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# HARMONIC SYNTHESIS OF SOLUTIONS OF ELLIPTIC EQUATION WITH PERIODIC COEFFICIENTS

by Victor P. PALAMODOV

## 0. Introduction.

Let $p = p(x, D)$ be a $s \times t$-matrix, whose entries are linear differential operators on $\mathbf{R}^n$ with $n$-periodic coefficients, *i.e.* $p(x + q, D) = p(x, D)$ for any $q \in \mathbf{Z}^n$, where we denote $D = (i\partial/\partial x_1, \ldots, i\partial/\partial x_n)$, $i = \sqrt{-1}$. Assuming that $p$ is an elliptic operator, we develop any solution of the system

$$p(x, D)u = 0, \tag{0.1}$$

which satisfies for some $a > 0$ the condition

$$u(x) = O(\exp(a|x|)), \quad |x| \to \infty, \tag{0.2}$$

in an integral over a variety of Floquet solutions. This development is similar to the exponential representation of solutions of (0.1) in the case of constant coefficients [1], [2]. A decomposition of this type was given by P. Kuchment [4] for the case $s = t$. Our approach gives a decomposition of solutions of (0.1) in a global integral over a series of holomorphic families $L_k = \{L_k(\lambda),\ \lambda \in N_k\}$, $k = 1, 2, \ldots$ of finite-dimensional representations $L_k(\lambda)$ of the translation group $\mathbf{Z}^n$. Here for each $k$ the parameter $\lambda$ runs

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over an irreducible analytic subset $N_k$ of the variety $\Lambda$ of all characters of the group in the space (0.2). Each representation $L_k(\lambda)$ consists of Floquet solutions of (0.1) with quasi-impulse $\lambda$ and contains only one Bloch solution. It may be thought as a Jordan cell for the given representation of the group $\mathbf{Z}^n$ in the space of solutions of (0.1).

To get such a decomposition we use a *global Noether operator* for the characteristic sheaf of the system (0.1). Note that a similar decomposition obtained in [4] is more involved, since there only the local Noether operators [9] were used. We prove in § 3 that any coherent analytic sheaf on arbitrary Stein space admits a global Noether operator. In § 5 we state an analog of Malgrange's approximation theorem.

## 1. Main result.

Let $\mathbf{C}^n$ the complex dual to $\mathbf{R}^n$ and $Z^n$ be the subgroup of integer vectors in $\mathbf{C}^n$. Then $\Lambda := \mathbf{C}^n/Z^n$ is the dual to $\mathbf{Z}^n$ complex Lie group and it is a Stein variety. There is a bilinear form $\Lambda \times \mathbf{Z}^n \to \mathbf{C}/\mathbf{Z}$, which is written as $\lambda \cdot q = \sum \zeta_j \cdot q_j$, where $\zeta := (\zeta_1, \dots, \zeta_n)$ is any pre-image of $\lambda$ under the canonical surjection $\chi : \mathbf{C}^n \to \Lambda$. For any $\lambda \in \Lambda$ the function $q \rightsquigarrow \exp(2\pi i \lambda \cdot q)$ is a character of the group $\mathbf{Z}^n$. This group is represented in the space $D'(\mathbf{R}^n)$ by translation operators $T_q f(x) = f(x+q)$, $q \in \mathbf{Z}^n$. Choose an euclidean norm $|\cdot|$ in $\mathbf{R}^n$ and denote by $\|\cdot\|$ the dual hermitian norm on $\mathbf{C}^n$. For any positive $a$ we denote by $\Lambda_a$ the image in $\Lambda$ of the strip $\|\operatorname{Im}\zeta\| < a/2\pi$, $\zeta \in \mathbf{C}^n$.

We assume that operator $p$ is included in an elliptic differential complex :

$$0 \to L_0 \xrightarrow{p_0} L_1 \xrightarrow{p_1} L_2 \longrightarrow \cdots \xrightarrow{p_m} L_{m+1} \longrightarrow \cdots, \quad p_0 = p, \tag{1.1}$$

where $L_i$, $i = 0, 1, \dots$ are sheaves of $C^\infty$-sections of some finite-dimensional trivial bundles on $\mathbf{R}^n$ and $p_0, p_1, \dots$ are differential operators with $n$-periodic coefficients.

THEOREM 1.1. — *Any solution $u$ of (0.1), which is defined on $\mathbf{R}^n$ and satisfies (0.2) for some $a > 0$, admits for any $b > a$ the following representation*

$$u(x) = \sum_k \sum_{j=1}^{r(k)} \int_{N_k} f_{kj}(\lambda, x) \mu_{kj}(\lambda), \tag{1.2}$$

*where*

i) $N_k$, $k = 1, 2, \ldots$ *are closed irreducible analytic subsets of* $\Lambda$, *associated to the characteristic sheaf* $M$ *of (0.1) (see § 3),*

ii) *for any* $k$, $f_{kj}(\lambda, x)$, $1, \ldots, r(k)$ *are smooth functions on* $N_k \times \mathbf{R}^n$, *which are holomorphic on* $\lambda \in N_k$ *and satisfy the equation (0.1) on* $x$;

iii) *for any* $k$, $\lambda \in N_k$ *the linear span* $L_k(\lambda)$ *of functions* $f_{kj}(\lambda, \cdot)$, $j = 1, \ldots, r(k)$ *is* $\mathbf{Z}^n$*-invariant and contains a unique invariant one-dimensional subspace; its character is equal to* $\exp(2\pi i \lambda \cdot x)$;

iv) $\mu_{kj}$ *are* $C^\infty$*-densities on the set* $\operatorname{reg} N_k$ *of regular points of* $N_k$; $\operatorname{supp} \mu_{kj} \subset \operatorname{reg} N_k \cap \Lambda_b$. *Moreover for arbitrary proper closed analytic subsets* $\Omega_k \subset N_k$, $k = 1, \ldots$ *there exist densities* $\mu_{kj}$, *which satisfies (1.2) such that* $\operatorname{supp} \mu_{kj} \subset N_k \cap \Lambda_b \setminus \Omega_k$ *for any* $j$ *and* $k$.

*Remark 1.2.* — Inversely for any densities $\mu_{kj}$, which fulfil iv), the second term of (1.2) is equal to $O(\exp(b|x|))$ at infinity and satisfies (0.1).

*Remark 1.3.* — The space $E_p$ of solutions of (0.1), which satisfy (0.2), is generally an infinite-dimensional non-unitary representation of $\mathbf{Z}^n$. The equation (1.2) may be considered as a decomposition of $E_p$ in an integral over the family of finite-dimensional subrepresentations $L_k(\lambda)$. But the densities $\mu_{kj}$ are far from being unique unlike the Stone-Naimark-Ambrose-Godement theorem for an unitary representation, where the spectral measure is unique.

Note that the representation $L_k(\lambda)$ is reducible, except for the case $\dim L_k(\lambda) = 1$, but is not decomposible, *i.e.* $L_k(\lambda)$ is not equal to a direct sum of some invariant subspaces.

*Remark 1.4.* — It follows from iii) that for any $k$ there exists an integer $d$ such that the identity

$$\left[T_q - \exp(2\pi i \lambda \cdot q)\right]^d f_{kj}(\lambda, \cdot) = 0, \quad \forall q \in \mathbf{Z}^n, \tag{1.3}$$

holds for any $\lambda \in N_k$ and $j = 1, \ldots r(k)$. This identity implies that for any $k, j$

$$f_{kj}(\lambda, x) = \sum_{|s|<d} x^s h_s(\lambda, x) \exp(2\pi i \lambda \cdot x),$$

where all the functions $h_s(\lambda, x)$ are $n$-periodic on $x$. Hence $f_{kj}(\lambda, x)$ is a *Floquet-solution* of (0.1) with quasi-impulse $\lambda$. Any generator, say $f_{k1}$, of the unique invariant one-dimensional subspace satisfies (1.3) with $d = 1$. It is called *Bloch-solution.*

## 2. Analytic lemmas.

Fix an integer $k$ and consider the following family of elliptic complexes with $n$-periodic coefficients (*cf.* [4]) :

$$W_*^k : 0 \longleftarrow W_0^{k(0)} \xleftarrow{p_0'(\zeta)} W_1^{k(1)} \xleftarrow{p_1'(\zeta)} W_2^{k(2)} \longleftarrow \cdots \tag{2.1}$$

where $W_i^k$ denotes the Sobolev space of sections of $L_i$ on the torus $T^n$, which are square-summable with its derivatives up to $k$-th degree,

$$p_i'(\zeta) := {}^t p_i(x, D + 2\pi\zeta),$$

where ${}^t p$ means the formally adjoint operator to $p$, $m_i$ is the order of $p_i$, $k(0) = 0$, $k(i) = k + m_0 + \cdots + m_{i-1}$, $i > 0$ and $\zeta$ runs over $\mathbf{C}^n$. This is a holomorphic family of Fredholm complexes since (1.1) is elliptic. Denote by $\mathcal{W}_i^k$ the sheaf of germs of holomorphic functions $f : \mathbf{C}^n \to \mathcal{W}_i^k$. Then (2.1) generates the following complex of analytic sheaves on $\mathbf{C}^n$

$$0 \longleftarrow \mathcal{W}_0^{k(0)} \xleftarrow{p_0'} \mathcal{W}_1^{k(1)} \xleftarrow{p_1'} \cdots \tag{2.2}$$

We define an action of the group $\mathbf{Z}^n$ on the sheaf $\mathcal{W}_i^k$ by the formula

$$T_\vartheta \psi(\zeta, x) = \exp(-2\pi i \vartheta \cdot x)\psi(\zeta + \vartheta, x), \quad \vartheta \in Z^n.$$

Let $\chi : \mathbf{C}^n \to \Lambda$ be the canonical projection; consider the sheaf $I_i^k | \Lambda$ of *invariant* sections of $\mathcal{W}_i^k$; a section of $I_i^k$ on an open set $V \subset \Lambda$ is identified with a section $\varphi$ of $\mathcal{W}_i^k$ on $\chi^{-1}(V)$ such that

$$\psi(\zeta + \vartheta, x) = \exp(2\pi i \vartheta \cdot x)\psi(\zeta, x), \forall \vartheta \in Z^n. \tag{2.3}$$

The following evident operator identity

$$p'(\zeta + \vartheta) = \exp(2\pi i \vartheta \cdot x) p'(\zeta) \exp(-2\pi i \vartheta \cdot x)$$

implies that (2.2) generates for any $k$ a sheaf complex

$$I_*^k : 0 \longleftarrow I_0^{k(0)} \xleftarrow{p_0'} I_1^{k(1)} \xleftarrow{p_1'} I_2^{k(2)} \longleftarrow \cdots$$

Denote by $H_* = \sum H_i$ the homology of this complex. All $H_i$ are coherent analytic sheaves on $\Lambda$, since (2.1) is a holomorphic Fredholm family (*cf.* [4]). This follows, for example, from [13, Lemma 4.3]. The embedding $I_*^{k+1} \to I_*^k$ induces for any $k$ sheaf morphisms $h^k : H_*^{k+1} \to H_*^k$.

LEMMA 2.1. — *For any $k$, $h^k$ is bijective.*

*Proof.* — Fix $\zeta$ and choose a parametrix $r$ for the complex (2.1). This is a pseudodifferential operator in the graded space of (2.1) of degree

1 and of order $-m_i$, when acting on the term $W_i^{k+\cdots}$, which satisfies the equation

$$p(\zeta)r + rp(\zeta) = \mathrm{id} + q,$$

where $q$ is a pseudodifferential operator of order $-1$ and of degree 0 as an endomorphism of the complex. It defines a morphism of complexes $q : \mathcal{W}_*^k \to \mathcal{W}_*^{k+1}$. This equation means that the compositions $eq$ and $qe$ are homotopic to the identity morphisms, where $e : \mathcal{W}_*^{k+1} \to \mathcal{W}_*^k$ is the natural embedding. This implies Lemma 2.1.

From now on we abbreviate the notation of $H_i^k$ to $H_i$.

LEMMA 2.2. — *For any $k$ and $a > 0$ there is a natural isomorphism*

$$H(\Gamma(\Lambda_a, I_*^k)) \cong \Gamma(\Lambda_a, H_*), \tag{2.4}$$

*where $H(K)$ means the homology group of a complex $K$.*

*Proof.* — Consider two spectral sequences for the functor $\Gamma(\Lambda_a, \cdot)$ and the complex $I_*^k$; both converge to the hyperhomology. For the first one we have $E^{pq} = H^p(H^q(\Lambda_a, I^k))$. This term vanishes for $q > 0$, since $H^q(\Lambda_a, I^k) = 0$, because $I_*^k$ is a holomorphic Banach sheaf [5]. Hence the hyperhomology is isomorphic to the left-hand side of (2.4). For the second spectral sequence we find $E_2^{pq} = H^p(\Lambda_a, H_q)$. These groups vanish for $p > 0$ as well, since $H_*$ is a coherent sheaf on a Stein space and $\Gamma(\Lambda_a, H_*) \cong E_2^{0*} \cong E_\infty$. This implies (2.4).

Now we pass in the spectrum $\Gamma_* := \Gamma(\Lambda_a, I_*^k)$ to the projective limit on $k$.

LEMMA 2.3. — *There is an isomorphism*

$$H(\Gamma(\Lambda_a, I_*)) \cong \Gamma(\Lambda_a, H_*), \quad I_* := I_*^\infty. \tag{2.5}$$

*Proof.* — A formal scheme is the same as in the previous lemma. We compare two standard spectral sequences for the hyperhomology of the functor Pr of projective limit and of the spectrum $\Gamma_*$. The term $E_2^{pq} = \mathrm{Pr}^p(H_q(\Gamma_*))$ vanishes for $p > 0$ since the spectrum $H_q(\Gamma_*)$ is constant in virtue of Lemma 2.2. The term $E_2^{0*} = E_\infty$ is equal to the right-hand side of (2.5).

For the second spectral sequence we have $E_2^{pq} = H_p(\mathrm{Pr}^q(\Gamma_*))$. These groups vanish for $q > 1$, since $\mathrm{Pr}^q = 0$. Evidently $\mathrm{Pr}^0(\Gamma_*) = \Gamma(\Lambda_a, I_*)$ hence $E_2^{*0}$ coincides with the left-hand side of (2.5). Now we verify that

$\mathrm{Pr}^1(\{\Gamma(\Lambda_a, I_*^k)\} = 0$. For this we need to show that the embedding of Fréchet spaces $\Gamma(\Lambda_a, I_*^{k+1}) \to \Gamma(\Lambda_a, I_*^k)$ has a dense image ([6]). This density property is easy to check, if we develop an arbitrary section of $I_*^k$ in Fourier series on $x$. Therefore the sequence $E_2$ degenerates to $E_2^{*0}$, which completes the proof of Lemma 2.3.

For an arbitrary positive $b$ we consider the space $S_b$ of $C^\infty$-functions $\varphi$ on $\mathbf{R}^n$, which satisfy the inequality

$$|D^i\varphi(x)| \leq C_{b',i} \exp(-b'|x|) \tag{2.6}$$

for any $i = (i_1, \ldots, i_n)$ and $b' < b$. For given $b'$ and $i$ take the minimal constant $C_{b',i} = C_{b',i}(\varphi)$. For any $b' > 0$ the functional $C_{b',i}(\varphi)$ is a norm on $S_b$. This family of norms makes $S_b$ a Fréchet space.

The cube $P = \{\xi \in \mathbf{R}^n, 0 \leq \xi_j < 1, j = 1, \ldots, n\}$ is a fundamental domain for the group $\mathbf{Z}^n$.

LEMMA 2.4 (*cf.* [3]). — *The formula*

$$\varphi(x) = \int_P \exp(-2\pi i\, \xi \cdot x)\psi(\xi, x)d\xi, \tag{2.7}$$

*defines for any $b > 0$ an operator $S : \Gamma(\Lambda_b, I) \to S_b$, which is a topological isomorphism, where the sheaf $I$ corresponds to the trivial line bundle $L$. The inverse operator $S^{-1}$ can be written as follows :*

$$\psi(\xi, x) = \sum_{q \in \mathbf{Z}^n} \exp(2\pi i\, \xi \cdot (x + q))\varphi(x + q). \tag{2.8}$$

It follows that for any differential operator $r$ with $n$-periodic coefficients there is a commutative diagram :

$$\begin{array}{ccc} S_b & \xrightarrow{{}^t r} & S_b \\ {\scriptstyle S}\uparrow & & {\scriptstyle S}\uparrow \\ \Gamma(\Lambda_b, I) & \xrightarrow{r'} & \Gamma(\Lambda_b, I) \end{array} \tag{2.9}$$

where the operator $r'$, generated by the family $r'(\zeta) = {}^t r(x, D + 2\pi\zeta)$ as above.

*Proof of Lemma 2.4.* — The integral (2.7) is evidently a bounded function of $x$ and moreover for arbitrary $\eta \in \mathbf{R}^n$, $\|\eta\| < b/2\pi$ we have

$$\varphi(x) = \int_{P+i\eta} \exp(-2\pi i\, \zeta \cdot x)\psi(\zeta, x)d\zeta,$$

because of Cauchy theorem and of (2.3). Hence $\varphi(x) = O(\exp(2\pi\eta \cdot x))$ for $x \to \infty$, which implies (2.6) for $i = 0$. The same conclusion is valid

for any derivative of $\varphi$, hence $\varphi$ is an element of $S_b$ and the operator $S$ is continuous.

For any function $\varphi \in S_b$ its inverse Fourier transform $\hat{\varphi}(\zeta)$ is a holomorphic function in the strip $\Lambda_b$, which decreases as fast as $O(|\zeta|^{-q})$, when $|\zeta| \to \infty$, for any $q$. Hence the inverse Fourier transformation may be written as follows :

$$\varphi(x) = \int_{\mathbf{R}^n} \exp(-2\pi i\, \xi \cdot x)\hat{\varphi}(\xi)d\xi = \sum_{\vartheta \in Z^n} \int_{P+\vartheta} \exp(-2\pi i\, \xi \cdot x)\hat{\varphi}(\xi)d\xi$$

$$= \int_P \exp(-2\pi i\, \xi \cdot x)\psi(\xi, x)d\xi,$$

where

$$\psi(\xi, x) = \sum_{\vartheta \in Z^n} \exp(-2\pi i \vartheta \cdot x)\hat{\varphi}(\xi + \vartheta). \tag{2.10}$$

It is easy to see that $\psi \in \Gamma(\Lambda_b, I)$ and the operator $R : \varphi \rightsquigarrow \psi$ is continuous. This formula means that $R$ is a right inverse to $S$. If we prove that $S$ is injective, Lemma will follow. Suppose that $S\psi = 0$ for an element $\psi \in \Gamma(\Lambda_b, I)$ and develop $\psi$ into a Fourier series on $x$ :

$$\psi(\zeta, x) = \sum_{k \in Z^n} \exp(2\pi i k \cdot x)\psi_k(\zeta).$$

Condition (2.3) implies that $\psi_k(\zeta + \vartheta) = \psi_{k-\vartheta}(\zeta)$, hence $\psi_k(\zeta) = \psi_0(\zeta - k)$. Therefore

$$0 \equiv \int_P \exp(-2\pi i\, \xi \cdot x)\psi(\xi, x)d\xi = \sum_k \int_P \exp(-2\pi i(\xi - k) \cdot x)\psi_0(\xi - k)d\xi$$

$$= \int_{\mathbf{R}^n} \exp(-2\pi i\, \xi \cdot x)\psi_0(\xi)d\xi.$$

It follows that $\psi_0(x) \equiv 0$ and therefore $\psi = 0$, q.e.d.

To find out an inverse formula we start from (2.10) and change the integration variables $y$ to $y + x$ :

$$\psi(\xi, x) = \sum_{\vartheta \in Z^n} \exp(-2\pi i \vartheta \cdot x) \int \exp(2\pi i(\xi + \vartheta) \cdot y)\varphi(y)dy$$

$$= \sum_{\vartheta} \int \exp(2\pi i(\xi \cdot (y + x) + \vartheta \cdot y)\varphi(y + x)dy = \sum \hat{\varphi}_{x\xi}(\vartheta),$$

where $\varphi_{x\xi}(y) := \exp(2\pi i \xi \cdot (y + x))\varphi(y + x)$. The right-hand side is equal to

$$\sum_{q \in Z^n} \varphi_{x\xi}(q) = \sum_q \exp(2\pi i\, \xi \cdot (x + q))\varphi(x + q),$$

since of the Poisson summation formula. The proof is complete.

## 3. Noether operators.

Let $A$ be a commutative algebra, $M$ be a $A$-module. A prime ideal $\mathfrak{p}$ of $A$ is called *associated to* $M$ ([7], [8]), if there exists an element $m \in M$, whose annulet ideal coincides with $\mathfrak{p}$. $M$ is called $\mathfrak{p}$-*coprimary*, if $\mathfrak{p}$ is the only ideal, associated to $M$; we denote it $\mathfrak{p}(M)$. The *Lasker-Noether decomposition* of $M$ is a representation of the zero submodule of $M$ in the form

$$0 = M_1 \cap \ldots \cap M_J, \tag{3.1}$$

where all $M/M_1, \ldots, M/M_J$ are coprimary $A$-modules. This decomposition is called *irreductible*, if no one of modules $M_j$ in (3.1) can be omitted and all the prime ideals $\mathfrak{p}_j = \mathfrak{p}(M/M_j)$, $j = 1, \ldots, J$ are different. If $A$ is a Noetherian algebra, any $A$-module $M$ of finite type admits an irreducible Lasker-Noether decomposition. The set of prime ideals $\{\mathfrak{p}_j, \ldots, \mathfrak{p}_J\}$ is defined uniquely.

Let $K$ be a field and $A$ be a commutative $K$-algebra, $M, N$ be $A$-modules. A $K$-linear mapping $\delta : M \to N$ is called a *differential operator of order* $\leq d$, if $(\operatorname{ad} b)^{d+1}\delta = 0$ for any $b \in A$, where $(\operatorname{ad} b)\gamma := \gamma b - b\gamma$.

DEFINITION 3.1 [9]. — *Let $\mathfrak{p}$ be an ideal associated to a $n$ $A$-module $M$; we call $\nu : M \to [A/\mathfrak{p}]^r$ a $\mathfrak{p}$-Noether operator, if*

i) *$\nu$ is a differential operator in $A$-modules and $r < \infty$,*

ii) *$\operatorname{Ker}\nu$ is a submodule of $M$ and $\mathfrak{p}$ is not associated to $\operatorname{Ker}\nu$.*

*A Noether operator for a module $M$ is a direct sum*

$$\nu = \sum \nu_j : M \to \sum_j [A/\mathfrak{p}_j]^{r(j)},$$

*where $\nu_j$ is a $\mathfrak{p}_j$-Noether operator, $j = 1, \ldots, J$ and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_J\} = \operatorname{Ass}(M)$.*

PROPOSITION 3.1. — *If $A$ is noetherian, then any Noether operator is injective.*

*Proof.* — We have $\operatorname{Ker}\nu = \cap \operatorname{Ker}\nu_j$ and $\operatorname{Ass}(\operatorname{Ker}\nu) \subset \operatorname{Ass}(M)$, since $\operatorname{Ker}\nu$ is a submodule of $M$. No one of ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_J$ is associated to $\operatorname{Ker}\nu$, according to Definition 3.1 ii). Hence $\operatorname{Ass}(\operatorname{Ker}\nu)$ is empty. This means that $\operatorname{Ker}\nu = 0$, because of existence of Lasker-Noether decomposition.

Recall that *analytic algebra* $A$ is a $\mathbf{C}$-algebra, which is isomorphic to a quotient of the algebra $\mathbf{C}\{z_1, \ldots, z_n\}$ for some $n$. It is a Noetherian algebra.

THEOREM 3.2. — *Let $A$ be an analytic algebra, $M, K, L$ be $A$-modules of finite type, $\lambda : M \to L$, $\kappa : M \to K$ be $A$-differential operators such that $\operatorname{Ker}\kappa$ is an $A$-submodule, $L$ is $\mathfrak{p}$-coprimary and $\lambda$ vanishes on $\operatorname{Ker}\kappa$. Then there exist an element $s \in A \setminus \mathfrak{p}$ and a differential operator $\sigma : K \to L$ such that $s\lambda = \sigma\kappa$.*

To check this statement we choose a Noether operator $\nu : K \to N$ for $K$ and apply the Unicity theorem of [9] to the composition $\nu\kappa : M \to N$.

Let $(X, O(X))$ be a complex analytic space, $M$, $N$ are $O(X)$-sheaves; a sheaf morphism $\delta : M \to N$ is called a *differential operator*, if for any point $x \in X$ the fibre morphism $\delta_x : M_x \to N_x$ is a differential operator over algebra $O_x(X)$; $\operatorname{ord}\delta := \sup \operatorname{ord}\delta_x$.

Fix a point $x \in \mathbf{C}^n$ and consider the analytic algebra $A := O_x(\mathbf{C}^n)$ of germs at $x$ of holomorphic functions. Let $G$ be an irreducible germ at $x$ of analytic set in $\mathbf{C}^n$. The ideal $I(G) \subset A$, consisting of function germs, which vanish on $G$, is prime; vice versa, any prime ideal in $A$ is equal to $I(G)$ for some irreducible germ $G$. We call such a germ $G$ *associated to* an $A$-module $M$, if so is the ideal $I(G)$. For example, for any analytic germ $Y$ in $\mathbf{C}^n$ the germs $G_1, \ldots, G_K$ associated to $A$-module $O(Y)$ are all the irreducible components of $Y$.

Now we pass to the global case and operate with closed irreducible analytic sets in $X$ instead of germs. Recall that a closed analytic subset $Y$ in a complex space $X$ is *irreducible*, if there is no proper open and closed analytic subset $Z \subset Y$.

DEFINITION 3.2. — *Let $(X, O(X))$ be a complex space, $M$ be a coherent analytic sheaf on $X$. We call an analytic subset $Y \subset X$ associated to $M$, if*

i) *$Y$ is closed and irreducible,*

ii) *for any point $x \in Y$ its germ $Y_x$ is an union of some irreducible germs $G_1, \ldots, G_K$, $K > 0$ associated to the $O_x(X)$-module $M_x$.*

A collection of all analytic sets associated to $M$ is denoted $\operatorname{Ass}(M)$. If $X$ is a Stein space, for any point $x \in X$ any germ $G$ associated to $M_x$ is a germ of a set $Y \in \operatorname{Ass}(M)$. This fact is contained in the following theorem

for semi-local situation and in [10] for the general case :

THEOREM 3.3 [9]. — *For any coherent sheaf $M$ on a complex space $X$ and any point $x \in X$ there exists a neighbourhood $U$ such that the set $\mathrm{Ass}(M|U)$ is finite and for any $Y \in \mathrm{Ass}(M|U)$ there exists an $O(X)$-differential operator*

$$\nu_Y : M \to \sum O(Y)$$

*where the direct sum is finite such that the operator*

$$\nu := \prod \nu_Y : M \longrightarrow \prod_{\mathrm{Ass}(M)} \sum O(Y)$$

*is a Noether operator for $M_x$ for each $x \in U$.*

DEFINITION 3.3. — *If $Y$ is an analytic set associated to $M$, we call $Y$-Noether operator for $M$ any differential operator $\nu : M \to \sum O(Y)$, where the direct sum is finite, such that for any point $x \in Y$ and any irreducible component $G$ of $Y_x$ the composition $\rho_G$ is a $G$-Noether operator, where $\rho_G : \sum O(Y) \to \sum O(G)$ is the restriction morphism.*

In fact the operators $\nu_Y$ in Theorem 3.2 are Noetherian. Now we prove the following

THEOREM 3.4. — *For any Stein space $X$, arbitrary coherent analytic sheaf $M$ on $X$ and any set $Y \in \mathrm{Ass}(M)$ there exists an $Y$-Noether operator*

$$\nu_Y : M \to \sum O(Y),$$

*which possesses the following property : there exists a holomorphic function $s \not\equiv 0$ on $X$ such that for any element $a \in \Gamma(Y, O(Y))$ there is an $O(Y)$-endomorphism $b$ of $\sum O(Y)$, which satisfies the equation*

$$s(\mathrm{ad}\, a)\nu_Y = b\nu_Y. \tag{3.2}$$

LEMMA 3.5. — *Let $M$ be a coherent analytic sheaf on a Stein space $X$, $Y$ be an irreducible component of* $\mathrm{supp}\, M$ *and $\delta : M \to \sum O(Y)$ be an $O(X)$-differential operator. Suppose that there exists a point $y \in Y$ and an irreducible component $W$ of $Y_y$ such that the composition $\partial := \rho_W \delta_y$ is an $W$-Noether operator with the following property : for any element $a \in O_y(Y)$ there exists an $O_y(Y)$-endomorphism $b$ of $\sum O_y(Y)$ such that*

$$\partial a = b\partial. \tag{3.3}$$

*Then $\delta$ is a $Y$-Noether operator for $M$.*

*Proof of Lemma 3.5.* — First we verify that the sheaf $K = \operatorname{Ker}\delta$ is an $O(X)$-subsheaf of $M$. For this we consider the sheaf $D(Y)$ of differential operators $e : M \to O(Y)$ of order $\leq \operatorname{ord}(\delta)$. It is a coherent sheaf ([9, Prop. 11.3]). Let $\delta_i : M \to O(Y)$, $i = 1, \ldots, r$ be the components of the operator $\delta$ and $I$ be the subsheaf of $D(Y)$, generated by $\delta_1, \ldots, \delta_r$. Take an arbitrary holomorphic function $a$ on $X$ and consider the subsheaf $A$ in $D(Y)$, generated by $I$ and operators $\delta_i a$, $i = 1, \ldots, r$, where $a$ is considered as an endomorphism of $M$. The sheaf $A/I$ is coherent and its support is contained in $Y$. The germ of $\operatorname{supp}(A/I)$ at $y$ does not contain the germ $W$ since of (3.3). Hence $\operatorname{supp}(A/I)$ is a proper analytic subset of $Y$. Choose arbitrary holomorphic function $s$, which belongs to the annulet ideal of $A/I$, but does not vanish identically on $Y$. All the operators $s\delta_i a$ are sections of the sheaf $I$, hence for any $i$ and any point $x \in Y$

$$s\delta_i a = \sum b_j \delta_j$$

with some functions germs $b_j$ at the point $x$. Therefore the equation $\delta(f) = 0$ for $f \in M_x$, $x \in Y$ implies that $s\delta(af) = 0$. This implies the equation $\delta(a'f) = 0$ for any $a' \in O_x(X)$, since functions $a \in \Gamma(X, O(X))$ are dense in $O_x(X)$ in $\mathfrak{m}_x$-adic topology and any differential operator is continuous with respect to this topology. Therefore $K$ is $O(X)$-sheaf.

This sheaf is coherent in virtue of [9, Th. 2]. Hence $\operatorname{supp} K$ is a closed analytic subset of $\operatorname{supp} M$. We need only to check that $\operatorname{supp} K$ does not contain $Y$. Since $Y$ is irreducible, it is sufficient to show that the germ of $\operatorname{supp} K$ at $y$ does not contain $W$. We have $(\operatorname{supp} K)_y = \operatorname{supp} K_y \subset \operatorname{supp} M_y$. At the other hand $\operatorname{supp} K_y$ is the union of all germs $V$, associated to the $O_y(X)$-module $K_y$. It follows from the condition of Lemma that the germ $W$ is not associated to the $O_y(X)$-module $K_y$. There is no other germ $V \supset W$ associated to $\operatorname{supp} K_y$, since $W$ is an irreducible component of $\operatorname{supp} M_y$. Hence $\operatorname{supp} K_y$ does not contain the germ $W$, q.e.d.

*Proof of Theorem 3.4.* — We may assume that $X = \operatorname{supp} M$. Otherwise we shrink $X$ to $\operatorname{supp} M$. Let $X_j$, $j \in J$ be the irreducible components of the space $X$ (*cf.* [11], ch. V]). Each of them is a Stein space and the covering $X = \cup X_j$ is locally finite. Therefore it is sufficient to prove Theorem for each sheaf $M \otimes O(X_j)|X_j$, $j \in J$ and we may suppose that $\operatorname{supp} X$ is an irreducible Stein space.

Fix a point $x \in X$ and an irreducible component $Y$ of the germ $X_x$. Since of Theorem 3.3 there exists a $Y$-Noether operator $\mu : M_x \to \sum O(Y)$.

Consider the coherent sheaf $D(X)$ of germs of differential operators $\delta : M \to O(X)$ of order $\leq \mathrm{ord}(\mu)$. Let $\delta_i$, $i = 1, \dots, r$ be its sections on $X$, which generate this sheaf at $x$. Consider the following operator

$$\delta : M \to [O(X)]^r \ ; \quad \delta(f) = (\delta_1(f), \dots, \delta_r(f)).$$

LEMMA 3.6. — *The composition $\delta_Y := \rho_Y \delta$ is an $Y$-Noether operator.*

*Proof.* — First we check that $\mathrm{Ker}\, \delta_Y$ is an $O_x$-submodule of $M_x$. Take arbitrary element $m \in \mathrm{Ker}\, \delta_x$ and function germ $a \in O_x$. Choose an embedding of the germ $X_x$ in $(\mathbf{C}^n, 0)$ and a function $b$ on the germ $(\mathbf{C}^n, 0)$ such that $\pi(b) = a$, where $\pi : O(\mathbf{C}^n) \to O_x(X)$ is the canonical surjection. Then we can write according to the Leibnitz formula (*cf.* [9, Prop. 3.1])

$$\delta_Y(am) = \sum (i!)^{-1} \pi(D_z^i b)(\mathrm{ad}\, z)^i \delta_Y(m)$$

where $z = (z_1, \dots, z_n)$ are coordinates on $\mathbf{C}^n$ and $(\mathrm{ad}\, z)^i$ means

$$(\mathrm{ad}\, z_1)^{i_1} \cdot \dots \cdot (\mathrm{ad}\, z_n)^{i_n}.$$

We have $(\mathrm{ad}\, z)^i \delta_Y = r_Y (\mathrm{ad}\, z)^i \delta$ and $(\mathrm{ad}\, z)^i \delta : M \to \sum O(X)$ is a differential operator of order $\leq \mathrm{ord}(\delta) \leq \mathrm{ord}(\mu)$. Hence $(\mathrm{ad}\, z)^i \delta_Y(m) = 0$ for each $i$. This implies that $\delta_Y(am) = 0$ and our assertion follows.

We have the inclusion $\mathrm{Ker}\, \mu \supset \mathrm{Ker}\, \delta_Y$, which follows from the fact that any component of $\mu$ belongs to $O_x(X)$-envelope of the set $\{\delta_i, i = 1, \dots, r\}$. This inclusion implies that the germ $Y$ is not associated to the $\mathrm{Ker}\, \delta_Y$, hence $\delta_Y$ is a $Y$-Noether operator for the module $M_x$.

Lemma 3.6 implies that $\delta$ is a $X$-Noether operator for $M$. Set $K := \mathrm{Ker}\, \delta$; this is a coherent subsheaf of $M$ and for any point $x$, $O_x(X)$-module $K_x$ has no associated germs $Z$, $\dim Z = \dim X$. Now we argue using the induction on the number $\dim \mathrm{supp}\, F$, hence may suppose that Theorem 3.4 is true for the sheaf $K$.

LEMMA 3.7. — *The equation* $\mathrm{Ass}(M) = \{X\} \cup \mathrm{Ass}(K)$ *holds.*

*Proof.* — One has the trivial inclusion $\mathrm{Ass}(K) \subset \mathrm{Ass}(M)$. For any point $x \in X$ any component of the germ $X_x$ belongs to $\mathrm{Ass}(M_x)$, since $\delta$ is a $X$-Noether operator for $M$. Hence it remains to check that for any point $x \in X$ any germ $Y \in \mathrm{Ass}(M_x)$, which is not a component of the germ $X_x$, belongs to $\mathrm{Ass}(K_x)$. Choose an element $m \in M_x$, whose annulet ideal is equal to $I(Y)$. We claim that $m \in K_x$. In fact the equation $am = 0$ implies

in view of Leibnitz formula that $a^k\delta(m) = 0$, for $k = \mathrm{ord}(\delta) + 1$. It follows that $\delta(m) = 0$, since $a^k$ is not a zero-divisor in $O_x(X)$. This means that $m \in K_x$, which implies that $Y \in \mathrm{Ass}(K_x)$, q.e.d.

To prove Theorem 3.4 we choose for any $Y \in \mathrm{Ass}(M)$ a regular point $y \in Y$ and a $Y_y$-Noether operator $\nu_Y$ for $M$. Consider the sheaf $D(Y)$ of all differential operators $M \to O(Y)$ of order $\leq k := \mathrm{ord}(\nu_Y)$. It is a coherent sheaf on Stein space $X$. Choose a finite set $\{\varepsilon_1, \ldots \varepsilon_r\} \subset \Gamma(X, D(Y))$, which $O_y(Y)$-envelope is equal to the stalk $D(Y)_y$. Consider the differential operator $\varepsilon(Y) : M \to \sum O(Y)$ with the components $\varepsilon_1, \ldots, \varepsilon_r$, and the sheaf $G = \mathrm{Ker}\,\varepsilon(Y)$. The stalk $G_y$ is an $O_y(Y)$-module of finite type and the set $\mathrm{Ass}(G_y)$ does not contain the germ $Y_y$. This can be proved by the arguments of Lemma 3.6. Hence the germ of $\varepsilon(Y)$ at $y$ is a $Y_y$-Noether operator, satisfying (3.3). Lemma 3.5 implies that $\varepsilon(Y)$ is a $Y$-Noether operator for $M$.

To check the property (3.2) we choose an arbitrary function $a \in \Gamma(Y, O(Y))$ and for any $i = 1, \ldots, r$ consider the operator $\varepsilon_i a : M \to O(Y)$. It belongs to the $O_y$-envelope of the operators $\varepsilon_1, \ldots, \varepsilon_r$ since of the construction. It follows that there exists a function $s \in \Gamma(Y, O(Y))$ such that the operator $s\varepsilon_i a$ belongs to $\Gamma(Y, O(Y))$-envelope of $\varepsilon_1, \ldots, \varepsilon_r$ (see proof of Lemma 3.5). Therefore the operator $s(\mathrm{ad}\, a)\varepsilon_i = s\varepsilon_i a - sa\varepsilon_i$ belongs to this envelope as well. This prove (3.2).

THEOREM 3.8. — *Let $M$ be a coherent sheaf on a Stein space $X$, $\nu_N$ for each $N \in \mathrm{Ass}(M)$ be a $N$-Noether operator for $M$ and $S(N)$ be an arbitrary proper closed analytic subset of $N$ such that* $\mathrm{sing}\, N \subset S(N)$. *Then for any open set $U \subset X$ the linear operator*

$$\nu := \prod \nu_N : \Gamma(U, M) \longrightarrow \prod \{ \sum \Gamma(U \cap N \setminus S(N), O(N)),\ N \in \mathrm{Ass}(M)\}$$

*is an isomorphism onto its image, when the second space is endowed with the topology induced from the distribution spaces $D'(U \cap N \setminus S(N))$.*

*Proof.* — Firstly we suppose that $U$ is a closed subspace of the open unit polydisc $\Delta$ in a coordinate space $\mathbf{C}^n$ and there is a morphism $\alpha : K \to L$ of free coherent $O(\mathbf{C}^n)$-sheaves on $\Delta$ such that $\mathrm{Cok}\,\alpha \cong M$. This implies the following exact sequence

$$\Gamma(\Delta, K) \xrightarrow{\alpha} \Gamma(\Delta, L) \xrightarrow{\pi} \Gamma(\Delta, M) \longrightarrow 0,$$

where the canonical surjection $\pi$ is an open operator. For any $N \in \mathrm{Ass}(M)$ the composition

$$\nu_N \pi : \Gamma(\Delta, L) \longrightarrow \prod_N \sum \Gamma(\Delta \cap N, O(N))$$

is a differential operator on $\Delta$. It may be written in the following explicit form :

$$\nu_N \pi(u) = \sum a_i D^i u,$$

where $a_i \in \Gamma(\Delta, L')$, $L' := \mathrm{Hom}(L, \sum O(N))$ and $a_i = 0$ if $|i| > \deg \nu_N$ (*cf.* [9]). Each operator $D^i$ acting on $\mathbf{C}^n$ is continuous since of the Cauchy inequality. Therefore $\nu_N \pi$ is continuous. The same is true for $\nu_N$, because $\pi$ is open.

In the general case the topology of $\Gamma(U, M)$ is the supremum of topologies, induced from the spaces $\Gamma(\Delta, \varphi_*(M|Y))$, where $(Y, \varphi)$ runs over a set of analytic polyhedrons, which covers $U$, and $\varphi : Y \to \Delta$ is a closed embedding. Hence the general case is reduced to the case $X = \Delta$, $M \cong \mathrm{Cok}\, \alpha$. It is obvious that $\nu$ is still continuous in this case. To prove the openness of $\nu$ we use

LEMMA 3.9. — *For any point $z \in \Delta$ and its neighbourhood $U \subset\subset \Delta$ there exist a neighbourhood $V \subset U$ of $z$, a neighbourhood $W(N)$ of $S(N)$ and a constant $C$ such that for any $f \in \Gamma(U, L)$ there exists a section $g \in \Gamma(V, K)$, which satisfies the inequality*

$$\begin{aligned} &\sup(|f + \alpha g|,\ z \in V) \\ &\leq C \max\big\{ \sup(|\nu_N f(z)|,\ z \in U \cap N \setminus W(N)),\ N \in \mathrm{Ass}(M)\big\}. \end{aligned} \tag{3.4}$$

The maximum in the right-hand side is well-defined since $U \cup N$ is empty, except for a finite subset of $\mathrm{Ass}(M)$. Lemma 3.9 implies Theorem 3.8, since we can choose a polyhedral covering for $X$, consisting of neighbourhoods $V$, which satisfy (3.4) and the sup-norm in the right-hand side is majorized by the topology induced from the distribution space $D'(U \cap N \setminus S(N))$.

*Proof of Lemma 3.9.* — In fact it is proved in [1, ch. IV] for a special Noether operator $\lambda$, $\lambda_N : M \to \sum O(N)$. We have for any $N$

$$\lambda_N = s^{-1} \sum \sigma_{N\Lambda} \nu_\Lambda\,, \tag{3.5}$$

where according to Theorem 3.2 $\sigma_{N\Lambda} : \sum O(\Lambda) \to \sum O(N)$ is a differential operator in a neighbourhood of $z$ and $\bar{s} \in O(X_z) \setminus I(N_z)$. Note that $\sigma_{N\Lambda} \neq 0$ only if $N \subset \Lambda$, since $\sigma_{N\Lambda}$ is a differential operator. Applying [1], we get the estimate

$$\begin{aligned} &\sup(|f + \alpha g|,\ z \in V) \\ &\leq C \max\big\{ \sup(|\lambda_N f|,\ z \in U' \cap N \setminus W(N)),\ N \in \mathrm{Ass}(M)\big\} \end{aligned} \tag{3.6}$$

for a section $g$ of the sheaf $K$, some neighbourhoods $V$, $U' \subset\subset U$ of $z$. We may assume that the set $S(N)$ contains $s^{-1}(0)$. Then we need to prove the inequality

$$\begin{aligned} &\sup\{|\lambda_N f|,\ z \in U' \cap N \setminus W(N)\} \\ &\le C' \max\{\sup(|\nu_\Lambda f|,\ z \in U \cap \Lambda \setminus W(\Lambda),\ \Lambda \supset N\}. \end{aligned} \tag{3.7}$$

Combining it with (3.6), we get (3.4). To prove (3.7) we use (3.5), the inequality $|s^{-1}| \le \text{const}$ on the set $U \cap N \setminus W(N)$ and the estimate

$$\sup\{|\sigma_{N\Lambda} h|,\ z \in U' \cap N \setminus W(N)\} \le C \sup\{|h|,\ z \in U \cap \Lambda \setminus W(\Lambda)\}$$

for any holomorphic function $h$ on $U \cap \Lambda$. To check this estimate we apply the Cauchy inequality. Lemma 3.9 and hence Theorem 3.8 are proved.

## 4. End of the proof of Theorem 1.1.

Now we apply Theorem 3.4 to the sheaf $M := H_0 \equiv \operatorname{Cok} p_0'$, denoting $N_1, N_2, \ldots$ all the elements of $\operatorname{Ass}(M)$. Thus for any $k$ there exists a $N_k$-Noether operator

$$\nu_k : \Gamma(\Lambda, M) \longrightarrow \sum \Gamma(N_k, O(N_k)),$$

possessing the property (3.2). Moreover Theorem 3.8 implies that the continuous operator

$$\nu = \prod \nu_k : \Gamma(\Lambda_b, M) \longrightarrow \prod_k \sum \Gamma(\Lambda_b \cap N_k, O(N_k))$$

is an open mapping onto its image, when the first space is equipped with the canonical topology and the second one is endowed with the topology induced from $\prod \sum D'(\Lambda_b \cap N_k \setminus \Omega_k)$, where $D'(\cdot)$ means the space of distributions and $\Omega_k$ is any proper closed analytic subset of $N_k$ such that $\operatorname{sing} N_k \subset \Omega_k$. We may assume that for any $k$ this set satisfies the condition : $s \not\equiv 0$ on $N_k \setminus \Omega_k$, where $s$ is holomorphic function, which appears in (3.2). Combining this mapping with the morphism $\pi : \Gamma(\Lambda_a, I_0) \to \Gamma(\Lambda_a, M)$, we get for any $a > 0$ the complex

$$\Gamma(\Lambda_b, I_1) \xrightarrow{p_0'} \Gamma(\Lambda_b, I_0) \xrightarrow{\nu\pi} \prod_k \sum \Gamma(\Lambda_b \cap N_k, O(N_k)). \tag{4.1}$$

It is exact, since $\operatorname{Ker} \pi = \operatorname{Im} p_0'$, because of Lemma 2.3 and of Proposition 3.1. The composition $\nu\pi$ is an open operator onto its image, since $\pi$ is open by the definition of the topology of $\Gamma(\Lambda_a, M)$ and $\nu$ is open, because of the aforesaid.

Now take an arbitrary solution $u$ of (0.1), which satisfies (0.2). It may be considered as a functional on $S_b$ for arbitrary $b > a$. This functional vanishes on $\operatorname{Im} {}^t p$. Let $S^*$ be the adjoint to the operator $S$ (see Lemma 2.4). Then $S^*(u)$ is a continuous functional on $\Gamma(\Lambda_b, I_0)$, which vanishes on the subspace $\operatorname{Im} p_0' = \operatorname{Ker} \nu\pi$, because of (2.9). Consider the operator

$$\rho : \Gamma(\Lambda_b, I_0)/\operatorname{Ker} \nu\pi \longrightarrow \operatorname{Im} \nu\pi,$$

generated by $\nu\pi$. It is a topological isomorphism, since $\nu\pi$ is open, hence we may consider a continuous functional $v := (\rho^{-1})^* S^*(u)$ on $\operatorname{Im} \nu\pi$. Applying Hahn-Banach theorem, we take a continuous extension $w$ of $v$ to the space $\prod \sum \Gamma(\Lambda_b \cap N_k, O(N_k))$. It can be written as a finite sum

$$w = \sum_k \sum_{j=1}^{r(k)} w_{kj},$$

where $w_{kj}$ is a continuous functional on $\Gamma(\Lambda_b \cap N_k, O(N_k))$. Then we use Hahn-Banach theorem once more to extend $w_{kj}$ to a continuous functional $\tilde{w}_{kj}$ on the space $D'(\Lambda_b \cap N_k \setminus \Omega_k)$ and write it as an integral

$$\tilde{w}_{kj}(f) = \int f \mu_{kj}$$

with a smooth density $\mu_{kj}$ such that $\operatorname{supp} \mu_{kj} \subset \Lambda_b \cap N_k \setminus \Omega_k$. Hence

$$u(\varphi) = v(\psi) = w(\nu\pi(\psi)) = \sum_{k,j} \int \nu_{kj} \pi(\psi) \mu_{kj}, \tag{4.2}$$

where $\varphi \in S_b$, $\psi := S^{-1}(\varphi)$ and

$$\nu_{kj} : \Gamma(\Lambda, M) \longrightarrow \Gamma(N_k, O(N_k)), \quad j = 1, \dots, r(k)$$

are components of $\nu_k$. The equality (4.2) coincides with (1.2) if we set

$$f_{kj}(\lambda, \varphi) := \delta_\lambda \delta_{kj}(\psi), \quad \delta_{kj} := \nu_{kj}\pi : \Gamma(\Lambda, I_0) \longrightarrow \Gamma(N_k, O(N_k)),$$

where $f_{kj}(\lambda, \varphi)$ means the value of the distribution $f_{kj}(\lambda, \cdot)$ on a test function $\varphi$ and $\delta_\lambda$ denotes the delta-distribution supported by the point $\lambda \in N_k$. The distribution $f_{kj}(\lambda, \cdot)$ satisfies (0.1) since it may be written in the form (4.2) with $\mu_{kj} := \delta_\lambda$. This a smooth function on $x \in \mathbf{R}^n$, since the equation (0.1) is elliptic. This solution is weakly holomorphic on $\zeta$ and therefore is a smooth function on $N_k \times \mathbf{R}^n$. This implies ii). Properties i) and iv) were proved earlier.

To check iii) we choose arbitrary $q \in \mathbf{Z}^n$ and compute for arbitrary $k$ and $j$

$$T_q f_{kj}(\lambda, \varphi) = f_{kj}(\lambda, T_{-q}(\varphi)) = \delta_\lambda \delta_{kj}(e_q \psi), \quad e_q(\lambda) := \exp(2\pi i \lambda \cdot q),$$

$$\delta_{kj}(e_q\psi) = e_q\delta_{kj}(\psi) + (\operatorname{ad} e_q)\delta_{kj}(\psi).$$

The operator $\gamma := s(\operatorname{ad} e_q)\delta_{kj}$ belongs to the linear span of operators $\delta_{ki}$, $i = 1, \dots, r(k)$ over algebra $\Gamma(\Lambda, O(\Lambda))$ since of (3.2) and $\operatorname{ord}\gamma < \operatorname{ord}\delta_{kj}$. We have

$$s(\lambda)\delta_\lambda\delta_{kj}(e_q \cdot \psi) = s(\lambda)e_q(\lambda)\delta_\lambda\delta_{kj}(\psi) + \delta_\lambda\gamma(\psi),$$

therefore for any $\lambda \in N_k \setminus \Omega_k$ whe have

$$T_q f_{kj}(\lambda, \varphi) = e_q(\lambda)f_{kj}(\lambda, \varphi) + s(\lambda)^{-1}\delta_\lambda\gamma(\psi),$$

hence

$$\big[T_q - \exp(2\pi i\lambda \cdot q)\big] f_{kj}(\lambda, \cdot) = g(\cdot),$$

where $g$ is an element of the linear span of functions $f_{ki}(\lambda, \cdot)$, $i = 1, \dots, r(k)$. Applying this computation to $g$ and so on, we come to (1.3), which implies iii). The proof is complete.

## 5. Approximation.

THEOREM 5.1. — *Suppose that a set $\Phi \subset \Lambda$ has a non-empty intersection with $N_k$ for each $k$. Then the set of Floquet solutions of (0.1) with quasi-impulses $\lambda \in \Phi$ is total in the space of all solutions, which satisfies (0.2) for some $a > 0$.*

*Remark.* — The similar result for differential equations with constant coefficients is due to Malgrange [12].

*Proof.* — The statement is equivalent to the following : for $\varphi \in \Gamma(\Lambda_a, I_0)$ the system of equations

$$\gamma_\lambda(\varphi) = 0, \qquad \lambda \in \Phi, \tag{5.1}$$

implies that $\varphi \in \operatorname{Im} p_0'$, if $\gamma_\lambda$ runs over the set of linear functionals over $\Gamma(\Lambda, I_0)$ supported at $\lambda$, which vanish on the image of the operator $p_0'$ in (4.1). To prove this implication we note that for any $k$, any $\lambda \in \Phi \cap N_k$ and any functional $\delta$ over $\sum\Gamma(N_k, O(N_k))$ supported at $\lambda$, the functional $\gamma_\lambda := \delta\nu_k\pi$ vanishes on $\operatorname{Im} p_0'$. Hence the system (5.1) implies the equation $\delta(\nu_k\pi\varphi) = 0$ for any $\delta$. This means that the image of $\nu_k\pi(\varphi)$ in $\sum\widehat{O}_\lambda(N_k)$ vanishes, where the symbol $\widehat{\ }$ denote the completion in $\mathfrak{m}_\lambda$-adic topology. If follows that the germ at $\lambda$ of the function $\nu_k\pi(\varphi)$ is equal to zero, since the canonical mapping $F_\lambda \to \widehat{F}_\lambda$ is an injection for any coherent sheaf $F$.

This implies the equalities $\nu_k \pi(\varphi) = 0$, $k = 1, \ldots$, since $N_k$ is irreducible for any $k$. Therefore $\varphi \in \operatorname{Im} p_0'$ because (4.1) is exact and Theorem 5.1 follows.

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