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POLYNOMIAL GROWTH AND SUBGROUPS OF $\mathrm{Out}(F_N)$

CROISSANCE POLYNOMIALE ET SOUS-GROUPES DE $\mathrm{Out}(F_N)$

Abstract. — This paper, which is the last of a series of three papers, studies dynamical properties of elements of $\mathrm{Out}(F_N)$, the outer automorphism group of a nonabelian free group F_N. We prove that, for every subgroup H of $\mathrm{Out}(F_N)$, there exists an element $\phi \in H$ such that, for every element g of F_N, the conjugacy class $[g]$ has polynomial growth under iteration of ϕ if and only if $[g]$ has polynomial growth under iteration of every element of H.

Résumé. — Dans cet article, nous étudions des propriétés dynamiques des éléments de $\mathrm{Out}(F_N)$, le groupe des automorphismes extérieurs d'un groupe non abélien libre F_N de rang $N \geqslant 2$. Nous montrons que, pour tout sous-groupe H de $\mathrm{Out}(F_N)$, il existe un élément $\phi \in H$, appelé *dynamiquement générique*, qui capture la croissance polynomiale de H au sens suivant. La classe de conjugaison d'un élément $g \in F_N$ est à croissance polynomiale sous itération de tous les éléments de H si, et seulement si, la classe de conjugaison de g est à croissance polynomiale sous itération de ϕ.

1. Introduction

Let $N \geqslant 2$. This paper, which is the last of a series of three papers [Gue21, Gue22b], studies the exponential growth of elements in $\mathrm{Out}(F_N)$. An outer automorphism

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$\phi \in \mathrm{Out}(F_N)$ is *exponentially growing* if there exist a conjugacy class $[g] \subseteq F_N$, a free basis $\mathfrak{B}$ of F_N and a constant $K > 0$ such that, for every $m \in \mathbb{N}^*$, we have

$$\ell_{\mathfrak{B}}(\phi^m([g])) \geqslant e^{Km}, \tag{1.1}$$

where $\ell_{\mathfrak{B}}(\phi^m([g]))$ denotes the length of a cyclically reduced representative of $\phi^m([g])$ in the basis $\mathfrak{B}$.

If $g \in F_N$ satisfies Equation (1.1), then g is said to be *exponentially growing under iteration of* ϕ. Otherwise, one can show, using for instance the technology of relative train tracks introduced by Bestvina and Handel [BH92], that g has *polynomial growth under iteration of* ϕ, replacing $\geqslant e^{Km}$ by $\leqslant (m+1)^K$ in Equation (1.1) (see also [Lev09] for a complete description of all growth types that can occur under iteration of an outer automorphism ϕ).

We denote by $\mathrm{Poly}(\phi)$ the set of elements of F_N which have polynomial growth under iteration of ϕ. If H is a subgroup of F_N, we set $\mathrm{Poly}(H) = \bigcap_{\phi \in H} \mathrm{Poly}(\phi)$. Note that $\mathrm{Poly}(\phi)$ and $\mathrm{Poly}(H)$ are invariant under conjugation. In this article, we prove the following theorem.

Theorem 1.1. — *Let $N \geqslant 2$ and let H be a subgroup of* $\mathrm{Out}(F_N)$*. There exists $\phi \in H$ such that* $\mathrm{Poly}(\phi) = \mathrm{Poly}(H)$.

In other words, there exists an element of H which encaptures all the exponential growth of H: there exists $\phi \in H$ such that if $g \in F_N$ has exponential growth for some element of H, then g has exponential growth for ϕ.

Theorem 1.1 has analogues in other contexts. For instance, one has a similar result in the context of the mapping class group of a closed, connected, orientable surface S equipped with a hyperbolic structure. Indeed, a consequence of the Nielsen–Thurston classification (see for instance [FM11, Theorem 13.2]) and the work of Thurston [FLP79, Proposition 9.21] is that the growth of the length of the geodesic representative of the homotopy class of an essential closed curve under iteration of an element of $\mathrm{Mod}(S)$ is either exponential or linear. Moreover, linear growth comes from twists about essential curves while exponential growth comes from pseudo-Anosov homeomorphisms of subsurfaces of S.

In [Iva92] (see also the work of McCarthy [McC85]), Ivanov proved that, for every subgroup H of $\mathrm{Mod}(S)$, up to taking a finite index subgroup of H, there exist finitely many homotopy classes of pairwise disjoint essential closed curves $C_1, \ldots, C_k$ elementwise fixed by H and such that, for every connected component S' of $S - \bigcup_{i=1}^k C_i$, the restriction $H|_{S'} \subseteq \mathrm{Mod}(S')$ is either the trivial group or contains a pseudo-Anosov element. One can then construct an element $f \in H$ such that the element $f|_{S'} \in \mathrm{Mod}(S')$ is a pseudo-Anosov whenever $H|_{S'} \subseteq \mathrm{Mod}(S')$ contains a pseudo-Anosov element.

In the context of $\mathrm{Out}(F_N)$, Clay and Uyanik [CU20] proved Theorem 1.1 when H is a subgroup of $\mathrm{Out}(F_N)$ such that $\mathrm{Poly}(H) = \{1\}$. Indeed, by a result of Levitt [Lev09, Proposition 1.4, Lemma 1.5], if $\phi \in \mathrm{Out}(F_N)$ and if $\mathrm{Poly}(\phi) \neq \{1\}$, there exist a nontrivial element $g \in F_N$ and $k \in \mathbb{N}^*$ such that $\phi^k([g]) = [g]$. In this context, Clay and Uyanik proved that, if H does not virtually preserve the conjugacy class of a nontrivial element of F_N, there exists an element $\phi \in H$ which is *atoroidal*: no power of ϕ fixes the conjugacy class of a nontrivial element of F_N.

Proof. — We now sketch the proof of Theorem 1.1. It is inspired by the proof of [CU20, Theorem A]. However, technical difficulties emerge due to the presence of elements of F_N with polynomial growth under iteration of elements of the considered subgroup of $\mathrm{Out}(F_N)$. The main difficulties are dealt with in the second article of the series [Gue22b]. Let H be a subgroup of $\mathrm{Out}(F_N)$. We first consider H-invariant *free factor systems* $\mathcal{F}$ of F_N, that is, $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$, where, for every $i \in \{1 \ldots, k\}$, $[A_i]$ is the conjugacy class of a subgroup A_i of F_N and there exists a subgroup B of F_N such that $F_N = A_1 * \ldots * A_k * B$. There exists a partial order on the set of free factor systems of F_N, where $\mathcal{F}_1 \leqslant \mathcal{F}_2$ if for every free factor A_1 of F_N such that $[A_1] \in \mathcal{F}_1$, there exists a free factor A_2 of F_N such that $[A_2] \in \mathcal{F}_2$ and A_1 is a subgroup of A_2. Hence we may consider a maximal H-invariant sequence of free factor systems

$$\varnothing = \mathcal{F}_0 \leqslant \mathcal{F}_1 \leqslant \ldots \leqslant \mathcal{F}_k = \{[F_N]\}.$$

The proof is now by induction on $i \in \{1, \ldots, k\}$: for every $i \in \{0 \ldots, k\}$, we construct an element $\phi_i \in H$ such that $\mathrm{Poly}(\phi_i|_{\mathcal{F}_i}) = \mathrm{Poly}(H|_{\mathcal{F}_i})$ (we define the meaning of the restrictions in Section 2.3). Let $i \in \{1, \ldots, k\}$ and suppose that we have constructed ϕ_{i-1}. There are two cases to consider. If the extension $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$ is *nonsporadic* (see the definition in Section 2.1) then the construction of ϕ_i from ϕ_{i-1} follows from the works of Handel–Mosher [HM20], Guirardel–Horbez [GH22] and Clay–Uyanik [CU18].

If the extension $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$ is *sporadic*, the construction of ϕ_i relies on the action of H on some natural (compact, metrizable) space that we introduced in [Gue21]. This space is called the *space of currents relative to* $\mathrm{Poly}(H|_{\mathcal{F}_{i-1}})$ and it is denoted by $\mathbb{P}\mathrm{Curr}(F_N, \mathrm{Poly}(H|_{\mathcal{F}_{i-1}}))$. It is defined as a subspace of the space of Radon measures on a natural space $\partial^2(F_N, \mathrm{Poly}(H|_{\mathcal{F}_{i-1}}))$, the double boundary of F_N relative to $\mathrm{Poly}(H|_{\mathcal{F}_{i-1}})$ (see Section 2.2 for precise definitions).

In [Gue22b], we proved that the element ϕ_{i-1} that we have constructed acts with a *North-South dynamics* on the space of relative currents $\mathbb{P}\mathrm{Curr}(F_N, \mathrm{Poly}(H|_{\mathcal{F}_{i-1}}))$: there exist two proper disjoint closed subsets of $\mathbb{P}\mathrm{Curr}(F_N, \mathrm{Poly}(H|_{\mathcal{F}_{i-1}}))$ such that every point of $\mathbb{P}\mathrm{Curr}(F_N, \mathrm{Poly}(H|_{\mathcal{F}_{i-1}}))$ which is not contained in these subsets converges to one of the two subsets under positive or negative iteration of ϕ_{i-1}. This North-South dynamics result allows us, applying classical ping-pong arguments similar to the one of Tits [Tit72], to construct the element $\phi_i \in H$ such that $\mathrm{Poly}(\phi_i|_{\mathcal{F}_i}) = \mathrm{Poly}(H|_{\mathcal{F}_i})$, which concludes the proof. □

The element constructed in Theorem 1.1 is in general not unique. Indeed, when the subgroup H of $\mathrm{Out}(F_N)$ is such that $\mathrm{Poly}(H) = \{1\}$, Clay and Uyanik [CU20, Theorem B] give necessary and sufficient conditions for H to contain a nonabelian free subgroup consisting in atoroidal elements.

We now outline some consequences of Theorem 1.1. The first one is a result concerning the periodic subset of a subgroup of $\mathrm{Out}(F_N)$. From Clay and Uyanik's theorem cited above, one can ask the following question. Let H be a subgroup of $\mathrm{Out}(F_N)$. If H is a subgroup of $\mathrm{Out}(F_N)$ such that H virtually fixes the conjugacy class of a nontrivial subgroup A of F_N, is it true that either H virtually fixes the conjugacy class of a nontrivial element $g \in F_N$ such that g is not contained in a conjugate of A, or there exists $\phi \in H$ such that the only conjugacy classes of elements of F_N virtually fixed by ϕ are contained in a conjugate of A?

Unfortunately, such a result is not true. Indeed, let $F_3 = \langle a, b, c\rangle$ be a nonabelian free group of rank 3. Let ϕ_a (resp. ϕ_b) be the automorphism of F_3 which fixes a and b and which sends c to ca (resp. c to cb), and let $H = \langle [\phi_a], [\phi_b]\rangle \subseteq \mathrm{Out}(F_3)$. Then every element $\phi \in H$ has a representative which fixes $\langle a, b\rangle$ and sends c to cg_ϕ with $g_\phi \in \langle a, b\rangle$. Thus, ϕ fixes the conjugacy class of $g_\phi c g_\phi c^{-1}$. However, there always exist $\phi' \in H$, such that ϕ' does not preserve the conjugacy class of $g_\phi c g_\phi c^{-1}$.

We denote by $\mathrm{Per}(H)$ the set of conjugacy classes of F_N fixed by a power of every element of H. In the above example, we constructed a subgroup H of $\mathrm{Out}(F_N)$ such that $\mathrm{Per}(H)$ contains the conjugacy class of a nonabelian subgroup of rank 2. This is in fact the lowest possible rank where a generalization of the theorem of Clay and Uyanik using $\mathrm{Per}(H)$ instead of $\mathrm{Poly}(H)$ cannot work, as shown by the following result, which is a consequence of Corollary 5.3 and Theorem 1.1.

Theorem 1.2. — *Let $N \geqslant 3$ and let $g_1, \ldots, g_k$ be nontrivial root-free elements of F_N. Let H be subgroup of $\mathrm{Out}(F_N)$ such that, for every $i \in \{1, \ldots, k\}$, every element of H has a power which fixes the conjugacy class of g_i. Then one of the following (mutually exclusive) statements holds.*

1. *There exists $g_{k+1} \in F_N$ such that $[\langle g_{k+1}\rangle] \notin \{[\langle g_1\rangle], \ldots, [\langle g_k\rangle]\}$ and whose conjugacy class is fixed by a power of every element of H.*
2. *There exists $\phi \in H$ such that $\mathrm{Per}(\phi) = \{[\langle g_1\rangle], \ldots, [\langle g_k\rangle]\}$.*

As proved by Ivanov [Iva92], Case (2) of Theorem 1.2 naturally occurs when we are working with a subgroup of a mapping class group of a compact, connected surface S whose fundamental group is identified with F_N. Finally, in Corollary 5.4, we prove a characterization of subgroups of the mapping class group of such a surface S using periodic conjugacy classes.

2. Preliminaries

2.1. Malnormal subgroup systems of F_N

Let N be an integer greater than 1 and let F_N be a free group of rank N. A *subgroup system of* F_N is a finite (possibly empty) set $\mathcal{A}$ whose elements are conjugacy classes of nontrivial (that is distinct from $\{1\}$) finite rank subgroups of F_N. Note that a subgroup system $\mathcal{A}$ is completely determined by the set of subgroups A of F_N such that $[A] \in \mathcal{A}$.

There exists a partial order on the set of subgroup systems of F_N, where $\mathcal{A}_1 \leqslant \mathcal{A}_2$ if for every subgroup A_1 of F_N such that $[A_1] \in \mathcal{A}_1$, there exists a subgroup A_2 of F_N such that $[A_2] \in \mathcal{A}_2$ and A_1 is a subgroup of A_2. In this case we say that $\mathcal{A}_2$ is an *extension* of $\mathcal{A}_1$.

The *stabilizer in* $\mathrm{Out}(F_N)$ *of a subgroup system* $\mathcal{A}$, denoted by $\mathrm{Out}(F_N, \mathcal{A})$, is the set of all elements $\phi \in \mathrm{Out}(F_N)$ such that $\phi(\mathcal{A}) = \mathcal{A}$. An element of $\mathrm{Out}(F_N, \mathcal{A})$ is called an *outer automorphism relative to* $\mathcal{A}$ or a *relative outer automorphism* if the context is clear. Note that ϕ might permute the conjugacy classes of subgroups of F_N contained in $\mathcal{A}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two subgroup systems, we set $\mathrm{Out}(F_N,\mathcal{A}_1,\mathcal{A}_2)=\mathrm{Out}(F_N,\mathcal{A}_1)\cap\mathrm{Out}(F_N,\mathcal{A}_2)$.

If $\mathcal{A}$ is a subgroup system of F_N, we denote by $\mathrm{Out}(F_N,\mathcal{A}^{(t)})$ the subgroup of $\mathrm{Out}(F_N)$ consisting in every element $\phi\in\mathrm{Out}(F_N)$ such that, for every subgroup P of F_N such that $[P]\in\mathcal{A}$, there exists $\Phi\in\phi$ such that $\Phi(P)=P$ and $\Phi|_P=\mathrm{id}_P$.

Recall that a subgroup A of F_N is *malnormal* if for every element $x\in F_N-A$, we have $xAx^{-1}\cap A=\{e\}$.

Definition 2.1 (Malnormal subgroup system, nonperipheral element). — *Let $\mathcal{A}$ be a subgroup system of F_N.*

(1) *The subgroup system $\mathcal{A}$ is* malnormal *if every subgroup A of F_N such that $[A]\in\mathcal{A}$ is malnormal and, for all subgroups A_1,A_2 of F_N such that $[A_1],[A_2]\in\mathcal{A}$, if $A_1\cap A_2$ is nontrivial then $A_1=A_2$.*
(2) *An element $g\in F_N$ is* $\mathcal{A}$-peripheral *(or simply* peripheral *if there is no ambiguity) if it is trivial or conjugate into one of the subgroups of $\mathcal{A}$, and* $\mathcal{A}$-nonperipheral *otherwise.*

An important class of examples of malnormal subgroup systems is given by the *free factor systems*. A *free factor system of F_N* is a (possibly empty) set $\mathcal{F}$ of conjugacy classes $\{[A_1],\ldots,[A_r]\}$ of nontrivial subgroups $A_1,\ldots,A_r$ of F_N such that there exists a subgroup B of F_N with $F_N=A_1*\ldots*A_r*B$. An ascending sequence of free factor systems $\mathcal{F}_1\leqslant\ldots\leqslant\mathcal{F}_i=\{[F_N]\}$ of F_N is called a *filtration of F_N*.

Definition 2.2 (Sporadic extension). —

(1) *An extension of free factor systems $\mathcal{F}_1\leqslant\mathcal{F}_2=\{[A_1],\ldots,[A_k]\}$ of F_N is* sporadic *if there exists $\ell\in\{1,\ldots,k\}$ such that, for every $j\in\{1,\ldots,k\}-\{\ell\}$, we have $[A_j]\in\mathcal{F}_1$ and if one of the following holds:*
   (a) *there exist subgroups B_1,B_2 of F_N such that $[B_1],[B_2]\in\mathcal{F}_1$ and $A_\ell=B_1*B_2$;*
   (b) *there exists a subgroup B of F_N such that $[B]\in\mathcal{F}_1$ and A_ℓ is an HNN extension of B over the trivial group (thus A_ℓ is isomorphic to $B*\mathbb{Z}$);*
   (c) *there exists $g\in F_N$ such that $\mathcal{F}_2=\mathcal{F}_1\cup\{[g]\}$ and $A_\ell=\langle g\rangle$.*

   Otherwise, the extension $\mathcal{F}_1\leqslant\mathcal{F}_2$ is nonsporadic.
(2) *A free factor system $\mathcal{F}$ of F_N is* sporadic *(resp.* nonsporadic*) if the extension $\mathcal{F}\leqslant\{[F_N]\}$ is sporadic (resp. nonsporadic).*

Given a free factor system $\mathcal{F}$ of F_N, a *free factor of* $(F_N,\mathcal{F})$ is a subgroup A of F_N such that there exists a free factor system $\mathcal{F}'$ of F_N with $[A]\in\mathcal{F}'$ and $\mathcal{F}\leqslant\mathcal{F}'$. A free factor of $(F_N,\mathcal{F})$ is *proper* if it is nontrivial, not equal to F_N and if its conjugacy class does not belong to $\mathcal{F}$.

In general, we will work in a finite index subgroup of $\mathrm{Out}(F_N)$ defined as follows. Let

$$\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})=\ker\Big(\mathrm{Out}(F_N)\to\mathrm{Aut}(H_1(F_N,\mathbb{Z}/3\mathbb{Z}))\Big).$$

For every $\phi\in\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, we have the following properties:

(1) any ϕ-periodic conjugacy class of free factor of F_N is fixed by ϕ [HM20, Theorem II.3.1];

(2) any ϕ-periodic conjugacy class of elements of F_N is fixed by ϕ [HM20, Theorem II.4.1].

Another class of examples of malnormal subgroup systems is the following one. Let $g \in F_N$ and let $\mathfrak{B}$ be a free basis of F_N. The length of the conjugacy class of g with respect to $\mathfrak{B}$ is

$$\ell_{\mathfrak{B}}([g]) = \min_{h \in [g]} \ell_{\mathfrak{B}}(h),$$

where $\ell_{\mathfrak{B}}(h)$ is the word length of h with respect to the basis $\mathfrak{B}$. An outer automorphism $\phi \in \mathrm{Out}(F_N)$ is *exponentially growing* if there exists $g \in F_N$ such that the length of the conjugacy class $[g]$ of g in F_N with respect to some basis of F_N grows exponentially fast under positive iteration of ϕ. One can show that if g is exponentially growing with respect to some free basis of F_N, then it is exponentially growing for every free basis of F_N.

If $\phi \in \mathrm{Out}(F_N)$ is not exponentially growing, one can show, using for instance the technology of train tracks due to Bestvina and Handel [BH92], that for every $g \in F_N$, the conjugacy class $[g]$ has polynomial growth under positive iteration of ϕ. In this case, we say that ϕ is *polynomially growing*. For an automorphism $\alpha \in \mathrm{Aut}(F_N)$, we say that α is *exponentially growing* if there exists $g \in F_N$ such that the word length of $[g]$ grows exponentially fast under iteration of $[\alpha] \in \mathrm{Out}(F_N)$. Otherwise, α is polynomially growing. The polynomial subgroup of α is the subgroup of F_N consisting in all elements $g \in F_N$ whose word length grows polynomially fast under iteration of α.

Let $\phi \in \mathrm{Out}(F_N)$ be exponentially growing. A subgroup P of F_N is a *polynomial subgroup* of ϕ if there exist $k \in \mathbb{N}^*$ and a representative α of ϕ^k such that $\alpha(P) = P$ and $\alpha|_P$ is polynomially growing. By [Lev09, Proposition 1.4], there exist finitely many conjugacy classes $[H_1], \ldots, [H_k]$ of maximal polynomial subgroups of ϕ. Moreover, the proof of [Lev09, Proposition 1.4] implies that the set $\mathcal{H} = \{[H_1], \ldots, [H_k]\}$ is a malnormal subgroup system (see [Gue22b, Section 2.1]). We denote this malnormal subgroup system by $\mathcal{A}(\phi)$.

Note that, if H is a subgroup of F_N such that $[H] \in \mathcal{A}(\phi)$, there exist $p \in \mathbb{N}^*$ and $\Phi^{-1} \in \phi^{-1}$ such that $\Phi^{-p}(H) = H$. By for instance [BFH05, Theorem 1.1], up to taking a larger p, the image of ϕ^p in $\mathrm{Out}(H)$ preserves a sequence $\mathcal{S}$ of free factor systems of H such that every extension of the sequence is sporadic. Hence the image of ϕ^{-p} in $\mathrm{Out}(H)$ preserves $\mathcal{S}$. This implies that H is a polynomially growing subgroup of ϕ^{-1}. Hence we have $\mathcal{A}(\phi) \leqslant \mathcal{A}(\phi^{-1})$. By symmetry, we have

$$\mathcal{A}(\phi) = \mathcal{A}\left(\phi^{-1}\right). \tag{2.1}$$

Moreover, for every element $\psi \in \mathrm{Out}(F_N)$, we have

$$\mathcal{A}\left(\psi\phi\psi^{-1}\right) = \psi(\mathcal{A}(\phi)).$$

In order to distinguish between the set of elements of F_N which have polynomial growth under positive iteration of ϕ and the associated malnormal subgroup system, we will denote by $\mathrm{Poly}(\phi)$ the former. We have $\mathrm{Poly}(\phi) = \mathrm{Poly}(\phi^{-1})$ by Equation (2.1). If H is a subgroup of $\mathrm{Out}(F_N)$, we set $\mathrm{Poly}(H) = \bigcap_{\phi \in H} \mathrm{Poly}(\phi)$.

Definition 2.3 (Atoroidal, expanding outer automorphism). — *Let $\mathcal{A}$ be a malnormal subgroup system of F_N and let $\phi \in \mathrm{Out}(F_N, \mathcal{A})$ be a relative outer automorphism.*

(1) *The outer automorphism ϕ is* atoroidal relative to $\mathcal{A}$ *if, for every $k \in \mathbb{N}^*$, the element ϕ^k does not preserve the conjugacy class of any $\mathcal{A}$-nonperipheral element.*
(2) *The outer automorphism ϕ is* expanding relative to $\mathcal{A}$ *if $\mathcal{A}(\phi) \leqslant \mathcal{A}$.*

Note that an expanding outer automorphism relative to $\mathcal{A}$ is in particular atoroidal relative to $\mathcal{A}$. When $\mathcal{A} = \varnothing$, the outer automorphism ϕ is expanding relative to $\mathcal{A}$ if and only if for every nontrivial element $g \in F_N$, the length of the conjugacy class $[g]$ of g in F_N with respect to some basis of F_N grows exponentially fast under iteration of ϕ. Therefore, using for instance a result of Levitt [Lev09, Corollary 1.6], the outer automorphism ϕ is expanding relative to $\mathcal{A} = \varnothing$ if and only if ϕ is atoroidal relative to $\mathcal{A} = \varnothing$.

Let $\mathcal{A} = \{[A_1], \ldots, [A_r]\}$ be a malnormal subgroup system and let $\mathcal{F}$ be a free factor system. Let $i \in \{1, \ldots, r\}$. By for instance [SW79, Theorem 3.14] for the action of A_i on one of its Cayley graphs, there exist finitely many subgroups $A_i^{(1)}, \ldots, A_i^{(k_i)}$ of A_i such that:

(1) for every $j \in \{1, \ldots, k_i\}$, there exists a subgroup B of F_N such that $[B] \in \mathcal{F}$ and $A_i^{(j)} = B \cap A_i$;
(2) for every subgroup B of F_N such that $[B] \in \mathcal{F}$ and $B \cap A_i \neq \{e\}$, there exists $j \in \{1, \ldots, k_i\}$ such that $A_i^{(j)} = B \cap A_i$;
(3) the subgroup $A_i^{(1)} * \ldots * A_i^{(k_i)}$ is a free factor of A_i.

Thus, one can define a new subgroup system as

$$\mathcal{F} \wedge \mathcal{A} = \bigcup_{i=1}^{r} \left\{ \left[A_i^{(1)}\right], \ldots, \left[A_i^{(k_i)}\right] \right\}.$$

Since $\mathcal{A}$ is malnormal, and since, for every $i \in \{1, \ldots, r\}$, the group $A_i^{(1)} * \ldots * A_i^{(k_i)}$ is a free factor of A_i, it follows that the subgroup system $\mathcal{F} \wedge \mathcal{A}$ is a malnormal subgroup system of F_N. We call it the *meet of $\mathcal{F}$ and $\mathcal{A}$*. If $\phi \in \mathrm{Out}(F_N, \mathcal{F}, \mathcal{A})$ then $\phi \in \mathrm{Out}(F_N, \mathcal{F} \wedge \mathcal{A})$.

2.2. Relative currents

In this section, we define the notion of *currents of F_N relative to a malnormal subgroup system $\mathcal{A}$*. The section follows [Gue21, Gue22b] (see the work of Gupta [Gup17] for the particular case of free factor systems and Guirardel and Horbez [GH19] in the context of free products of groups). It can be thought of as a functional space in which densely live the $\mathcal{A}$-nonperipheral elements of F_N.

Let $\partial_\infty F_N$ be the Gromov boundary of F_N. The *double boundary of F_N* is the metrisable locally compact, totally disconnected quotient topological space

$$\partial^2 F_N = (\partial_\infty F_N \times \partial_\infty F_N \setminus \Delta) / \sim,$$

where $\sim$ is the equivalence relation generated by the flip relation $(x, y) \sim (y, x)$ and Δ is the diagonal, endowed with the diagonal action of F_N. We denote by $\{x, y\}$ the equivalence class of (x, y).

Let T be the Cayley graph of F_N with respect to a free basis $\mathfrak{B}$. The boundary of T is naturally homeomorphic to $\partial_\infty F_N$ and the set $\partial^2 F_N$ is then identified with the set of unoriented bi-infinite geodesics in T. Let γ be a finite geodesic path in T. The path γ determines a subset in $\partial^2 F_N$ called the *cylinder set of* γ, denoted by $C(\gamma)$, which consists in all unoriented bi-infinite geodesics in T that contain γ. Such cylinder sets form a basis for the topology on $\partial^2 F_N$, and in this topology, the cylinder sets are both open and compact, hence closed (see for instance [Mar95, Section 5.4]). The action of F_N on $\partial^2 F_N$ has a dense orbit.

Let A be a nontrivial subgroup of F_N of finite rank. The induced A-equivariant inclusion $\partial_\infty A \hookrightarrow \partial_\infty F_N$ induces an inclusion $\partial^2 A \hookrightarrow \partial^2 F_N$. Let $\mathcal{A} = \{[A_1], \ldots, [A_r]\}$ be a malnormal subgroup system. Let

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^{r} \bigcup_{g \in F_N} \partial^2 \left(g A_i g^{-1}\right).$$

Definition 2.4 (Relative double boundary). — *Let $\mathcal{A}$ be a malnormal subgroup system. The* double boundary of F_N relative to $\mathcal{A}$ *is*

$$\partial^2(F_N, \mathcal{A}) = \partial^2 F_N - \partial^2 \mathcal{A}.$$

The double boundary of F_N relative to a malnormal subgroup system is a subset of $\partial^2 F_N$ which is invariant under the action of F_N on $\partial^2 F_N$ and inherits the subspace topology of $\partial^2 F_N$.

Lemma 2.5 ([Gue21, Lemmas 2.5, 2.6, 2.7]). — *Let $N \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system of F_N. The space $\partial^2(F_N, \mathcal{A})$ is an open subspace of $\partial^2 F_N$, hence is locally compact, and the action of F_N on $\partial^2(F_N, \mathcal{A})$ has a dense orbit.*

We can now define a *relative current*.

Definition 2.6 (relative current). — *Let $\mathcal{A}$ be a malnormal subgroup system of F_N. A* relative current on $(F_N, \mathcal{A})$ *is a (possibly zero) F_N-invariant nonnegative Radon measure μ on $\partial^2(F_N, \mathcal{A})$.*

The set $\mathrm{Curr}(F_N, \mathcal{A})$ of all relative currents on $(F_N, \mathcal{A})$ is equipped with the weak-$*$ topology: a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathrm{Curr}(F_N, \mathcal{A})^{\mathbb{N}}$ converges to a current $\mu \in \mathrm{Curr}(F_N, \mathcal{A})$ if and only if for every Borel subset $B \subseteq \partial^2(F_N, \mathcal{A})$ such that $\mu(\partial B) = 0$ (where ∂B is the topological boundary of B), the sequence $(\mu_n(B))_{n \in \mathbb{N}}$ converges to $\mu(B)$.

The group $\mathrm{Out}(F_N, \mathcal{A})$ acts on $\mathrm{Curr}(F_N, \mathcal{A})$ as follows. Let $\phi \in \mathrm{Out}(F_N, \mathcal{A})$ and let Φ be a representative of ϕ. The automorphism Φ acts diagonally by homeomorphisms on $\partial^2 F_N$. If $\Phi' \in \phi$, then the action of Φ' on $\partial^2 F_N$ differs from the action of Φ by a translation by an element of F_N. Let $\mu \in \mathrm{Curr}(F_N, \mathcal{A})$ and let C be a Borel subset of $\partial^2(F_N, \mathcal{A})$. Then, since ϕ preserves $\mathcal{A}$, we see that $\Phi^{-1}(C) \in \partial^2(F_N, \mathcal{A})$. Then we set

$$\phi(\mu)(C) = \mu\left(\Phi^{-1}(C)\right),$$

which is well-defined since μ is F_N-invariant.

Every conjugacy class of nonperipheral element $g \in F_N$ determines a relative current $\eta_{[g]}$ as follows. Suppose first that g is *root-free*, that is there do not exist $k \geqslant 2$ and $h \in F_N$ such that $g = h^k$. Let γ be a finite geodesic path in the Cayley graph T. Then $\eta_{[g]}(C(\gamma))$ is the number of axes in T of conjugates of g that contain the path γ. By [Gue21, Lemma 3.2], $\eta_{[g]}$ extends uniquely to a current in $\mathrm{Curr}(F_N, \mathcal{A})$ which we still denote by $\eta_{[g]}$. If $g = h^k$ with $k \geqslant 2$ and h root-free, we set $\eta_{[g]} = k\ \eta_{[h]}$. Such currents are called *rational currents*.

Let $\mu \in \mathrm{Curr}(F_N, \mathcal{A})$. The *support of* μ, denoted by $\mathrm{Supp}(\mu)$, is the support of the Borel measure μ on $\partial^2(F_N, \mathcal{A})$. We recall that $\mathrm{Supp}(\mu)$ is a *lamination of* $\partial^2(F_N, \mathcal{A})$, that is, a closed F_N-invariant subset of $\partial^2(F_N, \mathcal{A})$.

In the rest of the article, rather than considering the space of relative currents itself, we will consider the set of *projectivized relative currents*, denoted by

$$\mathbb{P}\mathrm{Curr}(F_N, \mathcal{A}) = (\mathrm{Curr}(F_N, \mathcal{A}) - \{0\}) \,/ \sim,$$

where $\mu \sim \nu$ if there exists $\lambda \in \mathbb{R}_+^*$ such that $\mu = \lambda\nu$. The projective class of a current $\mu \in \mathrm{Curr}(F_N, \mathcal{A})$ will be denoted by $[\mu]$. For every $\phi \in \mathrm{Out}(F_N, \mathcal{A})$, the action $\phi\colon \mu \mapsto \phi(\mu)$ is positively linear. Therefore, the action of $\mathrm{Out}(F_N, \mathcal{A})$ on $\mathrm{Curr}(F_N, \mathcal{A})$ induces an action on $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{A})$. We have the following properties.

Lemma 2.7. — [Gue21, Lemma 3.3] *Let $N \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system of F_N. The space $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{A})$ is compact.*

Proposition 2.8 ([Gue21, Theorem 1.2]). — *Let $N \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system of F_N. The set of projectivised rational currents associated with nonperipheral elements of F_N is dense in $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{A})$.*

2.3. Currents associated with an almost atoroidal outer automorphism of F_N

Let $N \geqslant 3$ and let $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$ be a free factor system of F_N. If $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ preserves $\mathcal{F}$, we denote by

$$\phi|_{\mathcal{F}} = ([\Phi_1|_{A_1}],\ \ldots,\ [\Phi_k|_{A_k}]) \in \prod_{i=1}^{k} \mathrm{Out}(A_i) \tag{2.2}$$

where, for every $i \in \{1,\ \ldots,\ k\}$, the element Φ_i is a representative of ϕ such that $\Phi_i(A_i) = A_i$. Note that the outer class of $\Phi_i|_{A_i}$ in $\mathrm{Out}(A_i)$ does not depend on the choice of Φ_i since A_i is a malnormal subgroup of F_N. Hence, for every $i \in \{1,\ \ldots,\ k\}$, we can naturally associate to ϕ the outer automorphism $[\Phi_i|_{A_i}] \in \mathrm{Out}(A_i)$ as in Equation (2.2), and this notation will be used from now on.

Note that, for every $i \in \{1,\ \ldots,\ k\}$, the element $[\Phi_i|_{A_i}]$ is expanding relative to the free factor system $\mathcal{F} \wedge \{[A_i]\} = \{[A_i]\}$, without additional assumption on ϕ. We will say that $\phi|_{\mathcal{F}}$ *is expanding relative to* $\mathcal{F}$.

Let

$$\mathrm{Poly}(\phi|_{\mathcal{F}}) = \bigcup_{i=1}^{k} \bigcup_{g \in F_N} g\mathrm{Poly}([\Phi_i|_{A_i}])g^{-1} \subseteq F_N.$$

If H is a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ which preserves $\mathcal{F}$, we set

$$\mathrm{Poly}(H|_{\mathcal{F}}) = \bigcap_{\phi \in H} \mathrm{Poly}(\phi|_{\mathcal{F}}).$$

We now define a class of outer automorphisms of F_N which we will study in the rest of the article.

Definition 2.9 (Almost atoroidal). — *Let $N \geqslant 3$ and let $\mathcal{F}$ be a free factor system of F_N. Let $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ be an outer automorphism preserving $\mathcal{F}$. The outer automorphism ϕ is* almost atoroidal relative to $\mathcal{F}$ *if* $\mathrm{Poly}(\phi) \neq \{[F_N]\}$ *and if ϕ is an atoroidal outer automorphism relative to $\mathcal{F}$ whenever the extension $\mathcal{F} \leqslant \{[F_N]\}$ is nonsporadic.*

Note that, if $\mathcal{F}$ is a sporadic free factor system, then $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F})$ is almost atoroidal relative to $\mathcal{F}$ if and only if $\mathrm{Poly}(\phi) \neq \{[F_N]\}$. Definition 2.9 is a subcase of a larger definition of almost atoroidality studied in [Gue22b, Definition 4.3].

Let $\mathcal{F} \leqslant \mathcal{F}_1 = \{[A_1], \ldots, [A_k]\}$ be two free factor systems of F_N. Let ϕ be an element of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1)$. We say that $\phi|_{\mathcal{F}_1}$ is *almost atoroidal relative to* $\mathcal{F}$ if, for every $i \in \{1, \ldots, k\}$, the outer automorphism $[\Phi_i|_{A_i}]$ defined in Equation (2.2) is almost atoroidal relative to $\mathcal{F} \wedge \{[A_i]\}$.

Let $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ be an almost atoroidal outer automorphism relative to $\mathcal{F}$. We now recall from [Gue22b] the definition and some properties of some subsets of the space $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$ associated with ϕ.

Definition 2.10 (Polynomially growing currents). — *Let $N \geqslant 3$ and let $\mathcal{F}$ be a free factor system of F_N. Let $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F})$ be an almost atoroidal outer automorphism relative to $\mathcal{F}$. The* space of polynomially growing currents associated with ϕ, *denoted by $K_{PG}(\phi)$, is the subspace of all currents in $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$ whose support is contained in $\partial^2\mathcal{A}(\phi) \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$.*

We will need the following result which gives the existence and properties of an approximation of the length function of the conjugacy class of an element of F_N in the context of the space of currents.

Proposition 2.11 ([Gue22b, Lemma 3.27, Lemma 3.28 (3)]). — *Let $N \geqslant 3$ and let $\mathcal{F}$ be a sporadic free factor system of F_N. Let $\phi \in \mathrm{Out}(F_N, \mathcal{F})$ be an almost atoroidal outer automorphism relative to $\mathcal{F}$. There exists a continuous, positively linear function*

$$\|.\|_{\mathcal{F}} \colon \mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}_+$$

such that the following holds.

(1) *There exist a basis $\mathfrak{B}$ of F_N and a constant $C \geqslant 1$ such that, for every $\mathcal{F} \wedge \mathcal{A}(\phi)$-nonperipheral element $g \in F_N$, we have $\|\eta_{[g]}\|_{\mathcal{F}} \in \mathbb{N}^*$ and*

$$\ell_{\mathfrak{B}}([g]) \geqslant C \ \left\|\eta_{[g]}\right\|_{\mathcal{F}}.$$

(2) *For every $\eta \in \mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$, if $\|\eta\|_{\mathcal{F}} = 0$, then $\eta = 0$.*

Proposition 2.12 ([Gue22b, Propositions 4.4, 4.12, 5.24]). — *Let $N \geqslant 3$ and let $\mathcal{F}$ be a sporadic free factor system of F_N ($\mathcal{F}$ might be equal to $\{[F_N]\}$). Let $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ be an almost atoroidal outer automorphism relative to $\mathcal{F}$. There exist two unique proper compact ϕ-invariant subsets $\Delta_\pm(\phi)$ of $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$ such that the following assertions hold.*

(1) *For every $[\mu] \in \Delta_+(\phi) \cup \Delta_-(\phi)$, the support of μ is contained in $\partial^2\mathcal{F}$.*
(2) *Let U_+ be a neighborhood of $\Delta_+(\phi)$, let U_- be a neighborhood of $\Delta_-(\phi)$, let V be a neighborhood of $K_{PG}(\phi)$. There exists $N \in \mathbb{N}^*$ such that for every $n \geqslant 1$ and every ($\mathcal{F} \wedge \mathcal{A}(\phi)$)-nonperipheral $w \in F_N$ such that $\eta_{[w]} \notin V$, one of the following holds*

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \quad \text{or} \quad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

The subsets $\Delta_+(\phi)$ and $\Delta_-(\phi)$ are called the *simplices of attraction and repulsion of ϕ*.

Let $\mathcal{F} \leqslant \mathcal{F}_1 = \{[A_1], \ldots, [A_k]\}$ be a sporadic extension of two free factor systems of F_N. Let ϕ be an element of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1)$. Let $i \in \{1, \ldots, k\}$. If $\phi|_{\mathcal{F}_1}$ is almost atoroidal relative to $\mathcal{F}$, we denote by $\Delta_\pm([A_i], \phi) \subseteq \mathbb{P}\mathrm{Curr}(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}([\Phi_i|_{A_i}]))$ the convexes of attraction and repulsion of $[\Phi_i|_{A_i}]$. If $\psi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ preserves the conjugacy class of A_i and $\mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}([\Phi_i|_{A_i}])$, then $\Delta_\pm([A_i], \psi\phi\psi^{-1}) = \psi(\Delta_\pm([A_i], \phi))$.

Let

$$\widehat{\Delta}_\pm(\phi) = \left\{[t\mu + (1-t)\nu] \;\middle|\; t \in [0,1], [\mu] \in \Delta_\pm(\phi), [\nu] \in K_{PG}(\phi), \|\mu\|_{\mathcal{F}} = \|\nu\|_{\mathcal{F}} = 1\right\}$$

be the *convexes of attraction and repulsion of ϕ*. We have the following results.

Theorem 2.13. — [Gue22b, Theorem 6.4] *Let $N \geqslant 3$ and let $\mathcal{F}$ be a sporadic free factor system of F_N. Let $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F})$ be an almost atoroidal outer automorphism relative to $\mathcal{F}$. Let $\widehat{\Delta}_\pm(\phi)$ be the convexes of attraction and repulsion of ϕ and $\Delta_\pm(\phi)$ be the simplices of attraction and repulsion of ϕ. Let $U_\pm$ be open neighborhoods of $\Delta_\pm(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$ and $\widehat{V}_\pm$ be open neighborhoods of $\widehat{\Delta}_\pm(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$. There exists $M \in \mathbb{N}^*$ such that for every $n \geqslant M$, we have*

$$\phi^{\pm n}\left(\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_\mp\right) \subseteq U_\pm.$$

Proposition 2.14 ([Gue22b, Corollary 6.5]). — *Let $N \geqslant 3$ and let $\mathcal{F}$ be a sporadic free factor system of F_N. Let $\phi \in \mathrm{Out}(F_N, \mathcal{F})$ be an almost atoroidal outer automorphism relative to $\mathcal{F}$. Let $\|.\|_{\mathcal{F}} \colon \mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}_+$ be the function given by Proposition 2.11.*

For every open neighborhood $\widehat{V}_- \subseteq \mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$ of $\widehat{\Delta}_-(\phi)$, there exists M in $\mathbb{N}^*$ and a constant $L_0 > 0$ such that, for every current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-$, and every $m \geqslant M$, we have

$$\|\phi^m(\mu)\|_{\mathcal{F}} \geqslant 3^{m-M} L_0 \|\mu\|_{\mathcal{F}}.$$

3. Nonsporadic extensions and fully irreducible outer automorphisms

Let $N \geqslant 3$ and let $\mathcal{F}$ and $\mathcal{F}_1 = \{[A_1], \ldots, [A_k]\}$ be two free factor systems of F_N with $\mathcal{F} \leqslant \mathcal{F}_1$ such that the extension $\mathcal{F} \leqslant \mathcal{F}_1$ is nonsporadic. Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ which preserves $\mathcal{F}$ and $\mathcal{F}_1$. We suppose that H is *irreducible with respect to* $\mathcal{F} \leqslant \mathcal{F}_1$, that is, there does not exist a proper, nontrivial free factor system $\mathcal{F}'$ of F_N preserved by H with $\mathcal{F} < \mathcal{F}' < \mathcal{F}_1$.

Suppose that there exists $\phi \in H$ such that $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. In this section, we show that there exists $\widehat{\phi} \in H$ such that $\mathrm{Poly}(\widehat{\phi}|_{\mathcal{F}_1}) = \mathrm{Poly}(H|_{\mathcal{F}_1})$.

The key point is to construct *fully irreducible outer automorphisms relative to* $\mathcal{F}$ in H in the following sense. Let $\phi \in \mathrm{Out}(F_N, \mathcal{F})$. We say that ϕ is *fully irreducible relative to* $\mathcal{F}$ if no power of ϕ preserves a proper free factor system $\mathcal{F}'$ of F_N such that $\mathcal{F} < \mathcal{F}'$. If $\phi \in \mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1)$, we say that $\phi|_{\mathcal{F}_1}$ is *fully irreducible relative to* $\mathcal{F}$ (resp. *atoroidal relative to* $\mathcal{F}$) if, for every $i \in \{1, \ldots, k\}$, the outer automorphism $[\Phi_i|_{A_i}]$ defined in Equation (2.2) is fully irreducible relative to $\mathcal{F} \wedge \{[A_i]\}$ (resp. atoroidal relative to $\mathcal{F} \wedge \{[A_i]\}$).

If H is a subgroup of $\mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1)$, we say that $H|_{\mathcal{F}_1}$ is *atoroidal relative to* $\mathcal{F}$ if there does not exist a conjugacy class of F_N which is H-invariant, $\mathcal{F}$-nonperipheral and $\mathcal{F}_1$-peripheral.

First, we recall some properties of fully irreducible outer automorphisms.

Proposition 3.1. — *Let $N \geqslant 3$ and let $\mathcal{F}$ be a nonsporadic free factor system of F_N. Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ which preserves $\mathcal{F}$ and such that H is irreducible with respect to the extension $\mathcal{F} \leqslant \{[F_N]\}$. Let $\phi \in H$ be a fully irreducible outer automorphism relative to $\mathcal{F}$.*

1. [Gue22b, Corollary 3.15] *There exists at most one (up to taking inverse) conjugacy class $[g]$ of root-free $\mathcal{F}$-nonperipheral element of F_N which has polynomial growth under iteration of ϕ. Moreover, the conjugacy class $[g]$ is fixed by ϕ.*
2. [GH22, Theorem 7.4] *One of the following holds:*
    - (a) *there exists $\psi \in H$ such that ψ is a fully irreducible, atoroidal outer automorphism relative to $\mathcal{F}$;*
    - (b) *if ϕ fixes the conjugacy class of a root-free $\mathcal{F}$-nonperipheral element g of F_N, then $[g]$ is fixed by H.*

Thus, there exists $\psi \in H$ such that ψ is fully irreducible relative to $\mathcal{F}$ and the conjugacy class of an $\mathcal{F}$-nonperipheral element $g \in F_N$ has polynomial growth under iteration of ψ if and only if it has polynomial growth under iteration of every element of H.

Hence Proposition 3.1 suggests that, if H is a subgroup of F_N which satisfies the hypotheses in Proposition 3.1, one step in order to prove Theorem 1.1 is to construct relative fully irreducible (atoroidal) outer automorphisms in H. This is contained in Theorem 3.3. First we need the following lemma, whose statement is similar to an argument appearing in the proof of [CU18, Theorem 6.6] (see also [HM20, Section IV.2.1]).

Lemma 3.2. — *Let $N \geqslant 3$ and let H be a subgroup of* $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. *Let*

$$\varnothing = \mathcal{F}_0 < \mathcal{F}_1 < \ldots < \mathcal{F}_k = \{[F_N]\}$$

be a maximal H-invariant sequence of free factor systems. Let

$$S = \{j \mid \text{the extension } \mathcal{F}_{j-1} \leqslant \mathcal{F}_j \text{ is nonsporadic}\}$$

and let $j \in S$. There exists a unique conjugacy class $[B_j]$ of a subgroup B_j in F_N such that $[B_j] \in \mathcal{F}_j$ and $[B_j] \notin \mathcal{F}_{j-1}$.

Proof. — There exists at least one such conjugacy class since $\mathcal{F}_{j-1} < \mathcal{F}_j$. Suppose towards a contradiction that there exist two distinct subgroups B_+ and B_- of F_N such that $[B_+] \neq [B_-]$, $[B_+], [B_-] \in \mathcal{F}_j$ and $[B_+], [B_-] \notin \mathcal{F}_{j-1}$. Then

$$\mathcal{F}'([B_-]) = (\mathcal{F}_j - \{[B_+]\}) \cup (\mathcal{F}_{j-1} \wedge \{[B_+]\})$$

is H-invariant and $\mathcal{F}_{j-1} < \mathcal{F}'([B_-]) < \mathcal{F}_j$, which contradicts the maximality hypothesis of the sequence of free factor systems. ☐

Theorem 3.3. — *Let $N \geqslant 3$ and let H be a subgroup of* $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. *Let*

$$\varnothing = \mathcal{F}_0 < \mathcal{F}_1 < \ldots < \mathcal{F}_k = \{[F_N]\}$$

be a maximal H-invariant sequence of free factor systems. There exists $\phi \in H$ such that for every $i \in \{1, \ldots, k\}$ such that the extension $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$ is nonsporadic, the element $\phi|_{\mathcal{F}_i}$ is fully irreducible relative to $\mathcal{F}_{i-1}$. Moreover, if $H|_{\mathcal{F}_i}$ is atoroidal relative to $\mathcal{F}_{i-1}$, one can choose ϕ so that $\phi|_{\mathcal{F}_i}$ is atoroidal relative to $\mathcal{F}_{i-1}$.

Proof. — The proof follows [CU18, Theorem 6.6] (see also [CU20, Corollary 3.4]). Let $S \subseteq \{0, \ldots, k\}$ be as in the statement of Lemma 3.2 and let $j \in S$. Let B_j be a subgroup of F_N given by Lemma 3.2. Let $A_{j,1}, \ldots, A_{j,s}$ be the subgroups of B_j with pairwise disjoint conjugacy classes such that $\mathcal{A}_{j-1} = \{[A_{j,1}], \ldots, [A_{j,s}]\} \subseteq \mathcal{F}_{j-1}$ and s is maximal for this property. Note that, for every $j \in S$, the free factor system $\mathcal{A}_{j-1}$ is a nonsporadic free factor system of B_j by Lemma 3.2 and since the extension $\mathcal{F}_{j-1} \leqslant \mathcal{F}_j$ is nonsporadic.

By [GH22, Theorem 7.1] (see also [HM20, Theorem D] for the finitely generated case), for every $j \in S$, there exists an element $\phi \in H$ such that $[\Phi_j|_{B_j}] \in \mathrm{Out}(B_j, \mathcal{A}_{j-1})$ is fully irreducible relative to $\mathcal{A}_{j-1}$. By Proposition 3.1 (2), for every $j \in S$ such that $H|_{\mathcal{F}_j}$ is atoroidal relative to $\mathcal{F}_{j-1}$, there exists $\phi \in H$ such that $[\Phi_j|_{B_j}] \in \mathrm{Out}(B_j, \mathcal{A}_{j-1})$ is fully irreducible and atoroidal relative to $\mathcal{A}_{j-1}$.

Let S_1 be the subset of S consisting in every $j \in S$ such that $H|_{\mathcal{F}_j}$ is atoroidal relative to $\mathcal{F}_{j-1}$, and let $S_2 = S - S_1$. By [GH22, Theorems 4.1, 4.2] (see also [Gup18, Hor16, Man14a, Man14b]), for every $j \in S_1$ (resp. $j \in S_2$) there exists a Gromov-hyperbolic space X_j (the *$\mathcal{Z}$-factor complex of B_j relative to $\mathcal{A}_{j-1}$* when $j \in S_1$ and the *free factor complex of B_j relative to $\mathcal{A}_{j-1}$* otherwise) on which $\mathrm{Out}(B_j, \mathcal{A}_{j-1})$ acts by isometries and such that $\phi_0 \in \mathrm{Out}(B_j, \mathcal{A}_{j-1})$ is a loxodromic element if and only if ϕ_0 is fully irreducible atoroidal relative to $\mathcal{A}_{j-1}$ (resp. fully irreducible relative to $\mathcal{A}_{j-1}$). The conclusion then follows from [CU18, Theorem 5.1]. ☐

4. Sporadic extensions and polynomial growth

Let $N \geqslant 3$ and let $\mathcal{F}$ and $\mathcal{F}_1 = \{[A_1], \ldots, [A_k]\}$ be two free factor systems of F_N with $\mathcal{F} \leqslant \mathcal{F}_1$. Suppose that the extension $\mathcal{F} \leqslant \mathcal{F}_1$ is sporadic. Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1)$.

In order to prove Theorem 1.1, we need to show that if $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$, there exists $\psi \in H$ such that $\mathrm{Poly}(\psi|_{\mathcal{F}_1}) = \mathrm{Poly}(H|_{\mathcal{F}_1})$.

Let $\phi \in H$ be such that $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. Note that, for every element g of $\mathrm{Poly}(\phi|_{\mathcal{F}})$, there exists a subgroup A of F_N such that $[A] \in \mathcal{F} \wedge \mathcal{A}(\phi)$ and $g \in A$. Conversely, for every subgroup A of F_N such that $[A] \in \mathcal{F} \wedge \mathcal{A}(\phi)$ and every element $g \in A$, we have $g \in \mathrm{Poly}(\phi|_{\mathcal{F}})$.

Thus $\mathcal{F} \wedge \mathcal{A}(\phi)$ is the natural malnormal subgroup system associated with the set $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. Thus, we see that H preserves $\mathcal{F} \wedge \mathcal{A}(\phi)$ and hence H acts by homeomorphisms on $\mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$.

We first need a general statement regarding the construction of an $\mathbb{R}$-tree equipped with an action of F_N stabilized by an exponentially growing outer automorphism.

Lemma 4.1. — *Let $\phi \in \mathrm{Out}(F_N)$ be an exponentially growing outer automorphism. Let $B_1, \ldots, B_n$ be subgroups of F_N such that, for every $i \in \{1, \ldots, n\}$, we have $[B_i] \in \mathcal{A}(\phi)$.*

(1) *Suppose that there exist distinct $k, \ell \in \{1, \ldots, n\}$ with $B_k \neq B_\ell$. Then there exist:*
   (a) *a finitely generated subgroup B of F_N containing every B_i with $i \in \{1, \ldots, n\}$;*
   (b) *an $\mathbb{R}$-tree T equipped with a minimal, isometric action of B with trivial arc stabilizers such that, for every $i \in \{1, \ldots, n\}$, the group B_i is elliptic in T;*
   (c) *distinct $i, j \in \{1, \ldots, n\}$ such that the point fixed by B_i in T is distinct from the point fixed by B_j.*

(2) *Suppose that there exist $g \in F_N$ and $k \in \{1, \ldots, n\}$ with $g \notin B_k$. Then there exist:*
   (a) *a finitely generated subgroup B of F_N containing g and every B_i with $i \in \{1, \ldots, n\}$;*
   (b) *an $\mathbb{R}$-tree T equipped with a minimal, isometric action of B with trivial arc stabilizers such that, for every $i \in \{1, \ldots, n\}$, the group B_i is elliptic in T;*
   (c) *$i \in \{1, \ldots, n\}$ such that the point fixed by B_i is not fixed by g.*

Note that, in the statement of Lemma 4.1 (2), the element g is not necessarily contained in $\mathrm{Poly}(\phi)$. In particular, the action of g on T might be loxodromic.

Proof. — We prove Assertion (1). By [Lev09, Lemma 1.2], there exists a nontrivial $\mathbb{R}$-tree T' equipped with a minimal, isometric action of F_N with trivial arc stabilizers and such that every polynomial subgroup of ϕ fixes a point in T'.

If there exist distinct $i, j \in \{1, \ldots, n\}$ such that B_i fixes a point in T' distinct from the point fixed by B_j, then the tree $T = T'$ satisfies the assertion of Lemma 4.1 (1).

Suppose that there exists a point x of T' fixed by every B_i with $i \in \{1, \ldots, n\}$. By [GaL95], there are only finitely many orbits of points in T' with nontrivial stabilizers. In particular, up to taking a power of ϕ, we may suppose that ϕ has a representative Φ_x which preserves $\mathrm{Stab}(x)$. Since $B_k \neq B_\ell$ and $B_k, B_\ell \subseteq \mathrm{Stab}(x)$, the automorphism $\Phi_x|_{\mathrm{Stab}(x)}$ is exponentially growing. By [GaL95], the rank of $\mathrm{Stab}(x)$ is less than N. An inductive argument replacing F_N and ϕ by $\mathrm{Stab}(x)$ and the outer class of $\Phi_x|_{\mathrm{Stab}(x)}$ concludes the proof of Assertion (2).

The proof of Assertion (2) is identical to the one of Assertion (1) replacing the fact that $B_k \neq B_\ell$ by the fact that $g \notin B_k$. ☐

Lemma 4.2. — *Let $N \geqslant 3$, let $\mathcal{F}$ be a sporadic free factor system of F_N and let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F})$ which is irreducible with respect to $\mathcal{F} \leqslant \{[F_N]\}$. Suppose that there exists $\phi \in H$ such that $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. If $\mathrm{Poly}(\phi) \neq \mathrm{Poly}(H)$, there exists an infinite subset $X \subseteq H$ such that for all distinct $\psi_1, \psi_2 \in X$, we have $\psi_1(K_{PG}(\phi)) \cap \psi_2(K_{PG}(\phi)) = \varnothing$.*

Proof. — Let $\mathcal{F} \wedge \mathcal{A}(\phi) = \{[A_1], \ldots, [A_r]\}$. Since

$$\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}}) \subseteq \mathrm{Poly}(H) \subsetneq \mathrm{Poly}(\phi),$$

we have $\mathcal{A}(\phi) \neq \mathcal{F} \wedge \mathcal{A}(\phi)$. By [Gue22b, Lemma 5.18 (7)], one of the following holds.

(i) There exist distinct $i, j \in \{1, \ldots, r\}$ such that, up to replacing A_i by a conjugate, we have $\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i], [A_j]\}) \cup \{[A_i * A_j]\}$.

(ii) There exist $i \in \{1, \ldots, r\}$ and an element $g \in F_N$ such that $\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i]\}) \cup \{[A_i * \langle g\rangle]\}$.

(iii) There exists $g \in F_N$ such that $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g\rangle]\}$.

By Definition 2.2, Assertion (ii) only occurs when the extension $\mathcal{F} \leqslant \{[F_N]\}$ is an HNN extension over the trivial group. In particular, we have $\mathcal{F} = \{[A]\}$ for some subgroup A of F_N and, up to changing the representative of $[A]$, we have $F_N = A * \langle g\rangle$ and $A_i \subseteq A$.

Case 1. — Suppose that there exist distinct $i, j \in \{1, \ldots, r\}$ such that

$$\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i], [A_j]\}) \cup \{[A_i * A_j]\}.$$

Since $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$ and $\mathrm{Poly}(\phi) \neq \mathrm{Poly}(H)$, there exists $\psi \in H$ such that, for every $n \in \mathbb{N}^*$, the element ψ^n does not preserve $[A_i * A_j]$ while preserving $[A_i]$ and $[A_j]$. Hence there exist a representative Ψ of ψ such that, for every $n \in \mathbb{N}^*$, there exists $g_n \in F_N - A_i * A_j$ such that $\Psi^n(A_i) = A_i$ and $\Psi^n(A_j) = g_n A_j g_n^{-1}$. Note that

$$g\Psi^n\left(A_i * A_j\right) g^{-1} = gA_ig^{-1} * gg_nA_jg_n^{-1}g^{-1}.$$

Claim 1. — *For every $n \in \mathbb{N}^*$ and every $g \in F_N$, there exist $t = t(g, n) \in F_N$ and $s = s(g, n) \in \{i, j\}$ such that*

$$(A_i * A_j) \cap \left(g\Psi^n(A_i * A_j)g^{-1}\right) \subseteq tA_st^{-1}.$$

Proof. — Let $n \in \mathbb{N}^*$. Note that, since $g_n \in F_N - A_i * A_j$ and since $A_i * A_j$ is a malnormal subgroup of F_N, $A_i * A_j$ is distinct from $gg_n(A_i * A_j)g_n^{-1}g^{-1}$ or from $g(A_i * A_j)g^{-1}$. Therefore, we can apply Lemma 4.1 (1) to ϕ and the polynomial subgroups $A_i * A_j$, $gg_n(A_i * A_j)g_n^{-1}g^{-1}$ and $g(A_i * A_j)g^{-1}$. Thus, there exist a subgroup B' of F_N containing the subgroups $A_i * A_j$, $gg_n(A_i * A_j)g_n^{-1}g^{-1}$ and $g(A_i * A_j)g^{-1}$ and an $\mathbb{R}$-tree T' equipped with a minimal, isometric action of B' with trivial arc stabilizers and such that the subgroups $A_i * A_j$, $gg_n(A_i * A_j)g_n^{-1}g^{-1}$ and $g(A_i * A_j)g^{-1}$ are elliptic but do not have a common fixed point. Let x_1 be the point in T' fixed by $A_i * A_j$, let x_2 be the point fixed by $g(A_i * A_j)g^{-1}$ and let x_3 be the point fixed by $gg_n(A_i * A_j)g_n^{-1}g^{-1}$.

Let

$$G = g\Psi^n\left(A_i * A_j\right)g^{-1} = gA_ig^{-1} * gg_nA_jg_n^{-1}g^{-1}.$$

Suppose first that $x_2 = x_3$. Then $x_1 \neq x_2$ by hypothesis. Note that the group $G \cap (A_i * A_j)$ fixes both x_1 and x_2. Since arc stabilizers are trivial, the intersection $G \cap (A_i * A_j)$ is trivial.

Thus, we may suppose that $x_2 \neq x_3$. Since arc stabilizers are trivial, by a standard ping pong argument, the points in T' fixed by elements of G are in the orbits of x_2 and x_3. Since arc stabilizers are trivial, and since G is the free product of gA_ig^{-1} and $gg_nA_jg_n^{-1}g^{-1}$, we see that $G \cap \mathrm{Stab}(x_2) = gA_ig^{-1}$ and $G \cap \mathrm{Stab}(x_3) = gg_nA_jg_n^{-1}g^{-1}$. Thus, elliptic elements in G are contained in conjugates of gA_ig^{-1} and conjugates of $gg_nA_jg_n^{-1}g^{-1}$. Since the intersection of G with $A_i * A_j$ is elliptic, it is contained in a conjugate of A_i or a conjugate of A_j. This proves Claim 1. ☐

Claim 1 implies that, for all distinct $m, n \in \mathbb{N}$ and every element $x \in F_N$, there exist $t = t(x, m, n) \in F_N$ and $s = s(x, m, n) \in \{i, j\}$ such that

$$\Psi^n\left(A_i * A_j\right) \cap \left(x\Psi^m(A_i * A_j)x^{-1}\right) \subseteq tA_st^{-1}.$$

By for instance [HM20, Fact I.1.2], for any subgroups A and B of F_N, we have the equalities $(\partial_\infty A) \cap (\partial_\infty B) = \partial_\infty(A \cap B)$ and $(\partial^2 A) \cap (\partial^2 B) = \partial^2(A \cap B)$. Thus, for all distinct $m, n \in \mathbb{N}$ and every $x \in F_N$, we have

$$\begin{aligned}
&\partial^2\left(\Psi^n(A_i * A_j)\right) \cap \partial^2\left(x\Psi^m(A_i * A_j)x^{-1}\right) \\
&\qquad = \partial^2\left(\Psi^n(A_i * A_j) \cap x\Psi^m(A_i * A_j)x^{-1}\right) \\
&\qquad \subseteq \partial^2(tA_st^{-1}) \\
&\qquad \subseteq \overline{\bigcup_{y \in F_N}\left(\partial^2\left(yA_iy^{-1}\right) \cup \partial^2\left(yA_jy^{-1}\right)\right)}.
\end{aligned}$$

By definition of $K_{PG}(\phi)$, we have $[\mu] \in K_{PG}(\phi)$ if and only if

$$\mathrm{Supp}(\mu) \subseteq \partial^2\mathcal{A}(\phi) \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)) = \partial^2\{[A_i * A_j]\} \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)).$$

Moreover, if $n \in \mathbb{N}$ and if $[\mu] \in \psi^n(K_{PG}(\phi))$, then

$$\mathrm{Supp}(\mu) \subseteq \partial^2\psi^n(\mathcal{A}(\phi)) \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)) = \partial^2\{[A_i * g_nA_jg_n^{-1}]\} \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)).$$

Let $n, m \in \mathbb{N}$ be distinct. Suppose towards a contradiction that

$$\psi^n(K_{PG}(\phi)) \cap \psi^m(K_{PG}(\phi)) \neq \varnothing$$

and let $[\mu] \in \psi^n(K_{PG}(\phi)) \cap \psi^m(K_{PG}(\phi))$. Thus, the support of μ is contained in

$$\left(\bigcup_{x,y \in F_N}\left(\partial^2(x(A_i * g_nA_jg_n^{-1})x^{-1})\right) \cap \left(\partial^2\left(y(A_i * g_mA_jg_m^{-1})y^{-1}\right)\right)\right) \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$$

and there exist $x, y \in F_N$ such that μ gives positive measure to

$$\left(\partial^2\left(x\left(A_i * g_nA_jg_n^{-1}\right)x^{-1}\right) \cap \partial^2\left(y\left(A_i * g_mA_jg_m^{-1}\right)y^{-1}\right)\right) \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)).$$

By F_N-invariance of μ, there exists $x \in F_N$ such that μ gives positive measure to

$$\begin{aligned}\partial^2\left(A_i * g_nA_jg_n^{-1}\right) \cap \partial^2\left(x\left(A_i * g_mA_jg_m^{-1}\right)x^{-1}\right) \cap \partial^2\left(F_N, \mathcal{F} \wedge \mathcal{A}(\phi)\right) \\ \subseteq \overline{\left(\bigcup_{y \in F_N} \partial^2\left(yA_iy^{-1}\right) \cup \partial^2\left(yA_jy^{-1}\right)\right)} \cap \partial^2(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))\end{aligned}$$

and the last intersection is empty by the definition of the relative boundary, a contradiction. ☐

Case 2. — Suppose that either there exist $i \in \{1, \ldots, r\}$ and an element $g \in F_N$ such that $\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i]\}) \cup \{[A_i * \langle g\rangle]\}$ or there exists $g \in F_N$ such that $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g\rangle]\}$.

In order to treat both cases simultaneously, in the case that there exists $g \in F_N$ such that $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g\rangle]\}$, we fix $A_i = \{e\}$.

Recall that we have $\mathcal{F} = \{[A]\}$ for some subgroup A of F_N and, up to changing the representative of $[A]$, we have $F_N = A * \langle g\rangle$ and $A_i \subseteq A$. In particular, since H preserves the extension $\mathcal{F} \leqslant \{[F_N]\}$, for every $\psi \in H$, there exist a unique representative Ψ_0 of ψ and $g_\psi \in A$ such that $\Psi_0(A) = A$ and $\Psi_0(g) = gg_\psi$.

Claim 2. — *There exists $\psi \in H$ such that, for every $n \in \mathbb{N}^*$, we have $g_{\psi^n} \notin A_i$.*

Proof. — First note that, since H is irreducible with respect to $\mathcal{F} \leqslant \{[F_N]\}$, the subgroup H does not preserve the free factor system $\mathcal{F} \cup \{[\langle g\rangle]\}$. Thus, there exists $\psi' \in H$ such that $g_{\psi'} \neq 1$.

Let S be the subset of H consisting in every element $\psi' \in H$ such that $g_{\psi'} \neq 1$. Note that, since $H \subseteq \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, for every $m \in \mathbb{N}^*$ and every $\psi' \in S$, we have $g_{\psi'^m} \neq 1$ as ψ'^m cannot fix the conjugacy class of g. Hence S is stable under taking powers. In particular, if A_i is trivial, any $\psi \in S$ satisfies the assertion of Claim 2. Similarly, the complement of S is stable under taking powers.

Note also that for every $\psi' \in S$, the elements g and $g_{\psi'}$ are contained in distinct factors of $A * \langle g\rangle$.

We now claim that there exists $\theta \in S$ such that one of the following holds:

(i) for any distinct $m, n \in \mathbb{N}^*$, we have $\Theta_0^n(A_i) \cap \Theta_0^m(A_i) = \{e\}$ (this is equivalent to the fact that, for all $m \neq n$, we have $\Theta_0^n(A_i) \neq \Theta_0^m(A_i)$);

(ii) for every $n \in \mathbb{N}^*$, we have $g_{\theta^n} \notin A_i$.

Indeed, for every element $\psi' \in S$, the automorphism Ψ'_0 acts naturally on the set of conjugates of A_i. If there exists $\psi' \in S$ such that A_i has an infinite orbit, then we may take $\theta = \psi'$, which satisfies (i).

Thus, we may suppose that, for every element $\psi \in S$, the element Ψ_0 has a power which preserves A_i. We now construct an element $\theta \in S$ which satisfies Assertion (ii).

Since $\mathrm{Poly}(H) \neq \mathrm{Poly}(\phi)$, there exists $\psi' \in H$ such that $A_i * \langle g\rangle \nsubseteq \mathrm{Poly}(\psi')$. We distinguish between two cases, according to whether $\psi' \in S$ or not.

If $\psi' \in S$, up to taking a power of ψ', we have $\Psi'_0(A_i) = A_i$ and $A_i * \langle g \rangle \nsubseteq \mathrm{Poly}(\psi')$.

Note that A_i is then contained in the polynomial subgroup of the automorphism Ψ'_0. As $A_i * \langle g \rangle \nsubseteq \mathrm{Poly}(\psi')$, for every $n \in \mathbb{N}^*$, we have $g_{\psi'^n} \notin A_i$. Thus, we may take $\theta = \psi'$.

So we may suppose that $\psi' \notin S$ and, for every $\theta' \in S$, that $A_i * \langle g \rangle \subseteq \mathrm{Poly}(\theta')$. Thus, there exists $\theta' \in S$ such that $\Theta'_0(A_i) = A_i$ and $g_{\theta'} \in A_i$. Moreover, we have $\Psi'_0(g) = g$ and, since $A_i * \langle g \rangle \nsubseteq \mathrm{Poly}(\psi')$, the subgroup A_i has an infinite orbit under iteration of Ψ'_0.

Then, for every $n \in \mathbb{N}^*$, we have

$$\Theta'_0 \Psi'^n_0 \Theta'^{-1}_0(g) = g g_{\theta' \psi'^n \theta'^{-1}} = g g_{\theta'} \Theta'_0 \left(\Psi'^n_0 (g_{\theta'^{-1}}) \right).$$

Since $g_{\theta'^{-1}} \in A_i$, we have $\Psi'^n_0(g_{\theta'^{-1}}) \notin A_i$ and $\Theta'_0(\Psi'^n_0(g_{\theta'^{-1}})) \notin A_i$. Since $g_{\theta'} \in A_i$, we have $g_{\theta' \psi'^n \theta'^{-1}} \notin A_i$. Therefore, the element $\theta' \psi' \theta'^{-1} \in S$ satisfies Assertion (ii). Hence we may take $\theta = \theta' \psi' \theta'^{-1}$. This proves the existence of θ.

Suppose first that θ satisfies Assertion (ii). Then we may set $\psi = \theta$, so that ψ satisfies the assertion of Claim 2. Otherwise, θ satisfies (i) and, up to taking a power of θ, we may suppose that $g_\theta \in A_i$.

We claim that θ^2 satisfies the assertion of Claim 2. Indeed, note that, for every $n \in \mathbb{N}^*$, we have

$$g_{\theta^{2n}} = h_0 \dots h_{2n-1},$$

where, for every $j \in \{0, \dots, 2n-1\}$, the element h_j is a nontrivial element of $\Theta^j_0(A_i)$, the fact that h_j is nontrivial following from the fact that $\theta \in S$.

Thus, in order to show that θ^2 satisfies the assertion of Claim 2, it suffices to show that, for every $m \in \mathbb{N}$, we have

$$\left\langle \Theta^j_0(A_i) \right\rangle_{j \in \{0, \dots, m\}} = A_i * \dots * \Theta^m_0(A_i). \tag{4.1}$$

We prove Equation (4.1) by induction on m, the result being trivial when $m = 0$.

Since θ satisfies Assertion (i), for any distinct $j, k \in \{0, \dots, m\}$, we have

$$\Theta^j_0(A_i) \neq \Theta^k_0(A_i).$$

In particular, we can apply Lemma 4.1 (1) to the outer class $[\Phi_0|_A] \in \mathrm{Out}(A)$ and the set $\{\Theta^j_0(A_i)\}_{j \in \{0, \dots, m\}}$ of polynomial subgroups of $[\Phi_0|_A]$. Thus, there exists a subgroup B' of A containing $\{\Theta^j_0(A_i)\}_{j \in \{0, \dots, m\}}$ and an $\mathbb{R}$-tree T' equipped with a minimal, isometric action of B' with trivial arc stabilizers, such that, for every $j \in \{0, \dots, m\}$, the group $\Theta^j_0(A_i)$ fixes a point x_j and there exist distinct k_1, k_2 such that $x_{k_1} \neq x_{k_2}$.

Since T' has trivial arc stabilizers, the groups $\mathrm{Stab}(x_0), \dots, \mathrm{Stab}(x_m)$ generate their free product. Since there exist k_1, k_2 such that $x_{k_1} \neq x_{k_2}$, for every $\ell \in \{0, \dots, m\}$, the group $\mathrm{Stab}(x_\ell)$ contains at most $m-1$ elements of the set $\{\Theta^j_0(A_i)\}_{j \in \{0, \dots, m\}}$. Thus, we can apply the induction hypothesis to conclude the proof of Equation (4.1) and thus the proof of Claim 2. ☐

Let $\psi \in H$ and g_ψ be as in the claim. We claim that, for every $n \in \mathbb{N}^*$, the conjugacy class $[g g_{\psi^n}]$ has exponential growth under iteration of ϕ. Indeed, recall the construction of Φ_0 above Claim 2. Since $g_\phi, g_{\psi^n} \in A$ and since $\Phi_0(A) = A$, for every $m \in \mathbb{N}$, the element $\Phi^m_0(g g_{\psi^n})$ is cyclically reduced. Hence $[g g_{\psi^n}]$ has exponential growth under iteration of ϕ if and only if gg_{ψ^n} has exponential growth under iteration of Φ_0. But the polynomial subgroup of Φ_0 is $A_i * \langle g\rangle$. Since $g_{\psi^n} \notin A_i$, the element gg_{ψ^n} has exponential growth under iteration of Φ_0. This proves the claim. In particular, for every $n \in \mathbb{N}^*$, no conjugate of gg_{ψ^n} is contained in $A_i * \langle g\rangle$.

Let $\Psi \in \psi$ be such that, for every $n \in \mathbb{N}^*$, there exists $h_{\psi^n} \in A$ with $\Psi^n(A_i) = A_i$ and $\Psi^n(g) = h_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}$. Note that, for every $n \in \mathbb{N}^*$, we have

$$\Psi^n\left(A_i * \langle g\rangle\right) = A_i * h_{\psi^n}\langle gg_{\psi^n}\rangle h_{\psi^n}^{-1}.$$

Claim 3. — *For every $n \in \mathbb{N}^*$ and every $a \in F_N$, there exists $t = t(n, a)$ such that*

$$\left(a\Psi^n(A_i * \langle g\rangle)a^{-1}\right) \cap (A_i * \langle g\rangle) \subseteq tA_it^{-1}.$$

Proof. — Let $n \in \mathbb{N}^*$ and let $a \in F_N$. First note that $ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1} \notin a(A_i * \langle g\rangle)a^{-1}$. Indeed, since $F_N = A * \langle g\rangle$, the element $h_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}$ can be written uniquely as a reduced product of elements in A and elements in $\langle g\rangle$. Since $h_{\psi^n}, g_{\psi^n} \in A$, if we have $h_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1} \in A_i * \langle g\rangle$, then $h_{\psi^n} \in A_i$ and $g_{\psi^n} h_{\psi^n}^{-1} \in A_i$. Therefore, we have $g_{\psi^n} \in A_i$, a contradiction. Thus, we have $ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1} \notin a(A_i * \langle g\rangle)a^{-1}$.

In particular, we can apply Lemma 4.1 (2) to ϕ, the polynomial subgroups $A_i * \langle g\rangle$, $a(A_i * \langle g\rangle)a^{-1}$ and the element $ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1}$. This shows that there exist a subgroup B' of F_N containing $A_i * \langle g\rangle$, $a(A_i * \langle g\rangle)a^{-1}$ and $ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1}$, and an $\mathbb{R}$-tree T' equipped with a minimal, isometric action of B' with trivial arc stabilizers and such that $A_i * \langle g\rangle$ fixes a point x_1 in T', $a(A_i * \langle g\rangle)a^{-1}$ fixes a point $x_2 = ax_1$ in T' and if $x_1 = x_2$, then $ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1}$ does not fix x_1.

Let $G = (a\Psi^n(A_i * \langle g\rangle)a^{-1}) \cap (A_i * \langle g\rangle)$. The group G fixes x_1. Let $h \in G$. Since we have $h \in a\Psi^n(A_i * \langle g\rangle)a^{-1}$, the element h can be written as a product of elements $s_0a_1b_1 \dots a_kb_ks_0^{-1}$ where the element s_0 is in $a\Psi^n(A_i * \langle g\rangle)a^{-1}$ and, for every $i \in \{1, \dots, k\}$, we have $a_i \in aA_ia^{-1}$ and $b_i \in \langle ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1}\rangle$. We suppose that $a_1b_1 \dots a_kb_k$ is a cyclic reduction of h when written in the free product $aA_ia^{-1} * \langle ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1}\rangle$. We will prove that h is a conjugate of a_1.

Suppose first that $ah_{\psi^n} gg_{\psi^n} h_{\psi^n}^{-1}a^{-1}$ fixes a point x in T'. We distinguish between two cases, according to x.

Suppose that $x = x_2$. Then $x_1 \neq x_2$. Recall that

$$a\Psi^n(A_i * \langle g\rangle)a^{-1} = a\left(A_i * h_{\psi^n}\langle gg_{\psi^n}\rangle h_{\psi^n}^{-1}\right)a^{-1}.$$

Thus $a\Psi^n(A_i * \langle g\rangle)a^{-1}$ fixes x_2 and h fixes both x_1 and x_2. Since T' has trivial arc stabilizers, we see that $h = e$.

Suppose now that $x \neq x_2$. Then the minimal tree in T' of the subgroup of F_N generated by $\mathrm{Stab}(x)$ and $\mathrm{Stab}(x_2)$ is simplicial and its vertex stabilizers are conjugates of $\mathrm{Stab}(x)$ and $\mathrm{Stab}(x_2)$. Recall that $a\Psi^n(A_i * \langle g\rangle)a^{-1}$ is a free product with one factor fixing x and the other factor fixing x_2. Thus, since arc stabilizers in T' are trivial, elliptic elements of $a\Psi^n(A_i * \langle g\rangle)a^{-1}$ are contained in conjugates of A_i or in conjugates of $h_{\psi^n}\langle gg_{\psi^n}\rangle h_{\psi^n}^{-1}$. Since h is elliptic in T', we see that h is conjugate to either a_1 or b_k.

Recall that we proved above Claim 3 that $gg_{\psi^n} \notin \mathrm{Poly}(\phi)$. Thus, no conjugate of gg_{ψ^n} is contained in $A_i * \langle g\rangle$. Since $h \in A_i * \langle g\rangle$, the element h is conjugate to a_1.

Finally, suppose that $ah_{\psi^n}gg_{\psi^n}h_{\psi^n}^{-1}a^{-1}$ is loxodromic. Then the minimal tree in T' of $\langle \mathrm{Stab}(x_2), ah_{\psi^n}gg_{\psi^n}h_{\psi^n}^{-1}a^{-1}\rangle$ is simplicial and its vertex stabilizers are either trivial or conjugates of $\mathrm{Stab}(x_2)$. Note that $a\Psi^n(A_i * \langle g\rangle)a^{-1}$ is a free product with one factor, A_i, fixing x_2 and the other factor being cyclic, generated by the loxodromic element $ah_{\psi^n}gg_{\psi^n}h_{\psi^n}^{-1}a^{-1}$. Thus, since arc stabilizers in T' are trivial, elliptic elements of the group $a\Psi^n(A_i * \langle g\rangle)a^{-1}$ are contained in conjugates of A_i. Since h fixes x_1, it is contained in a conjugate of A_i. Thus, in all cases, h is contained in a conjugate of A_i.

Therefore, every element of G is contained in a conjugate of A_i. Recall that $A_i * \langle g\rangle$ is a malnormal subgroup of F_N, so that every conjugate of A_i intersecting $A_i * \langle g\rangle$ nontrivially is a conjugate of A_i whose conjugator is in $A_i * \langle g\rangle$. Thus every element of G fixes a point in the Bass-Serre tree S of $A_i * \langle g\rangle$ associated with A_i. Since edge stabilizers in S are trivial, this implies that the group G fixes a point in S, hence is contained in a conjugate of A_i. This proves Claim 3. □

Claim 3 implies that, for all distinct $n, m \in \mathbb{N}^*$ and every $x \in F_N$, there exists $t = t(m, n, x)$ such that we have

$$\Psi^n(A_i * \langle g\rangle) \cap x\Psi^m(A_i * \langle g\rangle)x^{-1} \subseteq tA_it^{-1}$$

By [HM20, Fact I.1.2], we have

$$\partial^2\Psi^n(A_i * \langle g\rangle) \cap \partial^2\left(x\Psi^m(A_i * \langle g\rangle)x^{-1}\right) \subseteq \partial^2\left(tA_it^{-1}\right) \subseteq \overline{\bigcup_{y\in F_N} \partial^2\left(yA_iy^{-1}\right)}.$$

The rest of the proof is then similar to the one of Case 1. □

Lemma 4.3. — *Let $N \geqslant 3$, let $\mathcal{F}$ and $\mathcal{F}_1 = \{[A_1], \ldots, [A_k]\}$ be two free factor systems of F_N with $\mathcal{F} \leqslant \mathcal{F}_1$ such that the extension $\mathcal{F} \leqslant \mathcal{F}_1$ is sporadic. Let H be a subgroup of $\mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1) \cap \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ such that H is irreducible with respect to $\mathcal{F} \leqslant \mathcal{F}_1$. Suppose that there exists $\phi \in H$ such that $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. Suppose that $\mathrm{Poly}(\phi|_{\mathcal{F}_1}) \neq \mathrm{Poly}(H|_{\mathcal{F}_1})$. There exists $\psi \in H$ such that for every $i \in \{1, \ldots, k\}$, we have $\psi(K_{PG}([\Phi_i|_{A_i}])) \cap K_{PG}([\Phi_i|_{A_i}]) = \varnothing$, where $[\Phi_i|_{A_i}]$ is defined in Equation (2.2) of Section 2.3 and*

$$\Delta_+([A_i], \phi) \cap \psi(\Delta_-([A_i], \phi)) = \Delta_-([A_i], \phi) \cap \psi(\Delta_+([A_i], \phi)) = \varnothing.$$

Proof. — The proof follows [CU20, Lemma 5.1]. Recall that, since the extension $\mathcal{F} \leqslant \mathcal{F}_1$ is sporadic, there exists $\ell \in \{1, \ldots, k\}$ such that, for every $i \in \{1, \ldots, k\} - \{\ell\}$, we have $[A_i] \in \mathcal{F}$. By Lemma 4.2 applied to the image of H in $\mathrm{Out}(A_\ell)$ (which is contained in $\mathrm{IA}(A_\ell, \mathbb{Z}/3\mathbb{Z})$), there exists an infinite subset $X \subseteq H$ such that, for any distinct $h_1, h_2 \in X$, we have

$$h_1(K_{PG}\Big([\Phi_\ell|_{A_\ell}])\Big) \cap h_2(K_{PG}([\Phi_\ell|_{A_\ell}])) = \varnothing.$$

We now prove that there exist $h_1, h_2 \in X$ such that $h_2^{-1}h_1$ satisfies the assertion of Lemma 4.3. Note that, for any distinct $h_1, h_2 \in X$, we have

$$h_2^{-1}h_1(K_{PG}([\Phi_\ell|_{A_\ell}])) \cap K_{PG}([\Phi_\ell|_{A_\ell}]) = \varnothing.$$

Hence it suffices to find two distinct $h_1, h_2 \in X$ such that $\psi = h_2^{-1}h_1$ satisfies the second assertion of Lemma 4.3.

Let $i \in \{1, \ldots, k\}$ and let $[\mu]$ be an extremal point of $\Delta_+([A_i], \phi)$ or $\Delta_-([A_i], \phi)$. By [Gue22b, Lemma 4.13], the support $\mathrm{Supp}(\mu)$ contains the support of *only* finitely many projective currents $[\mu_1], \ldots, [\mu_s] \in \mathbb{P}\mathrm{Curr}(F_N, \mathcal{F} \wedge \mathcal{A}(\phi))$ such that, for every $t \in \{1, \ldots, s\}$, the support of μ_t is uniquely ergodic.

Let $E_\mu = \{[\mu_1], \ldots, [\mu_s]\}$. Let $E_\phi = \bigcup E_\mu$, where the union is taken over all i in $\{1, \ldots, k\}$ and extremal points of $\Delta_+([A_i], \phi)$ and $\Delta_-([A_i], \phi)$. The set E_ϕ is finite by [Gue22b, Lemma 4.7].

Since the set E_ϕ is finite, up to taking an infinite subset of X, we may suppose that, for every $s \in E_\phi$, either $h_1 s = h_2 s$ for every $h_1, h_2 \in X$ or for every distinct $h_1, h_2 \in X$, we have $h_1 s \neq h_2 s$. Let $E_1 \subseteq E_\phi$ be the subset for which the first alternative occurs and let $E_\infty = E_\phi - E_1$.

Let $h_1 \in X$ and, for every $s \in E_\infty$, let

$$X_s = \{h \in X \mid h_1 s = hs' \text{ for some } s' \in E_\infty\}.$$

Note that X_s is a finite set. Let $h_2 \in X - \bigcup_{s \in E_\infty} X_s$. For every $s, s' \in E_\infty$, we have $h_1 s \neq h_2 s'$. If there exists $s' \in E_1$ such that $h_1 s = h_2 s'$, then $s = h_1^{-1} h_2 s' = s'$, contradicting the fact that $s \in E_\infty$. Thus, for every $s \in E_\infty$, we have $h_2^{-1} h_1 s \notin E_\phi$ and for every $s \in E_1$, we have $h_2^{-1} h_1 s = s$. Let $\psi = h_2^{-1} h_1$. Then, for every $s \in E_\phi$, either $\psi(s) = s$ or $\psi(s) \notin E_\phi$. Moreover, by construction of X, for every $i \in \{1, \ldots, k\}$, we have $\psi(K_{PG}([\Phi_i|_{A_i}])) \cap K_{PG}([\Phi_i|_{A_i}]) = \varnothing$. Thus, ψ satisfies the first assertion of Lemma 4.3.

We now prove that ψ satisfies the second assertion. Let $i \in \{1, \ldots, k\}$, let $[\mu] \in \Delta_-([A_i], \phi)$ and suppose for a contradiction that we have $\psi([\mu]) \in \Delta_+([A_i], \phi)$. There exist extremal measures $\mu_1^-, \ldots, \mu_m^-$ of $\Delta_-([A_i], \phi)$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$ such that $\mu = \sum_{j=1}^m \lambda_j \mu_j^-$. Similarly, there exist extremal measures $\mu_1^+, \ldots, \mu_n^+$ of $\Delta_+([A_i], \phi)$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+$ such that $\psi(\mu) = \sum_{j=1}^n \alpha_j \mu_j^+$.

Thus, we have

$$\sum_{j=1}^m \lambda_j \psi(\mu_j^-) = \psi(\mu) = \sum_{j=1}^n \alpha_j \mu_j^+.$$

In particular, we have

$$\bigcup_{j=1}^m \mathrm{Supp}\left(\psi(\mu_j^-)\right) = \bigcup_{j=1}^n \mathrm{Supp}(\mu_j^+).$$

Let $\Lambda \subseteq \mathrm{Supp}(\mu_1^-)$ be the uniquely ergodic support of a current in E_ϕ. Let Ψ be a representative of ψ and let $\partial^2\Psi$ be the homeomorphism of $\partial^2 F_N$ induced by Ψ. Since uniquely ergodic laminations are minimal, there exists $j \in \{1, \ldots, n\}$ such that we have $\partial^2\Psi(\Lambda) \subseteq \mathrm{Supp}(\mu_j^+)$. Thus, we have $\psi([\mu_1^-|_\Lambda]) = [\mu_j^+|_\Lambda]$. This contradicts the fact that $[\mu_1^-|_\Lambda]$ and $[\mu_j^+|_\Lambda]$ are distinct elements of E_ϕ since $\Delta_+([A_i], \phi) \cap \Delta_-([A_i], \phi) = \varnothing$. $\square$

Proposition 4.4. — *Let $N \geqslant 3$, let $\mathcal{F}$ and $\mathcal{F}_1 = \{[A_1], \ldots, [A_k]\}$ be two free factor systems of F_N with $\mathcal{F} \leqslant \mathcal{F}_1$ such that the extension $\mathcal{F} \leqslant \mathcal{F}_1$ is sporadic. Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) \cap \mathrm{Out}(F_N, \mathcal{F}, \mathcal{F}_1)$ such that H is irreducible with respect to $\mathcal{F} \leqslant \mathcal{F}_1$. Suppose that there exists $\phi \in H$ such that $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. Suppose that $\mathrm{Poly}(\phi|_{\mathcal{F}_1}) \neq \mathrm{Poly}(H|_{\mathcal{F}_1})$. There exist $\psi \in H$ and a constant $M > 0$ such that, for all $m, n \geqslant M$, if $\theta = \psi\phi\psi^{-1}$, we have $\mathrm{Poly}(\theta^m \phi^n|_{\mathcal{F}_1}) = \mathrm{Poly}(H|_{\mathcal{F}_1})$.*

Proof. — The proof follows [CU20, Proposition 5.2]. Let $\psi \in H$ be an element given by Lemma 4.3 and let $\theta = \psi\phi\psi^{-1}$. For every $i \in \{1, \dots, k\}$, let Θ_i be a representative of θ such that $\Theta_i(A_i) = A_i$ and Φ_i be a representative of ϕ such that $\Phi_i(A_i) = A_i$. Note that, since for every $i \in \{1, \dots, k\}$, $[\Phi_i|_{A_i}]$ is almost atoroidal relative to $\mathcal{F}$, so is $[\Theta_i|_{A_i}]$. Moreover, for every $i \in \{1, \dots, k\}$, we have $K_{PG}([\Theta_i|_{A_i}]) = [\Psi_i|_{A_i}](K_{PG}([\Phi_i|_{A_i}]))$.

Let $i \in \{1, \dots, k\}$. Let $\mathcal{F} \wedge \{[A_i]\}$ be the free factor system of A_i induced by $\mathcal{F}$: it is the free factor system of A_i consisting in the intersection of A_i with every subgroup A of F_N such that $[A] \in \mathcal{F}$. It is well-defined by for instance [SW79, Theorem 3.14].

Claim. — *We have*

$$\widehat{\Delta}_+([A_i],\phi) \cap \psi\left(\widehat{\Delta}_-([A_i],\phi)\right) = \varnothing \text{ and } \widehat{\Delta}_-([A_i],\phi) \cap \psi\left(\widehat{\Delta}_+([A_i],\phi)\right) = \varnothing.$$

Proof. — We prove the first equality, the other one being similar. By Lemma 4.3, we have $\Delta_+([A_i],\phi) \cap \psi(\Delta_-([A_i],\phi)) = \varnothing$ and $\psi(K_{PG}([\Phi_i|_{A_i}])) \cap K_{PG}([\Phi_i|_{A_i}]) = \varnothing$.

Let $[\mu] \in \widehat{\Delta}_+([A_i],\phi) \cap \psi(\widehat{\Delta}_-([A_i],\phi))$. By definition, there exist $[\mu_1] \in \Delta_+([A_i],\phi)$, $[\nu_1] \in K_{PG}([\Phi_i|_{A_i}])$, $t \in [0,1]$, and $[\mu_2] \in \psi(\Delta_-([A_i],\phi)$, $[\nu_2] \in \psi(K_{PG}([\Phi_i|_{A_i}]))$ and $s \in [0,1]$ such that

$$[\mu] = [t\mu_1 + (1-t)\nu_1] = [s\mu_2 + (1-s)\nu_2].$$

Note that

$$\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi)) = \varnothing.$$

Moreover, since $\mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$, we have $\mathrm{Poly}(\theta|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}})$. Therefore, we see that $\mathcal{F} \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \psi(\mathcal{A}(\phi))$. Thus, we have

$$\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \psi\left(\partial^2\mathcal{A}(\phi)\right) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi)) = \varnothing.$$

Recall that, by Proposition 2.12, the supports of the currents in

$$\Delta_+([A_i],\phi) \cup \psi(\Delta_-([A_i],\phi))$$

are contained in $\partial^2(\mathcal{F} \wedge \{[A_i]\})$. Thus, we have

$$\begin{aligned}\mu_1&\left(\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))\right) \\ &= \mu_1\left(\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))\right) = 0.\end{aligned}$$

Since $\mathcal{F} \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \psi(\mathcal{A}(\phi))$, we also have

$$\begin{aligned}\mu_2&\left(\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))\right) \\ &= \mu_2\left(\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))\right) \\ &= \mu_2\left(\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \psi(\partial^2\mathcal{A}(\phi)) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))\right) = 0.\end{aligned}$$

Thus, if B is a measurable subset contained in $\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))$ and if $s, t < 1$, we have: $\mu(B) > 0$ if and only if $\nu_1(B) > 0$ if and only if $\nu_2(B) > 0$.

By definition, the supports of currents in $K_{PG}([\Phi_i|_{A_i}])$ are contained in the subset $\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))$ and the supports of currents in $\psi(K_{PG}([\Phi_i|_{A_i}]))$

are contained in $\psi(\partial^2\mathcal{A}(\phi))\cap\partial^2(A_i,\mathcal{F}\wedge\{[A_i]\}\wedge\mathcal{A}(\phi))$. Hence the support of ν_1 is contained in the support of ν_2. By definition of $\psi(K_{PG}([\Phi_i|_{A_i}]))$, this implies that

$$\nu_1\in K_{PG}([\Phi_i|_{A_i}])\cap\psi(K_{PG}([\Phi_i|_{A_i}]))=\varnothing.$$

Thus, we necessarily have $t=1$.

Thus, we have $[\mu]=[\mu_1]$ and the support of μ is contained in $\partial^2(\mathcal{F}\wedge\{[A_i]\})$. Since the support of ν_2 is contained in $\psi(\partial^2\mathcal{A}(\phi))\cap\partial^2(A_i,\mathcal{F}\wedge\{[A_i]\}\wedge\mathcal{A}(\phi))$ which is disjoint from $\partial^2(\mathcal{F}\wedge\{[A_i]\})$, we also have $s=1$. This implies that $[\mu_1]=[\mu_2]$ and that $\Delta_+([A_i],\phi)\cap\psi(\Delta_-([A_i],\phi))\neq\varnothing$, a contradiction. ☐

By the claim, there exist subsets $U,V,\widehat{U},\widehat{V}$ of $\mathbb{P}\mathrm{Curr}(A_i,(\mathcal{F}\wedge\{[A_i]\})\wedge\mathcal{A}(\phi))$ such that:

(1) $\Delta_+([A_i],\phi)\subseteq U$, $\widehat{\Delta}_+([A_i],\phi)\subseteq\widehat{U}$, $\Delta_-([A_i],\phi)\subseteq V$, $\widehat{\Delta}_-([A_i],\phi)\subseteq\widehat{V}$;
(2) $U\subseteq\widehat{U}$, $V\subseteq\widehat{V}$ and $U\cap K_{PG}(\phi)=V\cap K_{PG}(\phi)=\varnothing$;
(3) $\widehat{U}\cap\psi(\widehat{V})=\varnothing$ and $\widehat{V}\cap\psi(\widehat{U})=\varnothing$.

Note that Assertion (2) implies that $U\subsetneq\widehat{U}$ (resp. $V\subsetneq\widehat{V}$) since $K_{PG}(\phi)\subseteq\widehat{U}$ (resp. $K_{PG}(\phi)\subseteq\widehat{V}$). Let $\mathfrak{B}$ and $C>0$ be respectively the basis of F_N and the constant given by Proposition 2.11 (1). Let $M_0(\phi)$ (resp. $M_0(\theta^{-1})$) be the constant associated with ϕ, U and $\widehat{V}$ (resp θ^{-1}, $\psi(V)$ and $\psi(\widehat{U})$) given by Theorem 2.13. Let $M_1(\phi)$ and $L_1(\phi)$, (resp. $M_1(\theta)$ and $L_1(\theta)$) be the constants associated with $[\Phi_i|_{A_i}]$ and $\widehat{V}$ (resp. $[\Theta_i|_{A_i}]$ and $\psi(\widehat{V})$) given by Proposition 2.14. Similarly, let $M_1(\phi^{-1})$ and $L_1(\phi^{-1})$, (resp. $M_1(\theta^{-1})$ and $L_1(\theta^{-1})$) be the constants associated with $[\Phi_i|_{A_i}^{-1}]$ and $\widehat{U}$ (resp. $[\Theta_i|_{A_i}^{-1}]$ and $\psi(\widehat{U})$) given by Proposition 2.14. Let

$$M(i)=\max\Big\{M_0(\phi),M_0(\theta^{-1}),M_1(\phi),M_1(\theta),M_1(\phi^{-1}),M_1(\theta^{-1})\Big\}$$

and let

$$L=\min\Big\{L_1(\phi),L_1(\theta),L_1(\phi^{-1}),L_1(\theta^{-1})\Big\}>0.$$

Let $M(i)'$ be such that $3^{M(i)'}L^2>1$. Let

$$M=\max_{i\in\{1,\ldots,k\}}M(i)\quad\text{and}\quad M'=\max_{i\in\{1,\ldots,k\}}M(i)'.$$

Let $m,n\geqslant M+M'$ and let $\mu\in\mathrm{Curr}(A_i,\mathcal{F}\wedge\{[A_i]\}\wedge\mathcal{A}(\phi))$ be a nonzero current. We will prove that $[\mu]\notin K_{PG}(\theta^m\phi^n)$. This will imply that for every element $g\in F_N$ such that $\eta_{[g]}\in\mathrm{Curr}(A_i,\mathcal{F}\wedge\{[A_i]\}\wedge\mathcal{A}(\phi))$, we have $g\notin\mathrm{Poly}(\theta^m\phi^n)$. The proof is in two steps according to whether $[\mu]\in\widehat{V}$ or not.

• Suppose first that $[\mu]\notin\widehat{V}$. Then by Theorem 2.13, we have $\phi^n(\mu)\in U$. By Proposition 2.14, we have $\|\phi^n(\mu)\|_{\mathcal{F}}\geqslant 3^{n-M}L\|\mu\|_{\mathcal{F}}$. Since $U\cap\psi(\widehat{V})=\varnothing$, by Proposition 2.14, we have

$$\|\theta^m\phi^n(\mu)\|_{\mathcal{F}}\geqslant 3^{m-M}L\,\|\phi^n(\mu)\|_{\mathcal{F}}\geqslant 3^{m+n-2M}L^2\|\mu\|_{\mathcal{F}}.$$

Note that, by Theorem 2.13 applied to θ and the open subsets $\psi(V)$ $\psi(U)$, $\psi(\widehat{V})$ and $\psi(\widehat{U})$, we have $\theta^m\phi^n([\mu])\in\psi(U)\subseteq\psi(\widehat{U})$. Since $\widehat{V}\cap\psi(\widehat{U})=\varnothing$, we have

$\theta^m\phi^n([\mu]) \notin \widehat{V}$. Therefore, we can apply the same arguments replacing μ by $\theta^m\phi^n(\mu)$ and an inductive argument shows that, for every $n' \in \mathbb{N}^*$, we have

$$\left\|(\theta^m\phi^n)^{n'}(\mu)\right\|_{\mathcal{F}} \geqslant 3^{n'(m+n-2M-M')}\left(3^{M'}L^2\right)^{n'}\|\mu\|_{\mathcal{F}}.$$

Therefore, if μ is the current associated with an $\mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi)$-nonperipheral element $g \in A_i$ with $[\mu] \notin \widehat{V}$, for every $n' \geqslant 1$, by Proposition 2.11 (1) we have

$$\ell_{\mathfrak{B}}\Big((\theta^m\phi^n)^{n'}([g])\Big) \geqslant 3^{n'(m+n-2M-M')}\left(3^{M'}L^2\right)^{n'}C\|\mu\|_{\mathcal{F}} \geqslant 3^{n'(m+n-2M-M')}C.$$

Hence we have $g \notin \mathrm{Poly}([\Theta_i^m\Phi_i^n|_{A_i}])$.

- Suppose now that $[\mu] \in \widehat{V}$. As in the first case, This implies that $[\mu] \notin \psi(\widehat{U})$ and, by Theorem 2.13, that $\theta^{-m}([\mu]) \in \psi(V)$. By Proposition 2.14, we have $\|\theta^{-m}(\mu)\|_{\mathcal{F}} \geqslant 3^{m-M}L\|\mu\|_{\mathcal{F}}$. Since $\psi(V) \cap \widehat{U} = \varnothing$, we have $\theta^{-m}([\mu]) \notin \widehat{U}$ and

$$\left\|\phi^{-n}\theta^{-m}(\mu)\right\|_{\mathcal{F}} \geqslant 3^{n-M}L\left\|\theta^{-m}(\mu)\right\|_{\mathcal{F}} \geqslant 3^{n+m-2M-M'}\left(3^{M'}L^2\right)\|\mu\|_{\mathcal{F}}.$$

By Theorem 2.13, we have $\phi^{-n}\theta^{-m}(\mu) \in V$. As in the first case, since $\widehat{V} \cap \psi(\widehat{U}) = \varnothing$, we have $\phi^{-n}\theta^{-m}(\mu) \notin \psi(\widehat{U})$ and, for every $n' \in \mathbb{N}^*$, we have

$$\left\|\left(\phi^{-n}\theta^{-m}\right)^{n'}(\mu)\right\|_{\mathcal{F}} \geqslant 3^{n'(m+n-2M-M')}\left(3^{M'}L^2\right)^{2n'}\|\mu\|_{\mathcal{F}}.$$

Therefore as in the first case, replacing μ by the rational current associated with an $\mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi)$-nonperipheral element $g \in A_i$ with $[\mu] \in \widehat{V}$, we see that

$$g \notin \mathrm{Poly}\left([\Phi_i^{-n}\Theta_i^{-m}|_{A_i}]\right) = \mathrm{Poly}\left([\Theta_i^m\Phi_i^n|_{A_i}]\right).$$

Therefore, $\theta^m\phi^n|_{\mathcal{F}_1}$ is expanding relative to $\mathcal{F} \wedge \mathcal{A}(\phi)$. Thus, we have

$$\mathrm{Poly}(\theta^m\phi^n|_{\mathcal{F}_1}) = \mathrm{Poly}(\phi|_{\mathcal{F}}) = \mathrm{Poly}(H|_{\mathcal{F}}) \subseteq \mathrm{Poly}(H|_{\mathcal{F}_1}).$$

Since $\mathrm{Poly}(H|_{\mathcal{F}_1}) \subseteq \mathrm{Poly}(\theta^m\phi^n|_{\mathcal{F}_1})$, we have in fact $\mathrm{Poly}(H|_{\mathcal{F}_1}) = \mathrm{Poly}(\theta^m\phi^n|_{\mathcal{F}_1})$. This concludes the proof of Proposition 4.4. ☐

Proposition 4.5. — *Let $N \geqslant 3$ and let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Let*

$$\varnothing = \mathcal{F}_0 < \mathcal{F}_1 < \ldots < \mathcal{F}_k = \{[F_N]\}$$

be a maximal H-invariant sequence of free factor systems. Let $2 \leqslant i \leqslant k$. Suppose that $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$ is sporadic. Suppose that there exists $\phi \in H$ such that

(a) $\mathrm{Poly}(H|_{\mathcal{F}_{i-1}}) = \mathrm{Poly}(\phi|_{\mathcal{F}_{i-1}})$*;*
(b) *for every $j \in \{1, \ldots, k\}$, if the extension $\mathcal{F}_{j-1} \leqslant \mathcal{F}_j$ is nonsporadic, then $\phi|_{\mathcal{F}_j}$ is fully irreducible relative to $\mathcal{F}_{j-1}$ and if $H|_{\mathcal{F}_j}$ is atoroidal relative to $\mathcal{F}_{j-1}$, so is $\phi|_{\mathcal{F}_j}$.*

Then there exists $\widehat{\phi} \in H$ such that:

(1) $\mathrm{Poly}(H|_{\mathcal{F}_i}) = \mathrm{Poly}(\widehat{\phi}|_{\mathcal{F}_i})$*;*
(2) *for every $j \in \{1, \ldots, k\}$, if the extension $\mathcal{F}_{j-1} \leqslant \mathcal{F}_j$ is nonsporadic, then $\widehat{\phi}|_{\mathcal{F}_j}$ is fully irreducible relative to $\mathcal{F}_{j-1}$ and if $H|_{\mathcal{F}_j}$ is atoroidal relative to $\mathcal{F}_{j-1}$, so is $\widehat{\phi}|_{\mathcal{F}_j}$.*

Proof. — The proof follows [CU20, Proposition 5.3]. If $\mathrm{Poly}(H|_{\mathcal{F}_i}) = \mathrm{Poly}(\phi|_{\mathcal{F}_i})$, we may take $\widehat{\phi} = \phi$.

Otherwise, by Proposition 4.4, there exist $\psi \in H$ and a constant $M > 0$ such that, for every $m, n \geqslant M$, if $\theta = \psi\phi\psi^{-1}$, we have $\mathrm{Poly}(\theta^m\phi^n|_{\mathcal{F}_i}) = \mathrm{Poly}(H|_{\mathcal{F}_i})$. Therefore, for every $m, n \geqslant M$, the element $\widehat{\phi} = \theta^m\phi^n$ satisfies (1).

It remains to show that there exist $m, n \geqslant M$ such that $\theta^m\phi^n$ satisfies (2). Let

$$S = \{j \mid \text{the extension } \mathcal{F}_{j-1} \leqslant \mathcal{F}_j \text{ is nonsporadic}\}$$

and let $j \in S$.

Let X_j be the Gromov hyperbolic space equipped with an isometric action of H constructed in the proof of Theorem 3.3. Then $\psi \in H$ is a loxodromic element of X_j for every $j \in S$ if and only if ψ satisfies Hypothesis (b) of Proposition 4.5. In particular, the elements ϕ and θ are loxodromic elements of X_j.

Recall that two loxodromic isometries of a Gromov-hyperbolic space X are *independent* if their fixed point sets in $\partial_\infty X$ are disjoint and are *dependent* otherwise. Let $I \subseteq S$ be the subset of indices where for every $j \in I$, the elements ϕ and θ are independent and let $D = S - I$. By standard ping pong arguments (see for instance [CU18, Proposition 4.2, Theorem 3.1]), there exist constants $m, n_0 \geqslant M$ such that for every $n \geqslant n_0$ and every $j \in I$, the element $\theta^m\phi^n$ acts loxodromically on X_j. By [CU18, Proposition 3.4], there exists $n \geqslant n_0$ such that, for every $j \in D$ and every $j \in I$, the element $\theta^m\phi^n$ acts loxodromically on X_j.

Thus, for every $j \in S$ and every $n \geqslant n_0$, the element $\theta^m\phi^n$ satisfies Hypothesis (b). This concludes the proof of Proposition 4.5. □

5. Proof of the main result and applications

We are now ready to complete the proof of our main theorem.

Theorem 5.1. — *Let $N \geqslant 3$ and let H be a subgroup of $\mathrm{Out}(F_N)$. There exists $\phi \in H$ such that $\mathrm{Poly}(\phi) = \mathrm{Poly}(H)$.*

Proof. — Since $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is a finite index subgroup of $\mathrm{Out}(F_N)$ and since for every $\psi \in H$ and every $n \in \mathbb{N}^*$, we have $\mathrm{Poly}(\psi^k) = \mathrm{Poly}(\psi)$, we see that

$$\mathrm{Poly}(H) = \mathrm{Poly}(H \cap \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})).$$

Hence we may suppose that H is a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$.

Let

$$\varnothing = \mathcal{F}_0 < \mathcal{F}_1 < \ldots < \mathcal{F}_k = \{[F_N]\}$$

be a maximal H-invariant sequence of free factor systems. By Theorem 3.3, there exists $\phi \in H$ such that for every $j \in \{1, \ldots, k\}$ such that the extension $\mathcal{F}_{j-1} \leqslant \mathcal{F}_j$ is nonsporadic, the element $\phi|_{\mathcal{F}_j}$ is fully irreducible relative to $\mathcal{F}_{j-1}$ and if $H|_{\mathcal{F}_j}$ is atoroidal relative to $\mathcal{F}_{j-1}$, so is $\phi|_{\mathcal{F}_{j-1}}$.

We now prove by induction on $i \in \{0, \ldots, k\}$ that for every $i \in \{0, \ldots, k\}$, there exists $\phi_i \in H$ such that

(a) $\mathrm{Poly}(\phi_i|_{\mathcal{F}_i}) = \mathrm{Poly}(H|_{\mathcal{F}_i})$;

(b) for every $j \in \{1, \ldots, k\}$ such that the extension $\mathcal{F}_{j-1} \leqslant \mathcal{F}_j$ is nonsporadic, the element $\phi_i|_{\mathcal{F}_j}$ is fully irreducible relative to $\mathcal{F}_{j-1}$ and if $H|_{\mathcal{F}_j}$ is atoroidal relative to $\mathcal{F}_{j-1}$, so is $\phi_i|_{\mathcal{F}_{j-1}}$.

For the base case $i = 0$, we set $\phi_0 = \phi$.

Let $i \in \{1, \ldots, k\}$ and suppose that $\phi_{i-1} \in H$ has been constructed. We distinguish between two cases, according to the nature of the extension $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$.

Suppose first that the extension $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$ is nonsporadic. We set $\phi_i = \phi_{i-1}$. We claim that ϕ_i satisfies the hypotheses. Indeed, it clearly satisfies (b).

For (a), since $\mathrm{Poly}(\phi_{i-1}|_{\mathcal{F}_{i-1}}) = \mathrm{Poly}(H|_{\mathcal{F}_{i-1}})$, it suffices to show that for every element $g \in F_N$ which is $\mathcal{F}_i$-peripheral but $\mathcal{F}_{i-1}$-nonperipheral, if $g \in \mathrm{Poly}(\phi_i|_{\mathcal{F}_i})$, then $g \in \mathrm{Poly}(H|_{\mathcal{F}_i})$.

Note that, if $\phi_i|_{\mathcal{F}_i}$ is atoroidal relative to $\mathcal{F}_{i-1}$, by Proposition 3.1(1), we have $\mathrm{Poly}(\phi_i|_{\mathcal{F}_i}) = \mathrm{Poly}(\phi_i|_{\mathcal{F}_{i-1}})$. Hence we have $\mathrm{Poly}(H|_{\mathcal{F}_i}) = \mathrm{Poly}(\phi_i|_{\mathcal{F}_i})$. So we may suppose that $\phi_i|_{\mathcal{F}_i}$ is not atoroidal relative to $\mathcal{F}_{i-1}$.

Let $g \in \mathrm{Poly}(\phi_i|_{\mathcal{F}_i})$ be an element which is $\mathcal{F}_i$-peripheral but $\mathcal{F}_{i-1}$-nonperipheral. By Proposition 3.1(1), there exists at most one (up to taking inverse) $h \in F_N$ such that $g \in \langle h \rangle$ and $[h]$ is fixed by ϕ_i. By Proposition 3.1(2b), the conjugacy class of $[h]$ is fixed by H. Hence the conjugacy class of $[g]$ is fixed by H and $g \in \mathrm{Poly}(H|_{\mathcal{F}_i})$.

Suppose now that $\mathcal{F}_{i-1} \leqslant \mathcal{F}_i$ is a sporadic extension. If $\mathrm{Poly}(\phi_{i-1}|_{\mathcal{F}_i}) = \mathrm{Poly}(H|_{\mathcal{F}_i})$, we set $\phi_i = \phi_{i-1}$. Then ϕ_i satisfies (a) and (b). Suppose that $\mathrm{Poly}(\phi_{i-1}|_{\mathcal{F}_i}) \neq \mathrm{Poly}(H|_{\mathcal{F}_i})$. By Proposition 4.5, there exists $\widehat{\phi}_{i-1} \in H$ such that $\widehat{\phi}_{i-1}$ satisfies (a) and (b). Then we set $\phi_i = \widehat{\phi}_{i-1}$. This completes the induction argument. In particular, we have $\mathrm{Poly}(\phi_m) = \mathrm{Poly}(H)$. This concludes the proof of Theorem 5.1. ☐

We now give some applications of Theorem 5.1. The first one is a straightforward consequence using the fact that for every $\phi \in \mathrm{Out}(F_N)$, there exists a natural malnormal subgroup system associated with $\mathrm{Poly}(\phi)$.

Corollary 5.2. — *Let $N \geqslant 3$ and let H be a subgroup of $\mathrm{Out}(F_N)$ such that $\mathrm{Poly}(H) \neq \{1\}$. There exist nontrivial maximal subgroups $A_1, \ldots, A_k$ of F_N such that*

$$\mathrm{Poly}(H) = \bigcup_{i=1}^{k} \bigcup_{g \in F_N} gA_ig^{-1}$$

and $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$ is a malnormal subgroup system.

If H is a subgroup of $\mathrm{Out}(F_N)$ such that $\mathrm{Poly}(H) \neq \{1\}$, we denote by $\mathcal{A}(H)$ the malnormal subgroup system given by Corollary 5.2. If $\mathrm{Poly}(H) = \{1\}$, we set $\mathcal{A}(H) = \varnothing$.

The following result is a generalization of [CU20, Theorem A] regarding fixed conjugacy classes. For a subgroup system $\mathcal{A}$ of F_N, recall the definition of $\mathrm{Out}(F_N, \mathcal{A}^{(t)})$ above Definition 2.1. If $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, we denote by $\mathrm{Fix}(\phi)$ the set of conjugacy classes of maximal subgroups P of F_N such that $\phi \in \mathrm{Out}(F_N, \{[P]\}^{(t)})$. Note that, if P is a subgroup of F_N such that $[P] \in \mathrm{Fix}(\phi)$, then $P \subseteq \mathrm{Poly}(\phi)$. Moreover, by [Lev09, Lemma 1.5], if $\mathrm{Poly}(\phi) \neq \{1\}$, the set $\mathrm{Fix}(\phi)$ is nonempty. If H is a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, we denote by $\mathrm{Fix}(H)$ the set of conjugacy classes of maximal subgroups P of F_N such that $H \subseteq \mathrm{Out}(F_N, \{[P]\}^{(t)})$. The following result is a corollary of the existence of the malnormal subgroup system $\mathcal{A}(H)$ associated with a subgroup H of $\mathrm{Out}(F_N)$ constructed in Corollary 5.2.

Corollary 5.3. — *Let $N \geqslant 3$ and let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. One of the following (mutually exclusive) statements holds.*

(1) *There exist a (possibly empty) finite set $\mathcal{C}$ of conjugacy classes of maximal cyclic subgroups of F_N such that*
$$\mathrm{Fix}(H) = \mathcal{A}(H) = \mathcal{C}.$$
(2) *There exists a nonabelian free subgroup P of F_N such that*
$$H \subseteq \mathrm{Out}\left(F_N, \{[P]\}^{(t)}\right).$$

Proof. — First assume that H is finitely generated. Suppose that (1) does not hold. Let $\mathcal{A}(H) = \{[P_1], \ldots, [P_\ell]\}$, where for every $i \in \{1, \ldots, \ell\}$, P_i is a malnormal subgroup of F_N. Note that, for every $i \in \{1, \ldots, \ell\}$, since P_i is malnormal, we have a natural homomorphism $H \to \mathrm{Out}(P_i)$ whose image, denoted by $H|_{P_i}$, is contained in the set of polynomially growing outer automorphisms of P_i.

Note that, since Assertion (1) does not hold, there exists $i \in \{1, \ldots, \ell\}$ such that the rank of P_i is at least equal to 2. From now on we focus on this P_i and the subgroup $H|_{P_i}$ of $\mathrm{Out}(P_i)$.

Since H is finitely generated, up to taking a finite index subgroup of H, we can apply the Kolchin theorem for $\mathrm{Out}(F_N)$ (see [BFH05, Theorem 1.1]): there exists a $H|_{P_i}$-invariant sequence of free factor systems of P_i
$$\varnothing = \mathcal{F}_0^{(i)} < \mathcal{F}_1^{(i)} < \ldots < \mathcal{F}_{k_i}^{(i)} = \{[P_i]\}$$
such that, for every $j \in \{1, \ldots, k_i\}$, the extension $\mathcal{F}_{j-1}^{(i)} \leqslant \mathcal{F}_j^{(i)}$ is sporadic.

Since, for every $j \in \{1, \ldots, k_i\}$, the extension $\mathcal{F}_{j-1}^{(i)} \leqslant \mathcal{F}_j^{(i)}$ is sporadic, we have $k_i \geqslant 2$.

Let j_0 be the maximal integer such that $\mathcal{F}_{j_0-1}^{(i)}$ consists only in conjugacy classes of cyclic subgroups of P_i. The existence of j_0 follows from the following facts. First, we have $\mathcal{F}_{k_i}^{(i)} = \{[P_i]\}$ with P_i a nonabelian free subgroup. Moreover, since the extension $\varnothing \leqslant \mathcal{F}_1^{(i)}$ is sporadic, the free factor system $\mathcal{F}_1^{(i)}$ consists in the conjugacy class of a cyclic subgroup of P_i.

Since the extension $\mathcal{F}_{j_0-1}^{(i)} \leqslant \mathcal{F}_{j_0}^{(i)}$ is sporadic, by maximality of j_0, there exists a subgroup U_{j_0} of P_i such that $[U_{j_0}] \in \mathcal{F}_{j_0}^{(i)}$ and one of the following holds:

(a) there exist two subgroups B_1 and B_2 of P_i such that $\mathrm{rank}(B_1) = \mathrm{rank}(B_2) = 1$, $[B_1], [B_2] \in \mathcal{F}_{j_0-1}$ and $U_{j_0} = B_1 * B_2$;

(b) there exists a subgroup B of P_i such that $\mathrm{rank}(B) = 1$, $[B] \in \mathcal{F}_{j_0-1}$ and U_{j_0} is an HNN extension of B over the trivial group.

If Case (a) occurs, then H acts as the identity on U_{j_0} since $\mathrm{rank}(U_{j_0}) = 2$ and since every element of H fixes elementwise a set of conjugacy classes of generators of U_{j_0} (recall that the abelianization homomorphism $F_2 \to \mathbb{Z}^2$ induces an isomorphism $\mathrm{Out}(F_2) \simeq \mathrm{GL}(2, \mathbb{Z})$). Hence Assertion (2) holds.

If Case (b) occurs, let b be a generator of B and let $t \in U_{j_0}$ be such that $U_{j_0} = \langle b \rangle * \langle t \rangle$. Then, since $H \subseteq \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, for every element ψ of H, there exist $\Psi \in \psi$

and $k \in \mathbb{Z}$ such that $\Psi(b) = b$ and $\Psi(t) = tb^k$. In particular, for every $\psi \in H$, the automorphism Ψ fixes the group generated by b and tbt^{-1} and Assertion (2) holds. This concludes the proof when H is finitely generated.

Suppose now that H is not finitely generated and let $(H_m)_{m \in \mathbb{N}}$ be an increasing sequence of finitely generated subgroups of H such that $H = \bigcup_{m \in \mathbb{N}} H_m$. For every $m \in \mathbb{N}$, we have $H_m \subseteq \mathrm{Out}(F_N, \mathrm{Fix}(H_m)^{(t)})$ and for every $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leqslant m_2$, we have $\mathrm{Fix}(H_{m_2}) \subseteq \mathrm{Fix}(H_{m_1})$. By [GL15, Theorem 1.5], there exists $N \in \mathbb{N}$ such that, for every $m \geqslant N$, we have $\mathrm{Out}(F_N, \mathrm{Fix}(H_m)^{(t)}) = \mathrm{Out}(F_N, \mathrm{Fix}(H_N)^{(t)})$. In particular, we have $\mathrm{Fix}(H_N) = \mathrm{Fix}(H)$. The result now follows from the finitely generated case. □

The following result might be folklore as it is a consequence of the JSJ decomposition of F_N relative to a cyclic subgroup not contained in any free factor, but we did not find a precise statement in the literature. If S is a compact, connected surface, we denote by $\mathrm{Mod}(S)$ the group of homotopy classes of homeomorphisms that preserve the boundary of S.

Corollary 5.4. — *Let $N \geqslant 3$ and let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. The following assertions are equivalent:*

(1) *$\mathcal{A}(H) = \{[\langle g \rangle]\}$, where g is an element of F_N not contained in a proper free factor of F_N;*
(2) *there exist a connected, compact surface S with exactly one boundary component and an identification of $\pi_1(S)$ with F_N such that H is identified with a subgroup of $\mathrm{Mod}(S)$ and H contains a pseudo-Anosov element.*

Proof. — The implication (2)⇒(1) is well known and a proof can be found for instance in [Gue22a, Corollary 7.5.4]. Suppose that (1) holds. Let $\phi \in H$ be an element given by Theorem 5.1. Then $\mathcal{A}(\phi) = \mathcal{A}(H) = \{[\langle g \rangle]\}$. In particular, since $H \subseteq \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, the conjugacy class of g is fixed by every element of H. Let $f\colon G \to G$ be a CT map representing a power of ϕ (see the definition in [FH11, Definition 4.7]).

Claim. — *The graph G consists in a single stratum and this stratum is an EG stratum.*

Proof. — Let H_r be the highest stratum in G. Note that, since g is not contained in any proper free factor of F_N, the reduced circuit γ_g in G representing the conjugacy class of g has height r and is fixed by f.

We now prove that H_r is an EG stratum. Indeed, H_r is either a zero stratum, an EG stratum or an NEG stratum. The stratum H_r cannot be a zero stratum by [FH11, Definition 4.7 (6)]. Moreover, H_r cannot be a NEG stratum as otherwise by [CU20, Proposition 4.1], since γ_g has height r, the element g would be a basis element of F_N, contradicting the fact that g is not contained in any proper free factor of F_N. Hence H_r is an EG stratum.

By [HM20, Fact I.2.3], the stratum H_r is a geometric stratum in the sense of [HM20, Definition I.2.1]. By [HM20, Proposition I.2.18], the element ϕ fixes elementwise a finite set $\mathcal{C} = \{[g], [c_1], \ldots, [c_k]\}$ of conjugacy classes of elements of F_N. Since G is connected, by the definition of a geometric stratum and by [HM20, Proposition I.2.18 (5)], the stratum H_r is glued on G_{r-1} along closed paths in G_{r-1} whose associated reduced circuits represent the conjugacy classes $[c_1]$, ..., $[c_k]$. Thus, we have $k \geqslant 1$ whenever G_{r-1} is not reduced to a point. This implies that $\mathcal{C} = \{[g]\}$ if and only if G_{r-1} is reduced to a point, that is, if and only if G consists in the single stratum H_r. □

By the claim and [HM20, Fact I.2.3] (see also [BH92, Theorem 4.1]), the outer automorphism ϕ is *geometric*: there exist a connected, compact surface S with exactly one boundary component and an identification of $\pi_1(S)$ with F_N such that ϕ is identified with a pseudo-Anosov element of $\mathrm{Mod}(S)$. Moreover, the conjugacy class $[g]$ is identified with the conjugacy class in $\pi_1(S)$ of the element associated with the homotopy class of the boundary component of S. Since $[g]$ is fixed by every element of H, by the Dehn-Nielsen-Baer theorem (see for instance [FM11, Theorem 8.8] and [ZVC80, Theorem 5.6.2] for the orientable case and [Fuj02, Section 3] for the nonorientable case), the group H is identified with a subgroup of $\mathrm{Mod}(S)$. □

We finally state a proposition, whose proof can be found in [Gue22a, Proposition 7.5.6] in a more general setting, which allows us to compute the malnormal subgroup system $\mathcal{A}(H)$ associated with some subgroups H of $\mathrm{Out}(F_N)$. The definitions and properties associated with JSJ decompositions of F_N can be found for instance in [GL17], especially [GL17, Definitions 2.14, 5.13].

Proposition 5.5 ([Gue22a, Proposition 7.5.6]). — *Let $N \geqslant 3$ and let P be a finitely generated subgroup of F_N such that F_N is one-ended relative to P. Let T be the JSJ tree of F_N over cyclic groups relative to P. Suppose that $\mathrm{Out}(F_N, \{[P]\}^{(t)})$ is infinite. Every subgroup Q of F_N such that $[Q] \in \mathcal{A}(\mathrm{Out}(F_N, \{[P]\}^{(t)}))$ is either generated by stabilizers of some rigid vertices of T or is an extended boundary subgroup of the stabilizer of some flexible vertex of T.*

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