supp} T$. Take k, u, $v \in \mathcal{K}(G)$ with supports in U. Now $gf(Tk) = w \in L_1(G)$ and $w = 0$ on UU^{-1} ; hence $\langle \lambda(w)\, u , v \rangle = 0$, but if we take $\lambda(k)$ converging strongly to the identity then $gf \cdot (T\,\lambda(k)) = \lambda(w)$ converges strongly to $gf \cdot T$ which gives $\langle (gf \cdot T)\, u , v \rangle = 0$ for all u, $v \in \mathcal{K}(G)$ which supports in U, contradicting $1 \in \operatorname{supp}(gf \cdot T)$.

COROLLARY. – *Given* $T \in \mathrm{CONV}_p(G)$ *we have* $x \in \operatorname{supp} T$ *iff for all* $f \in A_p(G)$ *we have* : $f \cdot T = 0$ *implies* $f(x) = 0$.

For a pseudomeasure $S \in \mathrm{PM}_p(G) = A_p'(G)$ we have already defined the support in § 3. If A is any regular tauberian algebra of functions on G it is easy to see that for $S \in A'$ we have : $x \in \operatorname{supp} S$ iff for all $f \in A$ the statement $fS = 0$ implies the statement $f(x) = 0$. Since $\lambda : \mathrm{PM}_p(G) \to \mathrm{CONV}_p(G)$ is an A_p-module morphism we get

COROLLARY. – *If* $S \in PM_p(G)$ *then* supp S = supp $\lambda(S)$ *where* supp S *is defined in terms of the support of a linear functional on a regular tauberian algebra of functions and* supp $\lambda(S)$ *is defined in terms of the support of a convolution operator.*

9. Amenability and bounded approximate identities in A_p.

The following is folklore

THEOREM 6. – *If* G *is an amenable locally compact group then* $A_p(G)$ *has approximate identities of bound* 1 *for all* p, $1 < p < \infty$. *Conversely, if* $A_p(G)$ *has bounded approximate identities for any* p, $1 < p < \infty$, *then* G *is amenable.*

I shall indicate the details in the proof.

LEMMA 4. – *If* $A_p(G)$ *has bounded approximate identities then* $\left|\int k(x)\,dx\right| \leqslant |||k|||_p$ *for all* $k \in L_1(G)$.

Proof. – The convolution operator norm $|||k|||_p$ is equal to the norm of the linear functional $f \to \int f(x)\,k(x)\,dx$ on $A_p(G)$. If $A_p(G)$ has approximate identities of bound b then given and compact $K \subset G$ at $\varepsilon > 0$ there exists $f \in A_p(G)$ with $\|f\| \leqslant b$ such that $|1 - f| < \varepsilon$ on K. From this it follows that $\left|\int k(x)\,dx\right| \leqslant b\,|||k|||_p$. Applying the estimate to the n-fold convolution of k with itself we get

$$\left|\int k(x)\,dx\right|^n \leqslant b\,|||k|||_p^n\ ,$$

and hence the assertion.

Now Lemma 4 for p_0 implies the same assertion for p_0', and thence by convexity for $p = 2$. The assertion of Lemma 4 in case $p = 2$ is exactly the assertion that the identity representation of G is *weakly contained* in the regular representation on $L_2(G)$. This last implies (see the step i') $\Rightarrow$ ii') of [3 ; Proposition 18.3.6]) that the group G has the property

(P') The constant 1 can be approximated uniformly on compact sets by functions of the form $k * k^*$ with $k \in \mathcal{K}(G)$, where

$$k^*(x) = \overline{k}(x^{-1})\,.$$

Reiter [14 ; Ch. 8 § 3] shows the equivalence of (P′) with each of the properties (P_p), $1 \leqslant p < \infty$.

(P_p) Given a compact $K \subset G$ and $\varepsilon > 0$ there exists $\alpha \in L_p(G)$ with $\|\alpha\|_p = 1$ such that $\|\lambda(x)\alpha - \alpha\|_p < \varepsilon$ for all $x \in K$.

The properties (P_p) are equivalent to amenability in the sense of the existence of suitable invariant means. The details are given by Reiter [14 ; Ch. 8 § § 3-5]. The convenient point of departure for us is to view amenability in terms of the properties (P_p).

LEMMA 5. – *If* G *has property* (P_p) *then* $A_p(G)$ *has approximate identities of bound* 1.

Proof. – Given a compact set $K \subset G$ let F(K) be the convex hull of the of functions of the form $v * u$ where $u \in L_p(G)$, $v \in L_{p'}(G)$, $\|u\|_p \leqslant 1$, $\|v\|_{p'} \leqslant 1$, and supp u, supp $v \subset K$. The union of the sets F(K), as K ranges over the compact subsets of G, is dense in the unit ball $A_p(G)$. Hence it suffices to prove that for each compact K and each $\varepsilon > 0$ there exists $\varphi \in A_p(G)$ with $\|\varphi\|_1$ such that $\|\varphi f - f\| < 2\varepsilon$ for all $f \in F(K)$. For $1 < p \leqslant 2$ let $\alpha \in L_p(G)$ be the element corresponding to K^{-1} and ε in the definition of (P_p) ; if $p > 2$ use $\varepsilon/2p$. We may suppose $\alpha \geqslant 0$; otherwise replace α by $|\alpha|$. Put $\beta = \alpha^{p-1}$ and $\varphi = \beta * \check{\alpha}$. It is clear that $\varphi \in A_p(G)$ and $\|\varphi\| = 1$. We have to show that $\|\varphi(v * \check{u}) - v * \check{u}\| < 2\varepsilon$ where u and v are as in the definition on F(K). To do this we define U, $U' \in L_p(G ; L_p(G))$ by $U(x) = u(x)\alpha$, $U'(x) = u(x)\lambda(x^{-1})\alpha$, i.e. $U(x, \xi) = u(x)\alpha(\xi)$, $U'(x, \xi) = u(x)\alpha(x\xi)$. Similarly we define V, $V' \in L_{p'}(G ; L_{p'}(G))$ by $V(x) = v(x)\beta$, $V'(x) = v(x)\lambda(x^{-1})\beta$. Let

$$P : L_p(G ; L_p(G)) \otimes L_{p'}(G ; L_{p'}(G)) \to A_p(G)$$

be the canonical morphism. Then

$$P(U \otimes V) = v * \check{u} \quad \text{and} \quad P(U' \otimes V') = \varphi(v * \check{u}).$$

Therefore we have

$$\begin{aligned} \|\varphi(v * \check{u}) - (v * \check{u})\| &\leqslant \|U' \otimes V' - U \otimes V\| \\ &\leqslant \|(U' - U) \otimes V\| + \|U' \otimes (V' - V)\| \\ &\leqslant \|U' - U\|_p \, \|V\|_{p'} + \|U'\|_p \, \|V' - V\|_{p'} \end{aligned}$$

Now $\|V\|_{p'} = \|v\|_{p'}$ while $\|U' - U\|_p \leqslant \|u\|_p \cdot \sup \|\lambda(x^{-1})\alpha - \alpha\|_p$ with the supremum taken over $x \in \operatorname{supp} u$. Thus $\|U' - U\|_p \, \|V\|_{p'} < \varepsilon$ and a similar estimate holds for $\|U'\|_p \, \|V' - V\|_{p'}$ since

$$\|\lambda(x^{-1})\beta - \beta\|_{p'} < \varepsilon \quad \text{for} \quad x \in K.$$

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