ANNALES DE L'INSTITUT FOURIER

ANDRZEJ BIALYNICKI-BIRULA

On induced actions of algebraic groups

Annales de l'institut Fourier, tome 43, n° 2 (1993), p. 365-368

<http://www.numdam.org/item?id=AIF_1993__43_2_365_0>

© Annales de l'institut Fourier, 1993, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
43, 2 (1993), 365-368

ON INDUCED ACTIONS OF ALGEBRAIC GROUPS

by Andrzej BIALYNICKI-BIRULA(*)

Let H be a subgroup of an algebraic group G. Let Y be an algebraic space with an action of H (shortly an algebraic H-space). The aim of this note is to study some properties of $G \times_H Y$, defined as a quotient of $G \times Y$ by the action of H determined by $h(g,y) = (gh^{-1}, hy)$, for all $h \in H$, $g \in G$ and $y \in Y$. Left translations by elements of G on G determine an action of G on $G \times_H H$. The importance of the space $G \times_H Y$ follows from the fact that the map $Y \to G \times_H H$, which to $y \in Y$ attaches the image of $(1,y)$ in $G \times_H H$ solves the universal problem of H-equivariant morphisms of Y into G-spaces. In analogy with the theory of modules and representations we can say that the space $G \times_H Y$ is induced from the H-space Y by the group extension $H \subset G$ and that the action of G on $G \times_H Y$ is induced by the action of H on Y. In applications the notion is used for constructing a space with an action of G, when a space with an action of its subgroup H is given. Properties of $G \times_H Y$ in the case where Y is quasi-projective were studied in the classical paper [Se]. Though results presented here are perhaps predictable or even known, we hope that the paper will be useful as a reference.

In order to make our arguments more lucid, we are going to start with considering more general situations.

1. Let X and Y be two algebraic H-spaces. Then $X \times_H Y$ is defined as a quotient of the product $X \times Y$ by the action of H defined by $h(x,y) = (hx, hy)$, for $h \in H$. In general, neither the meaning of the notion

(*) supported by Polish KBN Grant GR-87.
Key words : Principal fiber bundles - Actions of algebraic groups - Algebraic spaces.
A.M.S. Classification : 14L30.

of quotient, nor its existence (when the meaning of the quotient has been already fixed) is clear. In the note we consider only the case when H is affine and X is a principal locally isotrivial H-fibration in the category of algebraic spaces. In this case we require the quotient $X \times_H Y$ to be an algebraic space, and the map $X \times Y \to X \times_H Y$ to be affine and a geometric quotient in the sense of [GIT]. If $X = G$, where G is an affine algebraic group containing H as its subgroup with an action of H by right translations, then by [Se] the above assumptions concerning H and X are satisfied.

THEOREM 1. — *Let H, X, Y be as above. Moreover assume that X is normal. Then $S \times_H Y$ exists in the category of algebraic spaces. If, moreover, X is an algebraic variety, Y is normal and can be covered by H-invariant open quasi-projective subsets, then $X \times_H Y$ is an algebraic variety.*

Proof. — The theorem will be proved in several steps.

1st step. Assume that the H-fibration on X is trivial *i.e.* $X = H \times U$, where U is an algebraic space. Then $X \times Y = H \times U \times Y \to U \times Y$ defined by $(h, x, y) \mapsto (x, h^{-1}y)$ satisfies desired conditions. Thus $X \times_H Y = U \times Y$.

2nd step. Assume that the H-fibration on X is isotrivial with the base space U. Since X is normal, U is normal and then there exists a Galois ramified cover $Z \to U$ such that $X \times_U Z$ is trivial. It follows from the 1st step that there exists $(X \times_U Z) \times_H Y$ and by Deligne's theorem [K] p. 183-4 there exists its quotient by the action of the Galois group (induced by the action on Z) in the category of algebraic spaces. The quotient can be identified with $X \times_H Y$. If moreover Y is normal quasi-projective and U is affine, then $(X \times_U Z) \times_H Y$ is normal quasi-projective. Because the quotient of a normal quasi-projective variety by an action of a finite group is quasi-projective, hence $X \times_H Y$ is also quasi-projective.

3rd step. Assume that X and Y are covered by H-invariant open subsets $\{U_i, i \in I\}$, $\{V_j, j \in J\}$, such that $U_i \times_H V_j$ exist, for all $i \in I$ and $j \in J$ (in the category of algebraic spaces). Then $X \times_H Y$ also exists (in the same category) and $\{U_i \times_H V_j\}$ form an open covering of $X \times_H Y$. Moreover if $U_i \times_H V_j$ are quasi-projective, then $X \times_H Y$ is an algebraic variety. Proof of this step is obvious.

4th step. Now we consider the general case. Notice first that the base space of the H-fibration given on X can be covered by open subsets $\{W_k\}$,

$k \in K$, such that for every $k \in K$, the inverse image U_k of W_k in X, as a principal H-fibration, is isotrivial. Then it follows from the 2nd step that, for every $k \in K$, $U_k \times_H Y$ exists and from the 3rd step that $X \times_H Y$ exists in the category of algebraic spaces. Moreover, if Y can be covered by H-invariant open quasi-projective subsets V_j, where $j \in J$, then by the second part of the 3rd step, we infer that $X \times_H Y$ is an algebraic variety.□

COROLLARY 2. — *Let Y be an algebraic space with an action of an algebraic group H and let G be an affine algebraic group containing H as its subgroup. Then $G \times_H Y$ is an algebraic space with an action of G induced by left translations on G. Moreover, if Y can be covered by H-invariant quasi-projective open subsets, then $G \times_H Y$ is an algebraic variety.*

THEOREM 3. — *Let G be a connected affine algebraic group and let H be its subgroup. Let Y be a normal algebraic space with an action of H. Then $G \times_H Y$ is an algebraic variety if and only if Y can be covered by H-invariant quasi-projective open subsets.*

Proof. — It follows from Corollary 2 that, if Y can be covered by H-invariant open quasi-projective subsets, then $G \times_H Y$ is an algebraic variety. Let us assume now that $G \times_H Y$ is an algebraic variety. Since G is connected, $G \times_H Y$ by Sumihiro Theorem [Su] can be covered by G-invariant open quasi-projective subsets. Intersecting these subsets with $H \times_H Y \subseteq G \times_H Y$ we obtain an H-invariant quasi-projective open covering of $H \times_H Y$. Since $Y \simeq H \times_H Y$ we obtain that Y can be covered by open quasi-projective H-invariant subsets. □

It follows from the above results and Sumihiro Theorem that whenever H is connected, any induced G-space from an algebraic normal H-variety is also an algebraic (normal) variety. However in case where H is a finite subgroup of a connected algebraic group and Y is an algebraic H-variety which can not be covered by H-invariant open quasi-projective open subsets, then the induced algebraic G-space is not an algebraic variety. For example, if two element group Z_2 acts on a normal algebraic variety Y in such a way that a Z_2-orbit is not contained in any affine open subset (see [H] or Chap. 4§ 3 in [GIT] for an example), then for any connected affine group G containing E_2 as a subgroup, $G \times_{Z_2} Y$ is an algebraic space but not an algebraic variety.

BIBLIOGRAPHY

[H] H. HIRONAKA, An example of a non-kahlerian deformation, Ann. of Math., 75 (1962), 190–208.

[K] D. KNUTSON, Algebraic spaces, Lecture Notes in Mathematics, 203, Springer-Verlag, 1971.

[GIT] D. MUMFORD, J. FOGARTY, Geometric Invariant Theory, 2nd edition, Ergeb. Math. 36, Springer-Verlag, 1982.

[Se] J.-P. SERRE, Espaces fibrés algébriques in Anneaux de Chow et Applications, Séminaire Chevalley, E.N.S. Paris, 1958.

[Su] H. SUMIHIRO, Equivariant completions I, J. Math. Kyoto Univ., 14 (1974), 1–28.

Manuscrit reçu le 23 avril 1992.

Andrzej BIALYNICKI-BIRULA,
Instytut Matematyki UW
Banacha 2
02097 Warszawa (Pologne).