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FRANCISCO BALIBREA
JOSE SALVADOR CÁNOVAS PEÑA
VÍCTOR JIMÉNEZ LÓPEZ

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COMMUTATIVITY AND NON-COMMUTATIVITY OF TOPOLOGICAL SEQUENCE ENTROPY

by F. BALIBREA, J.S. CÁNOVAS PEÑA
and V. JIMÉNEZ LÓPEZ

1. Introduction.

At the *"Thirty Years after Sarkovskii's Theorem. New Perspectives"* Conference, held in La Manga (Spain) in 1994, S. Kolyada and L'. Snoha announced that the equality $h(f \circ g) = h(g \circ f)$ holds true when f, g are continuous maps from a compact space X into itself and $h(q)$ denotes the topological entropy of q (see [9]). Algebra "teaches" us that the maps $f \circ g$ and $g \circ f$ are essentially different, so some people in the audience considered this result to be a rather surprising one.

In second thoughts, however, Kolyada and Snoha's formula is just natural. Notice that $f \circ g$ and $g \circ f$ are "more or less" topologically conjugate via the map g (and similarly for $g \circ f$ and $f \circ g$ via f) and then their dynamics "should" substantially coincide. For instance it is immediate to check that if $x \in X$ is a periodic point of $f \circ g$ with period k then $g(x)$ is a periodic point of $g \circ f$ with the same period, so the periodic structure of $f \circ g$ and $g \circ f$ is exactly the same. Adding to this the well known fact that (when $X = [0, 1]$) positive topological entropy is equivalent to the existence of periodic points of period not a power of two, we have just arrived to this weaker version of Kolyada and Snoha's result: $h(f \circ g) > 0$ if and only if $h(g \circ f) > 0$.

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Topological sequence entropy is a natural (and useful, see e.g. [5]) extension of topological entropy and one should reasonably expect that the commutativity property also holds for it. In fact in [2] we gave an elementary proof of Kolyada and Snoha's result and showed that $h_A(f \circ g) = h_A(g \circ f)$ is true if the maps f, g are onto (here $h_A(q)$ denotes the topological sequence entropy of q respect to the sequence A). Hence the first result of the present paper (Theorem 3.1), showing that if $X = [0,1]$ then $h_A(f \circ g) = h_A(g \circ f)$ is true without any additional restrictions on the maps f and g, will probably not surprise the reader. However, we will construct in Section 4 (Theorem 4.5) a compact subset X of the interval $[0,1]$ and continuous maps $f, g : X \to X$ such that $h_A(f \circ g) \neq h_A(g \circ f)$ for the sequence $A = (2^i)_{i=1}^{\infty}$. As a by-product we also show that the formula $h_A(f) = h_A(f|_{\cap_{n\geqslant 0} f^n(X)})$ (a standard one for topological entropy) is true for any continuous map f if $X = [0,1]$, but does not hold for the space X and the map f from the counterexample in Section 4 (cf. Theorem 3.6 and Remark 4.6).

Of course one may wonder whether comparing the dynamics of compositions $f \circ g$ and $g \circ f$ makes any special sense. As a justification let us remark that in his recent doctoral dissertation [10] A. Linero has studied some properties of the maps $F : [0,1]^2 \to [0,1]^2$ of the type $F(x,y) = (f(y), g(x))$, where f and g are continuous maps from the interval $[0,1]$ into itself. These maps have been long time used in Game Theory (see e.g. [4]). Although their dynamical nature may seem essentially "one-dimensional" they pose some interesting non-trivial problems (e.g. the structure of their ω-limit sets), and could be seen as a "bridge" towards more complicated two-dimensional dynamics. When studying maps of the above type the comparison between properties of $f \circ g$ and $g \circ f$ arises in a completely natural way.

2. Basic notation and definitions.

Throughout the paper $A = (a_i)_{i=1}^{\infty}$ will always denote a (non necessarily strictly) increasing unbounded sequence of positive integers, except in Section 4 when $A = (2^i)_{i=1}^{\infty}$.

T. N. T. Goodman introduced in [6] the notion of topological sequence entropy as an extension of the concept of topological entropy. If $f : X \to X$ is a continuous map on a compact topological space X, A is a sequence and $\mathcal{C}$ is a finite open cover of X we define the *topological sequence entropy of*

f relative to $\mathcal{C}$ (respect to the sequence A) as

$$h_A(f,\mathcal{C}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^{n} f^{-a_i}\mathcal{C}\right),$$

where $f^{-a}\mathcal{C} = \{f^{-a}(C) : C \in \mathcal{C}\}$, $\bigvee_{i=1}^{n} \mathcal{C}_i = \left\{\bigcap_{i=1}^{n} C_i : C_i \in \mathcal{C}_i\right\}$ and $\mathcal{N}(\mathcal{D})$ denotes the minimal possible cardinality of a subcover chosen from $\mathcal{D}$. We define the *topological sequence entropy of f (respect to the sequence A)* as

$$h_A(f) = \sup_{\mathcal{C}} h_A(f,\mathcal{C}).$$

When $A = (i)_{i=1}^{\infty}$ we get the standard topological entropy introduced by Adler, Konheim and McAndrew in [1].

If (X,d) is a compact metric space then there is a useful equivalent definition of topological sequence entropy, also introduced by Goodman in [6]. Let $f : X \to X$ be a continuous map, let A be a sequence, let Y be a subset of X and set $\epsilon > 0$. We say that a set $E \subset Y$ is *(A,n,ϵ,Y,f)–separated (by f)* if for any $x,y \in E$, $x \neq y$, there exists $k \in \{1,2,...,n\}$ such that $d(f^{a_k}(x), f^{a_k}(y)) > \epsilon$. Denote by $s_n(A,\epsilon,Y,f)$ the biggest cardinality of any (A,n,ϵ,Y,f)–separated set in Y. Define

$$s(A,\epsilon,Y,g) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(A,\epsilon,Y,g).$$

Now we define the *topological sequence entropy of f on the set Y (respect to the sequence A)* as

$$h_A(g,Y) = \lim_{\epsilon\to 0} s(A,\epsilon,Y,g).$$

It turns out that $h_A(g) = h_A(g,X)$.

As usual, $\mathbb{Z}$ will stand for the set of integer numbers, while if $Z \subset \mathbb{Z}$ then Z^n (resp. Z^∞) will denote the set of finite sequences of length n (resp. infinite sequences) of elements from Z. If $\theta \in \mathbb{Z}^n$ or $\alpha \in \mathbb{Z}^\infty$ then we will often describe them through their components as $(\theta_1,\theta_2,\ldots,\theta_n)$ or $(\alpha_i)_{i=1}^{\infty}$, respectively. The *shift map* $\sigma : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ is defined by $\sigma((\alpha_i)_{i=1}^{\infty}) = (\alpha_{i+1})_{i=1}^{\infty}$. If $f : X \to X$ is a map and $Y \subset X$ then $f|_Y$ will denote the restriction of f to Y. The cardinality of a set P will be denoted by Card$P$. $|I|$ will be the length of an interval I.

3. The interval case.

The main goal of this section to prove the following

THEOREM 3.1. — *Let $f, g : [0,1] \to [0,1]$ be continuous maps. Then $h_A(f \circ g) = h_A(g \circ f)$ for any sequence A.*

Let us begin with three general results. The first and second one were essentially proved in [2]; for completeness we give here their proof. Notice that the statement of Proposition 3.2 makes sense because since X is Hausdorff the spaces $\bigcap_{n\geqslant 0}(f\circ g)^n(X)$ and $\bigcap_{n\geqslant 0}(g\circ f)^n(X)$ are compact.

PROPOSITION 3.2. — *Let X be a compact Hausdorff space and let $f, g : X \to X$ be continuous maps. Suppose that $h_A(f \circ g) = h_A\left((f\circ g)|_{\cap_{n\geqslant 0}(f\circ g)^n(X)}\right)$ and $h_A(g\circ f) = h_A\left((g\circ f)|_{\cap_{n\geqslant 0}(g\circ f)^n(X)}\right)$ for any sequence A. Then $h_A(f \circ g) = h_A(g \circ f)$ for any sequence A. In particular, if f and g are onto then $h_A(f \circ g) = h_A(g \circ f)$ for any sequence A.*

Proof. — Put $A = (a_i)_{i=1}^{\infty}$ and let $\mathcal{D}$ be a finite open cover of $X_0 = \bigcap_{n\geqslant 0}(f\circ g)^n(X)$. Since $(f\circ g)|_{X_0}$ is surjective we have

$$\begin{aligned}
&h_{A+1}\left((f\circ g)|_{X_0}, \mathcal{D}\right)\\
&\quad= \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\left(\bigvee_{i=1}^{n}((f\circ g)|_{X_0})^{-a_i-1}\mathcal{D}\right)\\
&\quad= \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\left(((f\circ g)|_{X_0})^{-1}\left(\bigvee_{i=1}^{n}((f\circ g)|_{X_0})^{-a_i}\mathcal{D}\right)\right)\\
&\quad= \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\left(\bigvee_{i=1}^{n}((f\circ g)|_{X_0})^{-a_i}\mathcal{D}\right)\\
&\quad= h_A((f\circ g)|_{X_0}, \mathcal{D}),
\end{aligned}$$

where $A + 1 := (a_i + 1)_{i=1}^{\infty}$. Therefore

$$h_A((f\circ g)|_{X_0}) = h_{A+1}((f\circ g)|_{X_0})$$

and

$$h_A(f\circ g) = h_{A+1}(f\circ g).$$

On the other hand, if $\mathcal{C}$ is a finite open cover of X then

$$\begin{aligned} h_{A+1}(f\circ g,\mathcal{C}) &= \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\left(\bigvee_{i=1}^{n}(f\circ g)^{-a_i-1}(\mathcal{C})\right)\\ &= \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\left(g^{-1}\left(\bigvee_{i=1}^{n}(g\circ f)^{-a_i}(f^{-1}\mathcal{C})\right)\right)\\ &\leqslant \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\left(\bigvee_{i=1}^{n}(g\circ f)^{-a_i}(f^{-1}\mathcal{C})\right)\\ &= h_A(g\circ f, f^{-1}\mathcal{C})\\ &\leqslant h_A(g\circ f) \end{aligned}$$

and then

$$\begin{aligned} h_A(f\circ g) &= h_{A+1}(f\circ g)\\ &= \sup_{\mathcal{C}} h_{A+1}(f\circ g,\mathcal{C})\\ &\leqslant h_A(g\circ f). \end{aligned}$$

By a similar argument we obtain the reverse inequality. This finishes the proof. □

PROPOSITION 3.3. — *Let (X,d) be a compact metric space and let $f: X\to X$ be a continuous map. Let $Y\subset X$ and set $\epsilon>0$. Then*

$$s(A,\epsilon,Y,f)\geqslant s\left(\sigma^k(A),\epsilon,Y,f\right)\geqslant s(A,2\epsilon,Y,f)$$

for any k and any sequence A. In particular $h_A(f)=h_{\sigma^k(A)}(f)$ for any k and any sequence A.

Proof. — Let $A=(a_i)_{i=1}^{\infty}$ a sequence, fix n and k and let E_n, F_k and L_{n+k} respectively denote a $\left(\sigma^k(A),n,\epsilon,Y,f\right)$–separated set, an (A,k,ϵ,Y,f)–separated set and an $(A,n+k,2\epsilon,Y,f)$–separated set (all of them of the biggest possible cardinality).

Use the maximality of E_n to associate to each $x\in L_{n+k}$ a point $y_x\in E_n$ satisfying $d(f^{a_i}(x),f^{a_i}(y_x))\leqslant\epsilon$ for $i=k+1,...,n+k$. Notice that there are at most CardF_k points x from L_{n+k} associated to the same point $y\in E_n$; otherwise (because of the maximality of F_k) we could find two such points (let us call them x_1 and x_2) verifying $d(f^{a_i}(x_1),f^{a_i}(x_2))\leqslant\epsilon$ for any $1\leqslant i\leqslant k$. Since additionally $d(f^{a_i}(x_1),f^{a_i}(x_2))\leqslant d(f^{a_i}(x_1),f^{a_i}(y))+d(f^{a_i}(x_2),f^{a_i}(y))\leqslant 2\epsilon$ for any $k+1\leqslant i\leqslant n+k$, we arrive to a contradiction because both x_1 and x_2 belong to L_{n+k}.

Then we have

$$\begin{aligned} s(A, 2\epsilon, Y, f) &= \limsup_{n\to\infty} \frac{1}{n+k} \log \operatorname{Card} L_{n+k} \\ &\leqslant \limsup_{n\to\infty} \frac{1}{n+k} \log (\operatorname{Card} E_n \operatorname{Card} F_k) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Card} E_n \\ &= s\left(\sigma^k(A), \epsilon, Y, f\right). \end{aligned}$$

On the other hand, since any $\left(\sigma^k(A), n, \epsilon, Y, f\right)$–separated set is obviously an $(A, n+k, \epsilon, Y, f)$–separated set we get

$$\begin{aligned} s(A, \epsilon, Y, f) &= \limsup_{n\to\infty} \frac{1}{n+k} \log s_{n+k}(A, \epsilon, Y, f) \\ &\geqslant \limsup_{n\to\infty} \frac{1}{n} \log s_n\left(\sigma^k(A), \epsilon, Y, f\right) \\ &= s\left(\sigma^k(A), \epsilon, Y, f\right), \end{aligned}$$

which finishes the proof. □

PROPOSITION 3.4. — *Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. Suppose that there is a positive integer n_0 such that $\bigcap_{n\geqslant 0} f^n(X) = f^{n_0}(X)$. Then $h_A(f) = h_A\left(f\,|_{\cap_{n\geqslant 0} f^n(X)}\right)$ for any sequence A.*

Proof. — Let $\epsilon > 0$ and let $A = (a_i)_{i=1}^{\infty}$. Since A is increasing and unbounded, there exists a positive integer k_0 such that $f^{a_i}(x) \in f^{n_0}(X)$ for all $x \in X$ and $i > k_0$.

Let E be a $\left(\sigma^{k_0}(A), n, \epsilon, X, f\right)$–separated set. Since $f|_{f^{n_0}(X)}$ is surjective, for every $x \in X$ there exists $y_x \in f^{n_0}(X)$ such that $f^{a_i}(x) = f^{a_i}(y_x)$ if $i > k_0$. Define $F = \{y_x : x \in E\}$. Since E is separated by f, F and E has the same cardinality and F is a $(\sigma^{k_0}(A), n, \epsilon, f^{n_0}(X), f)$–separated set. This clearly implies $h_{\sigma^{k_0}(A)}(f) \leqslant h_{\sigma^{k_0}(A)}(f|_{f^{n_0}(X)})$. Since the reverse inequality is obvious it follows that $h_{\sigma^{k_0}(A)}(f) = h_{\sigma^{k_0}(A)}(f|_{f^{n_0}(X)})$. To conclude the proof it suffices to apply Proposition 3.3. □

Now we restrict ourselves to maps on the interval $[0, 1]$. According to Proposition 3.2, in order to prove Theorem 3.1 we just need to show that the topological sequence entropy of any map f and the topological sequence entropy of its restriction to the set $\bigcap_{n\geqslant 0} f^n([0, 1])$ are always the same.

First we prove an auxiliary lemma. In what follows we mean $[a,a]=\{a\}$.

LEMMA 3.5. — *Let* $f : [0,1] \to [0,1]$ *a continuous map, write* $\bigcap_{n=0}^{\infty} f^n([0,1]) = [a,b]$ *and suppose that* $f^n([0,1]) \neq [a,b]$ *for any* n. *Let* I_n^+ *and* I_n^- *be the (possibly empty) right and left-side components of* $f^n([0,1]) \setminus [a,b]$. *Then one of the following possibilities occurs:*

(a) *there exists* n_0 *such that* $I_{n_0}^- = \varnothing$ *and* $f(b) = b$;

(b) *there exists* n_0 *such that* $I_{n_0}^+ = \varnothing$ *and* $f(a) = a$;

(c) $I_n^- \neq \varnothing$ *and* $I_n^+ \neq \varnothing$ *for all* n *and* $f(\{a,b\}) \subset \{a,b\}$.

Proof. — Suppose that there exists n_0 with $I_{n_0}^- = \varnothing$ but $f(b) \neq b$. Since $f([a,b] \cup I_n^+) = [a,b] \cup I_{n+1}^+$ for any $n \geqslant n_0$, the uniform continuity of f implies $f(I_n^+) \subset [a,b]$ if n is large enough. On the other hand $f([a,b]) = [a,b]$ so we have in fact $f([a,b] \cup I_n^+) = [a,b]$. Hence $I_{n+1}^+ = \varnothing$ is n is large enough, a contradiction. The case (b) is analogous to this one.

Now suppose that $I_n^- \neq \varnothing$ and $I_n^+ \neq \varnothing$ for all n and for example $f(a) \in (a,b)$. If n is sufficiently large then $f(I_n^-) \subset [a,b]$ and $[a,b] \not\subset f(I_n^+)$. Further, since $f(I_n^- \cup [a,b] \cup I_n^+) = I_{n+1}^- \cup [a,b] \cup I_{n+1}^+$, $f(I_n^+)$ must intersect I_{n+1}^- and then $f(I_n^+) \subset I_{n+1}^- \cup [a,b]$. Hence $f(I_n^- \cup [a,b] \cup I_n^+) \subset I_{n+1}^- \cup [a,b]$ and $I_{n+1}^+ = \varnothing$, a contradiction. □

Finally:

THEOREM 3.6. — *Let* $f : [0,1] \to [0,1]$ *be a continuous map. Then* $h_A(f) = h_A\left(f\,|_{\cap_{n\geqslant 0} f^n([0,1])}\right)$ *for any sequence* A.

Proof. — If $\bigcap_{n\geqslant 0} f^n([0,1]) = \{a\}$ then all trajectories $(f^n(x))_{n=0}^{\infty}$ tend to a; applying [5] we get $h_A(f) = 0$ for any A and the proof is finished. Hence we can assume that $[a,b] = \bigcap_{n\geqslant 0} f^n([0,1])$ is a non-degenerate interval. In view of Proposition 3.4, we can also assume that $[a,b]$ is strictly included in $f^n([0,1])$ for any n. From now on we keep the notation from Lemma 3.5 and assume that we are in case (c) there with $f(a) = a$, $f(b) = b$; in the other cases one can argue in a similar fashion.

Let $A = (a_i)_{i=1}^{\infty}$ be a fixed arbitrary sequence and take $\epsilon > 0$. Since $f(a) = a$ and $f(b) = b$ there must exist a number n_0 such that $\max\{|I_{n_0}^+|, |I_{n_0}^-|\} < \epsilon/2$, $f(I_{n_0}^-) \subset I_{n_0+1}^- \cup [a,b]$ and $f(I_{n_0}^+) \subset I_{n_0+1}^+ \cup [a,b]$.

Now we construct a partition of the set $[0,a] \setminus I_{n_0}^-$ into intervals $P_1, ..., P_k$ of length smaller than $\epsilon/2$ just taking care that each of them belongs to some interval $I_i^- \setminus I_{i+1}^-$, $0 \leqslant i \leqslant n_0 - 1$. Notice that $P_r \cap f^m(P_r) = \varnothing$ for any $m \geqslant 1$ and any r because $f(I_n^-) \subset I_{n+1}^- \cup [a,b] \cup I_{n+1}^+$ for any $n \geqslant 0$. We can similarly construct a partition of $[b,1] \setminus I_{n_0}^+$ into intervals $M_1, ..., M_l$ of length smaller than $\epsilon/2$ so that $M_s \cap f^m(M_s) = \varnothing$ for any $m \geqslant 1$ and any s.

Fix now $m \geqslant 1$. We intend to compare the numbers $s_m(A, \epsilon, [0,1], f)$ and $s_m(A, \epsilon/2, [a,b], f)$. To do this our first step will be to assign to each $x \notin [a,b]$ a code $\alpha(x) = (B_1, B_2, \ldots, B_{r(x)})$, where $r(x)$ is the largest number i in the set $\{0, 1, ..., m\}$ with the property $f^{a_i}(x) \notin [a,b]$. (In particular $r(x) = 0$ means that $f^{a_i}(x) \in [a,b]$ for any i, so a point x may possibly have an "empty" code.) Namely, $B_i \in \{P_1, ..., P_k, M_1, ..., M_l, I_{n_0}^-, I_{n_0}^+\}$ is such that $f^{a_i}(x) \in B_i$ for any $1 \leqslant i \leqslant r(x)$. Notice that

(1)

if $\alpha(x) = \alpha(y)$ then $|f^{a_i}(x) - f^{a_i}(y)| < \epsilon/2$ for any $1 \leqslant i \leqslant r(x) = r(y)$.

We claim that if Λ is the set of possible codes for points from $[0,1] \setminus [a,b]$ then

$$\operatorname{Card} \Lambda \leqslant 1 + m2^{k+l+1} \tag{2}$$

(indeed this bound can be easily improved but we will not need it). Since there is just one "empty" code it suffices to check that the number of codes of a given length $1 \leqslant r \leqslant m$ is at most 2^{k+l+1}. This is easy to do because due to the way we have chosen n_0 and constructed our partitions above each of such codes must begin with a sequence of elements from $\{P_1, ..., P_k, M_1, ..., M_l\}$ (with each of these elements appearing at most once in the sequence) and finish with a string either of the type $I_{n_0}^-, I_{n_0}^-, \ldots, I_{n_0}^-$ or of the type $I_{n_0}^+, I_{n_0}^+, \ldots, I_{n_0}^+$. This amounts to a possible number of at most

$$2\left[\binom{k+l}{0} + \binom{k+l}{1} + \cdots + \binom{k+l}{k+l}\right] = 2^{k+l+1}$$

different codes.

For any $\alpha \in \Lambda$ let $Z(\alpha)$ denote the set of points from $[0,1] \setminus [a,b]$ having code α. We next prove that

$$s_m(A, \epsilon, Z(\alpha), f) \leqslant s_m(A, \epsilon/2, [a,b], f). \tag{3}$$

If the length of α is m then the statement is obvious: if $x \in Z(\alpha)$ then $\{x\}$ is an $(A, m, \epsilon, Z(\alpha), f)$–separated set with maximal cardinality by (1) so $s_m(A, \epsilon, Z(\alpha), f) = 1$.

Assume now that α has length $r < m$, let E be a fixed $(A, m, \epsilon/2, [a, b], f)$–separated set with maximal cardinality and let F be an $(A, m, \epsilon, Z(\alpha), f)$–separated set. We need to show that F has less cardinality than E. We will do this by associating to any point $x \in F$ a point $z_x \in E$ so that $z_{x_1} \neq z_{x_2}$ if $x_1 \neq x_2$. Namely, since $f|_{[a,b]}$ is surjective for any $x \in F$ there exists $y_x \in [a, b]$ such that $f^{a_{r+1}}(x) = f^{a_{r+1}}(y_x)$; since E is maximal, for this y_x there exists $z_x \in E$ such that $|f^{a_i}(y_x) - f^{a_i}(z_x)| < \epsilon/2$ for $1 \leqslant i \leqslant m$.

Indeed suppose that $z := z_{x_1} = z_{x_2}$ for some $x_1, x_2 \in F$. Notice that $|f^{a_i}(x_1) - f^{a_i}(z)| < \epsilon/2$ and $|f^{a_i}(x_2) - f^{a_i}(z)| < \epsilon/2$ (and hence $|f^{a_i}(x_1) - f^{a_i}(x_2)| < \epsilon$) for any $r < i \leqslant m$; on the other hand $|f^{a_i}(x_1) - f^{a_i}(x_2)| < \epsilon/2 < \epsilon$ for any $1 \leqslant i \leqslant r$ by (1). Since F is separated this is impossible unless $x_1 = x_2$.

We are finally ready to compare $s_m(A, \epsilon, [0, 1], f)$ and $s_m(A, \epsilon/2, [a, b], f)$. By (2) and (3),

$$\begin{aligned} s_m(A, \epsilon, [0, 1], f) &\leqslant s_m(A, \epsilon, [a, b], f) + \sum_{\alpha \in \Lambda} s_m(A, \epsilon, Z(\alpha), f) \\ &\leqslant s_m(A, \epsilon/2, [a, b], f) + \operatorname{Card} \Lambda s_m(A, \epsilon/2, [a, b], f) \\ &= \left(2 + m2^{l+k+1}\right) s_m(A, \epsilon/2, [a, b], f). \end{aligned}$$

Since the numbers k and l only depend on ϵ, we conclude $s(A, \epsilon, [0, 1], f) \leqslant s(A, \epsilon/2, [a, b], f)$ for any ϵ and hence $h_A(f) \leqslant h_A\left(f|_{[a,b]}\right)$. Since the converse inequality is obvious, the proof is finished. □

4. The general case.

This section is devoted to construct a counterexample for the commutativity formula in the setting of an appropiate compact metric space. To do this we will need some information concerning so-called *weakly unimodal maps of type* 2^∞.

Let $f : [0, 1] \to [0, 1]$ be a continuous map. We say that f is *weakly unimodal* if $f(0) = f(1) = 0$, it is non-constant and there is $c \in (0, 1)$ such that $f|_{(0,c)}$ and $f|_{(c,1)}$ are (non necessarily strictly) monotone. Recall

that a point $x \in [0,1]$ is said to be *periodic (for f)* if there exists a positive integer n such that $f^n(x) = x$. The smallest integer satisfying this condition is called the *period* of x. The map f is said to be *type* 2^∞ if it has periodic points of period 2^n for any $n \geqslant 0$ but no other periods.

Weakly unimodal maps of type 2^∞ (briefly, *w-maps*) were extensively studied in [8]. For these maps a kind of "symbolic dynamics" was introduced which could be seen as an extension of a standard tool in one-dimensional dynamics, "the adding machine" [7] (in our case we could better speak of a "substracting machine" as we will see later). Namely, in [8] was showed that for any w-map f it is possible construct a family $\{K_\alpha(f)\}_{\alpha\in\mathbb{Z}^\infty}$ (or simply $\{K_\alpha\}_{\alpha\in\mathbb{Z}^\infty}$ once there is no ambiguity on f) of pairwise disjoint (possibly degenerate) compact subintervals of $[0,1]$ satisfying the key properties (P1)-(P4) described below. As always when symbolic dynamics is concerned we hope to associate to any point a code (in this case, if $x \in K_\alpha$ then its code is α) so that we can deduce some valuable information about the dynamical behavior of the point from the combinatorial structure of its code. Although properties (P1)-(P4) comprise all the information we will need later, let us also remark for the sake of completeness that $\bigcup_{\alpha\in\mathbb{Z}^\infty} K_\alpha$ turns out to be strictly included in $[0,1]$ so *not* every point has a code. Hovewer (as shown in [8]), points without a code are not significant from a dynamical point of view: they are attracted by periodic orbits.

In what follows we denote $\mathbf{0} = (0,0,\ldots,0,\ldots)$ and $\mathbf{1} = (1,1,\ldots,1,\ldots)$, while if $\alpha \in \mathbb{Z}^\infty$ then $\alpha|_n \in \mathbb{Z}^n$ is defined by $\alpha|_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

The above-mentioned properties are the following:

(P1) The interval $K_{\mathbf{0}}$ contains all absolute maxima of f.

(P2) Define in $\mathbb{Z}^\infty$ the following total ordering: if $\alpha, \beta \in \mathbb{Z}^\infty$, $\alpha \neq \beta$ and k is the first integer such that $\alpha_k \neq \beta_k$ then $\alpha < \beta$ if either $\mathrm{Card}\{1 \leqslant i < k : \alpha_i \leqslant 0\}$ is even and $\alpha_k < \beta_k$ or $\mathrm{Card}\{1 \leqslant i < k : \theta_i \leqslant 0\}$ is odd and $\beta_k < \alpha_k$. Then $\alpha < \beta$ if and only if $K_\alpha < K_\beta$ (that is, $x < y$ for all $x \in K_\alpha$, $y \in K_\beta$).

(P3) Let $\alpha \in \mathbb{Z}^\infty$, $\alpha \neq \mathbf{0}$, and let k be the first integer such that $\alpha_k \neq 0$. Define $\beta \in \mathbb{Z}^\infty$ by $\beta_i = 1$ for $1 \leqslant i \leqslant k-1$, $\beta_k = 1 - |\alpha_k|$ and $\beta_i = \alpha_i$ for $i > k$ Then $f(K_\alpha) = K_\beta$. Also $f(K_{\mathbf{0}}) \subset K_{\mathbf{1}}$.

(P4) For any n and $\theta \in \mathbb{Z}^n$, let $K_\theta(f)$ (or just K_θ) be the least interval including all intervals K_α, $\alpha \in \mathbb{Z}^\infty$, such that $\alpha|_n = \theta$. Let $\alpha \in \mathbb{Z}^\infty$. Then $K_\alpha = \bigcap_{n=1}^{\infty} K_{\alpha|_n}$.

Additionally, for any fixed n you can easily check that the intervals K_θ, $\theta \in \mathbb{Z}^n$, are open and pairwise disjoint and (after replacing ∞ by n, $\mathbf{0}$ by $(0,0,\dots,\overset{n}{0})$ and $\mathbf{1}$ by $(1,1,\dots,\overset{n}{1})$) they also satisfy (P1)-(P3). For instance we have

$$K_{(0,3,1,-1,7,0)} < K_{(0,3,1,-1,7,1)}$$

and

$$K_{(0,3,1,-1,0,1)} < K_{(0,3,1,-1,0,0)},$$

while

$$f(K_{(7,0,0,1,2,0)}) = K_{(-6,0,0,1,2,0)}$$

and

$$f(K_{(0,0,0,1,6,0)}) = K_{(1,1,1,0,6,0)}.$$

In general, notice that if f maps K_θ over K_ϑ, $\theta, \vartheta \in \mathbb{Z}^n$, then $\sum_{i=1}^{n} |\vartheta_i| 2^{i-1} = \left(\sum_{i=1}^{n} |\theta_i| 2^{i-1}\right) - 1$ (except in the case $\theta = (0,0,\dots,0)$, $\vartheta = (1,1,\dots,1)$). This is the reason why we used the expression "substracting machine" before. Observe also that if $\theta \in \{-1,0,1\}^n$ and we put $|\theta| := (|\theta_1|, |\theta_2|, \dots, |\theta_n|)$ then $f^{2^n}(K_\theta) \subset K_{|\theta|}$; in particular $f^{2^n}(K_\theta) \subset K_\theta$ if $\theta \in \{0,1\}^n$.

The simple lemma below provides a last property of w-maps that will be used later. If $\alpha \in \mathbb{Z}^\infty$ then $K^-_{\alpha|_n}(f)$ and $K^+_{\alpha|_n}(f)$ (or simply $K^-_{\alpha|_n}$ and $K^+_{\alpha|_n}$) will denote the left and right-side components of $K_{\alpha|_n}(f) \setminus K_\alpha(f)$.

LEMMA 4.1. — *Let f be a w-map and let $\epsilon > 0$. Let $A_\epsilon = \{\alpha \in \mathbb{Z}^\infty : |K_\alpha| \geqslant \epsilon\}$. Then there exists a number n_ϵ such that for any $n \geqslant n_\epsilon$:*

(a) *if $\alpha \in A_\epsilon$ then $\max\{|K^+_{\alpha|_n}|, |K^-_{\alpha|_n}|\} < \epsilon$;*

(b) *if $\theta \in \mathbb{Z}^n$ and $\alpha|_n \neq \theta$ for any $\alpha \in A_\epsilon$ then $|K_\theta| < \epsilon$.*

Proof. — Let $\alpha \in \mathbb{Z}^\infty$. Since $(K_{\alpha|_n})_{n=1}^\infty$ decreases to K_α by (P4), if n is large enough then $\max\{|K^+_{\alpha|_n}|, |K^-_{\alpha|_n}|\} < \epsilon$. In particular, since A_ϵ is finite we have $\max\{|K^+_{\alpha|_n}|, |K^-_{\alpha|_n}|\} < \epsilon$ for all $\alpha \in A_\epsilon$ and all sufficient large n.

To finish the proof it then suffices to show that if n is large enough then $|K_\theta| < \epsilon$ for any $\theta \in \mathbb{Z}^n$ with the property $\alpha|_n \neq \theta$ for all $\alpha \in A_\epsilon$.

Suppose the contrary. Then there are a strictly increasing sequence $(n_j)_{j=1}^\infty$ and sequences $\theta^j \in \mathbb{Z}^{n_j}$ such that $|K_{\theta^j}| \geqslant \epsilon$ and $\alpha|_j \neq \theta^{n_j}$ for any

$\alpha \in A_\epsilon$. Let x_j be the midpoint of K_{θ^j}. It is clearly not restrictive to assume that $(x_j)_{j=1}^\infty$ converges to some x and $|x_j - x| < \epsilon/2$ for any j. Since for any fixed n all intervals K_θ, $\theta \in \mathbb{Z}^n$, are pairwise disjoint, this means that each pair K_{θ^j} and $K_{\theta^{j+1}}$ has non-empty intersection, which clearly implies $K_{\theta^{j+1}} \subset K_{\theta^j}$ for any j and hence the existence of an $\alpha \in \mathbb{Z}^\infty$ with $\alpha|_{n_j} = \theta^j$ for any j. Due to the definition of the intervals K_{θ^j}, α cannot belong to A_ϵ. However, $K_\alpha = \bigcap_{n=1}^\infty K_{\alpha|_n} = \bigcap_{j=1}^\infty K_{\theta^j}$ so $|K_\alpha| \geqslant \epsilon$, a contradiction. □

Let us finish our preparatory work with some additional notation. In the rest of this section, A will always denote the sequence $(2^i)_{i=1}^\infty$ and $\tilde{f}$ will denote a fixed w-map with the additional property that if $\alpha \in \mathbb{Z}^\infty$ then $K_\alpha(\tilde{f})$ is non-degenerate if and only if there is an $n \geqslant 0$ such that $\sigma^n(\alpha) = \mathbf{0}$ (recall that σ denotes the shift map; σ^0 is of course the identity map). An example of such a map is constructed in [8]; it is possible to show that the stunted tent map $\tilde{f}(x) = \max\{1 - |2x - 1|, \mu\}$ ($\mu \approx 0.8249...$) from [11] is also a w-map with this property. If $\theta \in \mathbb{Z}^n$ and $\vartheta \in \mathbb{Z}^m$ (with $m \leqslant \infty$) then $\theta * \vartheta \in \mathbb{Z}^{n+m}$ (where $n + \infty$ means ∞) will denote the sequence λ defined by $\lambda_i = \theta_i$ if $1 \leqslant i \leqslant m$ and $\lambda_i = \vartheta_{i-n}$ for any $i > m$. In particular we will consistently denote $2 * \{-1, 0, 1\}^\infty = \{2 * \alpha : \alpha \in \{-1, 0, 1\}^\infty\}$ and so on (here of course "2" denotes the one-component vector "(2)"). $\mathrm{Bd}Z$, $\mathrm{Cl}Z$ and $\mathrm{Int}Z$ will respectively denote the boundary, the closure and interior of Z.

We are now ready to construct our promised counterexample. First we have to choose our space X. We write

$$X_1 = \bigcup_{\alpha \in \{-1,0,1\}^\infty} \mathrm{Bd}K_\alpha$$

and

$$X_2 = \bigcup_{\alpha \in 2*\{-1,0,1\}^\infty} \mathrm{Bd}K_\alpha$$

and define $X = X_1 \cup X_2$. Let us emphasize that $\mathrm{Bd}K_\alpha$ consists of both endpoints of K_α if it is non-degenerate and of its only point if it is degenerate.

Next we need two appropriate maps $f, g : X \to X$. The first one is the restriction of the above-mentioned w-map $\tilde{f}$ to the set X. The map g is defined by fixing arbitrarily a point $x_0 \in X$ and putting $g(x) = \tilde{f}(x)$ if $x \in X_1$ and $g(x) = x_0$ if $x \in X_2$. The following lemma shows that at least the above choices make sense.

LEMMA 4.2. — *Both X_1 and X_2 (and hence X) are compact sets and both $f : X \to X$ and $g : X \to X$ are well defined continuous maps.*

Proof. — To begin with, note that

$$X_1 = \left(\bigcap_{n=1}^{\infty} \bigcup_{\theta \in \{-1,0,1\}^n} \mathrm{Cl} K_\theta \right) \Big\backslash \bigcup_{\alpha \in \{-1,0,1\}^\infty} \mathrm{Int} K_\alpha$$

by (P2) and (P4). Hence X_1 is compact. The compactness of X_2 can be similarly proved.

On the other hand recall that if $\mathbf{0} \neq \alpha \in \{-1,0,1\}^\infty \cup 2 * \{-1,0,1\}^\infty$ then $\tilde{f}$ carries the interval K_α onto K_β with β defined as in (P3) (and hence belonging to $\{-1,0,1\}^\infty$). Further $\tilde{f}$ is monotone on K_α because of (P1) so it maps the endpoints of K_α onto the endpoints of K_β. Similarly, since $K_{\mathbf{1}}$ is degenerate both endpoints of $K_{\mathbf{0}}$ are mapped onto its only point. The conclusion is that $f(X) \subset X_1$ and $g(X) \subset X_1 \cup \{x_0\}$ so both maps $f, g : X \to X$ are well defined (and are clearly continuous). Notice that the definition of g poses no additional problems since X_1 and X_2 are disjoint compact sets. □

Since $f(X) \subset X_1$ the map $f|_{X_1} : X_1 \to X_1$ is also well defined. In fact we have the following

LEMMA 4.3. — *With the notation above, $h_A(f|_{X_1}) = 0$.*

Proof. — Let $\epsilon > 0$ and take n_ϵ as in Lemma 4.1 (for $\tilde{f}$). Since K_α is non-degenerate if and only if $\sigma^n(\alpha) = \mathbf{0}$ for some n we can assume without loss of generality that n_ϵ is large enough so that if A_ϵ is defined as in Lemma 4.1 that $\alpha \in A_\epsilon$ implies $\sigma^{n_\epsilon}(\alpha) = \mathbf{0}$.

We intend to show that

$$s(\sigma^{n_\epsilon}(A), \epsilon, K_\theta \cap X_1, f) = 0 \tag{4}$$

for any $\theta \in \{-1,0,1\}^{n_\epsilon}$. In fact, according to Lemma 3.3 this would give $s(A, 2\epsilon, K_\theta \cap X_1, f) = 0$ for any $\theta \in \{-1,0,1\}^{n_\epsilon}$. Moreover, since $X_1 \subset \bigcup_{\theta \in \{-1,0,1\}^{n_\epsilon}} K_\theta$ we could reason as in Theorem 7.5 from [13] to deduce

$$s(A, 2\epsilon, X_1, f) \leqslant \max \{ s(A, 2\epsilon, K_\theta \cap X_1, f) : \theta \in \{-1,0,1\}^{n_\epsilon} \}$$

and hence get $s(A, 2\epsilon, X_1, f) = 0$. Since ϵ was arbitrarily chosen, this would imply the lemma.

Let $\theta \in \{-1,0,1\}^{n_\epsilon}$. Recall that property (P3) implies that $\tilde{f}^{2^m}(K_\theta) \subset K_{|\theta|}$ for any $m \geqslant n_\epsilon$. If $|\theta| \neq \alpha|_{n_\epsilon}$ for any $\alpha \in A_\epsilon$ then $|K_{|\theta|}| < \epsilon$ by Lemma 4.1 so any $(\sigma^{n_\epsilon}(A), n, \epsilon, K_\theta \cap X_1, f)$–separated set must consist of exactly one point; this gives (4).

Now assume that there is a (fixed) $\alpha \in A_\epsilon$ such that $|\theta| = \alpha|_{n_\epsilon}$. Let $n \geqslant 1$ and let E be a $(\sigma^{n_\epsilon}(A), n, \epsilon, K_\theta \cap X_1, f)$–separated set. We will show that $\mathrm{Card}E \leqslant 2n+1$. This obviously implies (4).

Put $q = f^{2^{n_\epsilon}}$ and write $L := \mathrm{Cl}K^-_{\alpha|_{n_\epsilon}}$ and $R := \mathrm{Cl}K^+_{\alpha|_{n_\epsilon}}$ (cf. Lemma 4.1). Since $f^m(X_1) \cap \mathrm{Int}K_\alpha = \varnothing$ for any m, we can associate to each point $x \in K_\theta \cap X_1$ a code $c(x) = (C_1, C_2, \ldots, C_n)$ of L's and R's given by $C_i = L$ or $C_i = R$ according to whether $q^i(x) \in L$ or $q^i(x) \in R$, $1 \leqslant i \leqslant n$. Since both L and R have length less than ϵ by Lemma 4.1, it turns out that if two different points x, y belong to E then $c(x) \neq c(y)$. Hence we just need to show that the number of possible codes for points from $K_\theta \cap X_1$ cannot exceed $2n+1$.

We will for example assume that $|\theta|$ has an even number of zeros (the other case is analogous). Hence property (P2) implies that if $(0,0,\ldots,0) \neq \vartheta \in \{0,1\}^n$ and i is the first number j for which $\vartheta_j = 1$ then we have $K_{|\theta|*\vartheta} \subset L$ (resp. $K_{|\theta|*\vartheta} \subset R$) if i is even (resp. odd).

Let $x \in K_\theta \cap X_1$. We must consider several possibilities. If $x \in K_{\theta*(-1)} \cup K_{\theta*1}$ then $q(x) \in K_{|\theta|*0}$ and $q^{2^i}(x) \in K_{|\theta|*1}$ by (P3). Hence $c(x) = (R, R, \ldots, R)$ by force. If $x \in K_{\theta*(0,-1)} \cup K_{\theta*(0,1)}$ then $q^2(x) \in K_{|\theta|*(0,0)}$ but $q^{2^i}(x) \in K_{|\theta|*(0,1)}$ for any $i > 1$ again by (P3), so $c(x) = (C, L, L, \ldots, L)$ (here "C" indistinctly means L or R). In general, if $x \in K_{\theta*(0,\ldots,0,-1)}^{\ i} \cup K_{\theta*(0,\ldots,0,1)}^{\ i}$, $1 \leqslant i \leqslant n$, it is rutinary to check that $c(x) = (L, R, L, R, \ldots, \overset{i-1}{C}, L, L, \ldots, L)$ if i is even and $c(x) = (L, R, L, R, \ldots, \overset{i-1}{C}, R, R, \ldots, R)$ if i is odd. Finally, if $x \in K_{|\theta|*(0,0,\ldots,0)}^{\ n}$ then $c(x) = (L, R, L, R, \ldots, C)$.

Since

$$K_\theta \cap X_1 \subset K_{|\theta|*(0,0,\ldots,0)}^{\ n} \cup \bigcup_{i=1}^{n} \left(K_{\theta*(0,\ldots,0,-1)}^{\ i} \cup K_{\theta*(0,\ldots,0,1)}^{\ i} \right),$$

there are at most $2n+1$ different codes as claimed. □

LEMMA 4.4. — *With the notation above,* $h_A(f) = \log 2$.

Proof. — We first show that $h_A(f) \geqslant \log 2$. To do this it suffices to find an (A, n, ϵ, X, f)–separated set E of cardinality 2^n for any fixed $n \geqslant 1$ and $\epsilon < |K_{\mathbf{0}}|$.

This is easy. Let E be a set containing exactly one point $x = x_\theta \in K_\theta \cap X$ for any $\theta \in 2 * \{-1, 1\}^n$, and let x_θ, x_ϑ be two different points from E. Let i be the last number $1 \leqslant j \leqslant n$ with the property $\theta_j = \vartheta_j$. Since $f^{2^i}(x_\theta) \in K_{(0,\ldots,0,\theta_{i+1},\ldots,\theta_{n+1})}$ and $f^{2^i}(x_\vartheta) \in K_{(0,\ldots,0,\vartheta_{i+1},\ldots,\vartheta_{n+1})}$ the interval $K_{\mathbf{0}}$ lies between $f^{2^i}(x_\theta)$ and $f^{2^i}(x_\vartheta)$ and then $|f^{2^i}(x_\theta) - f^{2^i}(x_\vartheta)| > \epsilon$. Hence E is an (A, n, ϵ, X, f)–separated set.

To get the inequality $h_A(f) \leqslant \log 2$ we must essentially repeat the proof of Lemma 4.3. Now the role of the set $\{-1, 0, 1\}^{n_\epsilon}$ is played by $\{-1, 0, 1\}^{n_\epsilon} \cup 2 * \{-1, 0, 1\}^{n_\epsilon - 1}$, when we extend the definition of $|\theta|$ for any $\theta \in 2 * \{-1, 0, 1\}^{n_\epsilon - 1}$ by writing $|\theta| := (0, |\theta_2|, \ldots, |\theta_{n_\epsilon}|)|$. Instead of (4) we must show now that

$$s(\sigma^{n_\epsilon}(A), \epsilon, K_\theta \cap X_1, f) \leqslant \log 2$$

for any $\theta \in \{-1, 0, 1\}^{n_\epsilon} \cup 2 * \{-1, 0, 1\}^{n_\epsilon - 1}$. This finally amounts to show that if θ is any of such sequences with the additional property that there is an $\alpha \in A_\epsilon$ with $|\theta| = \alpha|_{n_\epsilon}$ and E is a $(\sigma^{n_\epsilon}(A), n, \epsilon, K_\theta \cap X_1, f)$–separated set for some $n \geqslant 1$, then Card $E \leqslant 2^n$. Once we introduce the corresponding codes $c(x)$ we are done because trivially the number of possible sequences of L's and R's of length n cannot exceed 2^n. □

We are ready to obtain the main result of this section:

THEOREM 4.5. — *Let $A = (2^i)_{i=1}^{\infty}$, X and $f, g : X \to X$ be defined as above. Then $0 = h_A(f \circ g) < h_A(g \circ f) = \log 2$.*

Proof. — Notice that the $f \circ g$ is constant outside X_1. It is then easy to check that $h_A(f \circ g) = h_A((f \circ g)|_{X_1})$. Moreover, $(f \circ g)|_{X_1} = f^2|_{X_1}$ and then $h_A((f \circ g)|_{X_1}) = h_A(f^2|_{X_1}) = h_{2A}(f|_{X_1}) = h_{\sigma(A)}(f|_{X_1}) = h_A(f|_{X_1})$ by Proposition 3.3. Thus $h_A(f \circ g) = 0$ by Lemma 4.4.

On the other hand we clearly have $g \circ f = f^2$. By Proposition 3.3 again, $h_A(f^2) = h_A(f)$. Therefore $h_A(g \circ f) = \log 2$ by Lemma 4.4. □

Remark 4.6. — Recall that $f(X) \subset X_1$ so $\bigcap_{n=0}^{\infty} f^n(X) \subset X_1$. Therefore we get $h_A(f|_{\cap_{n=0}^{\infty} f^n(X)}) = 0$ by Lemma 4.3 while $h_A(f) = \log 2$ by Lemma 4.4. Hence the equality $h_A(f) = h_A(f|_{\cap_{n=0}^{\infty} f^n(X)})$ is not generally true (compare with Theorem 3.6).

Let A be a sequence, let X be a compact space and let $f : X \to X$ be a continuous map. If $\Omega(f)$ denotes the set of nonwandering points of f then $\Omega(f) \subset \bigcap_{n=0}^{\infty} f^n(X)$ as is well known, so the above example implies in particular that the formula $h_A(f) = h_A(f|_{\Omega(f)})$ does not necessarily hold. This fact was first pointed out by Szlenk in [12]; our counterexample is completely different to his and somewhat simpler. In fact the equality need not be true even in the case $X = [0,1]$: if $\tilde{f}$ is the w-map from Section 4 then it is possible to show that $h_A(\tilde{f}) = \log 2$ while $h_A(\tilde{f}|_{\Omega(\tilde{f})}) = 0$ (with of course $A = (2^i)_{i=1}^{\infty}$) [3].

Remark 4.7. — It seems that the commutativity formula for topological sequence entropy is essentially a one-dimensional phenomenon, the key being Lemma 3.5. In fact it can be analogously proved for circle maps, and we feel that it should even work for graph maps. On the other hand we conjecture that it is not necessarily true for maps of the square $[0,1]^2$.

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F. BALIBREA and V. JIMÉNEZ LÓPEZ,
Universidad de Murcia
Departamento de Matemáticas
Campus de Espinardo
Apto. de Correos 4021
30100 Murcia (Spain).
balibrea@fcu.um.es
vjimenez@fcu.um.es

J. S. CÁNOVAS PEÑA,
Universidad Politécnica de Cartagena
Departamento de Matemática Aplicada
Paseo de Alfonso XIII, 34–36
30203 Cartagena (Spain).
canovas@plc.um.es