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# $L^p$ ESTIMATES FOR SCHRÖDINGER OPERATORS WITH CERTAIN POTENTIALS

by Zhongwei SHEN (*)

## 0. Introduction.

In this paper we consider the Schrödinger differential operator

$$P = -\Delta + V(x) \quad \text{on} \quad \mathbb{R}^n, n \geq 3 \tag{0.1}$$

where $V(x)$ is a nonnegative potential. We will assume that $V$ belongs to the reverse Hölder class $B_q$ for some $q \geq n/2$. We are interested in the $L^p$ boundedness of the operators $(-\Delta+V)^{i\gamma}, \nabla(-\Delta+V)^{-1/2}, \nabla(-\Delta+V)^{-1}\nabla$ and $\nabla^2(-\Delta+V)^{-1}$ where $\gamma \in \mathbb{R}$.

Note that a nonnegative locally $L^q$ integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q (1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V dx\right) \tag{0.2}$$

holds for every ball $B$ in $\mathbb{R}^n$ ([G], [M]).

One remarkable feature about the $B_q$ class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on $n$ and the constant $C$ in (0.2), such that $V \in B_{q+\varepsilon}$ [G].

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We now state our main results in this paper.

THEOREM 0.3. — *Suppose $V \in B_q$ for some $q \geq n/2$. Then, for $1 < p \leq q$,*

$$\|\nabla^2(-\Delta + V)^{-1} f\|_p \leq C_p \|f\|_p$$

*where $C_p$ depends only on $p$, $n$ and the constant in (0.2).*

THEOREM 0.4. — *Suppose $V \in B_{n/2}$. Then, for $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$ is a Calderón–Zygmund operator.*

It is well known that Calderón–Zygmund operators are bounded on $L^p$ for $1 < p < \infty$.

THEOREM 0.5. — *Suppose $V \in B_q$ and $n/2 \leq q < n$. Then, for $1 < p \leq p_0$,*

$$\|\nabla(-\Delta + V)^{-1/2} f\|_p \leq C_p \|f\|_p$$

*where $(1/p_0) = (1/q) - (1/n)$.*

We remark that the ranges of $p$ in Theorems 0.3 and 0.5 are optimal. This can be shown by considering the potential $V(x) = |x|^\alpha$ where $-2 < \alpha < 0$. See Section 7.

It follows easily from Theorem 0.5 that, if $V \in B_q$, $n/2 \leq q < n$,

$$\|(-\Delta + V)^{-1/2} \nabla f\|_p \leq C_p \|f\|_p \quad \text{for} \quad p_0' \leq p < \infty \tag{0.6}$$

and

$$\|\nabla(-\Delta + V)^{-1} \nabla f\|_p \leq C \|f\|_p \quad \text{for} \quad p_0' \leq p \leq p_0 \tag{0.7}$$

where $p_0' = p_0/(p_0 - 1)$.

THEOREM 0.8. — *Suppose $V \in B_n$. Then*

$$\nabla(-\Delta + V)^{-1/2}, (-\Delta + V)^{-1/2} \nabla \text{ and } \nabla(-\Delta + V)^{-1} \nabla$$

*are Calderón–Zygmund operators.*

We also obtain the $L^p$ boundedness of the operators

$$V(-\Delta + V)^{-1}, \quad V^{1/2}(-\Delta + V)^{-1/2}, \quad V^{1/2}(-\Delta + V)^{-1} \nabla$$

and

$$V^{1/2} \nabla(-\Delta + V)^{-1}.$$

See Theorem 3.1, Theorem 5.10, Theorem 5.11 and Theorem 4.13 respectively. As a direct consequence of our $L^p$ estimates, we have

COROLLARY 0.9. — *Suppose $V \in B_q$ for some $q \geq n/2$. Assume that $-\Delta u + Vu = f$ in $\mathbb{R}^n$. Then*

$$\|\nabla^2 u\|_p \leq C_p \|f\|_p \text{ for } 1 < p \leq q,$$
$$\|Vu\|_p \leq C\|f\|_p \text{ for } 1 \leq p \leq q,$$
$$\|V^{1/2}\nabla u\|_p \leq C\|f\|_p \text{ for } 1 \leq p \leq p_1$$

*where $1/(p_1) = (3/2q) - (1/n)$ if $n/2 \leq q < n$; and $p_1 = 2q$ if $q \geq n$.*

COROLLARY 0.10. — *Suppose $V \in B_q$ for some $q \geq n/2$. Assume that $-\Delta u + Vu = \operatorname{div} \vec{g}$ in $\mathbb{R}^n$. Then*

$$\|\nabla u\|_p \leq C_p \|\vec{g}\|_p \text{ for } p_0' \leq p \leq p_0 (1 < p < \infty \text{ if } q \geq n),$$
$$\|V^{1/2} u\|_p \leq C\|\vec{g}\|_p \text{ for } p_0' \leq p \leq 2q (1 \leq p \leq 2q \text{ if } q \geq n)$$

*where $(1/p_0) = (1/q) - (1/n)$.*

We now recall that an operator $T$ taking $C_c^\infty(\mathbb{R}^n)$ into $L^1_{\text{loc}}(\mathbb{R}^n)$ is called a Calderón–Zygmund operator if

(a) $T$ extends to a bounded linear operator on $L^2(\mathbb{R}^n)$,

(b) there exists a kernel $K$ such that for every $f \in L_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \quad \text{a.e. on } \{\operatorname{supp} f\}^c,$$

(c) the kernel $K$ satisfies the Calderón–Zygmund estimates:

$$\text{(0.11)} \qquad \begin{cases} |K(x,y)| \leq \dfrac{C}{|x-y|^n}, \\ |K(x+h,y) - K(x,y)| \leq \dfrac{C|h|^\delta}{|x-y|^{n+\delta}}, \\ |K(x,y+h) - K(x,y)| \leq \dfrac{C|h|^\delta}{|x-y|^{n+\delta}} \end{cases}$$

for $x, y \in \mathbb{R}^n$, $|h| < |x-y|/2$ and for some $\delta > 0$. See [St2].

We remark that in his thesis, which inspired the work in this paper, J. Zhong [Z] studied the Schrödinger operator $-\Delta + V(x)$, assuming that $V$ is a nonnegative polynomial in $\mathbb{R}^n$. He showed that, for $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are Calderón–Zygmund operators with bounds depending only on the degree of $V$ and the dimension $n$.

Related results can also be found in [HN] and [T]. In [HN], Helffer and Nourrigat considered the case of nonnegative polynomials. They proved

the $L^2$ boundedness of $\nabla^2(-\Delta+V)^{-1}$ and $V^{1/2}\nabla(-\Delta+V)^{-1}$, based on a subelliptic estimate of Rothschild and Stein [RS]. In [T], the potential $V = |x|^2$ was considered in connection with the Hermite operator. Furthermore, it was pointed out by the referee that the $L^p$ $(1 < p < \infty)$ boundedness of the operator $(-\Delta + V)^{i\gamma}$ in Theorem 0.4 follows from a general result of W. Hebisch [H].

Note that, if $V$ is a nonnegative polynomial, then

$$\max_{x\in B} V(x) \le C\left(\frac{1}{|B|}\int_B V(x)dx\right) \tag{0.12}$$

for every ball $B$ in $\mathbb{R}^n$, where $C$ depends only on the degree of $V$ and $n$. It follows that $V$ satisfies (0.2), i.e., $V \in B_q$ for all $q$, $1 < q < \infty$, with the same constant as in (0.12). Hence, our Theorems 0.4 and 0.8 extend Zhong's results on the operator $(-\Delta+V)^{i\gamma}$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ to the general $B_q$ class. Clearly, Theorem 0.3 implies that $\nabla^2(-\Delta+V)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p$, $1 < p < \infty$, if $V$ satisfies (0.12). But it seems that extra conditions are needed to assure that the kernel function for the operator $\nabla^2(-\Delta+V)^{-1}$ satisfies the Calderón–Zygmund estimate (0.11).

It is interesting to notice that, if $V(x) = |x_n - \varphi(x')|^\alpha$, $\alpha > 0$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, then $V$ satisfies (0.12) with a constant depending only on $n, \alpha$ and $\|\nabla\varphi\|_\infty$. Also, if $V(x) = |x|^\alpha$, $\alpha q > -n$, then $V \in B_q$. In particular, $V \in B_{n/2}$ if $\alpha > -2$; and $V \in B_n$ if $\alpha > -1$.

The Schrödinger operator $-\Delta + V$ with nonnegative potentials is useful in the study of certain subelliptic operators. Indeed, by taking the partial Fourier transform in the $t$ variable, the operator $-\Delta_x - V(x)\partial_t^2$ is reduced to $-\Delta_x + V(x)\xi^2$. See [Sm].

We briefly sketch the argument we will use in this paper. First, we note that, in the case that $V = P(x)$ is a nonnegative polynomial, the weight function

$$M(x,V) = \sum_{|\alpha|\le k} |\partial_x^\alpha P(x)|^{1/(|\alpha|+2)} \tag{0.13}$$

plays a key role. In (0.13), $k$ = degree of $P(x)$. See [Sm], [Z]. Here, for $V \in B_{n/2}$, we define the function $m(x,V)$ by

$$\frac{1}{m(x,V)} = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}. \tag{0.14}$$

The function $m(x,V)$ was introduced in [Sh] for the potential $V$ satisfying the condition (0.12) to study the Neumann problem for the operator

$-\Delta+V(x)$ in the region above a Lipschitz graph. Note that, if $r = \dfrac{1}{m(x,V)}$, then

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy = 1.$$

In the case that $V = P(x) \geq 0$ is a polynomial, it can be shown that

$$m(x,V) \sim M(x,V).$$

Next, we use a lemma due to C. Fefferman and D.H. Phong [F] to show that, if $V \in B_{n/2}$,

$$|\Gamma(x,y)| \leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k} \cdot \frac{1}{|x-y|^{n-2}} \tag{0.15}$$

and, if $V \in B_n$,

$$|\nabla\Gamma(x,y)| \leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k} \cdot \frac{1}{|x-y|^{n-1}} \tag{0.16}$$

for any $k > 0$, where $\Gamma(x,y)$ denotes the fundamental solution for the operator $-\Delta + V(x)$ in $\mathbb{R}^n$. We remark that similar approaches were used in [Sm] and [Z]. Also see [Sh].

Estimates (0.15) and (0.16) are essential to deal with the kernels of operators in the part where $|x-y| > \dfrac{1}{m(x,V)}$. For the part where $|x-y| \leq \dfrac{1}{m(x,V)}$, the key observation is that, if $V \in B_q$, $q > (n/2)$,

$$|\Gamma(x,y) - \Gamma_0(x,y)| \leq \frac{C\{|x-y|m(x,V)\}^{2-(n/q)}}{|x-y|^{n-2}} \tag{0.17}$$

where $\Gamma_0(x,y)$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^n$.

Estimates like (0.15), (0.16) and (0.17) will enable us to control the operator $(-\Delta+V)^{i\gamma}$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ by the Hardy–Littlewood maximal function and the corresponding (maximal) singular integral operators associated with $-\Delta$.

Finally, we note that, to study $(-\Delta+V)^{i\gamma}$ and $\nabla(-\Delta+V)^{-1/2}$, by functional calculus, we actually need to deal with $\Gamma(x,y,\tau)$ and $\nabla\Gamma(x,y,\tau)$ where $\Gamma(x,y,\tau)$ is the fundamental solution for $-\Delta+V(x)+i\tau$, $\tau \in \mathbb{R}$. Moreover, if $V \in B_q$, $n/2 \leq q < n$, we do not have pointwise estimates for $\nabla\Gamma(x,y,\tau)$. But, these difficulties are overcome by our basic idea that, the operators associated with $-\Delta+V$ can be viewed as a perturbation of the corresponding operators associated with $-\Delta$ in the scale less than $\{m(x,V)\}^{-1}$.

The paper is organized as follows. In Section 1 we introduce the auxiliary function $m(x, V)$ and study its properties. We also state and prove the Fefferman–Phong Lemma (Lemma 1.9) under the assumption $V \in B_{n/2}$. In Section 2 we establish the estimate (0.15) on the fundamental solutions (Theorem 2.7). Section 3 is devoted to the proof of Theorem 0.3. In Section 4 we give the proof of Theorem 0.4. Theorem 0.5 is proved in Section 5, while Theorem 0.8 is proved in Section 6. A counterexample is given in Section 7.

Throughout this paper, unless otherwise indicated, we will use $C$ and $c$ to denote constants, which are not necessarily the same at each occurrence, which depend at most on the constant in (0.2) and the dimension $n$. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$, such that $c \leq A/B \leq C$.

Finally, the author would like to thank the referee for pointing out the relevance of Hebish's work, and for the valuable comments concerning the limitations on $p$ in Theorem 0.3 and 0.5, which lead to the counterexample in Section 7.

## 1. The auxiliary function $m(x, V)$.

Most of the results in this section were proved in [Sh] under the assumption (0.12). The extension to the case $V \in B_q$, $q > n/2$ is fairly straightforward. For the sake of completeness we provide the proofs.

Throughout the section we will assume that $V \in B_q$ for some $q > n/2$.

It is well known that $V \in B_q$, $q > 1$ implies that $V(x)dx$ is a doubling measure,

$$\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy. \tag{1.1}$$

In fact, if $V \in B_q$, $q > 1$, then $V$ is a Muckenhoupt $A_\infty$ weight (e.g., see [St2]).

LEMMA 1.2. — *There exists $C > 0$ such that, for $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left(\frac{R}{r}\right)^{\frac{n}{q}-2} \cdot \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

*Proof.* — By Hölder inequality,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} V dy \le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q dy\right)^{1/q}$$
$$\le \left(\frac{R}{r}\right)^{n/q}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} V^q dy\right)^{1/q}$$
$$\le C\left(\frac{R}{r}\right)^{n/q}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} V dy\right)$$

since $V \in B_q$.

The lemma then follows easily.

By Lemma 1.2 and the assumption $q > n/2$, we see that, for any $x \in \mathbb{R}^n$,

$$\lim_{r\to 0}\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = 0$$

and

$$\lim_{r\to\infty}\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = \infty.$$

DEFINITION 1.3. — *For $x \in \mathbb{R}^n$, the function $m(x,V)$ is defined by*

$$\frac{1}{m(x,V)} = \sup_{r>0}\left\{r: \frac{1}{r^{n-2}}\int_{B(x,r)} V dy \le 1\right\}.$$

Clearly, $0 < m(x,V) < \infty$ for every $x \in \mathbb{R}^n$ and if $r = \dfrac{1}{m(x,V)}$ then

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = 1.$$

Moreover, by Lemma 1.2, if

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V dy \sim 1, \quad \text{then } r \sim \frac{1}{m(x,V)}.$$

The following lemma is very useful to us.

LEMMA 1.4. — *There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that, for $x, y$ in $\mathbb{R}^n$,*

(a) $$m(x,V) \sim m(y,V) \text{ if } |x-y| \le \frac{C}{m(x,V)},$$

(b) $$m(y,V) \le C\{1+|x-y|m(x,V)\}^{k_0} m(x,V),$$

(c) $$m(y,V) \ge \frac{c\,m(x,V)}{\{1+|x-y|m(x,V)\}^{k_0/(k_0+1)}}.$$

*Proof.* — Let $r = \dfrac{1}{m(x,V)}$. Suppose $|x-y| \le Cr$. Since $Vdx$ is a doubling measure,

$$\frac{1}{r^{n-2}} \int_{B(y,r)} V dz \sim \frac{1}{r^{n-2}} \int_{B(x,r)} V dz = 1.$$

It follows that

$$m(y,V) \sim \frac{1}{r} = m(x,V).$$

To see part (b), suppose $|y-x| \sim 2^j r, j \ge 1$. Let $0 < r_1 < r$ and $2^k r_1 \sim 2^j r$. Then, by Lemma 1.2,

$$\begin{aligned}
\int_{B(y,r_1)} V dz &\le C(2^k)^{(n/q)-n} \int_{B(y,2^k r_1)} V dz \\
&\le C(2^k)^{(n/q)-n} \int_{B(y,2^j r)} V dz \\
&\le C(2^k)^{(n/q)-n} \int_{B(x,2^j r)} V dz \\
&\le C(2^k)^{(n/q)-n} \cdot C_0^j \cdot \int_{B(x,r)} V dz \text{ (by the doubling condition (1.1))} \\
&= C(2^k)^{(n/q)-n} \cdot C_0^j \cdot r^{n-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{r_1^{n-2}} \int_{B(y,r_1)} V dz &\le C(2^k)^{(n/q)-n} \cdot C_0^j \cdot \left(\frac{r}{r_1}\right)^{n-2} \\
&\le C(2^{(n/q)-n} C_0)^j \cdot \left(\frac{r}{r_1}\right)^{(n/q)-2} \\
&\le 1/2 \text{ if } r_1 \le C_1^{-j} r \text{ and } C_1 \text{ is large .}
\end{aligned}$$

Hence, by Definition 1.3,

$$\frac{1}{m(y,V)} \ge C_1^{-j} r.$$

It follows that

$$m(y,V) \le C_1^j m(x,V) \le C\{1 + |x-y| m(x,V)\}^{k_0} m(x,V), k_0 = \log_2 C_1.$$

Finally, we need to show part (c). We may assume that

$$|x-y| \ge \frac{1}{m(y,V)},$$

for otherwise it follows easily from part (a).

By part (b),

$$m(x,V) \le C\{1+|x-y|m(y,V)\}^{k_0} m(y,V)$$
$$\le C|x-y|^{k_0} m(y,V)^{k_0+1}.$$

Thus,

$$m(y,V) \ge \frac{c\,m(x,V)^{1/(k_0+1)}}{|x-y|^{k_0/(k_0+1)}} \ge \frac{c\,m(x,V)}{\{1+m(x,V)|x-y|\}^{k_0/(k_0+1)}}.$$

The proof is complete.

The following is an easy consequence of Lemma 1.4:

COROLLARY 1.5. — *There exist $C>0$, $c>0$ and $k_0>0$ such that, for any $x, y \in \mathbb{R}^n$,*

$$c\{1+|x-y|m(y,V)\}^{1/(k_0+1)} \le 1+|x-y|m(x,V)$$
$$\le C\{1+|x-y|m(y,V)\}^{k_0+1}.$$

Using Hölder inequality and the $B_q$ condition we see that

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}}dy \le \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy. \tag{1.6}$$

Similarly, if $V \in B_n$, hence $V \in B_{n+\varepsilon}$,

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-1}}dy \le \frac{C}{R^{n-1}} \int_{B(x,R)} V(y)dy. \tag{1.7}$$

LEMMA 1.8. — *There exist $C>0$ and $k_0>0$ such that, if $Rm(x,V) \ge 1$, then*

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V dy \le C\{Rm(x,V)\}^{k_0}.$$

*Proof.* — Let $r = \dfrac{1}{m(x,V)}$. Suppose $2^j r \le R < 2^{j+1}r$, $j \ge 0$. Then, by the doubling condition (1.1),

$$\int_{B(x,R)} V dy \le C_0^{j+1} \int_{B(x,r)} V dy = C_0^{j+1} r^{n-2}.$$

It follows that

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V dy \le C_0^{j+1} \left(\frac{r}{R}\right)^{n-2} \le C_0 (C_0 2^{2-n})^j$$
$$\le C(Rm(x,V))^{k_0}, k_0 = \log_2 C_0 + 2 - n.$$

We end this section with a lemma due to C. Fefferman and D.H. Phong [F], which plays the crucial role in the estimates of the fundamental solution for the Schrödinger operator $-\Delta + V(x)$.

LEMMA 1.9 (C. Fefferman-D.H. Phong). — *Let $u \in C_c^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^2 dx \le C\left\{\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^n} |u(x)|^2 V(x) dx\right\}.$$

*Proof.* — Fix $x_0 \in \mathbb{R}^n$, let $r_0 = \dfrac{1}{m(x_0,V)}$. Then

$$\int_B |\nabla u(x)|^2 dx \ge \frac{c}{r_0^{n+2}} \int_B \int_B |u(x) - u(y)|^2 dxdy$$

$$\int_B |u(x)|^2 V(x) dx \ge \frac{c}{r_0^n} \int_B \int_B |u(y)|^2 V(y) dxdy$$

where $B = B(x_0, r_0)$.

Adding two inequalities we obtain

$$\int_B |\nabla u|^2 dx + \int_B |u|^2 V dx \ge \frac{c}{r_0^n} \int_B \int_B \min_{y\in B}\left\{\frac{c_0}{r_0^2}, V(y)\right\} |u(x)|^2 dxdy$$

where $c_0 > 0$ is a constant to be determined later.

Recall that $V$ is an $A_\infty$ weight. Hence, there exists $\varepsilon > 0$ such that, for every ball $B$ in $\mathbb{R}^n$,

$$\left|\left\{x \in B : V(x) \ge \frac{\varepsilon}{|B|}\int_B V dy\right\}\right| \ge \frac{1}{2}|B|.$$

Now, let $c_0 = \varepsilon |B(0,1)|^{-1}$, then

$$\int_B \min_{y\in B}\left\{\frac{c_0}{r_0^2}, V(y)\right\} dy \ge c r_0^{n-2}.$$

It follows that

$$\int_B |u(x)|^2 m(x,V)^2 dx \le \frac{C}{r_0^2}\int_B |u|^2 dx \le C\left\{\int_B |\nabla u|^2 dx + \int_B |u|^2 V dx\right\}$$

where we have used part (a) of Lemma 1.4. Moreover,

$$\int_{B(x_0,r_0)} |u(x)|^2 m(x,V)^{2+n} dx \le C\left\{\int_{B(x_0,r_0)} |\nabla u(x)|^2 m(x,V)^n dx + \int_{B(x_0,r_0)} |u(x)|^2 V(x) m(x,V)^n dx\right\}.$$

To finish the proof, we integrate both sides of the above inequality in $x_0$ over $\mathbb{R}^n$ to obtain

$$\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^{2+n} \left( \int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \right) dx$$
$$\leq C \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 m(x,V)^n \left( \int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \right) dx \right.$$
$$\left. + \int_{\mathbb{R}^n} |u(x)|^2 V(x) m(x,V)^n \left( \int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \right) \right\}.$$

Finally, note that, by part (a) of Lemma 1.4,

$$\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \sim \left( \frac{1}{m(x,V)} \right)^n.$$

The lemma then follows easily.

## 2. Estimates of fundamental solutions.

This section is devoted to the estimates of fundamental solutions for the operator $-\Delta + (V + i\tau)$ on $\mathbb{R}^n$ where $\tau \in \mathbb{R}$.

We will assume that $V \in B_q$ for some $q > n/2$ throughout this section.

LEMMA 2.1. — *Suppose* $-\Delta u + (V + i\tau)u = 0$ *in* $B(x_0, 2R)$ *for some* $x_0 \in \mathbb{R}^n, R > 0$ *and* $\tau \in \mathbb{R}$. *Then*

$$\sup\{|u(x)| : x \in B(x_0, R)\}$$
$$\leq \frac{C_k}{\{1 + R|\tau|^{1/2}\}^k \{1 + Rm(x_0,V)\}^k} \left\{ \frac{1}{R^n} \int_{B(x_0,2R)} |u|^2 dy \right\}^{1/2}$$

*for any integer* $k > 0$, *where* $C_k$ *depends only on* $k, n$ *and the constant in the reverse Hölder inequality (0.2).*

*Proof.* — Since

$$\Delta(|u|^2) = 2Re\Delta u \cdot \overline{u} + 2|\nabla u|^2 = 2V|u|^2 + 2|\nabla u|^2 \geq 0,$$

$|u|^2$ is subharmonic. It follows that

$$\sup\{|u(x)| : x \in B(x_0, R)\} \leq C \left( \frac{1}{R^n} \int_{B(x_0,3R/2)} |u|^2 dy \right)^{1/2}. \tag{2.2}$$

By Caccioppoli's inequality,

$$\int_{B(x_0,3R/2)} |\nabla u|^2 dx + \int_{B(x_0,3R/2)} |u|^2 V dx + |\tau| \int_{B(x_0,3R/2)} |u|^2 dx \le \frac{C}{R^2} \int_{B(x_0,2R)} |u|^2 dx. \tag{2.3}$$

Now, let $\eta \in C_0^\infty(B(x_0, 3R/2))$ such that $\eta \equiv 1$ on $B(x_0, R)$ and $|\nabla \eta| \le C/R$. Applying Lemma 1.9 to the function $u\eta$, we obtain

$$\begin{aligned}\int_{B(x_0,R)} m(x,V)^2 |u|^2 dx &\le C \left\{ \int_{B(x_0,3R/2)} |\nabla u|^2 dx + \int_{B(x_0,3R/2)} |u|^2 V dx \right. \\ &\qquad \left. + \frac{1}{R^2} \int_{B(x_0,2R)} |u|^2 dx \right\} \\ &\le \frac{C}{R^2} \int_{B(x_0,2R)} |u|^2 dx\end{aligned}$$

where we used (2.3) in the last inequality.

Note that, by part (c) of Lemma 1.4, for $x \in B(x_0, R)$,

$$m(x,V) \ge \frac{c\, m(x_0,V)}{\{1 + Rm(x_0,V)\}^{k_0/(k_0+1)}}.$$

It follows that

$$\int_{B(x_0,R)} |u|^2 dx \le C \cdot \frac{\{1 + Rm(x_0,V)\}^{2k_0/(k_0+1)}}{\{m(x_0,V)R\}^2} \int_{B(x_0,2R)} |u|^2 dx.$$

Hence,

$$\int_{B(x_0,R)} |u|^2 dx \le \frac{C}{\{1 + Rm(x_0,V)\}^{2/(k_0+1)}} \int_{B(x_0,2R)} |u|^2 dx.$$

Clearly, by repeating above argument, we have

$$\int_{B(x_0,3R/2)} |u|^2 dx \le \frac{C_k}{\{1+Rm(x_0,V)\}^k} \int_{B(x_0,2R)} |u|^2 dx \text{ for any } k>0. \tag{2.4}$$

Similarly, by Caccioppoli's inequality (2.3),

$$\int_{B(x_0,3R/2)} |u|^2 dx \le \frac{C_k}{(1 + |\tau|^{1/2} R)^k} \int_{B(x_0,2R)} |u|^2 dx \text{ for any } k>0. \tag{2.5}$$

The lemma then follows from (2.2), (2.4) and (2.5).

Let $\Gamma(x, y, \tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V(x) + i\tau), \tau \in \mathbb{R}$. Clearly,

$$\Gamma(x, y, \tau) = \Gamma(y, x, -\tau).$$

Since $V \geq 0$ and $V \in L^{n/2}_{\mathrm{loc}}$, it is well known that

$$|\Gamma(x,y,\tau)| \leq \frac{C}{|x-y|^{n-2}} \quad \text{for} \quad x,y \in \mathbb{R}^n \tag{2.6}$$

where $C$ depends only on $n$.

Since $V \in B_{n/2}$ implies that $V \in B_q$ for some $q > n/2$, the following theorem follows easily from Lemma 2.1 and (2.6).

THEOREM 2.7. — *Suppose $V \in B_{n/2}$. Then, for any $x, y \in \mathbb{R}^n$,*

$$|\Gamma(x,y,\tau)| \leq \frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^k\{1+m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2}}$$

*where $C_k$ is a constant depending only on $n, k$ and the constant in (0.2).*

COROLLARY 2.8. — *Suppose $V \in B_{n/2}$. Then there exists $C > 0$ depending on $n$ and the constant in (0.2) such that, for every $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,*

$$\|m(\cdot,V)^2(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p.$$

*Proof.* — It follows from Theorem 2.7 that

$$\begin{aligned}\int_{\mathbb{R}^n} |\Gamma(x,y)|dy &\leq C\int_{\mathbb{R}^n} \frac{dy}{\{1+|x-y|m(x,V)\}^3|x-y|^{n-2}} \\ &\leq \frac{C}{m(x,V)^2}\end{aligned}$$

where $\Gamma(x,y) = \Gamma(x,y,0)$ is the fundamental solution for $-\Delta+V(x)$. Thus, if $f \in L^p(\mathbb{R}^n)$,

$$u(x) = (-\Delta+V)^{-1}f(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy,$$

we have

$$\begin{aligned}|u(x)| &\leq \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)|dy\right\}^{1/p'} \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)||f(y)|^p dy\right\}^{1/p} \\ &\leq \frac{C}{m(x,V)^{2/p'}} \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)||f(y)|^p dy\right\}^{1/p}, \quad p' = \frac{p}{p-1}.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^n} |m(x,V)^2 u(x)|^p dx \leq C\int_{\mathbb{R}^n} |f(y)|^p \left\{\int_{\mathbb{R}^n} m(x,V)^2|\Gamma(x,y)|dx\right\} dy.$$

Finally, note that, by part (b) of Lemma 1.4 and Theorem 2.7,

$$\int_{\mathbb{R}^n} m(x,V)^2|\Gamma(x,y)|dx \le C_k \int_{\mathbb{R}^n} \frac{(m(y,V))^2}{\{1+|x-y|m(y,V)\}^{k-2k_0}|x-y|^{n-2}}dx$$
$$\le C \quad \text{if we choose} \quad k = 2k_0 + 3.$$

Corollary 2.8 then follows.

*Remark* 2.9. — If $V$ satisfies (0.12), then $V(x) \le Cm(x,V)^2$ a.e. on $\mathbb{R}^n$. It follows from Corollary 2.8 that, for $1 \le p \le \infty$,

$$\|V(-\Delta+V)^{-1}f\|_p \le C\|f\|_p.$$

Also, for $1 < p < \infty$, by the $L^p$ boundedness of the Riesz transforms,

$$\|\nabla^2(-\Delta+V)^{-1}f\|_p \le C\|\Delta(-\Delta+V)^{-1}f\|_p$$
$$\le C\|V(-\Delta+V)^{-1}f\|_p + C\|f\|_p$$
$$\le C\|f\|_p.$$

## 3. The proof of Theorem 0.3.

In this section we give the proof of Theorem 0.3.

THEOREM 3.1. — *Suppose $V \in B_q$ for some $q \ge n/2$. Then, for $1 \le p \le q$,*

$$\|V(-\Delta+V)^{-1}f\|_p \le C\|f\|_p$$

*where $C$ depends only on $n$ and the constant in (0.2).*

*Proof.* — Let $f \in L^p(\mathbb{R}^n), 1 \le p \le q$ and

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy.$$

We need to show that

$$\|Vu\|_p \le C\|f\|_p.$$

Write

$$u(x) = \int_{|y-x|<r} \Gamma(x,y)f(y)dy + \int_{|y-x|\ge r} \Gamma(x,y)f(y)dy = u_1(x) + u_2(x)$$

where $r = \dfrac{1}{m(x,V)}$.

By the properties of $B_q$ class, $V \in B_{q_0}$ for some $q_0 > q$. We have

$$|u_1(x)| \le \int_{|y-x|<r} \frac{|f(y)|}{|x-y|^{n-2}} dy$$
$$\le Cr^{2-\frac{n}{q_0}} \left( \int_{|y-x|<r} |f(y)|^{q_0} dy \right)^{1/q_0}$$

where we have used Hölder inequality and the fact $q_0 > n/2$. Thus,

$$\int_{\mathbb{R}^n} |V(x)u_1(x)|^{q_0} dx \le C \int_{\mathbb{R}^n} \left\{ \int_{|y-x|<\frac{1}{m(x,V)}} |f(y)|^{q_0} dy \right\} V(x)^{q_0} m(x,V)^{n-2q_0} dx$$
$$= C \int_{\mathbb{R}^n} |f(y)|^{q_0} \left\{ \int_{|x-y|<\frac{1}{m(x,V)}} V(x)^{q_0}\ m(x,V)^{n-2q_0} dx \right\} dy.$$

Now, let $R = \dfrac{1}{m(y,V)}$. Then

$$\int_{|x-y|<\frac{1}{m(x,V)}} V(x)^{q_0} m(x,V)^{n-2q_0} dx \le CR^{2q_0-n} \int_{|x-y|\le CR} V(x)^{q_0} dx$$
$$\le CR^{2q_0} \left\{ \frac{1}{R^n} \int_{|x-y|\le CR} V(x) dx \right\}^{q_0}$$
$$\le C$$

where we used the part (a) of Lemma 1.4, (0.2) and Lemma 1.8.

Hence, we have proved that

$$\int_{\mathbb{R}^n} |V(x)u_1(x)|^{q_0} dx \le C \int_{\mathbb{R}^n} |f(x)|^{q_0} dx \quad \text{for some} \quad q_0 > q \ge n/2.$$

Next, note that, by (1.6),

$$\int_{\mathbb{R}^n} |V(x)u_1(x)| dx \le C \int_{\mathbb{R}^n} |f(y)| \left\{ \int_{|x-y|<\frac{1}{m(x,V)}} \frac{V(x)}{|x-y|^{n-2}} dx \right\} dy$$
$$\le C \int_{\mathbb{R}^n} |f(y)| dy.$$

Therefore, by interpolation,

$$\|Vu_1\|_p \le C\|f\|_p \quad \text{for} \quad 1 \le p \le q_0.$$

To finish the proof, we note that, by Theorem 2.7 and the Hölder inequality,

$$|u_2(x)| \le C \int_{|y-x|\ge r} \frac{|f(y)|dy}{\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}}$$
$$\le Cr^{2/p'} \left\{ \int_{|y-x|\ge r} \frac{|f(y)|^p dy}{\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}} \right\}^{1/p}$$

where $1 \le p \le q_0$ and $r = \dfrac{1}{m(x,V)}$.

Thus,

$$\int_{\mathbb{R}^n} |V(x)u_2(x)|^p dx \le C \int_{\mathbb{R}^n} |f(y)|^p \left\{ \int_{|y-x| \ge \frac{1}{m(x,V)}} \frac{|V(x)|^p dx}{m(x,V)^{2(p-1)} \{1 + |x-y| m(x,V)\}^k |x-y|^{n-2}} \right\} dy.$$

Now, fix $y \in \mathbb{R}^n$, let $R = \dfrac{1}{m(y,V)}$. By Lemma 1.4,

$$\int_{|y-x| \ge \frac{1}{m(x,V)}} \frac{|V(x)|^p dx}{m(x,V)^{2(p-1)} \{1 + |x-y| m(x,V)\}^k |x-y|^{n-2}}$$

$$\le C \int_{|y-x| \ge CR} \frac{|V(x)|^p dx}{R^{2(1-p)} (1 + R^{-1}|x-y|)^{k_1} |x-y|^{n-2}},$$

$$\text{(where } k_1 = \frac{k - 2(q-1)k_0}{k_0 + 1}\text{)}$$

$$\le C \sum_{j=1}^{\infty} \frac{1}{(2^j R)^n} \int_{|x-y| \le 2^j R} V^p dx \cdot (2^j)^{-k_1+2} \cdot R^{2p}$$

$$\le C \sum_{j=1}^{\infty} \left( \frac{1}{(2^j R)^n} \int_{|x-y| \le 2^j R} V(x) dx \right)^p \cdot (2^j)^{-k_1+2} \cdot R^{2p}$$

$$\le C \sum_{j=1}^{\infty} \left( \frac{1}{R^{n-2}} \int_{|x-y| \le R} V(x) dx \right)^p \cdot C^j \cdot (2^j)^{-k_1+2}$$

$$\le C \quad \text{if we choose } k \text{ sufficiently large.}$$

From this we have

$$\int_{\mathbb{R}^n} |V(x)u_2(x)|^p dx \le C \int_{\mathbb{R}^n} |f(x)|^p dx \quad \text{for} \quad 1 \le p \le q_0.$$

The theorem is then proved.

We are now in a position to give

*The proof of Theorem 0.3.* — Suppose $V \in B_q$ for some $q \ge n/2$. By Theorem 3.1

$$\|V(-\Delta + V)^{-1} f\|_p \le C \|f\|_p \quad \text{for} \quad 1 \le p \le q.$$

It follows that

$$\|\Delta(-\Delta + V)^{-1} f\|_p \le C \|f\|_p \quad \text{for} \quad 1 \le p \le q.$$

Then, by the $L^p$ boundedness of Riesz transforms, for $1 < p \le q$,

$$\|\nabla^2(-\Delta+V)^{-1}f\|_p \le C_p\|\Delta(-\Delta+V)^{-1}f\|_p \le C_p\|f\|_p.$$

## 4. The proof of Theorem 0.4.

In this section we give the proof of Theorem 0.4 stated in the Introduction. We also prove an $L^p$ estimate for the operator $V^{1/2}\nabla(-\Delta+V)^{-1}$ in the section (Theorem 4.13).

By the functional calculus, we may write, for $\gamma \in \mathbb{R}$,

$$(-\Delta+V)^{i\gamma} = -\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{i\gamma}(-\Delta+V+i\tau)^{-1}d\tau. \tag{4.1}$$

Thus,

$$\begin{aligned}(-\Delta+V)^{i\gamma}f(x) &= -\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{i\gamma}(-\Delta+V+i\tau)^{-1}f(x)d\tau \\ &= \int_{\mathbb{R}^n} K(x,y)f(y)dy\end{aligned}$$

where

$$K(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{i\gamma}\Gamma(x,y,\tau)d\tau. \tag{4.2}$$

Note that, by Theorem 2.7,

$$|K(x,y)| \le \frac{C_k e^{\pi|\gamma|/2}}{\{1+|x-y|m(x,V)\}^k}\cdot\frac{1}{|x-y|^n}, \text{ for any } k>0. \tag{4.3}$$

Let $\Gamma_0(x,y,\tau)$ denote the fundamental solution for the operator $-\Delta+i\tau$ in $\mathbb{R}^n$. It is well known that

$$\begin{cases} |\Gamma_0(x,y,\tau)| \le \dfrac{C_k}{(1+|\tau|^{1/2}|x-y|)^k}\cdot\dfrac{1}{|x-y|^{n-2}} \\ |\nabla_x\Gamma_0(x,y,\tau)| \le \dfrac{C_k}{(1+|\tau|^{1/2}|x-y|)^k}\cdot\dfrac{1}{|x-y|^{n-1}} \\ |\nabla_x^2\Gamma_0(x,y,\tau)| \le \dfrac{C_k}{(1+|\tau|^{1/2}|x-y|)^k}\cdot\dfrac{1}{|x-y|^{n}} \end{cases} \tag{4.4}$$

where $C_k$ is independent of $\tau \in \mathbb{R}$.

LEMMA 4.5. — *Suppose $V \in B_{n/2}$. Then there exists $C_k > 0$ such that*

$$|\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)| \le \frac{C_k}{(1+|\tau|^{1/2}|x-y|)^k}\cdot\frac{\{|x-y|m(x,V)\}^{2-(n/q_0)}}{|x-y|^{n-2}}$$

*for* $x, y \in \mathbb{R}^n$, $|x-y| \le \dfrac{1}{m(x,V)}$ *and for some* $q_0 > n/2$.

*Proof.* — Note that

$$-\Delta_x\{\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\} + i\tau\{\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\} = -V(x)\Gamma(x,y,\tau).$$

We have

$$\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau) = -\int_{\mathbb{R}^n} \Gamma_0(x,z,\tau)V(z)\Gamma(z,y,\tau)dz.$$

Now, let $R = |x-y|$ and suppose $R \le \dfrac{1}{m(x,V)}$. By Theorem 2.7,

$$\begin{aligned}
&|\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)| \\
&\quad \le C_k \int_{\mathbb{R}^n} \frac{(1+|\tau|^{1/2}|x-z|)^{-k}(1+|\tau|^{1/2}|z-y|)^{-k}V(z)\,dz}{|x-z|^{n-2}\{1+m(y,V)|z-y|\}^k|z-y|^{n-2}} \\
&\quad = C_k \int_{|z-x|<R/2} + C_k \int_{|z-y|<R/2} + C_k \int_{\substack{|z-x|\ge R/2 \\ |z-y|\ge R/2}} \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

Since $V \in B_{n/2}$, $V \in B_{q_0}$ for some $q_0 > n/2$. We obtain

$$\begin{aligned}
I_1 &\le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \cdot \frac{1}{R^{n-2}} \cdot \int_{B(x,R/2)} \frac{V(z)dz}{|z-x|^{n-2}} \\
&\le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \cdot \frac{1}{R^{n-2}} \cdot \frac{1}{R^{n-2}} \int_{B(x,R/2)} V(z)dz \\
&\le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \cdot \frac{\{Rm(x,V)\}^{2-(n/q_0)}}{R^{n-2}}
\end{aligned}$$

where we have used (1.6) in the second inequality and Lemma 1.2 in the third.

Similarly,

$$I_2 \le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \cdot \frac{\{Rm(x,V)\}^{2-(n/q_0)}}{R^{n-2}}.$$

Finally, we need to estimate $I_3$.

It is easy to see that

$$\begin{aligned}
I_3 &\le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \int_{|z-y|\ge R/2} \frac{V(z)dz}{|z-y|^{2n-4}\{1+m(y,V)|z-y|\}^k} \\
&\le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \left\{ \int_{r>|z-y|\ge \frac{R}{2}} \frac{V(z)dz}{|z-y|^{2n-4}} + r^k \int_{|z-y|\ge r} \frac{V(z)dz}{|z-y|^{2n-4+k}} \right\}
\end{aligned}$$

where $r = \dfrac{1}{m(y,V)} \sim \dfrac{1}{m(x,V)}$.

Using Hölder inequality and the $B_{q_0}$ condition (0.2), we obtain

$$\begin{aligned}
&\int_{r>|z-y|\geq R/2} \frac{V(z)dz}{|z-y|^{2n-4}} \\
&\qquad \leq C\left\{\int_{B(y,r)} V(z)^{q_0}dz\right\}^{1/q_0}\left\{\int_{R/2}^{r} t^{(n-1)-2(n-2)q_0'}dt\right\}^{1/q_0'} \\
&\qquad \leq C\left(\frac{R}{r}\right)^{2-(n/q_0)}\cdot\frac{1}{R^{n-2}}\cdot\left\{\int_{1/2}^{r/R} t^{(n-1)-2(n-2)q_0'}dt\right\}^{1/q_0'} \\
&\qquad \leq \frac{C\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}
\end{aligned}$$

where we have assumed, without loss of generality, that $(1/q_0) > (4-n)/n$ for $n \geq 3$, so $n-2(n-2)q_0' < 0$.

Also, using the doubling condition and taking $k$ sufficiently large, we see that

$$\begin{aligned}
r^k\int_{|z-y|\geq r}\frac{V(z)dz}{|z-y|^{2n-4+k}} &\leq Cr^k\sum_{j=1}^{\infty}\frac{1}{(2^jr)^{2n-4+k}}\int_{|z-y|\leq 2^jr}V(z)dz \\
&\leq Cr^{4-2n}\sum_{j=1}^{\infty}\frac{1}{(2^j)^{2n-4+k}}\cdot C_0^j\int_{|z-y|\leq r}V(z)dz \\
&\leq \frac{C}{r^{n-2}} \\
&\leq \frac{C\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}.
\end{aligned}$$

Thus,

$$I_3 \leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}.$$

The proof is then finished since $m(y,V) \sim m(x,V)$ when $|x-y| \leq \dfrac{1}{m(x,V)}$.

We need another lemma before we carry out the proof of Theorem 0.4.

LEMMA 4.6. — *Suppose $V \in B_{q_0}$, $(n/2) < q_0 < n$. Assume that $-\Delta u + (V+i\tau)u = 0$ in $B(x_0,2R)$. Then*

$$\left(\int_{B(x_0,R)}|\nabla u|^t dx\right)^{1/t} \leq CR^{(n/q_0)-2}\{1+Rm(x_0,V)\}^{k_0}\sup_{B(x_0,2R)}|u|$$

*where* $(1/t) = (1/q_0) - (1/n)$.

*Proof.* — Let $\eta \in C_0^\infty(B(x_0, 2R))$ such that $\eta = 1$ on $B(x_0, 3R/2)$, $|\nabla \eta| \le C/R$ and $|\nabla^2 \eta| \le C/R^2$.

Note that,

$$
\begin{aligned}
u(x)\eta(x) &= \int_{\mathbb{R}^n} \Gamma_0(x,y,\tau)\{-\Delta + i\tau\}(u\eta)(y)dy \\
&= \int_{\mathbb{R}^n} \Gamma_0(x,y,\tau)\{-Vu\eta - 2\nabla u \cdot \nabla \eta - u\Delta\eta\}dy \qquad (4.7) \\
&= \int_{\mathbb{R}^n} \Gamma_0(x,y,\tau)\{-Vu\eta + u\Delta\eta\}dy \\
&\qquad + 2\int_{\mathbb{R}^n} \nabla_y \Gamma_0(x,y,\tau) \cdot (\nabla\eta)u dy.
\end{aligned}
$$

Thus, for $x \in B(x_0, R)$,

$$
\begin{aligned}
|\nabla u(x)| &\le C\int_{B(x_0,2R)} \frac{V(y)|u(y)||\eta|}{|x-y|^{n-1}}dy + \frac{C}{R^{n+1}}\int_{B(x_0,2R)} |u(y)|dy \qquad (4.8) \\
&\le C \sup_{B(x_0,2R)} |u| \cdot \int_{B(x_0,2R)} \frac{V(y)|\eta(y)|}{|x-y|^{n-1}}dy \\
&\qquad + \frac{C}{R^{n+1}}\int_{B(x_0,2R)} |u(y)|dy.
\end{aligned}
$$

It then follows from the well known theorem on fractional integrals that

$$
\begin{aligned}
&\left(\int_{B(x_0,R)} |\nabla u(x)|^t dx\right)^{1/t} \\
&\le C \sup_{B(x_0,2R)} |u| \left(\int_{B(x_0,2R)} |V(x)|^{q_0} dx\right)^{1/q_0} + CR^{(n/q_0)-2} \sup_{B(x_0,2R)} |u| \\
&\le CR^{(n/q_0)-2} \sup_{B(x_0,2R)} |u| \left\{\frac{1}{R^{n-2}}\int_{B(x_0,2R)} V(x)dx + 1\right\}
\end{aligned}
$$

where $(1/t) = (1/q_0) - (1/n)$ and we have used (0.2) in the last inequality.

The desired estimate then follows from Lemma 1.8.

*Remark* 4.9. — If $V \in B_n$ and $-\Delta u + (V + i\tau)u = 0$ in $B(x_0, 2R)$, then

$$
\sup_{B(x_0,R)} |\nabla u| \le \frac{C}{R}\{1 + Rm(x_0, V)\}^{k_0} \cdot \sup_{B(x_0,2R)} |u|.
$$

Indeed, by (4.8) and (1.7), we have for $x \in B(x_0, R)$

$$|\nabla u(x)| \le C \sup_{B(x_0,2R)} |u| \cdot \frac{1}{R^{n-1}} \int_{B(x_0,2R)} V(y)dy + \frac{C}{R^{n+1}} \int_{B(x_0,2R)} |u(y)|dy$$

$$\le \frac{C}{R} \sup_{B(x_0,2R)} |u| \left\{ \frac{1}{R^{n-2}} \int_{B(x_0,2R)} V(y)dy + 1 \right\}$$

$$\le \frac{C}{R} \{1 + Rm(x_0, R)\}^{k_0} \cdot \sup_{B(x_0,2R)} |u|$$

where we used Lemma 1.8 in the last inequality.

*Remark* 4.10. — If $V \in B_{q_0}$ for some $q_0 > n$ and $-\Delta u + (V + i\tau)u = 0$ in $B(x_0, 2R)$, then, by (4.7) and the Calderón–Zygmund estimates,

$$\|\nabla^2(u\eta)\|_{q_0} \le C\|Vu\eta\|_{q_0} + CR^{(n/q_0)-2} \sup_{B(x_0,2R)} |u|$$

$$\le C \left\{ \left( \int_{B(x_0,2R)} V^{q_0} dy \right)^{1/q_0} + R^{(n/q_0)-2} \right\} \sup_{B(x_0,2R)} |u|$$

$$\le CR^{(n/q_0)-2} \left\{ \frac{1}{R^{n-2}} \int_{B(x_0,2R)} V(x)dx + 1 \right\} \sup_{B(x_0,2R)} |u|.$$

Hence, by Lemma 1.8,

$$\left( \int_{B(x_0,R)} |\nabla^2 u|^{q_0} dx \right)^{1/q_0} \le CR^{(n/q_0)-2} \{1 + Rm(x_0, V)\}^{k_0} \cdot \sup_{B(x_0,2R)} |u|.$$

We are now ready to give

*The proof of Theorem 0.4.* — Suppose $V \in B_{n/2}$. Then $V \in B_{q_0}$ for some $q_0 > n/2$. Note that

$$(-\Delta)^{i\gamma} f(x) = \int_{\mathbb{R}^n} K^0(x, y) f(y) dy$$

where

$$K^0(x, y) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} (-i\tau)^{i\gamma} \Gamma_0(x, y, \tau) d\tau.$$

It follows easily from Lemma 4.5 that

$$|K(x, y) - K^0(x, y)| \le Ce^{|\gamma|\pi/2} \cdot \frac{\{|x - y|m(x, V)\}^{2-(n/q_0)}}{|x - y|^n} \tag{4.11}$$

for $x, y \in \mathbb{R}^n$ and $|x - y|m(x, V) \le 1$.

We now claim that

$$(4.12)\quad |(-\Delta+V)^{i\gamma}f(x)| \le |(-\Delta)^{i\gamma}f(x)| + \sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K^0(x,y)f(y)dy\right| + Ce^{|\gamma|\pi/2}Mf(x)$$

where $Mf$ is the Hardy–Littlewood maximum function of $f$.

Since $(-\Delta)^{i\gamma}$ is a Calderón–Zygmund operator, the boundedness of $(-\Delta+V)^{i\gamma}$ on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ follows easily from (4.12). See [St2].

To see (4.12), we note that

$$\begin{aligned}(-\Delta+V)^{i\gamma}f(x) = (-\Delta)^{i\gamma}f(x) &+ \int_{|y-x|\le\frac{1}{m(x,V)}} \{K(x,y)-K^0(x,y)\}f(y)dy \\ &+ \int_{|y-x|>\frac{1}{m(x,V)}} K(x,y)f(y)dy \\ &+ \int_{|y-x|>\frac{1}{m(x,V)}} K^0(x,y)f(y)dy.\end{aligned}$$

This can be justified by using

$$(-\Delta+V)^{i\gamma}f(x) \equiv \lim_{\varepsilon\to 0^-}(-\Delta+V)^{i\gamma+\varepsilon}f(x)$$

and Lemma 4.5. By (4.11) and (4.3) it is not very hard to see that the second and the third term above are bounded in absolute value by $Ce^{\gamma\pi/2}Mf(x)$, while the last term is bounded by

$$\sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K^0(x,y)f(y)dy\right|$$

(4.12) is proved.

To finish the proof we need to show that the kernel $K(x,y)$ satisfies (0.11) for some $\delta>0$.

First, by (4.3),

$$|K(x,y)| \le \frac{C}{|x-y|^n}.$$

Next, we fix $x_0, y_0 \in \mathbb{R}^n, h\in\mathbb{R}^n$ and $|h|<|x_0-y_0|/4$. Let $R=|x_0-y_0|/4$ and $u(x)=\Gamma(x,y_0,\tau)$. It follows from the imbedding theorem of Morrey (see [GT], p. 163) and Lemma 4.6 that, for $(1/t)=(1/q_0)-(1/n)$,

$$\begin{aligned}|u(x_0+h)-u(x_0)| &\le C|h|^{1-(n/t)}\left(\int_{B(x_0,R)}|\nabla u|^t dx\right)^{1/t} \\ &\le C\left(\frac{|h|}{R}\right)^{2-(n/q_0)} \sup_{B(x_0,2R)}|u|\cdot\{1+Rm(x_0,V)\}^{k_0} \\ &\le C\left(\frac{|h|}{R}\right)^{2-(n/q_0)}\cdot\frac{1}{(1+|\tau|^{1/2}|x_0-y_0|)^3}\cdot\frac{1}{|x_0-y_0|^{n-2}},\end{aligned}$$

where we used Theorem 2.7 in the last inequality.

Thus, we have proved that, for $x, y \in \mathbb{R}^n$, $h \in \mathbb{R}^n$ and $|h| < |x-y|/4$,

$$|\Gamma(x+h,y,\tau) - \Gamma(x,y,\tau)| \le \frac{C}{\{1+|\tau|^{1/2}|x-y|\}^3} \cdot \frac{|h|^\delta}{|x-y|^{n-2+\delta}}$$

where $\delta = 2 - (n/q_0) > 0$. Clearly, this estimate also holds for $|x-y|/4 \le |h| < |x-y|/2$. It then follows from (4.2) that

$$|K(x+h,y) - K(x,y)| \le \frac{C|h|^\delta}{|x-y|^{n+\delta}}$$

whenever $x, y, h \in \mathbb{R}^n$ and $|h| < |x-y|/2$.

The estimate of $K(x, y+h) - K(x,y)$ can be carried out in the same manner.

The proof is complete.

We close this section with an $L^p$ estimate for the operator $V^{1/2}\nabla(-\Delta+V)^{-1}$.

THEOREM 4.13. — *Suppose $V \in B_{q_0}$ for some $q_0 \ge n/2$. Then, for $1 \le p \le p_0$,*

$$\|V^{1/2}\nabla(-\Delta+V)^{-1}f\|_p \le C\|f\|_p$$

*where $(1/p_0) = (3/(2q_0)) - (1/n)$ if $n/2 \le q_0 < n$ and $p_0 = 2q_0$ if $q_0 \ge n$.*

*Proof.* — Suppose $V \in B_{q_0}$ for some $q_0 \ge n/2$. Then $V \in B_{q_1}$ for some $q_1 > q_0$.

Let

$$Tf(x) = V(x)^{1/2} \int_{\mathbb{R}^n} \nabla_x \Gamma(x,y) f(y) dy.$$

The adjoint of $T$ is given by

$$T^* f(x) = \int_{\mathbb{R}^n} \nabla_y \Gamma(y,x) V^{1/2}(y) f(y) dy.$$

By duality, it suffices to show that

$$\|T^* f\|_p \le C\|f\|_p \quad \text{for} \quad p_0' \le p \le \infty. \tag{4.14}$$

To this end, we let $r = \dfrac{1}{m(x,V)}$ and consider the case $n/2 \le q_0 < q_1 < n$. We choose $t$ and $p_1$ such that $(1/t) = (1/q_1) - (1/n)$, $(1/p_1) = 1 - (3/2q_1) + (1/n)$. Thus,

$$\frac{1}{t} + \frac{1}{2q_1} + \frac{1}{p_1} = 1.$$

Hence, by Hölder inequality,

$$|T^*f(x)| \le \sum_{j=-\infty}^{+\infty} \int_{2^{j-1}<|y-x|\le 2^j r} |\nabla_y \Gamma(y,x)| V^{1/2}(y) |f(y)| dy$$

$$\le \sum_{j=-\infty}^{+\infty} \left( \int_{2^{j-1}r<|y-x|\le 2^j r} |\nabla_y \Gamma(y,x)|^t dy \right)^{1/t}$$

$$\left( \int_{|y-x|\le 2^j r} V^{q_1} dy \right)^{1/2q_1} \left( \int_{|y-x|\le 2^j r} |f(y)|^{p_1} dy \right)^{1/p_1}.$$

It follows from Lemma 4.6 and Theorem 2.7 that

$$\left( \int_{2^{j-1}r<|y-x|\le 2^j r} |\nabla_y \Gamma(y,x)|^t dy \right)^{1/t} \le C_k \cdot \frac{(2^j r)^{(n/q_1)-n}}{(1+2^j)^k}.$$

Thus,

$$|T^*f(x)| \le C_k \sum_{j=-\infty}^{+\infty} \frac{(2^j r)}{(1+2^j)^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V^{q_1} dy \right\}^{1/2q_1}$$

$$\cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} |f(y)|^{p_1} dy \right\}^{1/p_1}$$

$$\le C_k \{M(|f|^{p_1})(x)\}^{1/p_1} \sum_{j=-\infty}^{+\infty} \frac{(2^j r)}{(1+2^j)^k}$$

$$\cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2}.$$

Note that, by Lemma 1.2, if $j \le 0$,

$$(2^j r) \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2} \le C(2^j)^{1-(n/2q_1)}$$

and, by doubling condition (1.1), if $j > 0$,

$$(2^j r) \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2} \le C(2^j)^{k_0} \quad \text{for some } k_0 > 0.$$

Hence, choosing $k$ sufficiently large, we obtain

$$|T^*f(x)| \le C\{M(|f|^{p_1})(x)\}^{1/p_1}.$$

It follows that

$$\|T^*f\|_p \le C\|f\|_p \quad \text{for } p_1 < p \le \infty.$$

(4.14) then follows since $p_0' > p_1$.

Finally, we consider the case $q_1 \geq n$. Clearly we may then assume $q_1 > n$ because of self improvement of the $B_q$ class.

It is not difficult to see that Remark 4.9 implies

$$|\nabla_y \Gamma(x,y)| \leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k} \cdot \frac{1}{|x-y|^{n-1}}, \quad \text{if } V \in B_{q_1}, q_1 > n.$$

Thus,

$$\begin{aligned} |T^* f(x)| &\leq C_k \sum_{j=-\infty}^{+\infty} \frac{1}{(1+2^j)^k} \cdot \frac{1}{(2^j r)^{n-1}} \cdot \int_{B(x,2^j r)} V^{1/2}(y)|f(y)|dy \\ &\leq C_k \sum_{j=-\infty}^{+\infty} \frac{2^j r}{(1+2^j)^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V^{q_1} dy \right\}^{1/2q_1} \\ &\qquad \cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} |f(y)|^{p_1} dy \right\}^{1/p_1} \end{aligned}$$

where $(1/p_1) = 1 - (1/2q_1)$.

Therefore, as in the first case, we have

$$\begin{aligned} &|T^* f(x)| \\ &\leq C_k \{M(|f|^{p_1})(x)\}^{1/p_1} \sum_{j=-\infty}^{+\infty} \frac{(2^j r)}{(1+2^j)^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2} \\ &\leq C \{M(|f|^{p_1})(x)\}^{1/p_1}. \end{aligned}$$

Note that $p_1 = (2q_1)' < (2q_0)' = p_0'$. (4.14) follows from the boundedness of the Hardy–Littlewood maximal function as before.

The proof is complete.

## 5. The proof of Theorem 0.5.

By functional calculus, we may write

$$(-\Delta + V)^{-1/2} = -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-1/2} (-\Delta + V + i\tau)^{-1} d\tau. \tag{5.1}$$

Thus,

$$\nabla(-\Delta + V)^{-1/2} f(x) = \int_{\mathbb{R}^n} K_1(x,y) f(y) dy \tag{5.2}$$

where

$$K_1(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{-1/2}\nabla_x\Gamma(x,y,\tau)d\tau. \tag{5.3}$$

Let

$$Tf(x) = \int_{\mathbb{R}^n} K_1(y,x)f(y)dy. \tag{5.4}$$

To show Theorem 0.5, by duality, it is equivalent to prove that

$$\|Tf\|_p \le C_p\|f\|_p \quad \text{for} \quad p_0' \le p < \infty \tag{5.5}$$

where $p_0' = p_0/(p_0-1)$ and $(1/p_0) = (1/q) - (1/n)$. We write

$$\begin{aligned} Tf(x) = &\int_{|y-x|>r} K_1(y,x)f(y)dy \\ &+ \int_{|y-x|\le r}\{K_1(y,x) - K_1^0(y,x)\}f(y)dy \\ &+ \int_{|y-x|\le r} K_1^0(y,x)f(y)dy \end{aligned} \tag{5.6}$$

where $r = \dfrac{1}{m(x,V)}$ and $K_1^0(x,y)$ is the kernel for the operator $\nabla(-\Delta)^{-1/2}$.

LEMMA 5.7. — *Suppose $V \in B_{q_1}$ for some $q_1$, $(n/2) < q_1 < n$. Then*

$$\left|\int_{|y-x|>r} K_1(y,x)f(y)dy\right| \le C\left\{M(|f|^{p_1'})(x)\right\}^{1/p_1'}$$

*where $r = \dfrac{1}{m(x,V)}$ and $(1/p_1') = 1 - (1/p_1) = 1 - (1/q_1) + (1/n)$.*

*Proof.* — First, we fix $x_0, y_0 \in \mathbb{R}^n$. Let $u(y) = \Gamma(y,x_0,\tau)$ and $R = |x_0 - y_0|/4$. It follows from (4.8) that

$$|\nabla u(y_0)| \le C\int_{B(y_0,R)}\frac{V(y)|u(y)|}{|y-y_0|^{n-1}}dy + \frac{C}{R^{n+1}}\int_{B(y_0,2R)}|u(y)|dy.$$

Hence, by Theorem 2.7,

$$\begin{aligned} |\nabla_y\Gamma(y_0,x_0,\tau)| \le &\frac{C_k}{(1+|\tau|^{1/2}R)^k\{1+m(x_0,V)R\}^k} \\ &\cdot\left\{\frac{1}{R^{n-2}}\int_{B(y_0,R)}\frac{V(y)dy}{|y-y_0|^{n-1}} + \frac{1}{R^{n-1}}\right\}. \end{aligned}$$

Thus, by (5.3),

$$|K_1(y_0,x_0)| \le \frac{C_k}{\{1+m(x_0,V)R\}^k}\left\{\frac{1}{R^{n-1}}\int_{B(y_0,R)}\frac{V(y)}{|y-y_0|^{n-1}}dy + \frac{1}{R^n}\right\}.$$

Now, let $(1/p_1) = (1/q_1) - (1/n)$. For $j \geq 1$ integer, we use the fractional integral theorem to obtain

$$\left\{\int_{2^{j-1}r<|y-x_0|\leq 2^j r} |K_1(y,x_0)|^{p_1} dy\right\}^{1/p_1}$$

$$\leq \frac{C_k}{(2^j)^k}\left\{\frac{1}{(2^j r)^{n-1}}\left(\int_{|y-x_0|\leq 2^{j+1}r} V^{q_1} dy\right)^{1/q_1} + (2^j r)^{(n/p_1)-n}\right\}$$

$$\leq \frac{C_k}{(2^j)^k}\left\{(2^j r)^{(n/p_1)-n}\cdot C_0^j + (2^j r)^{(n/p_1)-n}\right\}$$

$$\leq \frac{C}{2^j}\cdot (2^j r)^{-n/p_1'} \quad \text{if } k \text{ is sufficiently large,}$$

where $r = \dfrac{1}{m(x_0,V)}$ and we have used the reverse Hölder inequality (0.2) and the doubling condition (1.1).

Finally, by Hölder inequality,

$$\left|\int_{|y-x_0|>r} K_1(y,x_0) f(y) dy\right| \leq \sum_{j=1}^{\infty}\int_{2^{j-1}r<|y-x_0|\leq 2^j r} |K_1(y,x_0)||f(y)| dy$$

$$\leq \sum_{j=1}^{\infty}\left\{\int_{2^{j-1}r<|y-x_0|\leq 2^j r} |K_1(y,x_0)|^{p_1} dy\right\}^{1/p_1}$$

$$\cdot\left\{\int_{|y-x_0|\leq 2^j r} |f(y)|^{p_1'} dy\right\}^{1/p_1'}$$

$$\leq C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}\sum_{j=1}^{\infty}\frac{1}{2^j}$$

$$= C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}.$$

LEMMA 5.8. — *Suppose $V \in B_{q_1}$ for some $q_1$, $n/2 < q_1 < n$. Then*

$$\left|\int_{|y-x|\leq r}\{K_1(y,x) - K_1^0(y,x)\} f(y) dy\right| \leq C\left\{M(|f|^{p_1'})(x)\right\}^{1/p_1'}$$

*where $r = \dfrac{1}{m(x,V)}$ and $(1/p_1') = 1 - (1/p_1) = 1 - (1/q_1) + (1/n)$.*

*Proof.* — First, we fix $x_0, y_0 \in \mathbb{R}^n$ such that $|x_0 - y_0| < \dfrac{1}{m(x_0,V)}$. Let $r = \dfrac{1}{m(x_0,V)}$ and $R = |x_0 - y_0|/4$. We claim that

$$(5.9)\qquad |K_1(y_0,x_0)-K_1^0(y_0,x_0)| \le \frac{C}{R^{n-1}}\left\{\int_{B(y_0,R)}\frac{V(z)dz}{|z-y_0|^{n-1}}+R^{-1}\left(\frac{R}{r}\right)^{2-(n/q_1)}\right\}.$$

Assume (5.9) for a moment, we give the proof of the lemma.

Let $j \le 0$ be an integer. It follows from the theorem on fractional integrals and (5.9) that, for $(1/p_1)=(1/q_1)-(1/n)$,

$$\begin{aligned}
&\left\{\int_{2^{j-1}r<|y-x_0|\le 2^j r}|K_1(y,x_0)-K_1^0(y,x_0)|^{p_1}dy\right\}^{1/p_1}\\
&\quad\le \frac{C}{(2^j r)^{n-1}}\left\{\left(\int_{B(x_0,2r)}V^{q_1}dz\right)^{1/q_1}+(2^j)^{2-(n/q_1)}(2^j r)^{(n/p_1)-1}\right\}\\
&\quad\le \frac{C}{(2^j r)^{n-1}}\cdot r^{(n/q_1)-2}\\
&\quad= C(2^j)^{2-(n/q_1)}(2^j r)^{-n/p_1'}
\end{aligned}$$

where we have used (0.2) in the last inequality.

Now, by Hölder inequality,

$$\begin{aligned}
&\int_{|y-x_0|\le r}|K_1(y,x_0)-K_1^0(y,x_0)||f(y)|dy\\
&\quad\le \sum_{j=-\infty}^{0}\left(\int_{2^{j-1}r<|y-x_0|\le 2^j r}|K_1(y,x_0)-K_1^0(y,x_0)|^{p_1}dy\right)^{1/p_1}\\
&\qquad\qquad\cdot\left(\int_{|y-x_0|\le 2^j r}|f(y)|^{p_1'}dy\right)^{1/p_1'}\\
&\quad\le C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}\sum_{j=-\infty}^{0}(2^j)^{2-(n/q_1)}\\
&\quad= C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}
\end{aligned}$$

since $2-(n/q_1)>0$.

It remains to prove (5.9).

To this end, recall that

$$\Gamma(y,x_0,\tau)-\Gamma_0(y,x_0,\tau)=-\int_{\mathbb{R}^n}\Gamma_0(y,z,\tau)V(z)\Gamma(z,x_0,\tau)dz$$

(see the proof of Lemma 4.5). Thus,

$$\begin{aligned}
&|\nabla_y\Gamma(y_0,x_0,\tau)-\nabla_y\Gamma(y_0,x_0,\tau)|\\
&\quad\le\int_{\mathbb{R}^n}|\nabla_y\Gamma_0(y_0,z,\tau)|V(z)|\Gamma(z,x_0,\tau)|dz\\
&\quad\le C_k\int_{\mathbb{R}^n}\frac{(1+|\tau|^{1/2}|y_0-z|)^{-k}(1+|\tau|^{1/2}|z-x_0|)^{-k}V(z)\,dz}{|y_0-z|^{n-1}\{1+m(x_0,V)|z-x_0|\}^k|z-x_0|^{n-2}}\\
&\quad= C_k\int_{|z-y_0|<R}+C_k\int_{|z-x_0|<R}+C_k\int_{\substack{|z-x_0|\ge R\\ |z-y_0|\ge R}}\\
&\quad= J_1+J_2+J_3
\end{aligned}$$

where $R=|x_0-y_0|/4$.

Clearly,

$$J_1\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\int_{B(y_0,R)}\frac{V(z)}{|z-y_0|^{n-1}}dz$$

and

$$\begin{aligned}
J_2&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-1}}\int_{B(x_0,R)}\frac{V(z)}{|z-x_0|^{n-2}}dz\\
&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-1}}\cdot\left(\frac{R}{r}\right)^{2-(n/q_1)}
\end{aligned}$$

where $r=\dfrac{1}{m(x_0,V)}$ and we have used (1.6) and Lemma 1.2 in the estimates of $J_2$.

Finally, note that

$$\begin{aligned}
J_3&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R}\int_{|z-y_0|\ge R}\frac{V(z)dz}{|z-y_0|^{2n-4}\{1+m(x_0,V)|z-x_0|\}^k}\\
&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-1}}\cdot\left(\frac{R}{r}\right)^{2-(n/q_1)}.
\end{aligned}$$

(See the estimate of $I_3$ in the proof of Lemma 4.5.) Thus, we have proved that

$$\begin{aligned}
&|\nabla_y\Gamma(y_0,x_0,\tau)-\nabla_y\Gamma_0(y_0,x_0,\tau)|\\
&\quad\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\left\{\int_{B(y_0,R)}\frac{V(z)}{|z-y_0|^{n-1}}dz+\frac{1}{R}\cdot\left(\frac{R}{r}\right)^{2-(n/q_1)}\right\}.
\end{aligned}$$

(5.9) now follows easily from (5.3) and above estimate by integration.

The proof is complete.

With Lemmas 5.7 and 5.8, the proof of Theorem 0.5 is easy.

*The proof of Theorem 0.5.* — Suppose $V \in B_q$ for some $q$, $n/2 \le q < n$. Then $V \in B_{q_1}$ where $q < q_1 < n$.

It follows from (5.6), Lemmas 5.7 and 5.8 that

$$|Tf(x)| \le C\left\{M(|f|^{p_1'})(x)\right\}^{1/p_1'} + 2\sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K_1^0(y,x)f(y)dy\right|$$

where $(1/p_1') = 1 - (1/p_1) = 1 - (1/q_1) + (1/n)$. Hence, by the standard Calderón–Zygmund theory,

$$\|Tf\|_p \le C_p\|f\|_p \quad \text{for} \quad p_1' < p < \infty.$$

(5.5) now follows since $p_1' < p_0'$. Therefore,

$$\|\nabla(-\Delta + V)^{-1/2}f\|_p \le C_p\|f\|_p \quad \text{for} \quad 1 < p \le p_0.$$

THEOREM 5.10. — *Suppose $V \in B_q$ for some $q \ge n/2$. Then*

$$\|V^{1/2}(-\Delta + V)^{-1/2}f\|_p \le C\|f\|_p \quad \textit{for} \quad 1 \le p \le 2q.$$

*Proof.* — Suppose $V \in B_q$ for some $q \ge n/2$. Then $V \in B_{q_0}$ for some $q_0 > n/2$.

Note that, by (5.1) and Theorem 2.7,

$$|V^{1/2}(-\Delta + V)^{-1/2}f(x)| \le C_k \int_{\mathbb{R}^n} \frac{V(x)^{1/2}|f(y)|dy}{\{1 + m(y,V)|x-y|\}^k|x-y|^{n-1}}.$$

Let

$$Sf(x) = \int_{\mathbb{R}^n} \frac{V(x)^{1/2}f(y)dy}{\{1 + m(y,V)|x-y|\}^k|x-y|^{n-1}}.$$

It follows from the proof of Theorem 4.13 that

$$|S^*f(x)| \le C\{M(|f|^{(2q_0)'})(x)\}^{1/(2q_0)'}$$

where $S^*$ denotes the adjoint of $S$. Hence,

$$\|S^*f\|_p \le C\|f\|_p \text{ for } (2q_0)' < p \le \infty.$$

So $\|Sf\|_p \le C\|f\|_p$ for $1 \le p < 2q_0$ by duality. The theorem then follows easily.

THEOREM 5.11. — *Suppose $V \in B_q$ for some $q$ and $n/2 \le q < n$. Then, for $p_0' \le p \le 2q$,*

$$\|V^{1/2}(-\Delta + V)^{-1}\nabla f\|_p \le C\|f\|_p \tag{5.12}$$

*where* $p_0' = p_0/(p_0 - 1)$ *and* $(1/p_0) = (1/q) - (1/n)$. *Moreover, if* $V \in B_q$ *for some* $q \geq n$, *then (5.12) holds for* $1 \leq p \leq 2q$.

*Proof.* — Note that

$$V^{1/2}(-\Delta + V)^{-1}\nabla = V^{1/2}(-\Delta + V)^{-1/2} \cdot (-\Delta + V)^{-1/2}\nabla.$$

The theorem then follows easily from Theorem 5.10 and Theorem 0.5 except for the case $p = 1$ and $q \geq n$. But, if $q \geq n$, we have

$$\begin{aligned} |V^{1/2}(-\Delta + V)^{-1}\nabla f(x)| &\leq \int_{\mathbb{R}^n} V(x)^{1/2}|\nabla_y \Gamma(x,y)|\,|f(y)|dy \\ &\leq C_k \int_{\mathbb{R}^n} \frac{V(x)^{1/2}|f(y)|dy}{\{1 + m(y,V)|x-y|\}^k |x-y|^{n-1}}. \end{aligned}$$

From this it is not hard to see that $\|V^{1/2}(-\Delta + V)^{-1}\nabla f\|_1 \leq C\|f\|_1$.

## 6. The proof of Theorem 0.8.

This section is devoted to the proof of Theorem 0.8. We will assume that $V \in B_n$ throughout the section.

From Remark 4.9 and Theorem 2.7 it is easy to see that

(6.1) $$|\nabla_x \Gamma(x,y,\tau)| + |\nabla_y \Gamma(x,y,\tau)| \leq \frac{C_k}{\{1 + |\tau|^{1/2}|x-y|\}^k \{1 + m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-1}}.$$

Also, since $\nabla_y \Gamma(x,y,\tau)$ is a solution to the equation $-\Delta u + (V + i\tau)u = 0$ in $\mathbb{R}^n \setminus \{y\}$ as a function of $x$, by the same token,

(6.2) $$|\nabla_x \nabla_y \Gamma(x,y,\tau)| \leq \frac{C_k}{\{1 + |\tau|^{1/2}|x-y|\}^k \{1 + m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^n}.$$

Recall that

$$\nabla(-\Delta + V)^{-1/2} f(x) = \int_{\mathbb{R}^n} K_1(x,y) f(y) dy$$

where $K_1(x,y)$ is given by (5.3).

We may also write

(6.3) $$\nabla(-\Delta + V)^{-1}\nabla f(x) = \int_{\mathbb{R}^n} K_2(x,y) f(y)$$

where

(6.4) $$K_2(x,y) = -\nabla_x \nabla_y \Gamma(x,y), \ \Gamma(x,y) = \Gamma(x,y,0).$$

Clearly, by (6.1) and (6.2)

$$|K_\ell(x,y)| \le \frac{C_k}{\{1+|x-y|m(x,V)\}^k} \cdot \frac{1}{|x-y|^n} \quad \text{for} \quad \ell = 1,2. \tag{6.5}$$

We now are in a positive to give

*The proof of Theorem 0.8.* — Suppose $V \in B_n$. Then $V \in B_{q_0}$ for some $q_0 > n$.

The boundedness of $\nabla(-\Delta+V)^{-1/2}, (-\Delta+V)^{-1/2}\nabla$ and $\nabla(-\Delta+V)^{-1}\nabla$ on $L^2(\mathbb{R}^n)$ is well known for any nonnegative potential $V$. We only need to show that the kernel $K_\ell(x,y)$ satisfies the Calderón–Zygmund condition (0.11). The result for $(-\Delta+V)^{-1/2}\nabla$ follows by duality.

We will give the details for the estimate

$$|K_2(x+h,y) - K_2(x,y)| \le \frac{C|h|^\delta}{|x-y|^{n+\delta}}, \delta > 0 \tag{6.6}$$

where $x, y \in \mathbb{R}^n$, $|h| < |x-y|/2$. The other estimates follow in a similar manner or follow from (6.5).

To see (6.6), we fix $x_0, y_0 \in \mathbb{R}^n$ and $h \in \mathbb{R}^n, |h| < |x_0 - y_0|/4$. Let $u(x) = \nabla_y \Gamma(x, y_0)$ and $R = |x_0 - y_0|/4$. It then follows from the imbedding theorem of Morrey and Remark 4.10 that

$$\begin{aligned}
|K_2(x_0+h,y_0) - K_2(x_0,y_0)| &= |\nabla u(x_0+h) - \nabla u(x_0)| \\
&\le C|h|^{1-(n/q_0)} \left\{ \int_{B(x_0,R)} |\nabla^2 u|^{q_0} dx \right\}^{1/q_0} \\
&\le C \left(\frac{|h|}{R}\right)^{1-(n/q_0)} \cdot \frac{1}{R} \cdot \{1 + Rm(x_0,V)\}^{k_0} \sup_{B(x_0,2R)} |u| \\
&\le C \left(\frac{|h|}{R}\right)^{1-(n/q_0)} \cdot \frac{1}{R^n} \\
&= \frac{C|h|^\delta}{|x_0-y_0|^{n+\delta}}
\end{aligned}$$

where $\delta = 1 - (n/q_0) > 0$ and we have used (6.1) in the last inequality.

(6.6) is then proved for $|h| < |x_0 - y_0|/4$, hence for $|h| < |x_0 - y_0|/2$ by (6.5).

The proof is finished.

## 7. A counterexample.

We end this paper with a counterexample which shows that the ranges of $p$ in Theorems 0.3 and 0.5 are optimal.

Consider the case

$$V(x) = \frac{1}{|x|^{2-\varepsilon}}$$

where $0 < \varepsilon < 2$. Let

$$v(x) = \sum_{m=0}^{\infty} \frac{(\frac{1}{\varepsilon})^{2m}|x|^{\varepsilon m}}{m!\Gamma(\frac{n-2}{\varepsilon} + m + 1)}.$$

A direct computation shows that $v$ satisfies the equation

$$-\Delta v + \frac{1}{|x|^{2-\epsilon}} v = 0 \text{ in } \mathbb{R}^n.$$

Next, let $u = \phi v$ where $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\phi = 1$ for $|x| \leq 1$. Then

$$-\Delta u + Vu = g \text{ in } \mathbb{R}^n,$$

where $g = -2\nabla v \cdot \nabla\phi - v\Delta\phi \in C_0^\infty(\mathbb{R}^n)$.

Now, given any $q_0 > n/2$. Let $\varepsilon = 2 - (n/q_0)$. Then

$$V(x) = \frac{1}{|x|^{2-\epsilon}} = \frac{1}{|x|^{n/q_0}} \in B_p \text{ for any } p < q_0.$$

We claim that the estimate in Theorem 0.3 does not hold for $p = q_0$. For otherwise it would imply that

$$\left\| \frac{1}{|x|^{n/q_0}} u \right\|_{q_0} < \infty.$$

This contradicts the fact that $u(x) \sim 1$ as $x \to 0$.

Similarly, if $n/2 < q_0 < n$ and the estimate in Theorem 0.5 held for $p = p_0$ where $(1/p_0) = (1/q_0) - (1/n)$, it would follow that

$$\|\nabla u\|_{p_0} = \|\nabla(-\Delta + V)^{-1} g\|_{p_0} \leq C\|(-\Delta + V)^{-1/2} g\|_{p_0} < \infty,$$

where we have used the fact that

$$|(-\Delta + V)^{-1/2} g(x)| \leq C \int_{\mathbb{R}^n} \frac{|g(y)|}{|x-y|^{n-1}} dy \leq \frac{C}{(1+|x|)^{n-1}}.$$

It is easy to see that this is not the case since $\nabla u(x) \sim |x|^{\varepsilon-1} = |x|^{1-(n/q_0)} = |x|^{-n/p_0}$ as $x \to 0$.

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