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AMENABLE GROUPS AND CELLULAR AUTOMATA

by T. G. CECCHERINI-SILBERSTEIN(*), A. MACHÌ(**) and F. SCARABOTTI

1. Introduction.

Both the notion of a cellular automaton and that of an amenable group were introduced by von Neumann ([vN1], [vN2]). The present paper is an attempt to study the connections between these two notions. Although we have barely scraped the surface of the subject we believe that these connections are in fact very deep.

In the classical situation a cellular automaton works on the lattice $U = \mathbb{Z}^2$ of integer points of Euclidean plane. If S is a finite set (the *set of states*) a *configuration* is a map $c : U \to S$. A *transition map* is a map $\tau : \mathcal{C} \to \mathcal{C}$ from the set $\mathcal{C}$ of all configurations into itself such that the state $\tau[c](x)$ at a point $x \in U$ only depends on the states $c(y)$ at the *neighbours* y's of x. In Moore's original paper ([Mo]) two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in U are neighbours if either $|x_1 - y_1| + |x_2 - y_2| = 1$ or $|x_1 - y_1| = 1 = |x_2 - y_2|$ whereas in the von Neumann setting ([vN1]) only the first possibility occurs. In other words, from von Neumann's point of view U may be regarded as the Cayley graph of the group $\mathbb{Z}^2$ with respect to the generating system $A = \{(1,0), (-1,0), (0,1), (0,-1)\}$, and in the other one as the Cayley graph of the same group with respect to the generating system $A \cup \{(1,1), (1,-1), (-1,1), (-1,-1)\}$.

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One often speaks of τ as being time: if c is the configuration of the universe at time t, then $\tau[c]$ is the configuration of the universe at time $t+1$. An *initial configuration* is a configuration at time $t = 0$. A configuration c not in the image of τ, namely $c \in \mathcal{C} \setminus \tau[\mathcal{C}]$, is called a Garden of Eden configuration, briefly a *GOE configuration*, this biblical terminology being motivated by the fact that GOE configurations may only appear as initial configurations.

Given a non-empty finite connected subset $F \subset U$, a *pattern* of *support* F is a map $p : F \to S$. A pattern p is called GOE if any configuration extending p outside its support F is a GOE configuration: $\tau(c)_{|F} \neq p$ for all $c \in \mathcal{C}$. It can be seen, using topological methods, that the existence of GOE patterns is equivalent to that of GOE configurations (see [MaMi] Sec. 5).

Two distinct patterns p_1 and p_2 with a common support F are said to be *mutually erasable*, briefly ME, if for all configurations c_1 and c_2 that are equal outside F and agree with p_1 and p_2, respectively, on F, we have $\tau(c_1) = \tau(c_2)$.

If we consider the existence of mutually erasable patterns as "non-injectivity" of the transition map τ then the equivalence of *surjectivity* and *injectivity* for maps of a finite set into itself or linear transformations of a finite dimensional vector space holds also in this setting: this is the content of the theorems of Moore and Myhill.

THEOREM 1 (Moore [Mo] and Myhill [My]). — *Let U denote the lattice $\mathbb{Z}^2$ of integral points in Euclidean plane, $\mathcal{C}$ the set of all configurations, and $\tau : \mathcal{C} \to \mathcal{C}$ a transition map. Then a necessary and sufficient condition for the existence of GOE patterns is the existence of mutually erasable patterns.*

Necessity is due to Moore and sufficiency to Myhill.

As shown in [MaMi], this theorem depends on the growth of the *universe*. In the classical case, the universe is the lattice $\mathbb{Z}^2$ of integer points of Euclidean plane, which has quadratic growth; it is easy to extend this result to $\mathbf{Z}^n$ and more generally to universes, i.e. Cayley graphs of groups (see Section 2 for the corresponding definitions), of polynomial growth.

In [MaMi] the theorems of Moore and Myhill are proved for universes of sub-exponential growth: the interest of this result relies on the existence of groups with non-polynomial sub-exponential growth, namely

Grigorchuk's intermediate growth groups ([G]). These groups are amenable, so it is natural to inquire if one can generalize these results to the wider class of amenable groups, thus extending the results of [MaMi] also to certain groups of exponential growth such as the Baumslag-Solitar groups $\langle a, b : a^{-1}ba = b^n \rangle$, $|n| \geq 2$ (see e.g. [CG]).

As remarked in [MaMi], the original proof of the theorems of Moore and Myhill relies on the following properties of the standard Cayley graph $\mathcal{G}$ of $\mathbb{Z}^2$:

(a) Given any square L in $\mathcal{G}$, there exists a nested sequence $B_1 \subset B_2 \subset \cdots$ of squares in $\mathcal{G}$ such that $\mathcal{G} = \bigcup_{k=1}^{\infty} B_k$ and such that each B_k is a disjoint union of copies of L.

(b) For any nested sequence $B_1 \subset B_2 \cdots$ of squares in $\mathcal{G}$ such that $\mathcal{G} = \bigcup_{k=1}^{\infty} B_k$ the ratio

$$\frac{\text{number of vertices of } \mathcal{G} \text{ at distance} \leq 1 \text{ of } \partial B_k}{\text{number of vertices in } B_k}$$

tends to zero when k tends to infinity.

The point is that a similar sequence exists also for universes which are the Cayley graphs of amenable groups (by Følner's theorem), so that the theorems of Moore and Myhill hold in this more general setting as well.

2. Cellular automata on Cayley graphs.

In this section we review some definitions and notations from [MaMi].

Let G be a finitely generated group and let $A = \{a_1, a_2, \ldots, a_N\}$ denote a finite and symmetric ($a \in A \rightarrow a^{-1} \in A$) set of generators. We assume that A does not contain the unit element e of G. The *Cayley graph* $\mathcal{G} = \mathcal{G}_A(G)$ of G relative to A is the graph with vertex set G and edge set the set of pairs $\{g, ag\}$ with $g \in G$ and $a \in A$. We denote by $B(g; n)$ the *ball* of radius n and center g consisting of all vertices in $\mathcal{G}$ whose graph distance from g is at most n.

A (deterministic) *cellular automaton* on a graph $\mathcal{G}$ is a triple $\mathcal{A} = (S, \mathcal{G}, f)$ where

(i) S is a finite set, $|S| = s > 1$, called the set of *states* or the *alphabet*;

(ii) $\mathcal{G}$ is the Cayley graph of a finitely generated group G;

(iii) $f : S^{B(e;1)} \ni \overline{s} = (s_e, s_{a_1}, s_{a_2}, \dots, s_{a_N}) \mapsto f(\overline{s}) \in S$ is a function, called the *local map*.

A *configuration* is a map $c : \mathcal{G} \to S$ that associates a state with each vertex of $\mathcal{G}$. We shall denote by $\mathcal{C}$ the set of all configurations.

The *transition map* is the function $\tau : \mathcal{C} \mapsto \mathcal{C}$ defined by

$$\tau[c](g) = f\left(c(g), c(a_1 g), c(a_2 g), \dots, c(a_N g)\right).$$

Note that the state at vertex g in the configuration $\tau[c]$ only depends on the pattern $c_{|B(g;1)}$ that is, on the states at the neighborhood $B(g;1)$ in the configuration c.

The notions of GOE configurations and of GOE or mutually erasable patterns are the same as those described in the Introduction for the group $\mathbb{Z}^2$.

Remark. — Observe that the Cayley graph $\mathcal{G} = \mathcal{G}_A(\mathbb{Z}^2)$, where $A = \{\pm(1,0), \pm(0,1)\}$ induces a tessellation of Euclidean plane into squares; in the classical framework one considers these squares, called *cells*, rather than the vertices of these squares (i.e. the vertices of the graph) like in [MaMi] and here. In this particular situation, the dual graph $\mathcal{G}^*$ of $\mathcal{G}$ (that is the graph whose vertices are the cells, and two cells are neighbour if they share a common edge) is clearly isomorphic to $\mathcal{G}$, so that the two points of view are the same. On the other side we shall consider groups whose Cayley graphs might not induce a tessellation of the plane (for instance free groups) or might even be non planar.

Given a subgraph F of $\mathcal{G}$ we denote by

$$F^+ = \bigcup_{g \in F} B(g;1), \quad F^- = \{g \in F \ : \ B(g;1) \subseteq F\} \ \text{ and } \partial F = F \setminus F^-$$

the *closure*, the *interior*, and the *boundary* of F, respectively.

Let F_1 and F_2 be two subgraphs of $\mathcal{G}$. We say that F_2 *contains* m *copies* of F_1 if there exist m isometric mappings (*embeddings*) $\sigma_j : F_1 \to F_2$ such that $\sigma_j(F_1) \cap \sigma_k(F_1) = \emptyset$ for $j \neq k$.

Moreover if c, c' are configurations we say that the pattern $c'_{|F_2}$ contains m copies of the pattern $c_{|F_1}$ if in addition

$$c'(\sigma_i(g)) = c(g) \ \text{ for all } g \in F_1.$$

For the proof of the main theorem of this paper we shall need the following lemmas from [MaMi]. The proofs are straightforward.

LEMMA 1. — *Let c be a configuration and let F be a subgraph of $\mathcal{G}$. Then the restriction $\tau(c)_{|F^-}$ only depends on $c_{|F}$ and not on c.*

LEMMA 2. — *Let F be a finite subgraph and suppose that F^+ is not the support of two mutually erasable patterns; then if two configurations c_1 and c_2 agree on $F^+ \setminus F^-$ and disagree on F^- then $\tau(c_1)_{|F} \neq \tau(c_2)_{|F}$.*

3. An example: Conway's game of life.

In order to give the reader an explicit example of a cellular automaton we present here the *game of life* of John H. Conway.

The set of states is $S = \{0, 1\}$; state 0 corresponds to *absence of life* while state 1 indicates *life*; therefore passing from 0 to 1 can be interpreted as *birth*, while passing from 1 to 0 corresponds to *death*. The universe $\mathcal{G}$ is the Cayley graph $\mathcal{G}_A(\mathbb{Z}^2)$ of $\mathbb{Z}^2$ in the sense of Moore (cf. Introduction), so that the ball $B((0,0);1)$ consists of the 9 points $(x_1, x_2) \in \mathbb{Z}^2$ with $|x_1| \leq 1$ and $|x_2| \leq 1$. The local map is defined as follows:

$$f(s_{(0,0)}, s_{(1,0)}, \ldots, s_{(-1,-1)}) = \begin{cases} 1 & \text{if} \begin{cases} \text{either} \sum\limits_{a \in A} s_a = 3 \\ \text{or} \sum\limits_{a \in A} s_a = 2 \text{ and } s_{(0,0)} = 1 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

In other words [BCG], Chapter 25, thinking of the transition map one has:

– *Birth.* A point that is *dead* at time t becomes *live* at time $t+1$ if and only if *three* of its neighbours are *live* at time t.

– *Survival.* A point that is *live* at time t will remain *live* at $t+1$ if and only if it has just *two or three live* neighbours at time t.

– *Death by overcrowding.* A point that is *live* at time t and has *four or more* of its neighbours *live* at t will be *dead* at time $t+1$.

– *Death by isolation.* A point that has *at most one live* neighbour at time t will be *dead* at time $t+1$.

In [BCG] an argument of Moore that we shall generalize in the course of the proof of the main theorem is used to show that there exist GOE

patterns with support a square of size 2.325.816.000; in other words, for the cellular automaton of the game of live, the transition map is not surjective.

PROBLEM (for *Lifenthusiasts*[1]). — Generalize the game of life over other universes such as *regular trees* (these can be thought of as Cayley graphs of free groups, if the degree is even, or, more generally of $\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ (the r-fold free product of the group with two elements) if the degree is r, with respect to the canonical sets of generators).

4. Amenable groups.

We recall the notion of amenability and the combinatorial characterization of discrete amenable groups due to Følner; for further details we refer to the monograph [Gr] or [CGH].

A discrete group G is *amenable* if it admits a G-invariant normalized measure, that is a map

$$\mu : \mathcal{P}(G) \to \mathbb{R}^+$$

where $\mathcal{P}(G)$ denotes the set of all subsets of G, such that: if $A, B \in \mathcal{P}(G)$, $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$ (*finite additivity*), $\mu(G) = 1$ (*normalization*) and $\mu(gA) = \mu(A)$ (*left invariance*), where if $g \in G$ and $A \in \mathcal{P}(G)$, $gA = \{ga : a \in A\}$.

Finite groups, abelian groups, solvable groups are all examples of amenable groups. It is easy to see that the free group $\mathbb{F}_2$ of rank 2 is non-amenable, and that subgroups of amenable groups are amenable [vN2]. Thus the class AG of amenable groups is contained in the class NF of groups which do not contain non-abelian free subgroups. Ol'shanski [O] has constructed groups in $NF \setminus AG$ showing that the inclusion is strict; other examples are given by the free Burnside groups $B(m, n)$ of rank $m \geq 2$ and odd exponent $n \geq 665$ ([A]). Følner's characterization of amenability is the following.

THEOREM 2 (Følner, [Gr]). — *A discrete group G is amenable if and only if given any finite subset $F \subset G$ and any $\varepsilon > 0$ there exists a finite subset K of G such that*

$$|FK \setminus K| < \varepsilon |K|.$$

[1] A word due to Robert T. Wainwright.

Remark. — It is easy to see that a finitely generated group of *sub-exponential growth* (that is $\limsup_{n\to\infty} \sqrt[n]{|B(e;n)|} = 1$) is amenable; indeed if G has sub-exponential growth, given $F \subset G$ finite and $\varepsilon > 0$, it suffices to take $K = B(e;n)$ with n large enough to satisfy Følner condition as in the above theorem. But there also exist amenable groups of exponential growth (see e.g. [CG]). Therefore Theorem 3 of the next section generalizes the main result of [MaMi].

5. The theorems of Moore and Myhill for amenable groups.

In this section we state and prove our main result.

THEOREM 3. — *Let G be a finitely generated amenable group and let $\mathcal{G}_A(G)$ denote the Cayley graph of G with respect to a finite and symmetric generating system A. Then for any cellular automaton $\mathcal{A} = (S, \mathcal{G}_A(G), f)$, there exist GOE patterns if and only if there exist ME patterns.*

Proof. — The proof is divided into two steps. In the first one (which is essentially Theorem 2 in [MaMi] and is included here for the sake of completeness), we show that, for any cellular automaton on the Cayley graph of a not necessarily amenable group, if a finite subgraph B contains m copies of a subgraph L which is support of a GOE pattern or of two mutually erasable patterns, respectively, then, under suitable conditions, B^+ is the support of two mutually erasable patterns or of a GOE pattern, respectively. In the second step we show that, by Følner's theorem, given a ball $L = B(e;n)$ of arbitrary radius n, a subgraph B containing m copies of L under the suitable condition we alluded to above always exists for a cellular automaton on the Cayley graph of an amenable group; since any finite subgraph containing the support of two mutually erasable patterns (of a GOE pattern) is itself the support of two mutually erasable patterns (of a GOE pattern) the assumption that L is a ball will not be restrictive, and the proof shall be complete.

Step 1. — Let L and B two finite subgraphs of $\mathcal{G}$ and assume that B contains m copies $\sigma_j(L)$ of L, $j = 1, 2, \ldots, m$. Let $\ell = |L|, b = |B|, s = |S|$ and $u = |\partial B|$ and suppose that

$$(*) \qquad s^{b-u} > (s^\ell - 1)^m s^{b-m\ell}.$$

Then

(i) If L is the support of two ME patterns, then B^- is the support of a GOE pattern;

(ii) if L is the support of a GOE pattern, then B^+ is the support of two ME patterns.

Proof of (i). — Let R (resp. R_j) be the equivalence relation defined on the set of patterns having support L (resp. $\sigma_j(L)$) by declaring equivalent two patterns if they are equal or mutually erasable. Under our assumptions on L the number of R-equivalence classes is at most $s^\ell - 1$. The R_j's induce an equivalence relation R^* on the set of patterns having support B as follows: $p_1 R^* p_2$ if $p_1(v) = p_2(v)$ for v not in one of the copies of L and $p_1|_{\sigma_j(L)}\ R_j\ p_2|_{\sigma_j(L)}$ for all j's. The number of equivalence classes of R^* is therefore at most $(s^\ell - 1)^m s^{b-m\ell}$. It is easy to see that two patterns with support B that are R^*-equivalent are either equal or mutually erasable. Therefore if $c_{1|B} R^* c_{2|B}$ Lemma 1 implies that $\tau(c_1)_{|B^-} = \tau(c_2)_{|B^-}$. The number of patterns having support B^- that are of the form $\tau(c)_{|B^-}$ for some $c \in \mathcal{C}$ is at most equal to the number of R^*-equivalence classes. The total number of patterns on B^- being s^{b-u}, $(*)$ implies that there exists a pattern on B^- that is not of the form $\tau(c)_{|B^-}$, that is, a GOE pattern.

Proof of (ii). — Assume the contrary. Lemma 2 applied to B^+ implies that the number of distinct patterns with support B of the form $\tau(c)_{|B}$ (and therefore not GOE) is at least equal to the number of distinct patterns on B^-; these are s^{b-u} in number. On the other hand, there can be no more than $(s^\ell - 1)^m s^{b-m\ell}$ patterns on B not containing at least one copy of the GOE under one of the σ_j's. Since a pattern containing a GOE pattern is also GOE, the number h of non-GOE patterns with support B satisfies the inequality

$$s^{b-u} \leq h \leq (s^\ell - 1)^m s^{b-m\ell}$$

which contradicts $(*)$.

Step 2. — Denote by $U = A \cup \{e\}$ the unit ball of G; then $L := U^n$ is the ball of radius n. By Følner's Theorem, given $\varepsilon > 0$ there exists a finite set $K = K(\varepsilon) \subset G$ such that

$$|U^n K \setminus K| < \varepsilon |K|.$$

Since $U^n K \supset U^{n-1} K \supset K$ and $U^n K \setminus K \supset U^n K \setminus U^{n-1} K$ one has

$$\frac{|U^n K \setminus U^{n-1} K|}{|U^n K|} \leq \frac{|U^n K \setminus K|}{|K|} < \varepsilon.$$

Setting $B = B(\varepsilon) = U^n K$ one has $B^- \supset U^{n-1}K$ so that $u = |B \setminus B^-| \leq |B \setminus U^{n-1}K| < \varepsilon b$, where $b = b(\varepsilon) = |B|$; hence $(1-\varepsilon)b < b - u$ and finally

$$(**) \qquad \frac{b}{b-u} < \frac{1}{1-\varepsilon} \to 1 \quad \text{when } \varepsilon \to 0^+.$$

On the other hand $U^n K = \bigcup_{k \in K} U^n k$ is a finite union of balls of radius n. Let $\{k_1, \ldots, k_m\}$ be a subset of K such that $U^n k_i \cap U^n k_j = \emptyset$ whenever $i \neq j$ and such that each $g \in U^n K$ belongs to $U^{3n} k_i$ for a suitable i (one chooses k_1 and eliminates all other k's such that $U^n k \cap U^n k_1 \neq \emptyset$; among all the remaining k's one chooses k_2 and repeats the operation inductively). Thus $U^n K \subset \bigcup_{i=1}^m U^{3n} k_i$ and with $\ell = |U^n| = |U^n k_i|$ one obtains

$$\frac{m\ell}{b} = \frac{|\bigcup_{i=1}^m U^n k_i|}{|U^n K|} \geq \frac{|\bigcup_{i=1}^m U^n k_i|}{|\bigcup_{i=1}^m U^{3n} k_i|} \geq \frac{m|U^n|}{m|U^{3n}|} = \frac{|U^n|}{|U^{3n}|}.$$

We are now able show that inequality $(*)$ holds provided ε is small enough. This inequality is equivalent, by taking logarithms, to

$$\frac{b}{b-u}\left[1 + \frac{m\ell}{b}\left(\frac{\log_s(s^\ell - 1)}{\ell} - 1\right)\right] < 1.$$

Now $m\ell/b \leq 1$ and $\log_s(s^\ell - 1)/\ell < 1$; thus the quantity in square brackets is positive and less than 1, and with $q := \dfrac{|U^n|}{|U^{3n}|}$ we have that

$$1 + q\left(\frac{\log_s(s^\ell - 1)}{\ell} - 1\right) =: t < 1$$

is an upper bound for it.

Choosing ε small enough, by $(**)$ one gets $\dfrac{b}{b-u} < s$ where $s = 1/t > 1$, and the proof is complete. □

We remark that, given a finite subgraph L which is the support of a GOE pattern (resp. two mutually erasable patterns), the above proof gives an upper bound for the size of a minimal subgraph B which contains L and supports two mutually erasable patterns (resp. a GOE pattern).

As in [MaMi], Sec. 5, we can restate Theorem 3 in the following form.

COROLLARY. — *Let G be a finitely generated amenable group and let $\mathcal{G}_A(G)$ denote the Cayley graph of G with respect to a finite and*

symmetric generating system A. Then for any cellular automaton $\mathcal{A} = (S, \mathcal{G}_A(G), f)$, there exist GOE configurations if and only if there exist ME patterns.

6. Cellular automata on groups with free subgroups.

In [MaMi], Sec. 6, examples of cellular automata on the Cayley graphs of the *non-amenable* groups $\mathbb{Z}_2 * \mathbb{Z}_3$ and $\mathbf{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ are given for which Theorem 3 fails to hold. The following examples, slight modifications of these, yield counterexamples for cellular automata on the free group $\mathbb{F}_2$, i.e. with the universe which is a 4-homogeneous tree.

Examples.

1. Let $\mathcal{G}$ be the Cayley graph of the free group $\mathbb{F}_2$ with respect to a free basis $\{a, b\}$; set $S_1 = \{0, 1\}$ and define the local map f_1 by

$$f_1(s_e, s_a, s_{a^{-1}}, s_b, s_{b^{-1}}) = \begin{cases} 1 \text{ if } \begin{cases} \text{either } s_a + s_{a^{-1}} + s_b + s_{b^{-1}} = 3 \\ \text{or } s_a + s_{a^{-1}} + s_b + s_{b^{-1}} \in \{1, 2\} \text{and } s_e = 1 \end{cases} \\ 0 \text{ otherwise.} \end{cases}$$

Let $p_1, p_2 : B(e; 2) \to S_1$ be the characteristic functions of the sets $B(e; 1)$ and $B(e; 1) \setminus \{e\}$; it is easy to check that p_1 and p_2 are mutually erasable patterns in $\mathcal{A}_1 = (S_1, \mathcal{G}, f_1)$. On the other side similar (and in fact even simpler) arguments as in [MaMi], p. 53–54, show the existence of no Garden of Eden patterns in $\mathcal{A}_1$. This shows that Moore's theorem does not hold for cellular automata on the free group $\mathbb{F}_2$.

2. We now define the second automaton $\mathcal{A}_2 = (S_2, \mathcal{G}, f_2)$. The set of states $S_2 = \{0, 1, 2, 3\}$ can be given the structure of a Klein group by setting $i + i = 0$ and $i + j = k$ if i, j and k are non zero and all different. The local map f_2 is defined by

$$f_2(s_e, s_a, s_{a^{-1}}, s_b, s_{b^{-1}}) = f(s_a, s_{a^{-1}}, s_b, s_{b^{-1}})$$

where f is as in (i)–(iii) of [MaMi], p. 55. The argument given there shows that for this cellular automaton there are GOE patterns but no mutually erasable patterns. Thus Myhill's theorem does not hold for cellular automata on the free group $\mathbb{F}_2$.

We now define the notion of *induction* for cellular automata.

DEFINITION. — *Let $\mathcal{A} = (S, \mathcal{G} = \mathcal{G}_A(H), f)$ be a cellular automaton on a group H with finite generating system $A = \{a_1, a_2, \ldots, a_n\}$. Suppose that H is a subgroup of a finitely generated group G and choose a (finite) generating system B for G containing A; set $\{b_1, b_2, \ldots, b_m\} = B \setminus A$. The cellular automaton $\widetilde{\mathcal{A}} = (\widetilde{S}, \widetilde{\mathcal{G}}, \tilde{f})$ defined by*

$$\widetilde{S} = S$$

$$\widetilde{\mathcal{G}} = \mathcal{G}_B(G)$$

$$\tilde{f}(s_e, s_{a_1}, s_{a_2}, \ldots, s_{a_n}, s_{b_1}, s_{b_2}, \ldots, s_{b_m}) = f(s_e, s_{a_1}, s_{a_2}, \ldots, s_{a_n})$$

is the induced cellular automaton.

Remark that although the edge-structure of $\widetilde{\mathcal{G}}$ depends on the choice of the generating system B for G, the dynamics of the induced automaton is independent of this choice and therefore $\widetilde{\mathcal{A}}$ is well defined. Also, denoting by $\{g_i\}_{i\in\mathbb{N}}$ a left transversal for H in G, with $g_0 = e$ the unit element, the universe $\widetilde{\mathcal{G}}$ is partitioned according to the corresponding partition $G = \coprod_{i\in\mathbb{N}} g_i H$ (disjoint union). Denoting by $\mathcal{C}_i = \{c : g_i H \to S = \widetilde{S}\}$ the set of configurations with domain the coset $g_i H$, so that in particular $\mathcal{C}_0$ equals the set $\mathcal{C}$ of configurations of the automaton $\mathcal{A}$, the mapping $\phi_i : \mathcal{C} \ni c \mapsto \phi_i(c) \in \mathcal{C}_i$ given by $\phi_i[c](g_i h) = c(h)$, $h \in H$, is bijective. Moreover given a configuration $c \in \widetilde{\mathcal{C}}$ of the automaton $\widetilde{\mathcal{A}}$ one has

$$\tilde{\tau}(c)_{|g_i H} = \phi_i[\tau(\phi_i^{-1}(c_{|g_i H})]$$

where $\tilde{\tau}$ (resp. τ) is the transition map of $\widetilde{\mathcal{A}}$ (resp. $\mathcal{A}$).

The main interest of the notion of induction is the following lemma whose proof is immediate.

LEMMA 3. — *Let $\widetilde{\mathcal{A}}$ be a cellular automaton induced from another cellular automaton $\mathcal{A}$. Then*

i) *there exist GOE patterns for $\mathcal{A}$ if and only if there exist GOE patterns for $\widetilde{\mathcal{A}}$;*

ii) *there exist ME patterns for $\mathcal{A}$ if and only if there exist ME patterns for $\widetilde{\mathcal{A}}$.*

From the previous lemma and examples we obtain:

THEOREM 4. — *Let G be a finitely generated group containing the free group of rank two. Then there exist finite and symmetric generating*

sets A_1, A_2 *for* G *and cellular automata* $\mathcal{A}_1 = (S_1, \mathcal{G}_{A_1}(G), f_1)$, $\mathcal{A}_2 = (S_2, \mathcal{G}_{A_2}(G), f_2)$ *such that*

i) *for* $\mathcal{A}_1$ *there are two ME patterns but no GOE patterns;*

ii) *for* $\mathcal{A}_2$ *there are GOE patterns but no ME patterns.*

In other words the theorems of Moore and Myhill fail to hold for cellular automata on groups containing the free group $\mathbb{F}_2$.

7. Concluding remarks.

We were unable to extend Theorem 4 to non-amenable groups with no free subgroups like, for instance, Ol'shanski's groups or the free Burnside groups $B(m,n)$, $m \geq 2$, $n \geq 665$, odd. The impossibility of such an extension would give a new characterization of amenable finitely generated groups.

On the other hand H. Furstenberg [F] suggested that entropy arguments may lead to an extension of Theorem 3 to a class of groups wider than that of amenable groups. In the appendix "Garden of Eden, entropy and surjunctivity" of a recent paper concerning endomorphisms of symbolic algebraic varieties, M. Gromov gives a new proof of our Theorem 3 based on entropy arguments (see [Gro]).

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