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POISSON STRUCTURES ON CERTAIN MODULI SPACES FOR BUNDLES ON A SURFACE

by Johannes HUEBSCHMANN(*)

Introduction.

Let X be a stratified space. A *stratified symplectic structure* on X in the sense of Sjamaar-Lerman [21] is a Poisson algebra of continuous functions on X which, on each stratum, restricts to a symplectic Poisson algebra of smooth functions. In the present paper we construct such structures and related ones on certain moduli spaces. It is the fifth of a series of papers about a program revealing the structure of these moduli spaces by means of the symplectic or more generally Poisson geometry of certain related classical constrained systems. Its predecessors are [7] – [10], and it will be followed by [11] and [12].

We explain briefly the moduli spaces : Let Σ be a closed surface, G a compact Lie group, not necessarily connected, with Lie algebra g, and $\xi: P \to \Sigma$ a principal G-bundle, having a connected total space P. Then a choice of Riemannian metric on Σ and *orthogonal structure* on g, that is, adjoint action invariant scalar product, determines a Yang-Mills functional on the space $\mathcal{A}(\xi)$ of connections on ξ; see [2] to which we refer for background and notation. We assume throughout that solutions of the corresponding Yang-Mills equations exist; this will be so for example when G is connected, cf. [2]. Then the *moduli space* $N(\xi)$ of gauge equivalence

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classes of central Yang-Mills connections is non empty; it is a compact space, including as special cases moduli spaces of flat connections and the Narasimhan-Seshadri-moduli spaces [17] of semi stable holomorphic vector bundles. In this paper we shall complete the proof of the following.

THEOREM 1. — *The decomposition of $N(\xi)$ according to orbit types of central Yang-Mills connections is a stratification (in the strong sense) and the data determine a stratified symplectic structure $(C^\infty(N(\xi)), \{\cdot,\cdot\})$ for it.*

The Poisson structure goes *beyond* usual symplectic geometry; it encapsulates the *mutual positions* of the *symplectic structures* on the strata.

In [8] we have shown that $N(\xi)$ is stratified by smooth manifolds and that the strata inherit symplectic structures from the data. In [9] we constructed a homeomorphism $\rho_\flat$, referred to as *Wilson loop mapping*, from $N(\xi)$ onto a certain representation space $\mathrm{Rep}_\xi(\Gamma, G)$ for the universal central extension Γ of the fundamental group π of Σ. While the space $N(\xi)$ *depends* on the choices of Riemannian metric on Σ the space $\mathrm{Rep}_\xi(\Gamma, G)$ does not. In [10] we constructed smooth structures $C^\infty(N(\xi))$ and $C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$ on these spaces, and we have shown that $\rho_\flat$ is a diffeomorphism with respect to these structures. In the present paper, proceeding somewhat more generally than needed for the proof of the above theorem, we construct Poisson structures on the algebras $C^\infty(N(\xi))$ and $C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$ involving as additional ingredient a coadjoint action invariant symmetric bilinear form on g^*, not necessarily positive definite nor non-degenerate, in such a way that the Wilson loop mapping identifies the Poisson structures. When the bilinear form on g^* is positive definite we obtain the Poisson algebra in the above Theorem. We now explain informally the Poisson brackets.

By construction, the space $\mathrm{Rep}_\xi(\Gamma, G)$ is the quotient $\mathrm{Hom}_\xi(\Gamma, G)/G$ of a certain space of homomorphisms $\mathrm{Hom}_\xi(\Gamma, G)$ of Γ into G determined by ξ; see Section 1 below for details. For $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, we denote by g^*_ϕ the dual g^* of the Lie algebra g, made into a π-module via ϕ and the coadjoint action. For every $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$, a choice of representative $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$ induces a linear map λ^*_ϕ from the real vector space $\Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ of differentials at $[\phi]$, with reference to $C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$, into the first homology group $H_1(\pi, g^*_\phi)$ of π with coefficients in g^*_ϕ, and λ^*_ϕ is an isomorphism if and only if $[\phi]$ is a non-singular point of $\mathrm{Rep}_\xi(\Gamma, G)$,

see (1.16) below. In view of [10] (7.10), λ_ϕ^* is independent of the choice of ϕ in the sense that, given $x \in G$, the corresponding linear map $\lambda_{x\phi}^*$ from $\Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ to $H_1(\pi, g_{x\phi}^*)$ equals the composite of λ_ϕ^* with the isomorphism $\mathrm{Ad}^*(x)_\flat$ induced by x, cf. (1.16) below. A coadjoint action invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on g^*, not necessarily positive definite nor non-degenerate, then gives rise to, for every $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, an intersection pairing $\langle \cdot, \cdot \rangle_\phi$ on $H_1(\pi, g_\phi^*)$, and the corresponding Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$ will then satisfy the formula

$$\{f, h\}[\phi] = \langle \lambda_\phi^*(df[\phi]), \lambda_\phi^*(dh[\phi]) \rangle_\phi \tag{0.1}$$

where $f, h \in C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$ and where $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$ is a representative of the point $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$. As a set function, the bracket $\{\cdot, \cdot\}$ is determined by (0.1). This formula is *intrinsic* in the sense that it does *not* involve choices except that of the representative ϕ which has been taken care of already by the discussion of the dependence of λ_ϕ^* on the choice of ϕ. It then remains to prove that (i) for every $f, h \in C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$, the bracket $\{f, h\}$ is an element of $C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$ and, the Leibniz rule and skew symmetry being obviously true for the bracket $\{\cdot, \cdot\}$, that (ii) this bracket satisfies the Jacobi identity. An appropriately reworded statement will be given in (2.1) below. Proofs will then be given in Section 2 except that Section 3 is devoted to the Jacobi identity. Our construction is entirely finite dimensional except that the *verification of the Jacobi identity* involves (i) the smooth open connected and dense stratum whose existence has been established in [8] and, furthermore, (ii) the local model constructed in [7], for the special case where the bilinear form on g^* is non-degenerate. The general case of an arbitrary symmetric bilinear 2-form on g^* is then handled by relating it to that of a certain associated non-degenerate 2-form. See Section 3 below for details. For a non-degenerate 2-form on g^*, the Jacobi identity on the smooth open connected and dense stratum can also be settled by the finite dimensional techniques in [22].

The intrinsic description (0.1) of the Poisson structure has the following consequence a proof of which will be given at the end of Section 2.

THEOREM 2. — *The induced action of the mapping class group of* Σ *respects the Poisson structure. More precisely, its subgroup of orientation preserving elements preserves the Poisson bracket on* $\mathrm{Rep}_\xi(\Gamma, G)$ *whereas the orientation reversing elements whereas the orientation reversing elements yield diffeomorphisms from* $\mathrm{Rep}_\xi(\Gamma, G)$ *to* $\mathrm{Rep}_{-\xi}(\Gamma, G)$ *which are*

compatible with the Poisson brackets; here $-\xi$ refers to the (topologically) "opposite" bundle (which may coincide with ξ).

When the bilinear form on g^* is non-degenerate but not necessarily positive definite the resulting Poisson structure is *symplectic* in the sense that its Casimir elements are the constants only and in fact then yields a structure of a *stratified symplectic space*, cf. Sections 4 below; this then completes the proof of a somewhat more general result than Theorem 1; see (4.1) below for details. In Section 5 we indicate how the twist flows constructed in [4] on the top stratum can be extended to the whole space as derivations of the smooth structure. Section 1 below is preparatory in character.

It has been known for a while, cf. e. g. Narasimhan-Seshadri [17], Atiyah-Bott [2], Goldman [3], that an orthogonal structure on the Lie algebra g gives rise to a symplectic structure on a certain *non-singular part* of spaces of the kind $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$. However, in general, these spaces come with singularites, and our Poisson structures include these singularities. One of the chief results of our earlier paper, [10] (6.2) and (6.3), says that, locally, a space of the kind $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ looks like the reduced space of a momentum mapping for a representation of a compact Lie group varying over the space. The result of the present paper says that this local picture is available even for a *globally* defined Poisson structure which, in the local model, then amounts to the Arms-Cushman-Gotay Poisson structure [1] on the reduced space for a representation of a compact Lie group. The local model, with the Poisson structure included, is made precise in (4.3) below. We hope to prove elsewhere that a suitable holomorphic quantization of such a globally defined Poisson structure then yields a finite dimensional complex vector space.

An illustration of our result is worked out in our paper [12] : For $G = \mathrm{SU}(2)$, in another guise, the moduli space $N(\xi)$ is that of semi stable holomorphic vector bundles on Σ (with reference to a choice of holomorphic structure) of rank 2, degree 0, and trivial determinant. This space and related ones have been studied extensively in the literature [18] – [20]. In particular, for genus $\ell \geq 2$, the complement $\mathcal{K}$ of the top stratum is known to be the *Kummer* variety of Σ associated with its Jacobien J and the canonical involution thereupon. In [12] we prove the following.

THEOREM 3. — *When Σ has genus $\ell \geq 2$, the Poisson algebra $(C^\infty(N(\xi)), \{\cdot,\cdot\})$ detects the Kummer variety $\mathcal{K}$ in $N(\xi)$ together with its*

$2^{2\ell}$ *double points. More precisely,* $\mathcal{K}$ *consists of the points of* $N(\xi)$ *where the rank of the Poisson structure is not maximal, the double points being those where the rank is zero.*

In particular, when Σ has genus two, the space $N(\xi)$ equals complex projective 3-space and $\mathcal{K}$ is the Kummer surface associated with the Jacobien of Σ, cf. Narasimhan-Ramanan [18]. In the literature, this case has been considered somewhat special since as a *space* $N(\xi)$ is then actually smooth. However, from our point of view, there is *no* exception. As a *stratified symplectic* space, $N(\xi)$ still has singularities, our algebra $C^\infty(N(\xi))$ is *not* that of smooth functions in the ordinary sense, and the Kummer surface $\mathcal{K}$ is the complement of the top stratum and hence still precisely the singular locus in the sense of stratified symplectic space; in particular, the symplectic structure on the top stratum does *not* extend to the whole space. It is interesting to observe that the symplectic stratification, that is, the one used exclusively in our approach, is *finer* than the standard complex analytic one on complex projective 3-space.

Recent work of Jeffrey and the author [13], [14] shows that the moduli spaces can be obtained by finite dimensional reduction from certain "extended moduli spaces". The theorem of Sjamaar and Lerman can then be applied to obtain the stratified symplectic structure on the moduli space. The approach of the current paper is elementary but less elegant than that in [13], [14]; yet it has its advantages : It yields at once the intrinsic formula (0.1) above from which the compatibility of the structure with the action of the mapping class group is deduced.

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1. Differentials.

Pick a base point Q of Σ and consider the standard presentation

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle, \quad r = [x_1, y_1] \cdot \ldots \cdot [x_\ell, y_\ell], \tag{1.1}$$

of the fundamental group $\pi = \pi_1(\Sigma, Q)$, the number ℓ being the genus of Σ; we denote by F the free group on $x_1, y_1, \ldots, x_\ell, y_\ell$ and by N the normal closure of r in F so that $\pi = F/N$. For an arbitrary commutative ground

ring R, the presentation $\mathcal{P}$ yields the object

$$\widehat{R(\mathcal{P})}\colon RF \xleftarrow{\partial_1^F} RF\,[x_1, y_1, \ldots, x_\ell, y_\ell] \xleftarrow{\partial_2^F} RF\,[r] \tag{1.2}$$

cf. [10] (5.2). Here $RF\,[x_1, y_1, \ldots, x_\ell, y_\ell]$ and $RF\,[r]$ refer to the free right RF-modules having $x_1, y_1, \ldots, x_\ell, y_\ell$ and r as bases, respectively, and, the elements of these modules being viewed as column vectors, the operators ∂_*^F are given by

$$\partial_2^F = \left[\frac{\partial r}{\partial x_1}, \cdots, \frac{\partial r}{\partial y_\ell}\right]^{\mathrm{t}} : RF \to (RF)^{2\ell},$$
$$\partial_1^F = [1 - x_1, \cdots, 1 - y_\ell] : (RF)^{2\ell} \to RF,$$

where $^{\mathrm{t}}$ refers to the transpose of a vector. Given a left RF-module V, with structure map χ from F to $\mathrm{Aut}(V)$, application of the functor $- \otimes_F V$ to $\widehat{R(\mathcal{P})}$ yields the sequence $\left(\widehat{R(\mathcal{P})} \otimes_{RF} V, \partial_*^\chi\right)$ which, in view of the obvious identifications $\widehat{R_2(\mathcal{P})} \otimes_{RF} V = V$, $\widehat{R_1(\mathcal{P})} \otimes_{RF} V = V^{2\ell}$, and $\widehat{R_0(\mathcal{P})} \otimes_{RF} V = V$, looks like

$$V \xleftarrow{\partial_1^\chi} V^{2\ell} \xleftarrow{\partial_2^\chi} V. \tag{1.3}$$

Here the operators ∂_*^χ depend on the RF-module structure on V whence the notation.

Modulo N, (1.2) yields the free resolution

$$R(\mathcal{P})\colon R\pi \xleftarrow{\partial_1} R\pi\,[x_1, y_1, \ldots, x_\ell, y_\ell] \xleftarrow{\partial_2} R\pi\,[r] \tag{1.4}$$

of the ground ring R, viewed as a trivial $R\pi$-module, in the category of *right* $R\pi$-modules. Thus when the left RF-module structure χ on V factors through a left $R\pi$-module structure on V, the sequence (1.3) is a chain complex

$$\mathbf{C}(\mathcal{P}, V)\colon C_0(\mathcal{P}, V) \xleftarrow{\partial_1^\chi} C_1(\mathcal{P}, V) \xleftarrow{\partial_2^\chi} C_2(\mathcal{P}, V) \tag{1.5}$$

computing the homology groups of π with coefficients in V.

We now take $R = \mathbf{R}$, the reals, and $V = g^*$, the dual of the Lie algebra g of G, with the corresponding structure of a *left* G-module given by the coadjoint action, that is, for a linear form u on g and $x \in G$,

$$(xu) = u \circ \mathrm{Ad}(x^{-1})\colon g \to \mathbf{R}. \tag{1.6}$$

Given a homomorphism ϕ from F to G, we write g^*_ϕ for g^*, viewed as a left F-module via ϕ and the coadjoint action of G on g^*; the sequence (1.3) then looks like

$$g^* \xleftarrow{\partial_1^\phi} (g^*)^{2\ell} \xleftarrow{\partial_2^\phi} g^*; \tag{1.7}$$

when ϕ is a homomorphism from F to G so that each $\phi(r_j)$ lies in the centre of G, the left F-module g^*_ϕ inherits a structure of a left π-module which we still denote by g^*_ϕ, and (1.7) computes the homology $H_*(\pi, g^*_\phi)$.

For $w \in G$, the operation of left translation from g to T_wG will be written L_w. The assignment to $\chi \in \mathrm{Hom}(F,G)$ of $(\chi(x_1), \ldots, \chi(y_\ell)) \in G^{2\ell}$ identifies $\mathrm{Hom}(F,G)$ with $G^{2\ell}$. The homomorphism χ being viewed as the point $\mathbf{w} = (\chi(\mathbf{x_1}), \ldots, \chi(\mathbf{y}_\ell))$ of $G^{2\ell}$, its operation of *left translation* is then an isomorphism L_χ from $g^{2\ell}$ to $T_\chi \mathrm{Hom}(F,G)$. At $\chi \in \mathrm{Hom}(F,G)$, with reference to the usual smooth structure, the vector space of differentials $T^*_\chi \mathrm{Hom}(F,G)$ is just the usual cotangent space, and the dual of L_χ is an isomorphism L^*_χ from $T^*_\chi \mathrm{Hom}(F,G)$ to $(g^*)^{2\ell}$. Henceforth we confuse in notation the relator r with its word map from $\mathrm{Hom}(F,G)$ to G; it is given by the assignment to $\chi \in \mathrm{Hom}(F,G)$ of $r(\chi) = [\chi(x_1), \chi(y_1)] \cdot \ldots \cdot [\chi(x_\ell), \chi(y_\ell)]$.

PROPOSITION 1.8. — *At a homomorphism χ from F to G, the cotangent map $T^*_\chi r$ from $T^*_{r(\chi)}G$ to $T^*_\chi \mathrm{Hom}(F,G)$ and the operation of left translation make commutative the diagram*

$$\begin{array}{ccc} T^*_\chi \mathrm{Hom}(F,G) & \xleftarrow{T^*_\chi r} & T^*_{r(\chi)}G \\ {\scriptstyle L^*_\chi}\downarrow & & \downarrow{\scriptstyle L^*_{r(\chi)}} \\ (g^*)^{2\ell} & \xleftarrow{\partial_2^\chi} & g^* \end{array}$$

where ∂_2^χ refers to the corresponding operator in (1.7).

Proof. — This follows at once from [10] (5.4). □

Recall that IK denotes the augmentation ideal of a discrete group K.

COROLLARY 1.9. — *At a homomorphism ϕ from F to G, having the property that $\phi(r)$ lies in the centre of G, left translation yields a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longleftarrow & \operatorname{coker}\left(T_\phi^* r\right) & \longleftarrow & T_\phi^*\mathrm{Hom}(F,G) & \xleftarrow{T_\phi^* r} & T_{r(\phi)}^* G \\ & & \downarrow & & \downarrow{\scriptstyle L_\phi^*} & & \downarrow{\scriptstyle L_{r(\phi)}^*} \\ 0 & \longleftarrow & I\pi \otimes_{\mathbf{R}\pi} g_\phi^* & \longleftarrow & (g^*)^{2\ell} & \xleftarrow{\partial_2^\phi} & g^* \end{array}$$

with exact columns and hence induces an isomorphism from $\operatorname{coker}(T_\phi^* r)$ *onto the space* $I\pi \otimes_{\mathbf{R}\pi} g_\phi^*$.

Proof. — In fact, the boundary operator ∂_1^ϕ from $C_1(\mathcal{P}, g_\phi^*)$ to $C_0(\mathcal{P}, g_\phi^*)$ induces an isomorphism from $C_1(\mathcal{P}, g_\phi^*)/\partial_2^\phi(C_2(\mathcal{P}, g_\phi^*))$ onto $I\pi \otimes_{\mathbf{R}\pi} g_\phi^*$. □

Next we pass to corresponding G-invariant objects.

LEMMA 1.10. — *Let* f *be a smooth real valued* G*-invariant function on* $\mathrm{Hom}(F,G)$ *and* ϕ *a homomorphism from* F *to* G*, having the property that* $\phi(r)$ *lies in the centre of* G*. Then under the dual left transformation* L_ϕ^* *from* $T_\phi^*\mathrm{Hom}(F,G)$ *to* $(g^*)^{2\ell}$ *the differential* $df(\phi) \in T_\phi^*\mathrm{Hom}(F,G)$ *goes to a cycle, that is,* $L_\phi^* df(\phi) = df(\phi) \circ L_\phi$ *lies in the subspace* $Z_1(\mathcal{P}, g_\phi^*)$ *of cycles in* $C_1(\mathcal{P}, g_\phi^*) = (g^*)^{2\ell}$ *and hence determines a class* $[df(\phi) \circ L_\phi] \in H_1(\pi, g_\phi^*)$.

Proof. — The value $\partial_1^\phi(df(\phi) \circ L_\phi)$ lies in $C_0(\mathcal{P}, g_\phi^*) = g^*$. Let $X \in g$ and

$$\mathbf{w} = (a_1, b_1, \ldots, a_\ell, b_\ell) = (\phi(x_1), \ldots, \phi(y_\ell)) \in G^{2\ell}.$$

In view of the description of (1.2) given above,

$$\left(\partial_1^\phi(df(\phi) \circ L_\phi)\right) X = (df(\phi) \circ L_\phi)\left(X - \mathrm{Ad}(a_1^{-1})X, \ldots, X - \mathrm{Ad}(b_\ell^{-1})X\right).$$

However, at ϕ which, in the present description, amounts to $\mathbf{w}$, the analytic path

$$t \longmapsto (\exp(-tX)a_1\exp(tX), \ldots, \exp(-tX)b_\ell\exp(tX))$$

has tangent vector

$$L_{\mathbf{w}}\left(X - \mathrm{Ad}(a_1^{-1})X, \ldots, X - \mathrm{Ad}(b_\ell^{-1})X\right) \in T_{\mathbf{w}}G^{2\ell} \cong T_\phi\mathrm{Hom}(F,G).$$

Since f is G-invariant, it is constant along this path whence its derivative along the tangent vector to this path is zero. Hence $(\partial_1^\phi df(\phi) \circ L_\phi)X = 0$. Since $X \in g$ was arbitrary, $\partial_1^\phi(df(\phi) \circ L_\phi) = 0$ as asserted. □

We now return to our principal bundle $\xi\colon P \to \Sigma$ over Σ with structure group G. As in the predecessors [7] – [10] to the present paper, we choose an invariant scalar product on g and a Riemannian metric on Σ. We then pick smooth closed curves $v_1, w_1, \ldots, v_\ell, w_\ell$ in Σ representing the generators $x_1, y_1, \ldots, x_\ell, y_\ell$ so that the standard cell decomposition of Σ with a single 2-cell e corresponding to r results; further, let $\widehat{Q} \in P$ be a base point so that $\xi(\widehat{Q}) = Q$. Then the holonomy along these paths yields the Wilson loop mapping ρ from $\mathcal{A}(\xi)$ to $\mathrm{Hom}(F, G)$, cf. Section 2 of [10]. The image $\rho(\mathcal{N}(\xi))$ in $\mathrm{Hom}(F, G)$ of the subspace $\mathcal{N}(\xi)$ of central Yang-Mills connections is a space $\mathrm{Hom}_\xi(\Gamma, G)$ of homomorphisms from the universal central extension Γ of π to G. See [9] and Section 3 of [10].

Let $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$. Recall that the Lie bracket on g induces a graded bracket $[\cdot, \cdot]_\phi$ on $H^*(\pi, g_\phi)$ that endows the latter with a structure of a graded Lie algebra. Further, the chosen orthogonal structure on g induces a graded non-degenerate bilinear pairing $(\cdot, \cdot)_\phi$ between $H^*(\pi, g_\phi)$ and $H^{2-*}(\pi, g_\phi)$ which, in degree 1, amounts to a symplectic structure σ_ϕ on $H^1(\pi, g_\phi)$; moreover, the assignment to $\eta \in H^1(\pi, g_\phi)$ of $\frac{1}{2}[\eta, \eta]_\phi \in H^2(\pi, g_\phi)$ yields a momentum mapping Θ_ϕ from $H^1(\pi, g_\phi)$ to $H^2(\pi, g_\phi)$, for the action of the stabilizer $Z_\phi \subseteq G$ of $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$ on $H^1(\pi, g_\phi)$. See Section 1 of [7] where this is spelled out for central Yang-Mills connections and Section 6 of [10]. Let $\mathrm{Hom}_\xi(\Gamma, G)^-$ denote the subspace of $\mathrm{Hom}_\xi(\Gamma, G)$ consisting of points ϕ so that the operation

$$[\cdot, \cdot]_\phi\colon H^1(\pi, g_\phi) \otimes H^1(\pi, g_\phi) \longrightarrow H^2(\pi, g_\phi) \tag{1.11.1}$$

is zero. Notice that $\mathrm{Hom}_\xi(\Gamma, G)^-$ depends on the chosen orthogonal structure on g.

PROPOSITION 1.12. — *The subspace* $\mathrm{Hom}_\xi(\Gamma, G)^-$ *is a smooth submanifold of* $\mathrm{Hom}(F, G)$. *Moreover, for every* $\phi \in \mathrm{Hom}_\xi(\Gamma, G)^-$, *the tangent space* $T_\phi \mathrm{Hom}_\xi(\Gamma, G)^-$ *coincides with the kernel of the derivative* $dr(\phi)$ *from* $T_\phi \mathrm{Hom}(F, G)$ *to* $T_{\exp(X_\xi)}G$, *and hence the spaces of differentials constitute an exact sequence*

$$0 \longleftarrow T^*_\phi \mathrm{Hom}_\xi(\Gamma, G)^- \longleftarrow T^*_\phi \mathrm{Hom}(F, G) \xleftarrow{dr^*(\phi)} T^*_{\exp(X_\xi)}G.$$

The proof will be given after that of (1.13) below. Let $\mathcal{N}^-(\xi)$ be the subspace of $\mathcal{N}(\xi)$ consisting of central Yang-Mills connections A with the

property that the operation

$$[\cdot,\cdot]_A\colon H^1_A(\Sigma,\mathrm{ad}(\xi))\otimes H^1_A(\Sigma,\mathrm{ad}(\xi))\longrightarrow H^2_A(\Sigma,\mathrm{ad}(\xi)) \tag{1.11.2}$$

is zero; by [7] (2.8), the space $\mathcal{N}^-(\xi)$ is a smooth submanifold of $\mathcal{A}(\xi)$ having at $A\in\mathcal{N}^-(\xi)$ tangent space $T_A\mathcal{N}^-(\xi)$ equal to the space of 1-cocycles $Z^1_A(\Sigma,\mathrm{ad}(\xi))$ in $\Omega^1(\Sigma,\mathrm{ad}(\xi))=T_A\mathcal{A}(\xi)$. The group of *based* gauge transformations is written $\mathcal{G}^Q(\xi)$.

PROPOSITION 1.13. — *The Wilson loop mapping passes to a smooth principal $\mathcal{G}^Q(\xi)$-fibre bundle $\rho^-\colon\mathcal{N}^-(\xi)\to\mathrm{Hom}_\xi(\Gamma,G)^-$. Consequently $\mathrm{Hom}_\xi(\Gamma,G)^-$ is a smooth submanifold of $\mathrm{Hom}(F,G)$ in such a way that, at every $\phi\in\mathrm{Hom}_\xi(\Gamma,G)^-$, left translation L_ϕ from $g^{2\ell}=C^1(\mathcal{P},g_\phi)$ to $T_\phi\mathrm{Hom}(F,G)$ identifies the space $Z^1(\pi,g_\phi)$ of 1-cocycles with the usual tangent space $T_\phi\mathrm{Hom}_\xi(\Gamma,G)^-$, viewed as a subspace of $T_\phi\mathrm{Hom}(F,G)$. Moreover $\mathrm{Hom}_\xi(\Gamma,G)^-$ is dense in $\mathrm{Hom}_\xi(\Gamma,G)$.*

Proof. — The action of $\mathcal{G}^Q(\xi)$ on $\mathcal{A}(\xi)$ and hence on $\mathcal{N}(\xi)$ is free, and the Wilson loop mapping ρ passes to a homeomorphism from $\mathcal{N}(\xi)/\mathcal{G}^Q(\xi)$ onto $\mathrm{Hom}_\xi(\Gamma,G)$. See our papers [9] and [10] for details. Moreover, given a central Yang-Mills connection A, with $\phi=\rho(A)\in\mathrm{Hom}_\xi(\Gamma,G)$, twisted integration yields an isomorphism from $H^*_A(\Sigma,\mathrm{ad}(\xi)$ onto $H^*(\pi,g_\phi)$ compatible with the relevant structure, cf. Section 4 of [10]. Consequently the operation (1.11.1) is zero if and only if (1.11.2) is zero whence the restriction ρ^- of the Wilson loop mapping ρ is a principal fibre bundle projection map, manifestly smooth, having at every $\phi\in\mathrm{Hom}_\xi(\Gamma,G)^-$ the asserted derivative. Finally, cf. our paper [8], the preimage $\mathcal{N}^{\mathrm{top}}(\xi)\subseteq\mathcal{N}(\xi)$ of the top stratum $N^{\mathrm{top}}(\xi)$ of $N(\xi)$ is contained in $\mathcal{N}^-(\xi)$, and, by [8] (1.4), the subspace $N^{\mathrm{top}}(\xi)$ is dense in $N(\xi)$. This implies that $\mathrm{Hom}_\xi(\Gamma,G)^-$ is dense in $\mathrm{Hom}_\xi(\Gamma,G)$. □

Proof of (1.12). — In view of [10] (5.4), which includes the statement dual to (1.8) above, (1.13) implies at once the statement of (1.12). □

COROLLARY 1.14. — *Let f be a smooth real valued function on $\mathrm{Hom}(F,G)$ that vanishes on $\mathrm{Hom}_\xi(\Gamma,G)^-$. Then its differential $df(\phi)\in T^*_\phi\mathrm{Hom}(F,G)$ at $\phi\in\mathrm{Hom}_\xi(\Gamma,G)^-$ passes to zero in $T^*_\phi\mathrm{Hom}_\xi(\Gamma,G)^-$.*

COROLLARY 1.15. — *Let f be a smooth real valued G-invariant function on $\mathrm{Hom}(F,G)$ that vanishes on $\mathrm{Hom}_\xi(\Gamma,G)^-$. Then, cf.* (1.10), *the homology class $[df(\phi)\circ L_\phi]$ in $H_1(\pi,g^*_\phi)$ determined by its differential $df(\phi)\in T^*_\phi\mathrm{Hom}_\xi(\Gamma,G)^-$ at $\phi\in\mathrm{Hom}_\xi(\Gamma,G)^-$ is zero.*

For intelligibility we recall that, by definition, $\mathrm{Rep}_\xi(\Gamma, G) = \mathrm{Hom}_\xi(\Gamma, G)/G$ and we reproduce the construction of the smooth structure $C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right)$; see Section 3 of [10] for details. Let I_ξ denote the ideal in the algebra $C^\infty\left(\mathrm{Hom}(F, G)\right)$ of smooth functions on $\mathrm{Hom}(F, G)$ that vanish on the subspace $\mathrm{Hom}_\xi(\Gamma, G)$ of $\mathrm{Hom}(F, G)$. The algebra of *Whitney smooth functions* $C^\infty\left(\mathrm{Hom}_\xi(\Gamma, G)\right) = C^\infty\left(\mathrm{Hom}(F, G)\right)/I_\xi$ endows $\mathrm{Hom}_\xi(\Gamma, G)$ with a *smooth structure*. We then *define* the *smooth structure* of $\mathrm{Rep}_\xi(\Gamma, G)$ to be the algebra

$$C^\infty\left(\mathrm{Rep}_\xi(\Gamma, G)\right) = \left(C^\infty(\mathrm{Hom}(F, G))\right)^G / I_\xi^G$$

of smooth G-invariant functions $\left(C^\infty(\mathrm{Hom}(F, G))\right)^G$ on $\mathrm{Hom}(F, G)$ modulo its ideal I_ξ^G of functions that vanish on $\mathrm{Hom}_\xi(\Gamma, G)$. By construction this is an algebra of functions on $\mathrm{Rep}_\xi(\Gamma, G)$ in an obvious fashion.

For $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$, the dual of the real vector space $\Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ of differentials at $[\phi]$ with reference to $C^\infty\mathrm{Rep}_\xi(\Gamma, G)$ is, by definition, the *Zariski tangent space* $T_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$. At a singular point $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, the statements of (1.14) and (1.15) are still true when the tangent space is replaced by the Zariski tangent space with reference to the smooth structure $C^\infty\mathrm{Hom}_\xi(\Gamma, G)$. This fact will not be needed here and we refrain from spelling out details; it follows from [10] (7.14).

Let $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$. By [10] (7.10), a choice of representative $\chi \in \mathrm{Hom}_\xi(\Gamma, G)$ induces a linear map λ_χ from $H^1(\pi, g_\chi)$ to $T_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ which is an isomorphism if and only if $[\phi]$ is a non-singular point of $\mathrm{Rep}_\xi(\Gamma, G)$; the dual

$$\lambda_\chi^* : \Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G) = T_{[\phi]}^*\mathrm{Rep}_\xi(\Gamma, G) \longrightarrow H_1(\pi, g_\chi^*) \tag{1.16}$$

furnishes then a linear map from $\Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ to $H_1(\pi, g_\chi^*)$ which is as well an isomorphism if and only if $[\phi]$ is a non-singular point of $\mathrm{Rep}_\xi(\Gamma, G)$, and [10] (7.10) entails at once the following.

PROPOSITION 1.17. — *The linear map λ_χ^* is independent of the choice of χ in the sense that, given $x \in G$, the corresponding linear map $\lambda_{x\chi}^*$ from $T_{[\phi]}^*\mathrm{Rep}_\xi(\Gamma, G)$ to $H_1(\pi, g_{x\chi}^*)$ coincides with the composite of λ_χ^* and the isomorphism $\mathrm{Ad}^*(x)_\flat$ from $H_1(\pi, g_\chi^*)$ onto $H_1(\pi, g_{x\chi}^*)$ induced by x.*

For a homomorphism ϕ from F to G having the property that $\phi(r)$ lies in the centre of G, the obvious non-degenerate real-valued bilinear pairing between $C^1(\mathcal{P}, g_\phi)$ and $C_1(\mathcal{P}, g_\phi^*)$ induces a non-degenerate bilinear pairing

between $Z^1(\pi, g_\phi)$ and $I\pi \otimes_{\mathbf{R}\pi} g_\phi^*$ which, in turn, induces the *cap* pairing $\cap$ between $H^1(\pi, g_\phi)$ and $H_1(\pi, g_\phi^*)$; it is manifestly non-degenerate and identifies $H_1(\pi, g_\phi^*)$ with the dual of $H^1(\pi, g_\phi)$. This observation yields at once a proof of the following.

PROPOSITION 1.18. — *At a class* $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$, *for every representative* $\chi \in [\phi]$, *the canonical evaluation pairing between* $T_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ *and* $T^*_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ *equals the composite*

$$T_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G) \otimes T^*_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G) \xrightarrow{\lambda_\chi \otimes \lambda_\chi^*} H^1(\pi, g_\chi) \otimes H_1(\pi, g_\chi^*) \xrightarrow{\cap} \mathbf{R}. \quad \square$$

Let A be a central Yang-Mills connection, and let $((\cdot,\cdot))_A$ denote the canonical evaluation pairing between $H^1_A(\Sigma, \mathrm{ad}(\xi))$ and $H^1_A(\Sigma, \mathrm{ad}^*(\xi))$ obtained from the wedge product of forms and integration in the usual way. We define the *dual twisted integration isomorphism*

$$\mathrm{Int}^*_A \colon H_1(\pi, g^*_{\rho(A)}) \longrightarrow H^1_A(\Sigma, \mathrm{ad}^*(\xi)) \tag{1.19}$$

by $(\mathrm{Int}_A \alpha) \cap u = ((\alpha, \mathrm{Int}^*_A u))_A$, for $\alpha \in H^1_A(\Sigma, \mathrm{ad}(\xi))$ and $u \in H_1(\pi, g^*_{\rho(A)})$. Moreover, we denote by λ^*_A the linear map from $T^*_{[A]}N(\xi)$ to $H^1_A(\Sigma, \mathrm{ad}^*(\xi))$ which is the dual of the linear map λ_A given in [10] (7.9).

PROPOSITION 1.20. — *For every central Yang-Mills connection* A, *the dual twisted integration isomorphism* (1.19) *makes commutative the diagram*

$$\begin{array}{ccc} H^1_A(\Sigma, \mathrm{ad}^*(\xi)) & \xleftarrow{\lambda^*_A} & T^*_{[A]}N(\xi) \\ {\scriptstyle \mathrm{Int}^*_A}\uparrow & & \uparrow{\scriptstyle d\rho_\flat[A]^*} \\ H_1(\pi, g^*_{\rho(A)}) & \xleftarrow{\lambda^*_{\rho(A)}} & T^*_{[\rho(A)]}\mathrm{Rep}_\xi(\Gamma, G). \end{array}$$

In particular, when A *represents a non-singular point, the cotangent map* $d\rho_\flat[A]^*$ *from* $T^*_{[\rho(A)]}\mathrm{Rep}_\xi(\Gamma, G)$ *to* $T^*_{[A]}N(\xi)$ *amounts to the isomorphism* (1.19).

Proof. — This follows at once from the commutativity of [10] (7.11). $\square$

2. Poisson structures.

Let $\langle \cdot, \cdot \rangle$ be a coadjoint action invariant symmetric bilinear form on g^*, not necessarily non-degenerate. For every $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, it induces an intersection pairing

$$\langle \cdot, \cdot \rangle_\phi : H_1(\pi, g^*_\phi) \otimes H_1(\pi, g^*_\phi) \longrightarrow \mathbf{R}. \tag{2.1.1}$$

For a smooth G-invariant function f on $\mathrm{Hom}(F, G)$, we write $[f] \in C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ for its image, obtained by restriction of f to $\mathrm{Hom}_\xi(\Gamma, G)$. For every f and h in $(C^\infty(\mathrm{Hom}(F, G)))^G$ and every $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, let

$$(f \bullet h)(\phi) = \langle [df(\phi) \circ L_\phi] \otimes [dh(\phi) \circ L_\phi] \rangle_\phi \tag{2.1.2}$$

where the notation $[df(\phi) \circ L_\phi]$ and $[dh(\phi) \circ L_\phi]$ indicates homology classes in $H_1(\pi, g^*_\phi)$, cf. (1.10). This furnishes a bilinear pairing

(2.1.3)

$$\bullet : (C^\infty(\mathrm{Hom}(F, G)))^G \otimes (C^\infty(\mathrm{Hom}(F, G)))^G \longrightarrow \mathrm{Map}(\mathrm{Hom}_\xi(\Gamma, G), \mathbf{R}).$$

By a *symplectic* Poisson structure we mean one whose Casimir elements are the constants only.

THEOREM 2.1. — *The pairing* (2.1.3) *induces a Poisson bracket*

$$\{\cdot, \cdot\} : C^\infty(\mathrm{Rep}_\xi(\Gamma, G)) \otimes C^\infty(\mathrm{Rep}_\xi(\Gamma, G)) \longrightarrow C^\infty(\mathrm{Rep}_\xi(\Gamma, G)) \tag{2.1.4}$$

which, for every f and h in $(C^\infty(\mathrm{Hom}(F, G)))^G$ and every $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, is calculated by the formula

$$\{[f], [h]\}[\phi] = \langle [df(\phi) \circ L_\phi] \otimes [dh(\phi) \circ L_\phi] \rangle_\phi . \tag{2.1.5}$$

When the bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate (but not necessarily positive definite) the Poisson bracket is symplectic.

Remark. — The description (2.1.5) of the Poisson bracket involves choices of representatives f and h of $[f], [h] \in C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$, respectively, *and* of a representative ϕ of the point $[\phi]$ of $\mathrm{Rep}_\xi(\Gamma, G)$. In a sense, the choice of ϕ amounts to introduction of local coordinates. In view of the construction of the linear map λ_ϕ from $H^1(\pi, g_\chi)$ to $T_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ in Section 7 of [10], it is clear that (2.1.5) amounts to the *intrinsic* description (0.1) given in the Introduction. The construction of λ_ϕ relies on the determination of appropriate Zariski tangent spaces given in Section 7 of [10]

while our construction of the Poisson bracket, in particular (2.1.5) above does *not*. The results in Section 7 of [10] are needed here *merely* to obtain the *description* (0.1) of the Poisson bracket.

We first give an outline of the proof of (2.1). Let f and h be smooth G-invariant functions on $\mathrm{Hom}(F,G)$; we shall establish the following facts.

(1) The function $f \bullet h$ is the restriction to $\mathrm{Hom}_\xi(\Gamma, G)$ of a smooth G-invariant function on $\mathrm{Hom}(F,G)$.

(2) When h is zero on $\mathrm{Hom}_\xi(\Gamma, G)$ so is $f \bullet h$.

(3) The bracket (2.1.4) satisfies the Leibniz rule.

(4) The bracket (2.1.4) satisfies the Jacobi identity.

The proof will proceed in steps. We shall construct a pairing

$$\diamondsuit : (C^\infty(\mathrm{Hom}(F,G))) \otimes (C^\infty(\mathrm{Hom}(F,G))) \longrightarrow C^\infty(\mathrm{Hom}(F,G))$$

satisfying the Leibniz rule; its construction will rely on a combinatorial description of the intersection pairing to be given in (2.3) – (2.9) and will be completed in (2.12) while its G-invariance will be established in (2.14). Statement (2.15) below will imply that the resulting pairing on $(C^\infty(\mathrm{Hom}(F,G)))^G$, combined with the projection from $(C^\infty(\mathrm{Hom}(F,G)))^G$ onto $C^\infty(\mathrm{Rep}_\xi(\Gamma,G))$, comes down to (2.1.3). In (2.17) we shall then show that $\diamondsuit$ passes to a pairing on $C^\infty(\mathrm{Rep}_\xi(\Gamma,G))$; this takes care of (2) in the above outline. The Jacobi identity and, moreover, the symplecticity of the Poisson bracket for a non-degenerate bilinear form on g^* will be established in the next Section.

We now begin working out the details. For a while we shall admit an arbitrary commutative ring R as ground ring. We proceed at first towards a description of the requisite intersection pairings. With reference to (1.4) above, let

$$\widetilde{R}(\mathcal{P}) = \mathrm{Hom}_{R\pi}(R(\mathcal{P}), R\pi). \tag{2.2}$$

With the notation $\widetilde{R}_{2-j}(\mathcal{P}) = \mathrm{Hom}_{R\pi}(R_j(\mathcal{P}), R\pi)$, for $0 \le j \le 2$, it looks like

$$\widetilde{R}(\mathcal{P}) \colon \widetilde{R}_2(\mathcal{P}) \xrightarrow{\partial_2} \widetilde{R}_1(\mathcal{P}) \xrightarrow{\partial_1} \widetilde{R}_0(\mathcal{P}), \tag{2.3.1}$$

and, in the standard way, the $R_j(\mathcal{P})$ coming as *right* $R\pi$-modules, each $\widetilde{R}_j(\mathcal{P})$ inherits a structure of a *left* $R\pi$-module by means of

$$(x\phi)(y) = x(\phi y), \quad x \in R\pi,\ y \in R_j(\mathcal{P}),\ j = 0,1,2. \tag{2.3.2}$$

Furthermore the canonical map from $R(\mathcal{P})$ to $\mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), R\pi)$ is an isomorphism of free resolutions of R in the category of right $R\pi$-modules and, for every left $R\pi$-module W, the assignment to $f \otimes w \in \mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), R\pi) \otimes_{R\pi} W$ of f_w given by the formula $f_w(y) = f(y)w$, for $y \in \widetilde{R}(\mathcal{P})$, yields an isomorphism of chain complexes from $\mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), R\pi) \otimes_{R\pi} W$ onto $\mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), W)$; hence the resulting composite

(2.3.3)

$$R(\mathcal{P}) \otimes_{R\pi} W \longrightarrow \Big(\mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), R\pi)\Big) \otimes_{R\pi} W \longrightarrow \mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), W)$$

is likewise an isomorphism of chain complexes. Poincaré duality may now be expressed in the following form.

PROPOSITION 2.3. — *The chain complex $\widetilde{R}(\mathcal{P})$ is a free resolution of the ground ring R in the category of left $R\pi$-modules in such a way that, for every left $R\pi$-module W, when $H^*(\pi, W)$ is calculated as $H^*(\mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), W))$ and $H_*(\pi, W)$ as $H_*(R(\mathcal{P}) \otimes_{R\pi} W)$, the Poincaré duality isomorphism*

$$[\pi] \cap - : H^*(\pi, W) \longrightarrow H_{2-*}(\pi, W), \quad 0 \leq j \leq 2,$$

that is, the cap product with the fundamental class $[\pi] \in H_2(\pi, R)$, is induced by the inverse of the canonical isomorphism (2.3.3).

Proof. — By construction, the homology of $\widetilde{R}(\mathcal{P})$ is the cohomology of π with values in $R\pi$. However, this cohomology is just the cohomology $H^*_f(\widetilde{\Sigma}, R)$ with finite cochains of the universal covering $\widetilde{\Sigma}$ and, by Poincaré duality, $H^*_f(\widetilde{\Sigma}, R)$ is isomorphic to $H_{2-*}(\widetilde{\Sigma}, R)$ whence $H^0(\pi, R\pi) = 0$, $H^1(\pi, R\pi) = 0$, $H^2(\pi, R\pi) \cong R$. Thus $\widetilde{R}(\mathcal{P})$ is a free resolution of the ground ring R in the category of left $R\pi$-modules.

To get our hands on Poincaré duality, let $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ be three free resolutions of R in the category of left $R\pi$-modules, let $\Delta: \mathbf{P}_1 \longrightarrow \mathbf{P}_2 \otimes \mathbf{P}_3$ be a diagonal map, and let W be a left $R\pi$-module and V a right $R\pi$-module. For every $\phi \in \mathrm{Hom}_{R\pi}(\mathbf{P}_2, W)$, we then have the chain map

$$- \cap \phi: V \otimes_{R\pi} \mathbf{P}_1 \longrightarrow V \otimes_{R\pi} (W \otimes \mathbf{P}_3),$$

defined as the composite

$$V \otimes_{R\pi} \mathbf{P}_1 \xrightarrow{\mathrm{Id}\otimes\Delta} V \otimes_{R\pi} (\mathbf{P}_2 \otimes \mathbf{P}_3) \xrightarrow{\mathrm{Id}\otimes\phi\otimes\mathrm{Id}} V \otimes_{R\pi} (W \otimes \mathbf{P}_3);$$

upon taking $V = R$ and writing $\overline{\mathbf{P}}_1 = R \otimes_{R\pi} \mathbf{P}_1$, we arrive at a chain map

$$\overline{\mathbf{P}}_1 \longrightarrow \mathrm{Hom}\,(\mathrm{Hom}_{R\pi}(\mathbf{P}_2, W), R \otimes_{R\pi} (W \otimes \mathbf{P}_3))$$

which assigns $a \cap -$ to $a \in \overline{\mathbf{P}}_1$. Moreover, write $\mathbf{P}_3^r$ for the free resolution of R in the category of right $R\pi$-modules obtained from $\mathbf{P}_3$ in the usual way, that is, as a chain complex in the category of R-modules, $\mathbf{P}_3^r = \mathbf{P}_3$, and the group π acts on the right in the obvious way so that $yx = x^{-1}y$ for every $x \in \pi$ and every $y \in \mathbf{P}_3$; then the chain map

$$R \otimes_{R\pi} (W \otimes \mathbf{P}_3) \longrightarrow \mathbf{P}_3^r \otimes_{R\pi} W$$

given by $1 \otimes w \otimes y \mapsto y \otimes w$, $\quad w \in W$, $y \in \mathbf{P}_3$, is an isomorphism. Consequently the assignment to $a \in \overline{\mathbf{P}}_1$ of $a \cap -$ yields a chain map

$$\overline{\mathbf{P}}_1 \longrightarrow \mathrm{Hom}\,(\mathrm{Hom}_{R\pi}(\mathbf{P}_2, W), \mathbf{P}_3^r \otimes_{R\pi} W)$$

which, by construction, induces the operation

$$H_i(\pi, R) \longrightarrow \mathrm{Hom}\,(H^*(\pi, W), H_{i-*}(\pi, W))$$

sending $a \in H_i(\pi, R)$ to $a \cap -$; in particular, with $a = [\pi] \in H_2(\pi, R)$, the fundamental class, we get the Poincaré duality isomorphism from $H^*(\pi, W)$ onto $H_{2-*}(\pi, W)$.

Let $\mathbf{P}_1$ be an arbitrary free resolution of R in the category of *left* $R\pi$-modules, $\mathbf{P}_2 = \widetilde{R}(\mathcal{P})$ cf. (2.2) above, and $\mathbf{P}_3 = R(\mathcal{P})^l$, the free resolution $R(\mathcal{P})$, converted into one in the category of *left* $R\pi$-modules by the same kind of construction as that used above for the passage from left to right modules. Then $R \otimes_{R\pi} (W \otimes \mathbf{P}_3)$ looks like $R(\mathcal{P}) \otimes_{R\pi} W$; further $\mathrm{Hom}_{R\pi}(\mathbf{P}_2, W) = \mathrm{Hom}_{R\pi}(\widetilde{R}(\mathcal{P}), W)$ amounts to $R(\mathcal{P}) \otimes_{R\pi} W$; and the cap product with the fundamental class boils down to the inverse of the canonical isomorphism (2.3.3) as asserted. □

Next we consider the lifted object $\widehat{\widetilde{R}(\mathcal{P})} = \mathrm{Hom}_{RF}(\widehat{R(\mathcal{P})}, RF)$. With the notation $\widehat{\widetilde{R}_{2-j}(\mathcal{P})} = \mathrm{Hom}_{RF}(\widehat{R_j(\mathcal{P})}, RF)$, it looks like

$$\widehat{\widetilde{R}(\mathcal{P})}\colon \widehat{\widetilde{R}_2(\mathcal{P})} \xrightarrow{\widehat{\partial}_2} \widehat{\widetilde{R}_1(\mathcal{P})} \xrightarrow{\widehat{\partial}_1} \widehat{\widetilde{R}_0(\mathcal{P})}, \tag{2.4}$$

and each $\widehat{\widetilde{R}_j(\mathcal{P})}$ inherits a structure of a left RF-module by means of a formula of the kind (2.3.2). Let $\Delta\colon \widetilde{R}(\mathcal{P}) \longrightarrow \widetilde{R}(\mathcal{P}) \otimes \widetilde{R}(\mathcal{P})$ be an $R\pi$-linear diagonal map for the free resolution $\widetilde{R}(\mathcal{P})$; as usual, the group ring $R\pi$

acts here on the tensor product $\widetilde{R}(\mathcal{P}) \otimes \widetilde{R}(\mathcal{P})$ through the diagonal map $\Delta\colon \pi \to \pi \times \pi$. We lift the diagonal map Δ to an RF-linear morphism $\widehat{\Delta}\colon \widehat{\widetilde{R}(\mathcal{P})} \longrightarrow \widehat{\widetilde{R}(\mathcal{P})} \otimes \widehat{\widetilde{R}(\mathcal{P})}$ of graded modules so that the diagram

$$\begin{array}{ccc} \widehat{\widetilde{R}(\mathcal{P})} & \xrightarrow{\widehat{\Delta}} & \widehat{\widetilde{R}(\mathcal{P})} \otimes \widehat{\widetilde{R}(\mathcal{P})} \\ \downarrow & & \downarrow \\ \widetilde{R}(\mathcal{P}) & \xrightarrow{\Delta} & \widetilde{R}(\mathcal{P}) \otimes \widetilde{R}(\mathcal{P}) \end{array}$$

is commutative. Thus $\widehat{\Delta}$ is a kind of diagonal map for $\widehat{\widetilde{R}(\mathcal{P})}$.

Let U be a left RF-module, with structure map χ from F to $\mathrm{Aut}(U)$; then the component $\widehat{\Delta}\colon \widehat{\widetilde{R}_2(\mathcal{P})} \longrightarrow \widehat{\widetilde{R}_1(\mathcal{P})} \otimes \widehat{\widetilde{R}_1(\mathcal{P})}$ induces a pairing

(2.5)
$$m_\chi\colon \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, U) \otimes \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, U) \longrightarrow \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_2(\mathcal{P})}, U \otimes U),$$

the tensor product $U \otimes U$ being equipped with the diagonal RF-module structure. Moreover, given another left RF-module V, with structure map θ from F to $\mathrm{Aut}(V)$, and a morphism $\alpha\colon U \to V$ of left RF-modules, by naturality, the resulting maps m_χ and m_θ are compatible in the sense that the corresponding diagram

(2.6)
$$\begin{array}{ccc} \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, U) \otimes \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, U) & \xrightarrow{m_\chi} & \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_2(\mathcal{P})}, U \otimes U) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, V) \otimes \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, V) & \xrightarrow{m_\theta} & \mathrm{Hom}_{RF}(\widehat{\widetilde{R}_2(\mathcal{P})}, V \otimes V) \end{array}$$

is commutative.

Taking into account the above isomorphism (2.3.3) we see that, $U^{2\ell}$ being identified with $R_1(\mathcal{P}) \otimes_{R\pi} U$ and $R_0(\mathcal{P}) \otimes_{R\pi} (U \otimes U)$ with $U \otimes U$, the pairing (2.5) looks like

$$m_\chi\colon U^{2\ell} \otimes U^{2\ell} \longrightarrow U \otimes U. \tag{2.7}$$

By construction, when the left RF-module structure χ on U comes from a left $R\pi$-module structure, the pairing (2.5) induces the cup pairing

$$\cup\colon H^1(\pi, U) \otimes H^1(\pi, U) \longrightarrow H^2(\pi, U \otimes U), \tag{2.8}$$

computed from the free resolution $\widetilde{R}(\mathcal{P})$ of R in the category of *left* $R\pi$-modules, and hence (2.7) induces the intersection pairing

$$H_1(\pi, U) \otimes H_1(\pi, U) \xrightarrow{\iota} H_0(\pi, U \otimes U), \tag{2.9}$$

computed from the *standard* free resolution (1.4) of R in the category of right $R\pi$-modules. In fact, with reference to the left π-module structure $\chi\colon \pi \to \mathrm{Aut}(U)$ on U and the corresponding one $\chi^{\otimes}\colon \pi \to \mathrm{Aut}(U \otimes U)$ on the tensor product $U \otimes U$, the diagram

$$\begin{array}{ccc} H^1(\pi,U) \otimes H^1(\pi,U) & \xrightarrow{\cup} & H^2(\pi, U \otimes U) \\ {\scriptstyle ([\pi]\cap -)\otimes([\pi]\cap -)}\downarrow & & \downarrow{\scriptstyle [\pi]\cap -} \\ H_1(\pi,U) \otimes H_1(\pi,U) & \xrightarrow{\iota} & H_0(\pi, U \otimes U) \end{array}$$

is commutative.

We now take $R = \mathbf{R}$, the reals. Given a homomorphism χ from F to G as before and taking $U = g^*_\chi$, with respect to the canonical identifications of $\mathrm{Hom}_{RF}(\widehat{\widetilde{R}_1(\mathcal{P})}, g^*_\chi)$ with $(g^*)^{2\ell}$ and of $\mathrm{Hom}_{RF}(\widehat{\widetilde{R}_2(\mathcal{P})}, g^*_\chi \otimes g^*_\chi)$ with $g^* \otimes g^*$, the resulting pairing (2.7) looks like $m_\chi\colon (g^*)^{2\ell} \otimes (g^*)^{2\ell} \longrightarrow g^* \otimes g^*$; it is clear that, on each connected component, the resulting map

$$m\colon \mathrm{Hom}(F,G) \longrightarrow \mathrm{Hom}\left((g^*)^{2\ell} \otimes (g^*)^{2\ell}, g^* \otimes g^*\right) \tag{2.10}$$

which sends $\chi \in \mathrm{Hom}(F,G)$ to m_χ is algebraic and hence smooth.

In the usual way, the group G acts on $\mathrm{Hom}(F,G)$ and $\mathrm{Hom}((g^*)^{2\ell} \otimes (g^*)^{2\ell}, g^* \otimes g^*)$; for an element β of the latter, for $x \in G$, and for $a, b \in (g^*)^{2\ell}$,

$$(x\beta)(a \otimes b) = x\left(\beta(x^{-1}a \otimes x^{-1}b)\right).$$

2.11. *The map m is G-equivariant.*

Proof. — Given $x \in G$, whatever $\chi \in \mathrm{Hom}(F,G)$, the induced linear map $Ad^*(x)$ from g^*_χ to $g^*_{x\chi}$ is an isomorphism of $\mathbf{R}F$-modules. By naturality, the diagram

$$\begin{array}{ccc} T^*_\chi \mathrm{Hom}(F,G) \otimes T^*_\chi \mathrm{Hom}(F,G) & \xrightarrow{m_\chi} & g^*_\chi \otimes g^*_\chi \\ {\scriptstyle Ad^*(x)\otimes Ad^*(x)}\downarrow & & {\scriptstyle Ad^*(x)\otimes Ad^*(x)}\downarrow \\ T^*_{x\chi} \mathrm{Hom}(F,G) \otimes T^*_{x\chi} \mathrm{Hom}(F,G) & \xrightarrow{m_{x\chi}} & g^*_{x\chi} \otimes g^*_{x\chi} \end{array}$$

is therefore commutative. This implies the assertion. □

For arbitrary smooth functions f and h on $\mathrm{Hom}(F,G)$ and every $\chi \in \mathrm{Hom}(F,G)$, let

$$(f \diamondsuit h)(\chi) = \langle m_\chi\left((df(\chi) \circ L_\chi) \otimes (dh(\chi) \circ L_\chi)\right)\rangle. \tag{2.12}$$

This yields a bilinear pairing $\diamondsuit$ on $C^\infty(\mathrm{Hom}(F,G))$ with values in $C^\infty(\mathrm{Hom}(F,G))$. We list some of its properties.

2.13. *It satisfies the Leibniz rule* $f\diamondsuit(hk) = h(f\diamondsuit k) + k(f\diamondsuit h)$, *whatever smooth functions* f, h, k *on* $\mathrm{Hom}(F,G)$.

This follows at once from the construction of $\diamondsuit$ in terms of differentials. For smooth G-invariant functions f and h on $\mathrm{Hom}(F,G)$, we now spell out (2.14), (2.15), and (2.17) below.

2.14. *The function* $f\diamondsuit h$ *is also* G*-invariant.*

Proof. — Let $x \in G$ and $\chi \in \mathrm{Hom}(F,G)$. Because f and h are G-invariant real valued functions, $df(x\chi) = \mathrm{Ad}^*(x)df(\chi)$ and $dh(x\chi) = \mathrm{Ad}^*(x)dh(\chi)$. In view of (2.11),

$$\begin{aligned}(f\diamondsuit h)(x\chi) &= \langle m_{x\chi}\,((df(x\chi)\circ L_{x\chi})\otimes(dh(x\chi)\circ L_{x\chi}))\rangle\\ &= \langle m_{x\chi}\,(\mathrm{Ad}^*(x)(df(\chi)\circ L_\chi)\otimes \mathrm{Ad}^*(x)(dh(\chi)\circ L_\chi))\rangle\\ &= \langle(\mathrm{Ad}^*(x)\otimes \mathrm{Ad}^*(x))m_\chi\,((df(\chi)\circ L_\chi)\otimes(dh(\chi)\circ L_\chi))\rangle\\ &= \langle m_\chi\,((df(\chi)\circ L_\chi)\otimes(dh(\chi)\circ L_\chi))\rangle\\ &= (f\diamondsuit h)(\chi),\end{aligned}$$

where we exploited the fact that the given bilinear form $\langle\cdot,\cdot\rangle$ on g^* is coadjoint action invariant. □

Recall from (1.10) that $df(\chi)\circ L_\chi \in Z_1(\mathcal{P}, g^*_\chi)$ and $dh(\chi)\circ L_\chi \in Z_1(\mathcal{P}, g^*_\chi)$, whatever $\chi \in \mathrm{Hom}(F,G)$.

2.15. *For every* $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, *the value* $(f\diamondsuit h)(\phi)$ *equals the right-hand side* $\langle[df(\phi)\circ L_\phi]\otimes[dh(\phi)\circ L_\phi]\rangle_\phi$ *of* (2.1.2) *and hence depends only on the classes* $[df(\phi)\circ L_\phi] \in H_1(\pi, g^*_\phi)$ *and* $[dh(\phi)\circ L_\phi] \in H_1(\pi, g^*_\phi)$. *Consequently the pairing* $\diamondsuit$, *combined with the projection from* $(C^\infty(\mathrm{Hom}(F,G)))^G$ *onto* $C^\infty(\mathrm{Rep}_\xi(\Gamma,G))$, *comes down to the pairing* (2.1.3).

This is a consequence of the following.

2.16. *For an arbitrary left* π*-module* U, *with structure map* χ *from* π *to* $\mathrm{Aut}(U)$, *the restriction of the composite* $U^{2\ell}\otimes U^{2\ell} \to \mathbf{R}$ *of* (2.7) *with a* π*-invariant bilinear form on* U *to the subspace* $Z_1(R(\mathcal{P}),U)\otimes Z_1(R(\mathcal{P}),U)$

of $(R_1(\mathcal{P}) \otimes_{R\pi} U) \otimes (R_1(\mathcal{P}) \otimes_{R\pi} U)$ *factors through the corresponding* $\mathbf{R}$*-valued intersection pairing on* $H_1(\pi, U) \otimes H_1(\pi, U)$.

Proof. — This is just another way of saying that, with reference to the free resolution $\widetilde{R}(\mathcal{P})$, the pairing induced by the diagonal map, when restricted to the cocycles, factors through cohomology. □

2.17. *If* h *is zero on* $\mathrm{Hom}_\xi(\Gamma, G)$, *so is the function* $f \diamondsuit h$. *In other words, for general* h, *the set function* $\{f, h\}$ *defined on* $\mathrm{Rep}_\xi(\Gamma, G)$ *by the bracket* (2.1.4) *is well defined on* $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$.

Proof. — Since h is zero on $\mathrm{Hom}_\xi(\Gamma, G)$, at $\phi \in \mathrm{Hom}_\xi^-(\Gamma, G)$, by virtue of (1.14), $[dh(\phi) \circ L_\phi] \in H_1(\pi, g_\phi^*)$ is zero. Consequently the value

$$(f \diamondsuit h)(\phi) = \langle [df(\phi) \circ L_\phi] \otimes [dh(\phi) \circ L_\phi] \rangle_\phi$$

is zero for every $\phi \in \mathrm{Hom}_\xi^-(\Gamma, G)$. Since $\mathrm{Hom}_\xi^-(\Gamma, G)$ is dense in $\mathrm{Hom}_\xi(\Gamma, G)$, cf. (1.4) above, the function $f \diamondsuit h$ is zero on $\mathrm{Hom}_\xi(\Gamma, G)$. □

We can now prove compatibility with the action of the mapping class group since this does not rely on the Jacobi identity.

Proof of Theorem 2. — Let ϕ be an element of $\mathrm{Hom}_\xi(\Gamma, G)$ and consider an automorphism β of Γ. The composite $\phi\beta$ is a homomorphism from Γ to G which lies in $\mathrm{Hom}_{\beta^*\xi}(\Gamma, G)$, where $\beta^*\xi$ equals ξ (topologically) if β preserves the orientation and equals the (topologically) "opposite" bundle otherwise (which may coincide with ξ). The automorphism β induces a commutative diagram

$$\begin{array}{ccc} \Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G) & \xrightarrow{\lambda_\phi^*} & H_1(\pi, g_\phi^*) \\ {\scriptstyle \beta_\sharp}\uparrow & & {\scriptstyle \beta_\sharp}\uparrow \\ \Omega_{[\phi\beta]}\mathrm{Rep}_{\beta^*\xi}(\Gamma, G) & \xrightarrow{\lambda_{\phi\beta}^*} & H_1(\pi, g_{\phi\beta}^*). \end{array}$$

Comparison with the formula (0.1) shows that the induced diffeomorphism $\beta^\sharp$ from $\mathrm{Rep}_{\beta^*\xi}(\Gamma, G)$ to $\mathrm{Rep}_\xi(\Gamma, G)$ is compatible with the brackets (2.1.4), taken on both $\mathrm{Rep}_{\beta^*\xi}(\Gamma, G)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ if necessary. Thus the induced action of the group of orientation preserving outer automorphisms of Γ on $\mathrm{Rep}_\xi(\Gamma, G)$ preserves the bracket. However the group of outer automorphisms of Γ is isomorphic to the mapping class group of Σ in such a way that the action corresponds to that of the mapping class group on $\mathrm{Rep}(\Gamma, G)$. The latter, restricted to orientation preserving mapping classes, preserves each subspace of the kined $\mathrm{Rep}_\xi(\Gamma, G)$.

3. The Jacobi identity.

Let $\langle \cdot, \cdot \rangle_g$ be an orthogonal structure on g, that is, an adjoint action invariant scalar product. After a choice of Riemannian metric and orientation on Σ, the theory established in [2] and in our earlier papers [7] – [10] is available. In particular, the Wilson loop mapping $\rho_\flat$ from $N(\xi)$ to $\mathrm{Rep}(\Gamma, G)$ restricts to a diffeomorphism from $N^{\mathrm{top}}(\xi)$ onto $\mathrm{Rep}^{\mathrm{top}}_\xi(\Gamma, G)$ whose derivative at a point $[A] \in N^{\mathrm{top}}(\xi)$ amounts to the twisted integration mapping Int_A from $H^1_A(\Sigma, \mathrm{ad}(\xi))$ to $H^1(\pi, g_\phi)$; see e. g. [10] (7.11) for details. Moreover the data determine a symplectic structure on $N^{\mathrm{top}}(\xi)$ which, for $[A] \in N^{\mathrm{top}}(\xi)$, on the tangent space amounts to the symplectic structure σ_A on $H^1_A(\Sigma, \mathrm{ad}(\xi))$ induced by the data, cf. [2] and our earlier paper [7]; with $\phi = \rho(A)$, under the twisted integration mapping, this structure then passes to the symplectic structure σ_ϕ on $H^1(\pi, g_\phi)$ mentioned in Section 1. In this way, the orthogonal structure on g gives rise to a symplectic structure on $\mathrm{Rep}^{\mathrm{top}}_\xi(\Gamma, G)$.

Henceforth we denote the given coadjoint action invariant symmetric bilinear form on g^* by $\langle \cdot, \cdot \rangle_{g^*}$. It induces a 2-tensor ω on $N^{\mathrm{top}}(\xi)$. In fact, let A be a central Yang-Mills connection representing a point of the top stratum $N^{\mathrm{top}}(\xi)$, and let $\phi = \rho(A) \in \mathrm{Hom}_\xi(\Gamma, G)$. By means of the isomorphism λ^*_A from $T^*_{[A]}N(\xi)$ onto $H^1_A(\Sigma, \mathrm{ad}^*(\xi))$, cf. Section 1 above, at $[A]$, the tensor ω then amounts to the 2-form ω_A on $H^1_A(\Sigma, \mathrm{ad}^*(\xi))$ induced by $\langle \cdot, \cdot \rangle_{g^*}$ via the wedge product of forms and integration. Further, by (1.20), the cotangent map of the diffeomorphism from $N^{\mathrm{top}}(\xi)$ onto $\mathrm{Rep}^{\mathrm{top}}_\xi(\Gamma, G)$ boils down to the dual twisted integration mapping (1.19); moreover the composite

$$H_1(\pi, g^*_\phi) \otimes H_1(\pi, g^*_\phi) \xrightarrow{\mathrm{Int}^*_A \otimes \mathrm{Int}^*_A} H^1_A(\Sigma, \mathrm{ad}^*(\xi)) \otimes H^1_A(\Sigma, \mathrm{ad}^*(\xi)) \xrightarrow{\omega_A} \mathbf{R}$$

coincides with the intersection pairing (2.1.1) induced by the given 2-form on g^*. On the other hand, via the isomorphism λ^*_ϕ given in (1.16), the latter pairing amounts just to the 2-tensor at $[\phi] \in \mathrm{Rep}^{\mathrm{top}}_\xi(\Gamma, G)$ which corresponds to the bracket (2.1.3) induced by $\langle \cdot, \cdot \rangle_{g^*}$.

When the bilinear form $\langle \cdot, \cdot \rangle_{g^*}$ is positive definite, it induces an isomorphism between g and g^* and hence an orthogonal structure $\langle \cdot, \cdot \rangle_g$ on g so that under this isomorphism the two 2-forms correspond, and the above remarks apply; moreover the 2-forms σ_A and ω_A then correspond to each other under adjointness, and the resulting bracket is just the corresponding symplectic Poisson structure. Since A is arbitrary, this shows that then

the bracket (2.1.3) coincides with the symplectic Poisson bracket induced by the symplectic structure on $\mathrm{Rep}_\xi^{\mathrm{top}}(\Gamma, G)$ determined by the orthogonal structure $\langle \cdot, \cdot \rangle_g$ on g and hence in particular satisfies the Jacobi identity. By [8] (1.4), the top stratum $N^{\mathrm{top}}(\xi)$ is dense in $N(\xi)$, and hence $\mathrm{Rep}_\xi^{\mathrm{top}}(\Gamma, G)$ is dense in $\mathrm{Rep}_\xi(\Gamma, G)$; consequently for a positive definite 2-form on g^* the bracket (2.1.3) satisfies the Jacobi identity everywhere on $\mathrm{Rep}_\xi(\Gamma, G)$. This establishes Theorem 2.1 in this special case.

To handle the case of a general 2-form $\langle \cdot, \cdot \rangle_{g^*}$ we recall that, by structure theory, the Lie algebra g decomposes uniquely into a direct sum of its centre z and the simple ideals in the semi simple Lie algebra $[g, g]$. This implies that g^* decomposes as a direct sum

$$g^* = g_+^* \oplus g_-^* \oplus g_0^* \tag{3.1}$$

of G-modules, together with coadjoint action invariant scalar products $\langle \cdot, \cdot \rangle_+^*$ and $\langle \cdot, \cdot \rangle_-^*$ on g_+^* and g_-^*, respectively, so that the 2-form on g^* decomposes as

$$\langle \cdot, \cdot \rangle_{g^*} = \langle \cdot, \cdot \rangle_+^* - \langle \cdot, \cdot \rangle_-^*,$$

and so that g_0^* is its null space. We note that, even when G is not connected, the decomposition (3.1) is one of G-modules; this relies on the uniqueness of the decomposition of g since $\langle \cdot, \cdot \rangle_{g^*}$ is assumed G-invariant. Picking a coadjoint action invariant scalar product $\langle \cdot, \cdot \rangle_0^*$ on g_0^*, we obtain a coadjoint action invariant scalar product

$$\langle \cdot, \cdot \rangle'_{g^*} = \langle \cdot, \cdot \rangle_+^* + \langle \cdot, \cdot \rangle_-^* + \langle \cdot, \cdot \rangle_0^* \colon g^* \otimes g^* \longrightarrow \mathbf{R}$$

on g^*; it induces an isomorphism from g^* onto g which, in turn, identifies $\langle \cdot, \cdot \rangle'_{g^*}$ with an orthogonal structure

$$\langle \cdot, \cdot \rangle'_g \colon g \otimes g \longrightarrow \mathbf{R} \tag{3.2}$$

on g. By construction, then, the direct sum decomposition (3.1) passes to a direct sum decomposition

$$g = g_+ \oplus g_- \oplus g_0 \tag{3.3}$$

of G-modules; the forms $\langle \cdot, \cdot \rangle_+^*$ and $\langle \cdot, \cdot \rangle_-^*$ pass to corresponding forms

(3.4)
$$\langle \cdot, \cdot \rangle_+ \colon g_+ \otimes g_+ \longrightarrow \mathbf{R}, \quad \langle \cdot, \cdot \rangle_- \colon g_- \otimes g_- \longrightarrow \mathbf{R}, \quad \langle \cdot, \cdot \rangle_0 \colon g_0 \otimes g_0 \longrightarrow \mathbf{R};$$

and the orthogonal structure (3.2) decomposes as

$$\langle \cdot, \cdot \rangle'_g = \langle \cdot, \cdot \rangle_+ + \langle \cdot, \cdot \rangle_- + \langle \cdot, \cdot \rangle_0 \colon g \otimes g \longrightarrow \mathbf{R}. \tag{3.5}$$

Finally, the 2-form $\langle \cdot, \cdot \rangle_{g^*}$ on g^* passes to the adjoint action invariant symmetric bilinear form

$$\langle \cdot, \cdot \rangle_g = \langle \cdot, \cdot \rangle_+ + (-\langle \cdot, \cdot \rangle_-)$$

on g. In other words, all the relevant structure is preserved.

With reference to the decomposition (3.3), the Lie algebra bundle $\mathrm{ad}(\xi)$ decomposes into a direct sum of corresponding Lie algebra bundles ζ_+, ζ_-, and ζ_0. Hence the graded vector space $\Omega^*(\Sigma, \mathrm{ad}(\xi))$ of $\mathrm{ad}(\xi)$-valued forms decomposes as well into the direct sum of $\Omega^*(\Sigma, \zeta_+)$, $\Omega^*(\Sigma, \zeta_-)$, and $\Omega^*(\Sigma, \zeta_0)$ and, whatever connection $\widetilde{A}$, the operator of covariant derivative $d_{\widetilde{A}}$ preserves the decompositions. When we divide out the appropriate groups of translations, we obtain the affine spaces

$$\begin{aligned}
\mathcal{A}(\xi)_+ &= \mathcal{A}(\xi) / (\Omega^*(\Sigma, \zeta_-) \oplus \Omega^*(\Sigma, \zeta_0)), \\
\mathcal{A}(\xi)_- &= \mathcal{A}(\xi) / (\Omega^*(\Sigma, \zeta_+) \oplus \Omega^*(\Sigma, \zeta_0)), \\
\mathcal{A}(\xi)_0 &= \mathcal{A}(\xi) / (\Omega^*(\Sigma, \zeta_+) \oplus \Omega^*(\Sigma, \zeta_-)),
\end{aligned}$$

together with a canonical isomorphism of affine spaces from $\mathcal{A}(\xi)$ to $\mathcal{A}(\xi)_+ \times \mathcal{A}(\xi)_- \times \mathcal{A}(\xi)_0$. The decomposition (3.5) of the orthogonal structure on g entails that the relevant structure made explicit in Section 1 of our paper [7] decomposes accordingly. In particular, the Lie algebra $\mathbf{g}(\xi) = \mathbf{\Omega^0}(\mathbf{\Sigma}, \mathrm{ad}(\xi))$ of infinitesimal gauge transformations decomposes into a direct sum of Lie ideals $\mathbf{g}(\xi)_+ = \mathbf{\Omega^0}(\mathbf{\Sigma}, \zeta_+)$, $\mathbf{g}(\xi)_- = \mathbf{\Omega^0}(\mathbf{\Sigma}, \zeta_-)$, and $\mathbf{g}(\xi)_0 = \mathbf{\Omega^0}(\mathbf{\Sigma}, \zeta_0)$.

We now consider the Yang-Mills theory on ξ, with the orthogonal structure (3.2) playing the role of the orthogonal structure on g at the beginning of this Section. The same construction as that of the weakly symplectic structure σ on $\mathcal{A}(\xi)$ resulting from the latter yields weakly symplectic structures σ_+, σ_-, σ_0, on respectively $\mathcal{A}(\xi)_+$, $\mathcal{A}(\xi)_-$, $\mathcal{A}(\xi)_0$, resulting from the corresponding orthogonal structures (3.4). Moreover the space $\mathcal{N}(\xi) \subseteq \mathcal{A}(\xi)$ of central Yang-Mills connections accordingly decomposes into a product $\mathcal{N}(\xi)_+ \times \mathcal{N}(\xi)_- \times \mathcal{N}(\xi)_0$. The results in our paper [7] imply that, near a point of the top stratum $N^{\mathrm{top}}(\xi)$, the space $N(\xi)$ locally decomposes into a product of suitable submanifolds of $N(\xi)_+$, $N(\xi)_-$, and $N(\xi)_0$. On taking on these spaces locally the Poisson structures coming from respectively σ_+, $-\sigma_-$, and the zero structure on $N(\xi)_0$, we locally recover the bracket (2.1.3) for an arbitrary 2-form $\langle \cdot, \cdot \rangle_{g^*}$. This implies that the bracket (2.1.3) satisfies the Jacobi identity in the general case.

When the form $\langle \cdot, \cdot \rangle_{g^*}$ is non-degenerate, in the above discussion, the objects labelled $-_0$ do not occur. This implies that the resulting Poisson structure (2.1.3) is then symplectic. The proof of Theorem 2.1 is now complete.

Remark 3.6. — The argument may be extended to show that, when the point $[A]$ of $N(\xi)$ represented by A is non-singular, the Kuranishi map yields Darboux coordinates for the Poisson structure on $N(\xi)$ near $[A]$.

4. Stratified symplectic structures.

We remind the reader that the spaces $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ are stratified by connected components of orbit types; see Section 2 of our paper [8] for details.

THEOREM 4.1. — *Suppose the given coadjoint action invariant symmetric bilinear form on g^* is non-degenerate. Then the Poisson bracket* (2.1.3) *restricts to a symplectic Poisson bracket on each stratum. In other words, this Poisson structure endows the spaces $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ with a structure of a stratified symplectic space.*

Remark 4.2. — We note that, in the statement of the theorem, there is *no need* to assume the 2-form to be positive definite.

Proof of (4.1). — Clearly this is a local statement, and we can argue in terms of the local model of a neighborhood of a point $[A]$ of $N(\xi)$ given in [7] (2.32), with the following slight modification which is needed when the given 2-form on g^* is not positive definite : Let A be a central Yang-Mills connection, and consider the decomposition of $H^1_A(\Sigma, \mathrm{ad}(\xi))$ into the direct sum of $H^1_A(\Sigma, \zeta_+)$ and $H^1_A(\Sigma, \zeta_-)$ where the notation is that in the previous Section. Denote by σ^+_A and σ^-_A the symplectic structures on $H^1_A(\Sigma, \zeta_+)$ and $H^1_A(\Sigma, \zeta_-)$ induced by the 2-forms $\langle \cdot, \cdot \rangle_+$ and $\langle \cdot, \cdot \rangle_-$, respectively, cf. (3.4), and consider the mappings

$$\Theta^+_A \colon H^1_A(\Sigma, \zeta_+) \longrightarrow H^2_A(\Sigma, \zeta_+), \quad \Theta^-_A \colon H^1_A(\Sigma, \zeta_-) \longrightarrow H^2_A(\Sigma, \zeta_-),$$

given by the assignments to $\alpha \in H^1_A(\Sigma, \zeta_+)$ and $\beta \in H^1_A(\Sigma, \zeta_-)$ of $\Theta^+_A(\alpha) = \frac{1}{2}[\alpha, \alpha]_A$ and $\Theta^-_A(\beta) = -\frac{1}{2}[\beta, \beta]_A$, respectively. Their direct sum Θ_A yields a momentum mapping for the action of the stabilizer Z_A of A in

$\mathcal{G}(\xi)$ on $H^1_A(\Sigma, \mathrm{ad}(\xi))$, with the symplectic structure $\sigma_A^+ + (-\sigma_A^-)$; in fact it is the unique momentum mapping having the value zero at the origin. Now the Marsden-Weinstein reduced space $H_A = \Theta_A^{-1}(0)/Z_A$ is a local model of $N(\xi)$ near $[A]$, cf. [7], [10]. By the main result of Sjamaar-Lerman [21], the decomposition of H_A into connected components of orbit types is a stratification in such a way that each stratum inherits a symplectic structure. Moreover, write $H^1_A = H^1_A(\Sigma, \mathrm{ad}(\xi))$ for short, and consider the algebra

$$C^\infty(H_A) = \left(C^\infty(H^1_A)\right)^{Z_A} / I_A^{Z_A}$$

of smooth Z_A-invariant functions $\left(C^\infty(H^1_A)\right)^{Z_A}$ on H^1_A modulo its ideal $I_A^{Z_A}$ of functions that vanish on the zero locus $\Theta_A^{-1}(0)$. This algebra endows H_A with a smooth structure; see [10] (6.1.2) for more details. By Arms-Cushman-Gotay [1], the symplectic Poisson bracket on $C^\infty(H^1_A)$ passes to a Poisson bracket $\{\cdot,\cdot\}_A$ on $C^\infty(H_A)$ which, in turn, restricts to the symplectic Poisson bracket on each stratum of H_A. This establishes the assertion of the Theorem locally.

By [10] (6.2), near $[A] \in N(\xi)$, with the smooth structure $C^\infty(H_A)$, the space H_A is a local model of $N(\xi)$, with its smooth structure $C^\infty(N(\xi))$ introduced in Section 4 of [10]. However, by construction, near $[A] \in N(\xi)$, the algebra $C^\infty(H_A)$ with the Poisson bracket $\{\cdot,\cdot\}_A$ is in fact a local model of $N(\xi)$ with the Poisson bracket on $C^\infty(N(\xi))$ induced from (2.1.3) via the Wilson loop mapping $\rho_\flat$ from $N(\xi)$ to $\mathrm{Rep}_\xi(\Gamma, G)$. This completes the proof. □

The proof also establishes the following.

ADDENDUM 4.3. — *For every central Yang-Mills connection A, near $[A] \in N(\xi)$, the Poisson algebra $(C^\infty(H_A), \{\cdot,\cdot\}_A)$ yields a local model of $N(\xi)$ with its Poisson structure and likewise, near the point $\rho_\flat[A]$, a local model of $\mathrm{Rep}_\xi(\Gamma, G)$ with its Poisson structure, where $\rho_\flat$ refers to the Wilson loop mapping from $N(\xi)$ to $\mathrm{Rep}_\xi(\Gamma, G)$. More precisely, the choice of A (in its class $[A]$) induces a Poisson diffeomorphism of an open neighborhood W_A of $[0] \in H_A$ onto an open neighborhood U_A of $[A] \in N(\xi)$, where W_A and U_A are endowed with the induced smooth and Poisson structures, and a similar statement holds for $\mathrm{Rep}_\xi(\Gamma, G)$ near the point $\rho_\flat[A]$.*

5. Twist flows.

Let f be a smooth invariant real function on G, and let C be a closed curve in Σ, having starting point Q. The homotopy class of C induces a homomorphism $[C]$ from $\mathbf{Z}$ to the fundamental group π of Σ, and the association $\phi \mapsto f(\phi([C](1))) \in \mathbf{R}$ induces a real valued function f^C on $\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)/G$. Let ξ be a flat G-bundle over Σ. Since $[C]$ lifts to a homomorphism from $\mathbf{Z}$ to the free group F on the chosen generators, f^C yields a function in $C^\infty(\mathrm{Rep}_\xi(\pi, G))$, and hence $\{f^C, \cdot\}$ is a derivation of $C^\infty(\mathrm{Rep}_\xi(\pi, G))$. On each stratum it amounts of course to a smooth vector field. The corresponding flow on the *non-singular* part of $\mathrm{Rep}_\xi(\pi, G)$ has been studied by Goldman in [4], referred to as a *twist flow*. However, the derivation $\{f^C, \cdot\}$ in fact integrates to a "twist flow" on the whole space $\mathrm{Rep}_\xi(\pi, G)$, that is, an action of the real line on this space preserving the smooth structure. The argument in Section 3 of [21], applied to our local model constructed in [7], shows that this is locally so, and using the partition of unity established in our paper [10], we conclude that these twist flows in fact exist globally and are unique. We hope to give the details at another occasion.

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