ANNALES DE L'INSTITUT FOURIER

KATHRYN E. HARE

Central sidonicity for compact Lie groups

Annales de l'institut Fourier, tome 45, n° 2 (1995), p. 547-564

<http://www.numdam.org/item?id=AIF_1995__45_2_547_0>

© Annales de l'institut Fourier, 1995, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
45, 2 (1995), 547-564

CENTRAL SIDONICITY FOR COMPACT LIE GROUPS

by Kathryn E. HARE (*)

0. Introduction.

Suppose G is a compact group with dual object $\widehat{G}$. It is well known that if G is an abelian group, then every infinite subset of $\widehat{G}$ contains an infinite Sidon set [8]. In contrast, there are non-abelian groups which admit no infinite central Sidon sets [11]. For central p-Sidon sets the situation is quite different; even in the non-abelian setting these are plentiful. Indeed, Dooley [3] showed that every compact, connected group admits an infinite central p-Sidon set for all $p > 1$, however he was unable to determine if every infinite set contains an infinite central p-Sidon subset.

The main result of our paper answers this question affirmatively. In fact, we prove formally more. We study certain weighted generalizations of Sidon sets, introduced in [5], called (central) (a,p)-Sidon sets, which arise by considering classical Sidonicity with the Fourier transform weighted by the a'th powers of the representation degree: (central) $(1,p)$-Sidon sets are (central) p-Sidon sets. We prove that every infinite subset of the dual of a compact, connected group contains an infinite subset which is central (a,p)-Sidon for all $p \geq 1$ and $a < 2p-1$. Our method is essentially constructive: we show that certain "lacunary-like" sets have the desired property.

(*) This research was partially supported by NSERC. It was carried out in part while the author enjoyed the hospitality of the University of New South Wales. She is grateful to David Wilson for many helpful conversations.
Key words: Sidon set – Lacunary set – Representations of compact Lie groups.
Math. classification: 43A46 – 43A80 – 22E46.

When G is a compact, simply-connected, semisimple Lie group of rank ℓ, the dual object can be identified with the set of dominant weights and consequently with $(\mathbf{Z}^+)^\ell$. Our examples of central (a,p)-Sidon sets in the duals of these groups correspond to Sidon sets in $\mathbf{Z}^\ell$. A natural question to ask is if all Sidon sets in $(\mathbf{Z}^+)^\ell$ correspond to Sidon-type sets in $\widehat{G}$. We show that such sets are always central $(0,1)$-Sidon, but need not be central $(a,1)$-Sidon for any $a > 0$, and that there are central $(a,1)$-Sidon sets in $\widehat{G}$ which do not correspond to Sidon sets in $(\mathbf{Z}^+)^\ell$.

1. Preliminaries.

If G is a compact group, $\widehat{G}$ will denote a maximal set of pairwise inequivalent, unitary, irreducible representations of G. The degree of $\sigma \in \widehat{G}$ will be denoted by d_σ.

The following generalization of Sidonicity was introduced in [5].

DEFINITION. — *Let $a \in \mathbf{R}$, $1 \le p < \infty$. A subset E of $\widehat{G}$ is called a* (central) (a,p)-Sidon set *if there is a constant $\kappa(a,p)$ so that whenever $f = \sum_{\sigma \in E} d_\sigma \operatorname{Tr} A_\sigma \sigma$ is a (central) trigonometric polynomial on G, then*

$$\|\hat{f}\|_{(a,p)} \equiv \Big(\sum d_\sigma^a \operatorname{Tr} |A_\sigma|^p\Big)^{1/p} \le \kappa(a,p)\|f\|_\infty.$$

(Central) $(1,p)$-Sidon sets are usually called (central) p-Sidon and (central) 1-Sidon sets are simply referred to as (central) Sidon sets.

Obviously if E consists of representations of bounded degree there is no distinction between (a,p)-Sidonicity for different values of a; if G is abelian then central p-Sidon and p-Sidon properties coincide; and (for all groups) it is easier to be (central) (a,p)-Sidon as a decreases or p increases. There are other relationships between (a,p)-Sidon sets. For this paper we only need note that since $\ell^q \subset \ell^p$ if $q < p$, then any central (a,q)-Sidon set is central (b,p)-Sidon provided $(b+1)/p \le (a+1)/q$. In particular any set which is central $(a,1)$-Sidon for all $a < 1$ is also central (b,p)-Sidon for all $p \ge 1$ and $b < 2p - 1$.

One reason for the interest in (a,p)-Sidon sets is the scarcity of (central) Sidon sets: a compact, connected group admits an infinite central Sidon set if and only if it is not a semisimple Lie group [11], [12].

It is seen in [5] that if G is an infinite compact, connected group then $\widehat{G}$ is never central $(0,1)$-Sidon, but there are examples where $\widehat{G}$ is $(-\varepsilon,1)$-Sidon for any given $\varepsilon > 0$. Also, every central $(1+\varepsilon,1)$-Sidon set for $\varepsilon > 0$ is a set of representations of bounded degree; consequently our interest (when $p = 1$) is in the range $0 \leq a \leq 1$.

There are a number of equivalent characterizations of (central) (a,p)-Sidonicity (see [5]). For example, analogous to [6], 37.2 we have

PROPOSITION 1.1. — *Let G be a compact group. A subset E of $\widehat{G}$ is central $(a,1)$-Sidon if and only if whenever $\phi \in \ell^\infty(E)$ there is a central measure μ on G with*

$$\hat{\mu}(\sigma) \equiv \int_G \frac{\operatorname{Tr}\sigma}{d_\sigma} d\mu = \frac{\phi(\sigma)}{d_\sigma^{1-a}} \quad \text{for all } \sigma \in E.$$

Next we recall some notation and basic facts from Lie theory. The reader is referred to [7] or [14] for more details. Let G denote a compact, simply-connected, semisimple Lie group of rank ℓ, T^ℓ a maximal torus for G and t its Lie algebra. Let Φ denote the set of roots for (G,T^ℓ) and Φ^+ the positive roots relative to a fixed base Δ. To each $\lambda = (n_1,\ldots,n_\ell) \in \mathbf{Z}^\ell$ we associate the weight $\lambda = \sum_{j=1}^{\ell} n_j\lambda_j$ where λ_j are the fundamental dominant weights relative to Δ, and we denote by Λ^+ the set of all dominant weights i.e. the set of all λ with non-negative integer coefficients. We view Φ as a subset of it^*. The lattice of weights Λ is isomorphic to $\widehat{T^\ell}$: $\lambda = \sum n_j\lambda_j$ determines a character on T^ℓ by the map: $\exp H \mapsto e^{\lambda(H)} = e^{\Sigma n_j\lambda_j(H)}$ for $H \in t$. The set $\widehat{G}$ is in a $1-1$ correspondence with Λ^+; $\sigma_\lambda \in \widehat{G}$ is indexed by its highest weight $\lambda \in \Lambda^+$. Thus if E is a subset of $(\mathbf{Z}^+)^\ell$, then E indexes a subset of $\widehat{G}$ in a canonical way, and we refer to this subset of $\widehat{G}$ by E as well. It should be clear from the context which set is actually meant. A partial order is defined on Λ by the positive roots: $\mu \prec \sigma$ if and only if $\sigma - \mu$ is a non-negative integral sum of positive roots. The Weyl group will be denoted by W and the weights of $\sigma \in \Lambda^+$ by

$$\Pi(\sigma) \equiv \{\mu \in \Lambda : w(\mu) \prec \sigma \quad \text{for all } w \in W\}.$$

The set $\Pi(\sigma)$ consists of all $\mu \in \Lambda^+$ with $\mu \prec \sigma$, together with all their Weyl-conjugates. Lastly, we set $\rho = \sum_{j=1}^{\ell} \lambda_j$; ρ is also half the sum of the positive roots.

One reason for the success in studying central (a,p)-Sidon sets is that there are formulas for $\operatorname{Tr}\sigma$ restricted to the torus. One of these is the Weyl character formula:

$$\operatorname{Tr}\sigma(x) = \frac{\sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)}}{q(x)}, \quad x \in T^\ell$$

where

$$\begin{aligned} q(x) &= \sum_{w\in W} \det(w) e^{iw(\rho)(x)} \\ &= e^{-i\rho(x)} \prod_{\alpha\in\Phi^+} (e^{i\alpha(x)} - 1). \end{aligned}$$

Related to this is the Weyl dimension formula which states:

$$d_\sigma = \prod_{\alpha\in\Phi^+} \frac{\langle \sigma+\rho, \alpha\rangle}{\langle \rho, \alpha\rangle}.$$

A final fact which we will record here is that the weights in $\Pi(\sigma)$ correspond to the irreducible subrepresentations of $\sigma|_{T^\ell}$, and so we also have the formula

$$\operatorname{Tr}\sigma(x) = \sum_{\mu\in\Pi(\sigma)} m_\sigma(\mu) e^{i\mu(x)}, \quad x \in T^\ell$$

where $m_\sigma(\mu)$ is the multiplicity of μ in $\sigma|_{T^\ell}$.

2. Main result.

In [3], Dooley constructs in the dual of any compact, connected, semisimple Lie group examples of infinite sets which are p-Sidon for all $p > 1$. By making the obvious modifications to his proof these examples can be seen to be central (a, p)-Sidon for all $p \geq 1$ and $a < 2p-1$. Consequently, every compact, connected group G admits infinite central (a, p)-Sidon sets for any a and p as above. The main objective of this section is to prove that these thin sets can be found in any infinite subset of $\widehat{G}$.

We first construct examples in the case when G is semisimple.

THEOREM 2.1. — *Suppose G is a compact, simply-connected, semisimple Lie group of rank ℓ. Choose $0 < t < \ell/|\Phi^+|$ and $1-t/2 \leq a < 1$. There is a constant $C = C(t, G)$ so that if $\{\sigma_j\}$ is any set of representations in $\widehat{G}$ whose degrees, $d_{\sigma_j} \equiv d_j$, satisfy*

(1) $d_j^{t/4} \geq 4^j$ *for* $j \geq 1$,

(2) $d_j^{(1-a)/2} \geq 4(C \log C d_j)^\ell$ *for* $j \geq 1$, *and*

(3) $d_j \geq C(d_{j-1})^{2\ell/t}$ *for* $j \geq 2$,

then $\{\sigma_j\}$ *is a central* $(a, 1)$*-Sidon set.*

It is useful to prove two lemmas.

LEMMA 2.2. — *There is a constant* $C_1 = C_1(G)$ *so that if* $\sigma \in \widehat{G}$ *and* $\mu \in \Pi(\sigma)$ *with* $\mu = \sum_{i=1}^{\ell} \mu_i \lambda_i$ *and* $\sigma = \sum_{i=1}^{\ell} \sigma_i \lambda_i$, *then*

$$\max_i |\mu_i| \leq C_1 \max_i \sigma_i \leq C_1 d_\sigma.$$

Proof. — The second inequality is immediate from the Weyl dimension formula so we only need prove the first.

Suppose $\Delta = \{\alpha_j\}_{j=1}^{\ell}$. Then each $\lambda_k = \sum_{j=1}^{\ell} a_{kj}\alpha_j$ for some $a_{kj} = a_{kj}(G) \geq 0$, and because $\lambda_k \neq 0$ there is an index j_k such that $a_{kj_k} > 0$. Since $\mu \prec \sigma$ with respect to the partial order induced by the positive roots,

$$\sum_{i=1}^{\ell} \mu_i a_{ij} \leq \sum_{i=1}^{\ell} \sigma_i a_{ij} \text{ for all } j.$$

If μ is a dominant weight, taking $j = j_k$ above we get

$$\begin{aligned} 0 \leq \mu_k &\leq \frac{1}{a_{kj_k}} \sum_{i=1}^{\ell} \sigma_i a_{ij_k} \\ &\leq C(G) \max_i \sigma_i. \end{aligned}$$

Otherwise $\mu = w(v)$ for some dominant weight $v \in \Pi(\sigma)$ and $w \in W$. Since the Weyl action is linear,

$$\max_i |\mu_i| \leq C'(G) \max_i v_i$$

for some constant $C'(G)$. Now take $C_1 = CC'$. □

LEMMA 2.3. — *There is a constant* $C_2 = C_2(t, G)$ *so that if* $\sigma \in \widehat{G}$ *and* $\mu \in \Pi(\sigma)$, *then* $m_\sigma(\mu) \leq C_2 d_\sigma^{1-t}$.

Proof. — This is a straight forward calculation. We begin with the fact that

$$m_\sigma(\mu) = \int_{T^\ell} \operatorname{Tr} \sigma(x) e^{-i\mu(x)} dx.$$

By the Weyl character formula and standard inequalities

$$\begin{aligned} m_\sigma(\mu) &\le \int_{T^\ell} \left| \frac{\sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)}}{q(x)} \right| dx \\ &\le \left\| \sum_{w\in W} \det(w) e^{iw(\sigma+\rho)(x)} \right\|_\infty^t \|q^{-t}\|_{L^1(T^\ell)} \| \operatorname{Tr} \sigma|_{T^\ell} \|_\infty^{1-t} \\ &\le |W|^t \|q^{-t}\|_{L^1(T^\ell)} d_\sigma^{1-t}. \end{aligned}$$

Since $q^{-t} \in L^1(T^\ell)$ for any $t < \ell/|\Phi^+|$ ([13]) the proof is complete. □

Proof of Theorem 2.1. — Throughout the proof we will use the following notation: $m_j(\mu) \equiv m_{\sigma_j}(\mu)$; $\Pi_j \equiv \Pi(\sigma_j)$; and

$$B_j \equiv \left\{ \sum_{i=1}^{\ell} m_i \lambda_i : |m_i| \le C_1 d_j,\ m_i \in \mathbf{Z} \right\} \subseteq \Lambda.$$

For C_1 and C_2 as in the lemmas, put

$$C = \max\left\{ \left((2C_1+1)^\ell C_2 \right)^{2/t},\ \sup_N \left(\frac{\left\| \sum_{n=-N}^{N} e^{inx} \right\|_{L^1[0,2\pi)}}{\log N} \right) \right\}.$$

Lemma 2.2 obviously implies $\Pi_j \subseteq B_j$. The key idea of the proof (which we make precise below) is that "most" of Π_j, counted by multiplicity, lies outside B_{j-1}. This we are able to obtain from Lemma 2.3 and property (3). To be precise we have, if $k > j$,

$$\begin{aligned} \sum_{\mu\in\Pi_k\cap B_j} m_k(\mu) &\le |B_j| \max_{\mu\in\Pi_k} m_k(\mu) \\ (*) \qquad &\le (2C_1 d_j + 1)^\ell C_2 d_k^{1-t} \\ &\le d_k^{1-t/2}, \end{aligned}$$

and thus

$$\begin{aligned} \sum_{\mu\in\Pi_k\setminus B_j} m_k(\mu) &= \sum_{\mu\in\Pi_k} m_k(\mu) - \sum_{\mu\in\Pi_k\cap B_j} m_k(\mu) \\ (**) \qquad &\ge d_k - d_k^{1-t/2} \\ &\ge \frac{1}{2} d_k. \end{aligned}$$

Let D_n be the ℓ-dimensional Dirichlet kernel supported by B_n (thinking now of B_n as a subset of $\mathbf{Z}^\ell$ rather than of Λ),

$$D_n(x_1, \dots, x_\ell) = \prod_{k=1}^{\ell} \sum_{j=-C_1 d_n}^{C_1 d_n} e^{ij(x_k)},$$

and let $H_n = D_n - D_{n-1}$. Then $\widehat{H}_n = \chi_{B_n \setminus B_{n-1}}$ and $\|H_n\|_1 \le 2(C \log C d_n)^\ell$.

Suppose $f = \sum_{j=1}^{N} d_j a_j \operatorname{Tr} \sigma_j$ is a central trigonometric polynomial with $\|f\|_\infty \le 1$. With our notation

$$f|_{T^\ell}(x) = \sum_{j=1}^{N} d_j a_j \sum_{\mu \in \Pi_j} m_j(\mu) e^{i\mu(x)}.$$

Notice that $H_n * \sum_{\mu \in \Pi_j} m_j(\mu) e^{i\mu(x)} = 0$ if $j < n$ (here the convolution is over T^ℓ), and so if $n \le N$,

$$\begin{aligned} 1 \ge \frac{\|f|_{T^\ell} * H_n\|_\infty}{\|H_n\|_1} &\ge \frac{|f|_{T^\ell} * H_n(0)|}{\|H_n\|_1} \\ &= \frac{\left| \sum_{j=n}^{N} d_j a_j \sum_{\mu \in \Pi_j} m_j(\mu) e^{i\mu(x)} * H_n(0) \right|}{\|H_n\|_1} \\ &\ge \frac{\left| \sum_{j=n}^{N} d_j a_j \sum_{\mu \in \Pi_j \cap (B_n \setminus B_{n-1})} m_j(\mu) \right|}{2(C \log C d_n)^\ell}. \end{aligned}$$

An application of the triangle inequality yields

$$d_n |a_n| \sum_{\mu \in \Pi_n \setminus B_{n-1}} m_n(\mu) \le 2(C \log C d_n)^\ell + \sum_{j=n+1}^{N} d_j |a_j| \sum_{\mu \in \Pi_j \cap (B_n \setminus B_{n-1})} m_j(\mu).$$

Combined with our estimates (*) and (**), and property (2), this gives

$$d_n^2 |a_n| \le d_n^{(1-a)/2} + 2 \sum_{j=n+1}^{N} d_j^{2-t/2} |a_j|.$$

For $j = 1, 2, \ldots, N$ set $S_j = \sum_{k=0}^{j-1} d_{N-k}^{2-t/2} |a_{N-k}|$ and set $S_0 = 0$. This gives

$$d_n^2 |a_n| \le d_n^{(1-a)/2} + 2S_{N-n},$$

and since (1) guarantees $d_{N-j}^{t/2} \ge 2$,

$$S_{j+1} \le 2S_j + d_{N-j}^{\varepsilon - t/2}$$

where $\varepsilon = (1-a)/2$. By induction,

$$S_j \le \sum_{i=1}^{j} 2^{i-1} d_{N-j+i}^{\varepsilon - t/2} \text{ for } j = 1, 2, \ldots, N.$$

Property (1) also ensures $S_j \leq 1$, thus

$$d_n^2 |a_n| \leq d_n^{(1-a)/2} + 2.$$

It is now easy to see that $\{\sigma_j\}$ is central $(a,1)$-Sidon:

$$\begin{aligned} \|\hat{f}\|_{(a,1)} &= \sum_{n=1}^{N} d_n^{1+a} |a_n| \\ &\leq \sum_{n=1}^{N} \frac{1}{d_n^{(1-a)/2}} + \frac{2}{d_n^{1-a}}, \end{aligned}$$

and this sum is bounded over N since $\{d_n\}$ is lacunary and $a < 1$. □

Remark. — An application of the Weyl dimension formula shows that if $\sigma_j = \sum_{i=1}^{\ell} \sigma_{ji}\lambda_i$, then $\{(\sigma_{j1}, \ldots, \sigma_{j\ell})\}_j$ is the union of a finite set and a dissociate set in $\mathbf{Z}^\ell$, and hence is a Sidon set in the dual of the torus.

COROLLARY 2.4. — *If G is a compact, simply-connected, semisimple Lie group, then every infinite subset of $\widehat{G}$ contains an infinite central (a,p)-Sidon set for all $p \geq 1$ and $a < 2p-1$.*

Proof. — As remarked in the first section it suffices to prove this for $p = 1$ and all $a < 1$.

Let ℓ=rank G and fix $0 < t < \ell/|\Phi^+|$. Set $a_1 = 1 - \frac{t}{2}$ and choose an increasing sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n < 1$ and limit one. Let $E \subseteq \widehat{G}$ be infinite. Since $\widehat{G}$ contains only finitely many representations of any given degree we can choose an infinite subset $\{\sigma_j\}$ of E satisfying (where C is as in the theorem):

(1) $d_j^{t/4} \geq 4^j$ for $j \geq 1$,

(2) $d_j^{(1-a_j)/2} \geq 4(C \log C d_j)^\ell$ for $j \geq 1$, and

(3) $d_j \geq C(d_{j-1})^{2\ell/t}$ for $j \geq 2$.

Choose $a < 1$. Then $a \leq a_j$ for all $j \geq J$ and by the theorem $\{\sigma_j\}_{j=J}^{\infty}$ is a central $(a,1)$-Sidon set. It is easy to see from Proposition 1.1 that the union of a finite set and a central $(a,1)$-Sidon set is again central $(a,1)$-Sidon, and therefore $\{\sigma_j\}_{j=1}^{\infty}$ is central $(a,1)$-Sidon for any $a < 1$. □

Remark. — As noted previously, these groups admit no infinite central Sidon sets. It is unknown if they admit infinite central $(2p-1,p)$-Sidon sets for any $p > 1$.

The next step towards our main result is to consider the case when G is an infinite product group.

THEOREM 2.5. — *Let $G = \prod_{j=1}^{\infty} G_j$ be a product of compact, simply-connected, semisimple Lie groups, and suppose σ_j is a non-trivial representation of G_j. Then $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is a central $(a,1)$- Sidon set for all $a < 1$.*

Proof. — Suppose $f = \sum_{j=1}^{N} d_j a_j \operatorname{Tr} \sigma_1 \times \cdots \times \sigma_j$ where $d_j = d_{\sigma_1} \cdots d_{\sigma_j}$. Without loss of generality assume $\|f\|_\infty \leq 1$. For the duration of this proof we will use the following notation: $m_j(\mu) \equiv m_{\sigma_j}(\mu)$; $m_j \equiv m_j(0)$; $\Pi_j' \equiv \Pi(\sigma_j)\backslash\{0\}$; $T^{G_j} \equiv$ torus of G_j; and $T_N = T^{G_1} \times \cdots \times T^{G_N}$. With this notation we have

$$f|_{T_N}(x_1, \ldots, x_N) = \sum_{j=1}^{N} d_j a_j \prod_{k=1}^{j} \left(m_k + \sum_{\mu \in \Pi_k'} m_k(\mu) e^{i\mu(x_k)} \right) \text{ for } x_k \in T^{G_k}.$$

Viewing f as a function on T_N, we can read off the Fourier coefficients:

$$\hat{f}(0, \ldots, 0) = \sum_{j=1}^{N} d_j a_j m_1 \cdots m_j;$$

$$\hat{f}(\mu_1, \ldots, \mu_k, 0, \ldots, 0) = \left(d_k a_k + \sum_{j=k+1}^{N} d_j a_j m_{k+1} \cdots m_j \right) m_1(\mu_1) \cdots m_k(\mu_k)$$

$$\text{if } \mu_i \in \Pi(\sigma_i) \text{ for } i = 1, \ldots, k-1, \ \mu_k \in \Pi_k';$$

and $\hat{f}(\mu_1, \ldots, \mu_N) = 0$ otherwise.

For $x_j = (x_{j1}, \ldots, x_{j\ell(j)}) \in T^{G_j}$ (here $\ell(j) = \operatorname{rank} G_j$), and M very large, let

$$H_j(x_j) = \prod_{k=1}^{\ell(j)} \sum_{n=-M}^{M} \left(1 - \frac{|n|}{M+1}\right) e^{in(x_{jk})},$$

and for $n \leq N$ let

$$K_n(x_1, \ldots, x_N) = \prod_{j=1}^{n-1} H_j(x_j)(H_n(x_n) - 1).$$

Observe that if $\mu_j \in \widehat{T^{G_j}}$ for $j = 1, \ldots, N$, and $\hat{K}_n(\mu_1, \ldots, \mu_N) \neq 0$, then $\mu_j = 0$ for $j > n$ and $\mu_n \neq 0$. Consider the convolution of K_n and $f|_{T_N}$. If M is chosen sufficiently large then

$$2 \geq |f * K_n(0)| \geq \frac{1}{2} \left| d_n a_n + \sum_{j=n+1}^{N} d_j a_j m_{n+1} \cdots m_j \right| \sum_{n}{}' m_1(\mu_1) \cdots m_n(\mu_n)$$

where $\sum_n'$ denotes the sum over all $\mu_j \in \Pi(\sigma_j)$ for $j = 1, \ldots, n-1$, and $\mu_n \in \Pi_n'$. Clearly

$$\sum_n' m_1(\mu_1) \cdots m_n(\mu_n) = d_{\sigma_1} \cdots d_{\sigma_{n-1}}(d_{\sigma_n} - m_n)$$
$$= d_{n-1}(d_{\sigma_n} - m_n).$$

Thus

$$|d_n a_n| \leq \frac{4}{d_{n-1}(d_{\sigma_n} - m_n)} + \left| \sum_{j=n+1}^{N} d_j a_j m_{n+1} \cdots m_j \right|.$$

Furthermore,

$$\left| \sum_{j=n+1}^{N} d_j a_j m_{n+1} \cdots m_j \right| = \left| d_{n+1} a_{n+1} + \sum_{j=n+2}^{N} d_j a_j m_{n+2} \cdots m_j \right| m_{n+1}$$
$$\leq \frac{4 m_{n+1}}{d_n(d_{\sigma_{n+1}} - m_{n+1})}$$

(where the empty sum and m_{N+1} equal 0). Thus

$$|d_n a_n| \leq \frac{4}{d_{n-1}(d_{\sigma_n} - m_n)} + \frac{4 m_{n+1}}{d_n(d_{\sigma_{n+1}} - m_{n+1})}.$$

In [4] Gallagher proves that if σ is any non-trivial representation of a compact, simply-connected, semisimple Lie group then $\operatorname{Tr}\sigma$ has a root, say x, in the maximal torus. Evaluating $\operatorname{Tr}\sigma$ at x we derive the formula

$$m_\sigma(0) = -\sum_{\mu \in \Pi(\sigma) \setminus \{0\}} m_\sigma(\mu) e^{i\mu(x)},$$

from which one readily sees that $m_\sigma(0) \leq d_\sigma/2$. Hence $|d_n a_n| \leq 12/d_n$, and so

$$\|\hat{f}\|_{(a,1)} = \sum_{j=1}^{N} d_n^{1+a} |a_n| \leq \sum_{j=1}^{N} \frac{12}{d_n^{1-a}}.$$

Since $d_n \geq 2^n$, this sum converges provided $a < 1$, and thus $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is a central $(a,1)$-Sidon set for all $a < 1$. □

This set of representations is independent in the sense that if

$$\int_G \prod_{j=1}^{N} (\operatorname{Tr}\sigma_1 \times \cdots \times \sigma_j)^{\varepsilon_j} \neq 0$$

for some $N \in \mathbf{N}$ and $\varepsilon_j = 0, \pm 1$ for $j = 1, \ldots, N$, then necessarily all $\varepsilon_j = 0$. This independence condition is not sufficient to be Sidon [1]. It is not sufficient for central-Sidon either as the next example demonstrates.

Example 2.6. — Suppose $G_j = SU(2)$, $G = \prod_{j=1}^{\infty} G_j$ and $\sigma_j = 2\lambda_1$. The set $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is not central $(2p-1,p)$-Sidon for any $p \geq 1$.

Proof. — It is well known that the torus of $SU(2)$ is the circle group T and that $\operatorname{Tr} 2\lambda_1|_T = 1 + e^{ix} + e^{-ix}$ ([6], 29.25). Therefore $\operatorname{Tr} 2\lambda_1|_T$ takes on precisely the values in [-1,3]. Let

$$f_N = \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p} 3^j} \operatorname{Tr} \sigma_1 \times \cdots \times \sigma_j;$$

$\|\hat{f}_N\|_{(2p-1,p)} = \sum_{j=1}^{N} \frac{1}{j}$ which diverges as $N \to \infty$.

Being a central function, $\|f_N\|_\infty = \|f_N|_T\|_\infty$, and from the remark above the latter equals

$$\sup_{w_i \in [-\frac{1}{3},1]} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}} w_1 \cdots w_j \right|.$$

We will now prove that this supremum is bounded over N which certainly suffices to prove $\{\sigma_1 \times \cdots \times \sigma_j\}_{j=1}^{\infty}$ is not central $(2p-1,p)$-Sidon.

Set $j_1 = 1$ and inductively define j_k to be the least integer greater than j_{k-1} with

$$(-1)^{j_k} w_1 \cdots w_{j_k} (-1)^{j_{k-1}} w_1 \cdots w_{j_{k-1}} \leq 0.$$

Consider first the alternating sum

$$\sum_k \frac{(-1)^{j_k}}{j_k^{1/p}} w_1 \cdots w_{j_k}.$$

Since $\frac{|w_1 \cdots w_{j_k}|}{j_k^{1/p}}$ decreases to zero, this sum is bounded in absolute value by $\frac{1}{j_1^{1/p}} = 1$.

If $j \notin \{j_i\}$ then $(-1)^j w_1 \cdots w_j$ and $(-1)^{j-1} w_1 \cdots w_{j-1}$ have the same sign. This can occur only if $w_j < 0$, but then $|w_1 \cdots w_j| \leq \frac{1}{3}|w_1 \cdots w_{j-1}|$. As $|w_i| \leq 1$ for all i, it follows that

$$\left| \sum_{j \notin j_i} (-1)^j \frac{w_1 \cdots w_j}{j^{1/p}} \right| \leq \sum_{k=1}^{\infty} \frac{1}{3^k} \leq \frac{1}{2}.$$

These estimates clearly combine to give

$$\sup_{w_i \in [-\frac{1}{3},1]} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j^{1/p}} w_1 \cdots w_j \right| \leq \frac{3}{2}. \qquad \square$$

THEOREM 2.7. — *Let $G = \prod_\alpha G_\alpha$ be a product (possibly finite) of compact, simply-connected, simple Lie groups. Then any infinite subset of $\hat{G}$ contains an infinite central (a,p)-Sidon set for all $p \geq 1$ and $a < 2p-1$.*

First we introduce some notation and prove a lemma.

Notation. — Let $\sigma_j \in \widehat{G}$. Then $\sigma_j = \times \sigma_{j\alpha}$ where $\sigma_{j\alpha} \in \widehat{G}_\alpha$ and only finitely many $\sigma_{j\alpha}$ are non-trivial. Denote by $\operatorname{supp}\sigma_j$ the set $\{\alpha : \sigma_{j\alpha} \neq 1\}$. We will say σ_j is orthogonal to σ_k, and write $\sigma_j \perp \sigma_k$, if $\operatorname{supp}\sigma_j \cap \operatorname{supp}\sigma_k$ is empty. Recall that Parker [9] has shown that if $\{\sigma_j\}$ consists of mutually orthogonal, non-trivial representations then $\{\sigma_j\}$ is a central Sidon set.

LEMMA 2.8. — *Let $a \leq 1$ and suppose $\{\sigma_j\}$ is a central $(a,1)$-Sidon set in $\widehat{G}$. Suppose $\{\tau_j\} \subseteq \widehat{G}$ and $\tau_j \perp \sigma_k$ for all j,k. Then $\{\tau_j \times \sigma_j\}$ is another central $(a,1)$-Sidon set.*

Proof. — This is an easy consequence of the fact that
$\|f\|_\infty \geq \sup\left\{|f(x)| : x = (x_\alpha) \text{ and } x_\alpha = 1 \text{ if } \alpha \in \bigcup_j \operatorname{supp}\tau_j\right\}$. □

Proof of Theorem 2.7. — It suffices to show that any countably infinite set, $E = \{\sigma_j\}_{j=1}^\infty$, contains an infinite subset which is central $(a,1)$-Sidon for all $a < 1$.

Suppose first $\{\sigma_{j\alpha} : \sigma_j \in E\}$ is infinite for some α. By Corollary 2.4 we can find an infinite subset of $\{\sigma_{j\alpha}\}_{j=1}^\infty$ which is a central $(a,1)$-Sidon subset of $\widehat{G}_\alpha$, for all $a < 1$. The corresponding subset of E has the same property.

So we may assume $\{\sigma_{j\alpha} : \sigma_j \in E\}$ is finite for each α.

Case 1. — For each index α, $\{\sigma_j : \sigma_{j\alpha} \neq 1\}$ is finite.

Set $j_1 = 1$ and inductively assume mutually orthogonal representations $\sigma_{j_1}, \ldots, \sigma_{j_n} \subseteq E$ have been picked. Since there are only finitely many representations σ_j with $\sigma_{j\alpha} \neq 1$ for $\alpha \in \bigcup_{k=1}^n \operatorname{supp}\sigma_k$, we can choose $\sigma_{j_{n+1}}$ orthogonal to each of $\sigma_{j_1}, \ldots, \sigma_{j_n}$. By Parker [9] $\{\sigma_{j_k}\}_k$ is central Sidon.

Case 2. — $\{\sigma_j : \sigma_{j\alpha} \neq 1\}$ is infinite for some α, say $\alpha = \alpha_1$.

Since $\{\sigma_{j\alpha_1} : \sigma_j \in E\}$ is finite there must be a non-trivial representation ϕ_1 of G_{α_1}, with $\phi_1 = \sigma_{j\alpha_1}$ for all $\sigma_j \in F_1$, an infinite subset of E. Select $\sigma_{j_1} \in F_1$.

If $\{\sigma_j \in F_1 : \sigma_{j\alpha} \neq 1\}$ is finite for all $\alpha \notin \operatorname{supp}\sigma_{j_1}$, then by arguments similar to case 1 we can obtain an infinite subset of F_1 of the form $\{\tau_k \times w_k\}_{k=2}^{\infty}$ where $\operatorname{supp}\tau_k \subseteq \operatorname{supp}\sigma_{j_1}$ and the representations w_k are non-trivial, mutually orthogonal, and all orthogonal to σ_{j_1}. By [9] and the lemma this set is central Sidon.

Otherwise we repeat the argument to produce infinite sets $F_n \subset F_{n-1}$ ($F_0 = E$), representations $\sigma_{j_n} \in F_n$ and ϕ_n orthogonal to σ_{j_k} for $k \leq n-1$, and an index α_n with the property that $\sigma_{j\alpha_n} = \phi_n$ for all $\sigma_j \in F_n$. If $\{\sigma_j \in F_n : \sigma_{j\alpha} \neq 1\}$ is finite for all $i \notin \bigcup_{k=1}^{n} \operatorname{supp}\sigma_{j_k}$ we quit this process and produce an infinite central Sidon set in F_n by standard arguments. Otherwise, as in the first step of case 2, we choose F_{n+1}, α_{n+1}, ϕ_{n+1} and $\sigma_{j_{n+1}}$ with the properties above.

If this process never stops we produce an infinite set $\{\sigma_{j_n}\} \subseteq E$. By construction $\sigma_{j_n} = \phi_1 \times \cdots \times \phi_n \times \tau_n$ where $\phi_n \perp \phi_1 \times \cdots \times \phi_j \times \tau_j$ for all $n > j$. From Theorem 2.5 $\{\phi_1 \times \cdots \times \phi_n\}_{n=1}^{\infty}$ is central $(a,1)$-Sidon for all $a < 1$ and hence so is $\{\sigma_{j_n}\}$.

In either case we can find an infinite central $(a,1)$-Sidon subset of E and thus the proof of the theorem is complete. □

The main result will now be seen to follow from the structure theorem ([10], 6.5.6): If G is a compact, connected group then there is a continuous epimorphism $\phi : T \times \prod_{\alpha} G_\alpha \to G$ where T is a compact abelian group and each G_α is a compact, simply-connected, simple Lie group.

THEOREM 2.9. — *If G is a compact, connected group then any infinite subset of $\widehat{G}$ contains an infinite central (a,p)-Sidon set for all $p \geq 1$ and $a < 2p-1$.*

We need only one additional lemma whose proof is obvious.

LEMMA 2.10. — *If $\phi : H \to G$ is a continuous epimorphism of compact groups then $E \subseteq \widehat{G}$ is a (central) (a,p)-Sidon set if and only if the same is true for $E \circ \phi = \{\sigma \circ \phi : \sigma \in E\} \subseteq \widehat{H}$.*

Proof of Theorem 2.9. — Let $E \subset \widehat{G}$ be an infinite set and let $\phi : T \times \prod_{\alpha} G_\alpha \to G$ be the structure theorem epimorphism. Since ϕ is onto $E \circ \phi$ is also infinite. For $\sigma \circ \phi \in E \circ \phi$, write $\sigma \circ \phi = \tau_\sigma \times \psi_\sigma$ where $\tau_\sigma \in \widehat{T}$ and $\psi_\sigma \in \prod \widehat{G_\alpha}$. If $\{\tau_\sigma : \sigma \in E\}$ is infinite, then since T is an

abelian group there is an infinite Sidon subset of $\{\tau_\sigma\}$, and by Lemma 2.8 the corresponding subset of $E \circ \phi$ is central Sidon. If $\{\psi_\alpha\}$ is infinite we appeal to Theorem 2.7 and Lemma 2.8 to obtain an infinite central $(a, 1)$-Sidon set for all $a < 1$. In either case the corresponding infinite subset of E has the required property. □

COROLLARY 2.11. — *Suppose G is a compact, connected group. Any infinite subset of $\widehat{G}$ contains an infinite set which is central p-Sidon for all $p > 1$.*

Remark. — This answers the open problem left in [3].

3. Central $(0, 1)$-Sidon sets.

In this section we investigate the relationship between weighted central Sidonicity for a Lie group G and Sidonicity for its abelian torus. This investigation is motivated in part by the fact that both Dooley's examples [3] of central p-Sidon sets and our examples from Theorem 2.1 correspond to Sidon sets in $\mathbf{Z}^{\text{rank}G}$.

THEOREM 3.1. — *Let G be a compact, simply-connected, semisimple Lie group of rank ℓ, with torus T^ℓ. If $E \subseteq (\mathbf{Z}^+)^\ell$ is a Sidon set for T^ℓ, then E viewed as a subset of $\widehat{G}$ is central $(0, 1)$-Sidon.*

Proof. — Let $f = \sum_{\sigma \in E} d_\sigma a_\sigma \operatorname{Tr} \sigma$ be a central trigonometric polynomial on G. Since $|q(x)| \leq |W|$, the Weyl character formula implies

$$\|f\|_\infty \geq \frac{1}{|W|} \sup_{x \in T^\ell} \left| \sum_{\sigma \in E} d_\sigma a_\sigma \sum_{w \in W} \det(w) e^{iw(\sigma+\rho)(x)} \right|.$$

Because the representations $\sigma + \rho$, $\sigma \in \widehat{G}$, belong to the fundamental Weyl chamber, the weights $w(\sigma + \rho)$ are distinct as w varies over W and σ over E ([7], ch. 10). Furthermore, the family of Sidon sets in an abelian group is closed under linear transformations and finite unions ([8], p. 44) so this set of distinct elements, $\bigcup_{w \in W} \{w(\sigma + \rho) : \sigma \in E\}$, forms a Sidon set in $\mathbf{Z}^\ell$ (with the natural identification). With these observations it is straight forward to check that E is central $(0, 1)$-Sidon. □

Our next result shows that Theorem 3.1 cannot be improved. Recall that $SU(2)$ has one fundamental weight so its dual can be identified with $\mathbf{Z}^+$. The degree of the representation indexed by n is $n+1$.

PROPOSITION 3.2. — *There is a Sidon set in* $\mathbf{Z}$, *which is contained in* $\mathbf{Z}^+$, *and is not a central* $(a,1)$*-Sidon set in* $\widehat{SU(2)}$ *for any* $a > 0$.

Proof. — Let E be any infinite Sidon subset of $\mathbf{Z}$ contained in $\{2,3,4,\ldots\}$ and disjoint from $E-2$. Certainly $E \cup E-2$ is a Sidon set in $\mathbf{Z}$. If it was a central $(a,1)$-Sidon set in $\widehat{SU(2)}$ for some $a > 0$, by Proposition 1.1 there would be a measure μ on $SU(2)$ satisfying

$$\hat{\mu}(n) = \begin{cases} \dfrac{1}{(n+1)^{1-a}} & \text{for } n \in E \\ 0 & \text{for } n \in E-2. \end{cases}$$

Coifman and Weiss [2] have shown that μ is a measure on $SU(2)$ if and only if

$$\sum_{n\geq 2}((n+1)\hat{\mu}(n) - (n-1)\hat{\mu}(n-2))\cos n\theta$$

represents a measure v on T. But for $n \in E$, $\hat{v}(n) = (n+1)^a$ which tends to infinity, so this is an impossibility. □

It is natural to ask if the converse to Theorem 3.1 is true. It is not.

THEOREM 3.3. — *There are subsets of* $\mathbf{Z}^+$ *containing arbitrarily long arithmetic progressions which are central* $(a,1)$*-Sidon sets in* $\widehat{SU(2)}$, *for all* $a < 1$; *consequently a central* $(a,1)$*-Sidon set need not be a Sidon set in* $\mathbf{Z}$.

Proof. — The second statement follows from the first since sets containing arbitrarily long arithmetic progressions are never Sidon sets in $\mathbf{Z}$ ([8], p. 77). We follow the strategy of [3] to produce examples of central $(a,1)$-Sidon sets with this property.

Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of positive integers, σ_j the representation of $SU(2)$ indexed by $2n_j$, and let $f = \sum_{j=1}^{N}(2n_j+1)a_j \operatorname{Tr}\sigma_j$ be a central trigonometric polynomial on $SU(2)$. It is well known ([6], 29.25) that for $t_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \in T$,

$$\operatorname{Tr}\sigma_j(t_\theta) = \sin\Big(n_j + \frac{1}{2}\Big)\theta / \sin\frac{\theta}{2} = D_{n_j}(\theta),$$

where D_n is the n'th Dirichlet kernel. Thus

$$\|f\|_\infty = \|f|_T\|_\infty = \sup_{\theta\in[0,2\pi)} \left|\sum_{j=1}^{N}(2n_j+1)a_j D_{n_j}(\theta)\right|.$$

For even integers $a < b$, let F_{ab} denote the translated Fejér kernel with transform supported on (a,b). One easily sees that

(i) if $k < a$ then $F_{ab} * D_k = 0$; while

(ii) if $b \le k$ then $F_{ab} * D_k(0) = F_{ab}(0) = \dfrac{b-a}{2}$.

To simplify notation we write $F'_j = F_{n_{j-1},n_j}$ (taking $n_0 = 0$), $B_j = (n_{N-j} - n_{N-j-1})/2$ and $X_j = (2n_{N-j}+1)|a_{N-j}|$. With this notation

$$(*) \qquad \|\hat{f}\|_{(a,1)} = \sum_{j=0}^{N-1}(2n_{N-j}+1)^a X_{N-j}.$$

Without loss of generality we may assume $\|f\|_\infty = 1$, so, for $m = N-k$,

$$\begin{aligned} 1 \ge |f * F'_m(0)| &= \left|\sum_{j=1}^{N}(2n_j+1)a_j F'_m * D_{n_j}(0)\right| \\ &\ge (2n_m+1)|a_m|\Big(\frac{n_m - n_{m-1}}{2}\Big) - \sum_{j>m}(2n_j+1)|a_j|\Big(\frac{n_m-n_{m-1}}{2}\Big) \\ &\ge X_k B_k - \sum_{j=0}^{k-1} X_j B_k. \end{aligned}$$

Thus

$$X_k \le \frac{1}{B_k} + \sum_{j=0}^{k-1} X_j,$$

and simplifying this yields the estimate

$$X_k \le \frac{1}{B_k} + \sum_{j=0}^{k-1}\frac{2^j}{B_{k-1-j}}.$$

Obviously there are many ways to choose a sequence $\{n_j\}$ containing arbitrarily long arithmetic progressions, and yet have X_k sufficiently small so that $(*)$ bounded over all N and all $a < 1$. One choice, whose verification is routine, and is left for the reader, is to set $n_{2^k+i} = A^{A^k}(1+i)$ for $i = 0, 1, \ldots, 2^k - 1$, where A is sufficiently large. □

There is however a partial converse to Theorem 3.1. We state it for $SU(2)$, the context in which we will apply it to show the failure of the

union property, but similar results hold more generally for all compact, simply-connected, semisimple Lie groups.

PROPOSITION 3.4. — *Suppose E and $E-2$ are disjoint subsets of $\mathbf{Z}^+$ and that $E \cup E-2$ is a central $(0,1)$-Sidon set in $\widehat{SU(2)}$. Then E is a Sidon set in $\mathbf{Z}$.*

Proof. — Let $\phi \in \ell^\infty(E)$. Since $E \cup E-2$ is central $(0,1)$-Sidon, there exists a central measure μ on $SU(2)$ with $\hat{\mu}(n) = \phi(n)/(n+1)$ for $n \in E$ and $\hat{\mu} = 0$ on $E-2$. As in Proposition 3.2, [2] implies that there is a measure ν on T with $\hat{\nu}(\pm n) = (n+1)\hat{\mu}(n) - (n-1)\hat{\mu}(n-2)$ if $n \in \mathbf{Z}^+$. For $n \in E$, $\hat{\nu}(n) = \phi(n)$, and consequently E is a Sidon set in $\mathbf{Z}$. □

In contrast to the situation for abelian groups it is known that the union of two central Sidon sets need not be central Sidon [12]. This extends to central $(a,1)$-Sidon sets.

PROPOSITION 3.5. — *The union of two sets which are central $(a,1)$-Sidon for all $a < 1$, need not be a central $(0,1)$-Sidon set.*

Proof. — Consider the example $E = \{n_j\}$, where $n_{2^k+i} = A^{A^k}(1+i)$ for $i = 0, 1, \cdots, 2^k - 1$ and A sufficiently large. This example is seen in Theorem 3.3 to be a non-Sidon set in $\mathbf{Z}^+$ which is a central $(a,1)$-Sidon set for all $a < 1$. The set $E-2$ clearly has the same properties and is disjoint from E. By the previous proposition their union is not central $(0,1)$-Sidon. □

Remark. — Our understanding of weighted Sidon sets is much less satisfactory in the non-central case. It is known that any set of representations whose degrees tend to infinity sufficiently fast is $(-\varepsilon, 1)$-Sidon for any given $\varepsilon > 0$, and that a compact Lie group admits no $(\varepsilon, 1)$-Sidon set for $\varepsilon > 0$ [5], but we do not know if any of our examples of central $(a,1)$-Sidon sets, or any other infinite sets in the dual of a compact, simple-connected semisimple Lie group, are $(0,1)$-Sidon.

BIBLIOGRAPHY

[1] D.I. CARTWRIGHT and J.R. MCMULLEN, A structural criterion for the existence of infinite Sidon sets, Pacific J. Math., 96 (1981), 301–317.

[2] R. COIFMAN and G. WEISS, Central multiplier theorems for compact Lie groups, Bull. Amer. Math. Soc., 80 (1974), 124–126.

[3] A.H. DOOLEY, Central lacunary sets for Lie groups, J. Aust. Math. Soc., 45 (1988), 30–45.

[4] P. GALLAGHER, Zeroes of group characters, Math. Z., 87 (1965), 363–364.

[5] K. HARE and D. WILSON, Weighted p-Sidon sets, J. Aust. Math. Soc., to appear.

[6] E. HEWITT and K. ROSS, Abstract harmonic analysis II, Springer-Verlag, New York, 1970.

[7] J.E. HUMPHREYS, Introduction to Lie algebras and representation theory, Springer-Verlag, New York, 1972.

[8] J. LOPEZ and K. ROSS, Sidon sets, Lecture Notes Pure Appl. Math., No.13, Marcel Dekker, New York, 1975.

[9] W.A. PARKER, Central Sidon and central $\Lambda(p)$ sets, J. Aust. Math. Soc., 14 (1972), 62–74.

[10] J.F. PRICE, Lie groups and compact groups, London Math. Soc. Lecture Note Series No.25, Cambridge Univ. Press, Cambridge, 1977.

[11] D.L. RAGOZIN, Central measures on compact simple Lie groups, J. Func. Anal., 10 (1972), 212–229.

[12] D. RIDER, Central lacunary sets, Monatsh. Math., 76 (1972), 328–338.

[13] R. STANTON and P. TOMAS, Polyhedral summability of Fourier series on compact Lie groups, Amer. J. Math., 100 (1978), 477–493.

[14] V.S. VARADARAJAN, Lie groups, Lie algebras and their representations, Springer-Verlag, New York, 1984.

Manuscrit reçu le 30 juin 1994.

Kathryn E. HARE,
Dept. of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1 (Canada).