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A MCCOOL WHITEHEAD TYPE THEOREM FOR FINITELY GENERATED SUBGROUPS OF $\mathsf{Out}(F_n)$

UN THÉORÈME DU TYPE MCCOOL WHITEHEAD POUR LES SOUS-GROUPES DE TYPE FINI DE $\mathsf{Out}(F_n)$

ABSTRACT. — S. Gersten announced an algorithm that takes as input two finite sequences $\vec{K} = (K_1, \ldots, K_N)$ and $\vec{K}' = (K'_1, \ldots, K'_N)$ of conjugacy classes of finitely generated subgroups of F_n and outputs:
  (1) YES or NO depending on whether or not there is an element $\theta \in \mathsf{Out}(F_n)$ such that $\theta(\vec{K}) = \vec{K}'$ together with one such θ if it exists and
  (2) a finite presentation for the subgroup of $\mathsf{Out}(F_n)$ fixing $\vec{K}$.
S. Kalajdžievski published a verification of this algorithm. We present a different algorithm from the point of view of Culler–Vogtmann's Outer space. New results include that the subgroup of $\mathsf{Out}(F_n)$ fixing $\vec{K}$ is of type VF, an equivariant version of these results, an application, and a unified approach to such questions.

Résumé. — S. Gersten a annoncé un algorithme qui prend en entrée deux suites finies $\vec{K} = (K_1, \ldots, K_N)$ et $\vec{K}' = (K_1', \ldots, K_N')$ de classes de conjugaison de sous-groupes de type fini de F_n et renvoie :

(1) OUI ou NON selon qu'il existe ou non un élément $\theta \in \mathsf{Out}(F_n)$ tel que $\theta(\vec{K}) = \vec{K}'$, avec un tel θ s'il existe et

(2) une présentation finie du sous-groupe de $\mathsf{Out}(F_n)$ fixant $\vec{K}$.

S. Kalajdžievski a publié une vérification de cet algorithme. Nous présentons un algorithme différent avec le point de vue de l'outre-espace de Culler–Vogtmann. Les nouveaux résultats incluent que le sous-groupe de $\mathsf{Out}(F_n)$ fixant $\vec{K}$ est de type VF, une version équivariante de ces résultats, une application et une approche unifiée de ces questions.

1. Introduction

Let $\mathcal{K}$ be the set of conjugacy classes of finitely generated subgroups of F_n and let $X_{\mathcal{K}}$ be the set of finite sequences of elements of $\mathcal{K}$. For our first main result, we consider the action of $\mathsf{Out}(F_n)$ on $X_{\mathcal{K}}$. For $\vec{K} := (K_1, \ldots, K_N) \in X_{\mathcal{K}}$, the subgroup $GMc(\vec{K})$ of $\mathsf{Out}(F_n)$ fixing $\vec{K}$ in $X_{\mathcal{K}}$ is called a *generalized McCool group of F_n*. See Section 2 for more on terminology.

THEOREM 1.1. — *There is an algorithm that takes as input two finite sequences $\vec{K} = (K_1, \ldots, K_N)$ and $\vec{K}' = (K_1', \ldots, K_N')$ of conjugacy classes of finitely generated subgroups of F_n and outputs:*

(1) YES *or* NO *depending on whether or not there is an element $\theta \in \mathsf{Out}(F_n)$ such that $\theta(\vec{K}) = \vec{K}'$ together with one such θ if it exists and*

(2) *a finite presentation for $GMc(\vec{K})$.*

Further,

(3) *$GMc(\vec{K})$ is of type* VF.

If one replaces $\mathcal{K}$ with the set $\mathcal{C}$ of conjugacy classes of elements of F_n, then this theorem is classical: (1) is due to J.H.C. Whitehead [Whi36b, Whi36a] (who also computes a finite generating set for the stabilizer of the elements); (2) is due to McCool [McC75]; and (3) is due to Culler and Vogtmann [CV86].

Items (1) and (2) of Theorem 1.1 were announced by Gersten [Ger84] and proved by Kalajdžievski [Kal92]. Gersten's result plays an important role in the solution [FH19] of the conjugacy problem for UPG elements of $\mathsf{Out}(F_n)$. A motivation for this paper was an accessible proof of Gersten's result from the point of view of Culler–Vogtmann's Outer space.

The subgroup of $\mathsf{Out}(G)$ fixing a given finite sequence of conjugacy classes of elements in F_n has been termed an *elementary McCool group of F_n* by Guirardel–Levitt [GL15]. See also Section 2. They have announced [Lev] a structure theorem for the stabilizers of points in the boundary of Outer space where the key terms are elementary McCool groups of F_n. In particular, they show that these stabilizers are of type VF, that is, some finite index subgroup has a finite Eilenberg–MacLane space. Note that, up to finite index, an elementary McCool group is a generalized McCool group.

For a group G, $\mathsf{Out}(F_n)$ acts on the set $\mathsf{Hom}(\mathsf{G}, \mathsf{Out}(F_n))$ of representations via $\theta(\alpha) = i_\theta \circ \alpha$ where i_θ denotes conjugation by θ. For our second main result, we consider the action of $\mathsf{Out}(F_n)$ on the product of $\mathsf{Hom}(\mathsf{G}, \mathsf{Out}(F_n))$ and $X_\mathcal{K}$, obtaining the following generalization of Theorem 1.1.

THEOREM 1.2 (Generalized Theorem 1.1). — *For any finite group G, there is an algorithm that takes as input two homomorphisms $\alpha, \alpha' : \mathsf{G} \to \mathsf{Out}(F_n)$ and two finite sequences $\vec{K} = (K_1, \ldots, K_N)$ and $\vec{K}' = (K_1', \ldots, K_N')$ of conjugacy classes of finitely generated subgroups of F_n and outputs:*

(1) YES *or* NO *depending on whether or not there is an element $\theta \in \mathsf{Out}(F_n)$ such that $\theta^{-1}\alpha(g)\theta = \alpha'(g)$ for all $g \in \mathsf{G}$ and such that $\theta(\vec{K}) = \vec{K}'$. Further the algorithm finds one such θ if it exists.*

(2) *a finite presentation for the subgroup of $\mathsf{Out}(F_n)$ fixing $\vec{K}$ and commuting with each element of $\alpha(\mathsf{G})$.*

Further,

(3) *The subgroup of $\mathsf{Out}(F_n)$ fixing $\vec{K}$ and commuting with each element of $\alpha(\mathsf{G})$ is* VF. *In the terminology of Section 2, equivariant generalized McCool groups of F_n are of type* VF.

Remark 1.3. — Even though Theorem 1.2 implies Theorem 1.1, we include a proof of the latter to serve as a less technical warm-up for that of Theorem 1.2.

If one replaces $\mathcal{K}$ with $\mathcal{C}$, then Theorem 1.2(1) is due to Krstić, Lustig, and Vogtmann [KLV01, Theorem 1.1]. They go on in that same paper to extend the solution to the conjugacy problem for Dehn twists [CL99] to all linear growth elements of $\mathsf{Out}(F_n)$. Theorem 1.2 should be useful in extending the solution of the UPG conjugacy problem to all polynomially growing elements of $\mathsf{Out}(F_n)$. See [Kal92, Theorem A] for some progress toward Theorem 1.2(3).

Following [CV86] and [KV93], we associate a subcomplex of the spine of Outer space to each pair $(\vec{K}, \alpha)$ (or just $\vec{K}$ in the context of Theorem 1.1) and prove that this subcomplex is contractible. We follow the proofs in [Vog17] (which revisits [CV86]) and [KV93] closely. In those papers, $\mathcal{K}$ is replaced by $\mathcal{C}$ and the construction of the subcomplex is based upon the length $\|\tau\|_w$ of the realization of $w \in \mathcal{C}$ in a marked graph τ. We replace this with the volume $\|\tau\|_K$ of the Stallings graph determined by $K \in \mathcal{K}$ and a marked graph τ; see Section 5. Most of the arguments in the proof of contractibility are unaffected by this change and we do not repeat their proofs here. The elements of the proofs that are affected are these. In the non-equivariant case ($\mathsf{G} = \{1\}$), i.e. Theorem 1.1, [Vog17, Lemma 11.2] must be replaced and we do this in Lemma 6.9. In the equivariant case, i.e. Theorem 1.2, we replace [KV93, Lemmas 7.1 and 7.2] with Lemmas 8.12 and 8.11 respectively. This is the most technical part of the paper and is carried out in Section 8.

With contractibility in hand, we deduce all three items in our main theorems at once in a simple, but seemingly new, way. For example, we do not use peak reduction. The keys here are Definition 4.1 and Proposition 4.2. See also Proposition 6.10.

We end the paper with an application. Suppose K is the conjugacy class of a finitely generated subgroup of F_n that is its own normalizer. There is then a natural

restriction map from $GMc(K)$ to $\mathsf{Out}(K)$. The image $Ext(F_n; K)$ of this map is the subgroup of $\mathsf{Out}(K)$ of elements that extend to $\mathsf{Out}(F_n)$.

THEOREM 1.4 (special case). — *With K as above, $Ext(F_n; K)$ is commensurable to a finite product of generalized McCool groups. In particular, it is of type* VF.

Acknowledgements

2. McCool Groups and Generalized McCool Groups

In this section, added at the suggestion of the referee and motivated by [GL15], we slightly generalize Theorem 1.2 in Theorem 2.4. A reader interested only in the proofs of Theorems 1.1 and 1.2 can skip this section. Some terminology is needed.

DEFINITION 2.1 ([GL15, Section 2]). — *Let G be a group and H a subgroup. Say that $\phi \in \mathsf{Out}(G)$ acts trivially on, respectively preserves, the conjugacy class $[H]$ of H in G if there is a representative $\Phi \in \mathsf{Aut}(G)$ for ϕ such that Φ is the identity on H, respectively such that $\Phi(H) = H$. Note that if ϕ acts trivially on $[H]$ then ϕ preserves $[H]$, but not necessarily vice versa.*

DEFINITION 2.2 (cf. [GL15, Definition 2.1]). — *Let G and G be groups, $\alpha : \mathsf{G} \to \mathsf{Out}(G)$ a homomorphism, and $\vec{H} = \{[H_i] : i \in I\}$ and $\vec{K} = \{[K_j] : j \in J\}$ indexed sets of conjugacy classes of subgroups of G. We denote by $\mathsf{Out}(G; \vec{H}^{(t)}, \vec{K}, \alpha)$ the subgroup of $\mathsf{Out}(G)$ consisting of outer automorphisms ϕ that:*

- *act trivially on each element of $\vec{H}$;*
- *preserve each element of $\vec{K}$; and*
- *commute with each element of $\alpha(\mathsf{G})$.*

If G is trivial, we often write $\mathsf{Out}(G; \vec{H}^{(t)}, \vec{K})$ instead of $\mathsf{Out}(G; \vec{H}^{(t)}, \vec{K}, \alpha)$. If I is finite, then we call $\vec{H}$ a finite sequence.

THEOREM 2.3 ([GL15, first part of Theorem 3.4]). — *If G is a toral relatively hyperbolic group and $\vec{H}$, $\vec{K}$ are finite sequences of finitely generated subgroups, with every element of $\vec{K}$ abelian, then $\mathsf{Out}(G; \vec{H}^{(t)}, \vec{K})$ is of type* VF.

THEOREM 2.4 (Generalized Theorem 1.2). — *There is an algorithm that takes as input:*

- *a finite group G and two homomorphisms $\alpha, \alpha' : \mathsf{G} \to \mathsf{Out}(F_n)$;*
- *two finite sequences $\vec{H}, \vec{H}'$ of conjugacy classes of finitely generated subgroups of F_n; and*
- *two finite sequences $\vec{K}, \vec{K}'$ of conjugacy classes of finitely generated subgroups of F_n*

and outputs:

(1) YES *or* NO *depending on whether or not there is an element* $\theta \in \mathsf{Out}(F_n)$ *such that*
  - $\theta^{-1}\alpha(g)\theta = \alpha'(g)$ *for all* $g \in \mathsf{G}$;
  - $\theta(\vec{H}) = \vec{H}'$; *and*
  - $\theta(\vec{K}) = \vec{K}'$.

  Further the algorithm finds one such θ *if it exists.*

(2) *a finite presentation for* $\mathsf{Out}(F_n; \vec{H}^{(t)}, \vec{K}, \alpha)$.

Further,

(3) $\mathsf{Out}(F_n; \vec{H}^{(t)}, \vec{K}, \alpha)$ *is of type* VF.

After a lemma, we prove the Theorem 2.4.

LEMMA 2.5. — *There is an algorithm with input a finite sequence*

$$\vec{H} = ([H_1], [H_2], \dots, [H_k])$$

of conjugacy classes of subgroups of F_n *and output a finite set* $\{[g_i]\}$ *of conjugacy classes of elements of* F_n *such that* $\phi \in \mathsf{Out}(F_n)$ *acts trivially on each* $[H_j]$ *iff* ϕ *fixes each* $[g_i]$.

Proof. — Suppose we have a finite sequence $\vec{C}$ of elements of conjugacy classes of elements of $\cup_j H_j$. Use Whitehead [Whi36b, Whi36a] or McCool [McC75] to compute a finite generating set for the stabilizer of $\vec{C}$. Check whether each generator acts trivially on each H_j. If not, concatenate to $\vec{C}$ an element that is moved. In this way we get a strictly decreasing chain of elementary McCool groups of F_n. By [GL15, Theorem 1.5 and Corollary 1.6] such a chain is bounded and so this process terminates. □

Remark 2.6. — [GL15, Corollary 1.6] shows that such a finite set $\{g_i\}$ exists (even if $\vec{H}$ is an arbitrary set of not necessarily finitely generated subgroups and in the setting of toral relatively hyperbolic groups). The point of Lemma 2.5 is that, in its setting, finding such elements can easily be made algorithmic.

Proof of Theorem 2.4. —
(1) follows by applying Theorem 1.2(1) to the concatenation $\vec{H}\vec{K}$ of $\vec{H}$ and $\vec{K}$.
(2), (3): Apply the algorithm of Lemma 2.5 to $\vec{H}$ to obtain the finite set $\{[g_i]\}$. Let $\vec{H}_0 := ([\langle g_1 \rangle], [\langle g_2 \rangle], \dots)$ be the corresponding finite sequence of conjugacy classes of cyclic subgroups of F_n. $\widehat{\mathsf{O}} := \mathsf{Out}(F_n; \emptyset, \vec{H}_0\vec{K}, \alpha)$ acts on $\{[g_i], [g_i^{-1}]\}$ with kernel $\mathsf{O} := \mathsf{Out}(F_n; \vec{H}^{(t)}, \vec{K}, \alpha)$. By Theorem 1.2(3), $\widehat{\mathsf{O}}$, and hence also O, has type VF thus proving (3). Using the finite presentation of $\widehat{\mathsf{O}}$ provided by Theorem 1.2(2), we can compute the image of $\widehat{\mathsf{O}}$ in the permutation group of $\{[g_i], [g_i^{-1}]\}$ and then use the Reidemeister–Schreier theorem (see for example the section on finite presentations and finite index subgroups of [FH19] in this setting), to finitely present O. □

One could ask about generalizations of Theorem 2.4.

QUESTIONS 2.7. — *Under what conditions is* $\mathsf{Out}(F_n; \vec{H}^{(t)}; \vec{K}; \alpha)$ *of type* VF? *Is there a sequence* $\vec{K} = \{[K_j] : j \in \mathbb{N}\}$ *of conjugacy classes of finitely generated subgroups of* F_n *such that* $\mathsf{Out}(F_n; \emptyset, \vec{K})$ *does not have type* VF? *Does* $\mathsf{Out}(F_n; \emptyset, ([K_1])) > \mathsf{Out}(F_n; \emptyset, ([K_1], [K_2])) > \dots$ *always stabilize?*

Remark 2.8. —

- Using [GL15, Corollary 1.6], Theorem 2.4(3) remains true if the condition on $\vec{H}$ is replaced by the assumption that $\vec{H}$ is an arbitrary indexed set of conjugacy classes of (not necessarily finitely generated) subgroups of F_n.

- For $n \geqslant 3$, there is a finite sequence $\vec{K}$ of conjugacy classes of (infinitely generated) subgroups of F_n so that $\mathsf{Out}(F_n; \emptyset, \vec{K})$ does not have type VF. Indeed, the Torelli group $\mathcal{T}_n$ of $\mathsf{Out}(F_n)$ is torsion free and does not have type F [BBM07] (for $n = 3$, this is Krstić and McCool [KM97] and, for $3 \leqslant n < 100$, this is Smillie–Vogtmann [SV87]). $\mathcal{T}_n$ is the kernel of the action of $\mathsf{Out}(F_n)$ on $H_1(F_n; \mathbb{Z})$ and so equivalently on the abelian group of homomorphisms $F_n \to \mathbb{Z}$, which is isomorphic to $\mathbb{Z}^n$. If f is a homomorphism from F_n to $\mathbb{Z}$ and $\phi \in \mathsf{Out}(F_n)$ preserves its kernel, then there is an induced $\overline{\phi} \in \mathsf{Out}(\mathbb{Z})$ such that $f \circ \phi = \overline{\phi} \circ f$, and so $f \circ \phi = \pm f$. It follows that if $\{f_1, f_2, \dots, f_n\}$ is a basis then $\mathcal{T}_n$ has index 2^n in $\mathsf{Out}(F_n; \emptyset, (\mathrm{Ker}(f_1), \mathrm{Ker}(f_2), \dots))$. Hence the latter group does not have type VF.

We end this section with a bit more terminology.

DEFINITION 2.9 (cf. [GL15, Definitions 1.1, 1.2, and 2.1]). — *Let* G, G, $\vec{H}$, $\vec{K}$, *and* α *be as in Definition 2.2. Additionally, let* $\vec{C}$ *denote a set conjugacy classes of elements of* G.

- $Mc_G(\vec{C})$ *denotes the subgroup of* $\mathsf{Out}(G)$ *fixing each* $[c] \in \vec{C}$. *Equivalently,* $Mc_G(\vec{C}) = \mathsf{Out}(G; \vec{H}^{(t)}, \emptyset)$ *where* $\vec{H} = \{[\langle c \rangle] \mid [c] \in \vec{C}\}$. *If* $\vec{C}$ *is finite, then we say that* $Mc_G(\vec{C})$ *is an elementary McCool group of* G *(or of* $\mathsf{Out}(G)$*).*
- $Mc_G(\vec{H}) := \mathsf{Out}(G; \vec{H}^{(t)}, \emptyset)$. *If* $\vec{H}$ *is finite with every element finitely generated then we say that* $Mc_G(\vec{H})$ *is a McCool group of* G *(or of* $\mathsf{Out}(G)$*).*
- $GMc_G(\vec{K}) := \mathsf{Out}(G; \emptyset, \vec{K})$. *If* $\vec{K}$ *is finite with every element finitely generated then we say that* $GMc_G(\vec{H})$ *is a generalized McCool group of* G *(or of* $\mathsf{Out}(G)$*).*
- *If* G *is not trivial, then equivariant elementary, equivariant, or equivariant generalized McCool groups are defined analogously.*
- *If* $G = F_n$ *then we usually suppress the subscript.*

Remark 2.10. —

- Guirardel–Levitt [GL15, Corollary 1.6] show that if G is toral relatively hyperbolic then $Mc(\vec{H})$ is a McCool group and also that any McCool group is an elementary McCool group.
- $Mc_G(\vec{K})$ is a normal subgroup of $GMc_G(\vec{K})$. In case $G = F_n$, Theorem 9.1 shows that the quotient $Ext(F_n; \vec{K})$ is of type VF.
- There are generalized McCool groups of F_n that are not commensurable in $\mathsf{Out}(F_n)$ to a McCool group of F_n. For example, there are $\phi \in \mathsf{Out}(F_n)$, $n \geqslant 4$,

preserving a free factor F but without any non-trivial periodic conjugacy classes of non-trivial elements of F_n. In this case, $GMc([F])$ is such. Indeed, an iterate of ϕ appears in a given finite index subgroup of $GMc([F])$. Since an iterate doesn't fix any non-trivial conjugacy classes, it is not contained in an elementary McCool group of F_n. Using the first bulleted item of this remark, an iterate is therefore not contained in a McCool group of F_n.

Remark 2.11. — The notations $Mc_G(\vec{C})$ and $Mc_G(\vec{H})$ and the definitions *elementary McCool group* and *McCool group* correspond with those in [GL15] with the small differences that Guirardel and Levitt use sets and we use indexed sets and that they use $\mathcal{C}, \mathcal{H}, \mathcal{K}$ for our $\vec{C}, \vec{H}, \vec{K}$ and we use $\mathcal{C}$ (respectively $\mathcal{K}$) for the set of conjugacy classes of elements (respectively finitely generated subgroups) of F_n.

Remark 2.12. — Except in this section, we only consider finite $\vec{K}$ with every element finitely generated, i.e. $GMc(\vec{K})$ will be a generalized McCool group. The notation $Mc(\cdot)$ and $GMc(\cdot)$ and the terminology *elementary McCool, McCool*, and *generalized McCool group* don't return until the last section, Section 9.

3. Background

Most arguments in this paper take place in the spine of Outer space, a space introduced by Culler and Vogtmann in their seminal paper [CV86]. In this section we quickly recall key definitions and notation needed for these arguments. See also [Vog17, KV93].

R_n denotes a specific rose, i.e. a graph with one vertex and n edges, whose fundamental group has been identified with F_n. A *marked graph* is a pair (G, g) consisting of a graph G and a homotopy equivalence $g : R_n \to G$. G is required to have no vertices of valence $\leqslant 2$. Marked graphs $\tau = (G, g)$ and $\tau' = (G', g')$ are *equivalent* if there is cellular isomorphism[1] $h : G \to G'$ such that $h \circ g$ and g' are homotopy equivalent. We write $[(G, g)]$ for the equivalence class, but often blur the distinction between a marked graph and its equivalence class by omitting the brackets.

A homotopy equivalence is a *(forest) collapse* if it is of the form $c : G \to G/F$ where $F \subset G$ is a forest and G/F is the quotient space obtained by collapsing components of F. We also say that $(G/F, c \circ g)$ is obtained from (G, g) by a *forest collapse*. The *spine* of Outer space is defined to be the geometric realization of the partial order induced by forest collapse on the set of equivalence classes of marked graphs. $\mathsf{Out}(F_n)$ acts on the spine: if $h : R_n \to R_n$ realizes $\theta \in \mathsf{Out}(F_n)$ then $(G, g) \cdot \theta := (G, g \circ h)$. This is a right action. We will have one occasion (in the proof of Proposition 6.10) to use the left action defined as $\theta \cdot (G, g) := (G, g) \cdot \theta^{-1}$.

L_n denotes the $\mathsf{Out}(F_n)$-subcomplex of the spine spanned by graphs with no separating edge. We usually suppress the subscripts n. $\mathcal{R}$ denotes the set of equivalence classes of marked roses. Since every marked graph collapses to a rose, L is the union of the stars of the elements of $\mathcal{R}$, i.e. $L = \cup\{St(\rho) \mid \rho \in \mathcal{R}\}$. Note that $\mathsf{Out}(F_n)$ acts

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[1] Recall that a map between CW-complexes is *cellular* if it takes k-skeleta to k-skeleta. It is a *cellular isomorphism* if it is a cellular homeomorphism with a cellular inverse.

transitively on $\mathcal{R}$. Indeed, if $(R, r) \in \mathcal{R}$ and $h : R \to R_n$ is a cellular isomorphism then $h \circ r : R_n \to R_n$ represents some $\theta \in \mathsf{Out}(F_n)$ and $(R_n, id) \cdot \theta = (R, r)$.

If $f : R \to R'$ is a homotopy equivalence of roses of the form $R \leftarrow G \to R'$ where each map is an edge collapse then we say that $\rho' := (R', f \circ r) \in \mathcal{R}$ is obtained from $\rho := (R, r) \in \mathcal{R}$ by a *Whitehead move*. Note that $St(\rho) \cap St(\rho') \neq \emptyset$ and that, for any $\theta \in \mathsf{Out}(F_n)$, $\rho' \cdot \theta$ is obtained from $\rho \cdot \theta$ by a Whitehead move.

4. Presentation Families

The main work in proving Theorems 1.1 and 1.2 is proving that certain subcomplexes of the spine of Outer space are contractible. In this section we describe how to complete the proofs once this has been done.

In the model case for the following definition, $G = \mathsf{Out}(F_n)$, X is the set $X_{\mathcal{C}}$ of finite sequences of conjugacy classes of elements in F_n, L is the subcomplex of the spine of Outer space spanned by marked graphs without separating edges and, for $x \in X$, L_x is the subcomplex of L that is the union of the stars of the roses in which the length of x is minimized.

DEFINITION 4.1. — *A G-set X has a presentation family if:*
  (1) *there is an algorithm that takes as input $g_1, g_2, g_3 \in G$ and outputs* YES *or* NO *depending on whether or not $g_1 g_2 = g_3$.*
  (2) *there is an algorithm that takes as input $g \in G$ and $x_1, x_2 \in X$ and outputs* YES *or* NO *depending on whether or not $g x_1 = x_2$.*
  (3) *there is a simplicial, properly discontinuous action of G on a simplicial complex L with the following properties.*
    (a) *G acts without inverting edges of L.*
    (b) *For all vertices, v, w of L, there is an algorithm that decides if there exists $g \in G$ such that $gv = w$; If* YES *then the algorithm outputs the finite set $G_{v,w} = \{g \mid gv = w\}$.*
  (4) *For each $x \in X$ there is a contractible G_x-invariant subcomplex L_x of L with the following properties where $G_x := \{g \in G \mid gx = x\}$.*
    (a) *there is an algorithm that takes as input $x \in X$ and outputs a finite fundamental domain D_x for the G_x-action on L_x; more precisely, D_x is a finite subcomplex of L_x such that $G_x D_x = L_x$ and we are given the set V of vertices of D_x together with the finite subset of 2^V consisting of the simplices of D_x.*
    (b) *$\theta(L_x) = L_{\theta(x)}$ for all $\theta \in G$.*

In this case, we say that $\{L_x \mid x \in X\}$ is a presentation family in L for the action of G on X.

Recall that a group has type VF if some finite index subgroup has a finite Eilenberg–MacLane space.

PROPOSITION 4.2. — *Suppose the G-set X has a presentation family and G has a torsion-free subgroup of finite index. Then there is an algorithm that takes as input $x, y \in X$ and outputs*

(1) YES *or* NO *depending on whether or not there is $g \in G$ such that $gx = y$ together with g if it exists;*

(2) *a finite presentation for $G_x := \{g \in G | gx = x\}$.*

Further,

(3) G_x *is of type* VF.

Proof. — We adopt the notation of Definition 4.1. We are assuming that the action of G on L is properly discontinuous and that G has a finite index torsion-free subgroup, say G'. It follows that the action of G' on L, and hence the action of $G' \cap G_x$ on L_x, is free. Proposition 4.2 (3) therefore follows from Definition 4.1 (4a), which implies that the action of $G' \cap G_x$ on L_x is cocompact.

We use K. Brown [Bro84] to give a finite presentation for G_x based on its action on L_x. Our actions do not invert edges and so we use the simplification of Brown's presentation given in [AFV08, Theorem 2]. The only things needed to make this algorithmic are:

- an explicit description of a fundamental domain D_x for this action;
- for each $v, w \in D_x$, the finite set $\{g \in G_x \mid gv = w\}$; and
- for each $v \in D_x$, a presentation for the finite group $\{g \in G_x \mid gv = v\}$.

These are provided, respectively, by:

- Definition 4.1 (4a);
- Definition 4.1 (3b) and the ability to determine if a given $g \in G$ is contained in G_x (Definition 4.1 (2));
- Definition 4.1 (1), which allows us to compute its multiplication table (and hence a presentation).

This completes the proof of Proposition 4.2 (2).

It remains to show that it is algorithmic to check, given $x, y \in X$, whether there is $g \in G$ such that $g(x) = y$ and to find such a g if it exists. Let $D_x \subset L_x$ and $D_y \subset L_y$ be the domains given in Definition 4.1 (4a). Definition 4.1 (4b) implies that if $g(x) = y$ then $g(L_x) = L_y$. In this case, (by pre-composing with an element of G_x and post-composing with an element of G_y) there is such a $g \in G$ taking a vertex of D_x to one of D_y. Using Definition 4.1 (3b) we then check, for each g taking a vertex of D_x to a vertex of D_y, whether g takes x to y. If we find one, then this is the desired g; if not then there is no g taking x to y. This completes the proof of Proposition 4.2 (1). □

5. Stallings Graphs

Recall $\mathcal{C}$ denotes the set of conjugacy classes of elements of F_n. The constructions in [CV86], [KV93] and [Vog17] are based on the length function for an element of $\mathcal{C}$ in a marked graph τ. In this section we generalize this to a volume function for an element of $\mathcal{K}$ in τ where $\mathcal{K}$ is the set of finitely generated subgroups of F_n.

Each $w \in \mathcal{C}$ is represented in a marked graph $\tau = (G, g)$ by a unique circuit, which we view as an immersion $w_\tau : S^1 \to G$. Subdivide S^1 so that w_τ maps each edge in S^1 to an edge in G and define the *length* $\|\tau\|_w$ of w to be the number of

edges in the subdivided S^1. Equivalently, $\|\tau\|_w$ is the number of edges of G (counted with multiplicity) crossed by the circuit w_τ. If $\tau = (G, g)$ and $\tau' = (G', g')$ represent the same vertex in the spine of Outer space then there is a cellular isomorphism $h : G \to G'$ such that $w_{\tau'} = h w_\tau : S^1 \to G'$ and so $\|\tau'\|_w = \|\tau\|_w$. We may therefore view $\|\tau\|_w$ as an invariant of the point in the spine represented by τ.

Fix a rose $\rho = (R, r)$ and let H be the set of half-edges of R, equivalently the set of oriented edges of R. For each $w \in \mathcal{C}$, divide S^1, thought of as the domain of w_ρ, into *pieces* $\mathsf{Y} = \{Y\}$ by snipping each edge at its midpoint. Each Y is a topological arc that is composed of two half-edges; using the half-edges, label the endpoints of Y by elements of H. To construct the *star graph for ρ and w* (see [Vog17, Section 9]), start with $2n$ vertices V labeled by the elements of H and then add $\|\rho\|_w$ edges, one for each $Y \in \mathsf{Y}$, with endpoints attached to V according to the labeling on the endpoints of Y.

We now modify the definitions of length function and star graph so that they apply to all elements of $\mathcal{K}$, in fact all elements of $X_\mathcal{K}$, and not just those of rank one.

A *Stallings graph (over G)* is a map $m : \Gamma \to G$ such that:
- Γ and G are finite graphs with no vertices of valence $\leqslant 2$;
- G is connected;
- the components $(\Gamma_1, \ldots, \Gamma_N)$ of Γ are ordered; and
- m takes each open edge of Γ homeomorphically to an open edge of G and the induced maps on the stars of vertices are injective.

We view each edge of Γ as *labeled* by its image edge in G. Stallings graphs $m : \Gamma \to G$ and $m' : \Gamma' \to G'$ are *equivalent* if there is a cellular isomorphism $h : G \to G'$ with a lift $\eta : \Gamma \to \Gamma'$, also a cellular isomorphism, that restricts to order preserving maps $\eta_i : \Gamma_i \to \Gamma'_i$ on components.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \eta\ } & \Gamma' \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m'} \\ G & \xrightarrow{\ h\ } & G' \end{array}$$

Example 5.1. — Label the edges of the rose R_2 by the letters a and b. The circle Γ with three edges labeled *aab* mapping to R_2 preserving labels gives a Stallings graph $m : \Gamma \to R_2$. Similarly the circle Γ' labeled *bba* gives $m' : \Gamma' \to R_2$. Note that m and m' are equivalent, but the two-component Stallings graphs given by $\Gamma \sqcup \Gamma$ and $\Gamma \sqcup \Gamma'$ are not equivalent.

Recall from the Section 1 that $X_\mathcal{K}$ denotes the set of finite sequences in $\mathcal{K}$. A *marked Stallings graph* is a pair (m, τ) where $m : \Gamma \to G$ is a Stallings graph and $\tau = (G, g)$ is a marked graph. We say (m, τ) *represents* $\vec{K} = (K_1, K_2, \ldots, K_N) \in X_\mathcal{K}$ if K_i is the image of the induced monomorphism $(m_i)_* : \pi_1(\Gamma_i) \to \pi_1(G)$ (where $\pi_1(G)$ has been identified with F_n via τ). For each $\vec{K} \in X_\mathcal{K}$ and marked graph $\tau = (G, g)$, there is a marked Stallings graph $(\vec{K}_\tau \to G, \tau)$ that represents $\vec{K}$, constructed as follows. If $p : \tilde{G} \to G$ is the universal cover of G and T_{A_i} is the minimal subtree for a subgroup A_i representing K_i then $(\vec{K}_\tau)_i$ is the quotient of T_{A_i} by the action of A_i and $(K_\tau)_i \to G$ is the immersion induced from p. We sometimes abuse notation and write $\vec{K}_\tau$ for the Stallings graph $\vec{K}_\tau \to G$.

PROPOSITION 5.2. — *Let* $\tau = (G, g), \tau' = (G', g')$ *be marked graphs representing elements in the spine of Outer space and* $\vec{K}, \vec{K}' \in X_{\mathcal{K}}$.

(1) *Marked Stallings graphs* $(\Gamma \xrightarrow{m} G, \tau)$ *and* $(\Gamma' \xrightarrow{m'} G, \tau)$ *represent the same element of* $X_{\mathcal{K}}$ *iff there is a label-preserving (i.e.* η *is a lift of* id_G*) cellular isomorphism* $\Gamma \xrightarrow{\eta} \Gamma'$ *inducing* $\eta_i : \Gamma_i \to \Gamma'_i$.

(2) *If* $[\tau] = [\tau']$ *then* $\vec{K}_\tau$ *and* $\vec{K}_{\tau'}$ *are equivalent. The equivalence class of* $\vec{K}_\tau$ *is denoted* $\vec{K}_{[\tau]}$.

(3) *For* $\phi \in \mathsf{Out}(F_n)$, $\vec{K}_{[\tau]\phi} = (\phi(\vec{K}))_{[\tau]}$

(4) *There is* $\phi \in \mathsf{Out}(F_n)$ *such that* $[\tau'] \cdot \phi = [\tau]$ *and* $\phi(\vec{K}) = \vec{K}'$ *if and only if* $\vec{K}_{[\tau]} = \vec{K}'_{[\tau']}$.

Proof. —

(1) $\implies$: Add forests to Γ_i and Γ'_i to create covering spaces $\widehat{\Gamma}_i$ and $\widehat{\Gamma}'_i$ of G and G' respectively. Since Γ_i and Γ'_i have the same images in $\pi_1(G)$, by covering space theory there is a lift $\widehat{\eta}_i : \widehat{\Gamma}_i \to \widehat{\Gamma}'_i$ of id_G. Set η_i to be the induced map $\Gamma_i \to \Gamma'_i$.

$\impliedby$: $m = m' \circ \eta$ and so m and m' have the same images in $\pi_1(G)$.

(2): If $[\tau] = [\tau']$, then by definition there is a cellular isomorphism $h : G \to G'$ so that $\tau' \simeq h \circ \tau$. Again by covering space theory there is the desired lift $\vec{K}_\tau \to \vec{K}_{\tau'}$.

(3): If $f : R_n \to R_n$ induces ϕ and if (m, τ) represents $\phi(\vec{K})$ then $(m, (G, g \circ f))$ represents $\vec{K}$.

(4): $\implies$: Using (3), $\vec{K}'_{[\tau']} = (\phi^{-1}\vec{K}')_{[\tau'\phi]} = \vec{K}_{[\tau]}$.

$\impliedby$: Since $\vec{K}_{[\tau]} = \vec{K}'_{[\tau']}$, we have the right hand commuting square in the diagram below where h and η are cellular isomorphisms. Further, we may choose f so that the left rectangle homotopy commutes.

$$\begin{array}{ccccc}
R_n & \xrightarrow{g'} & G' & \xleftarrow{m'} & \vec{K}'_{\tau'} \\
f\uparrow & & h\downarrow & & \downarrow\eta \\
R_n & \xrightarrow{g} & G & \xleftarrow{m} & \vec{K}_\tau
\end{array}$$

Define $\phi \in \mathsf{Out}(F_n)$ to be induced by f. In particular, $[\tau']\phi = [\tau]$. The marked Stallings graph (m, τ) represents $\vec{K}$ and $(m', g \circ f)$ represents $\phi^{-1}\vec{K}'$. Since the images in $\pi_1(R_n)$ of the maps induced by the homotopy equivalences $f^{-1} \circ g'^{-1} \circ m'$ and $g^{-1} \circ m$ are equal, $\phi(\vec{K}) = \vec{K}'$. □

The *volume* of a connected graph Γ is the cardinality of its edge set. The *volume* of a Stallings graph $\Gamma \to G$ is the sequence of volumes of the components of Γ. Equivalent graphs have equal volumes. By Proposition 5.2(2), the volume $\|\tau\|_{\vec{K}}$ of $\vec{K}_\tau$ depends only on $[\tau]$. When $\vec{K}$ has only one component, say $\vec{K} = (K)$, we sometimes view $\vec{K}$ as an element of $\mathcal{K}$ and write for example $\|\tau\|_K := \|\tau\|_{\vec{K}}$. Example 5.1 shows that the map $\vec{K}_{[\tau]} \mapsto ((K_1)_{[\tau]}, \ldots, (K_N)_{[\tau]})$ from equivalence classes of Stallings graphs representing elements of $X_{\mathcal{K}}$ to sequences of equivalence classes representing elements of $\mathcal{K}$ is not injective. However, $\|\tau\|_{\vec{K}} = (\|\tau\|_{K_1}, \ldots, \|\tau\|_{K_N})$. By Proposition 5.2(3), we have:

Remark 5.3. — $\|\tau \cdot \phi\|_{\vec{K}} = \|\tau\|_{\phi(\vec{K})}$ for all marked graphs τ, all $\vec{K} \in X_{\mathcal{K}}$ and all $\phi \in \mathsf{Out}(F_n)$.

We now define the modified star graph. Fix a rose $\rho = (R, r)$ and a $K \in \mathcal{K}$. Divide K_ρ into *pieces* $\mathsf{Y} = \{Y\}$ by snipping each edge at its midpoint. Denote the set of valence one vertices of Y by ∂Y. Each piece Y is a tree that is naturally the union of distinct half-edges meeting at a common vertex of K_ρ. We use those half-edges to label the elements of ∂Y by distinct elements of H. To construct the modified star graph of ρ and K, start with $2n$ vertices V labeled by the elements of H and then add each $Y \in \mathsf{Y}$, with elements of ∂Y attached to H according to their labels. It is clear that if $a \in F_n$, K is the conjugacy class of the cyclic subgroup $\langle a \rangle$, and w is the conjugacy class of a then the modified star graph for ρ and K is the same as the star graph for ρ and w.

We conclude this section by showing that some standard facts about $\|\tau\|_w$ and star graphs also hold for $\|\tau\|_K$ and modified star graphs.

Suppose that $\rho = (R, r)$ is a rose and that $\tau = (G, g) \in St(\rho)$, i.e. there is a marking-preserving map $G \to R$ that collapses a maximal tree T. There is a resulting collapse map $K_\tau \to K_\rho$. The edges of G that are collapsed by $G \to R$, i.e. the edges of T, are *new* edges and the others are *old*. Similarly, K_ρ consists of old edges and $K_\tau \to K_\rho$ collapses new edges. We have a commuting diagram

$$
\begin{array}{ccc}
K_\tau & \longrightarrow & K_\rho \\
\downarrow & & \downarrow \\
G & \longrightarrow & R
\end{array}
$$

If $\{e_i\}$ is the set of new edges in G and $|e_i|_{\tau,K}$ is the number of edges in K_τ labeled by either e_i or $\bar{e}_i$ then

$$
\|\tau\|_K = \|\rho\|_K + \sum_i |e_i|_{\tau,K}
$$

Following [Vog17, Definition 8.1], we define a *non-trivial ideal edge* α to be a partition of H into two subsets of cardinality $\geqslant 2$ that separate some pair $\{e, \bar{e}\} \subset H$. Each such α determines a marked graph $\tau \in St(\rho)$ and a collapse map $\tau \to \rho$ that collapses a single edge, say A. There is an induced collapse map $K_\tau \to K_\rho$. If we snip the old edges of K_τ at their midpoints, the resulting components are indexed by the pieces $Y \in \mathsf{Y}$ of K_ρ. If $\widetilde{Y}$ is the component corresponding to Y, then the map $\widetilde{Y} \to Y$ either is a cellular isomorphism or collapses a single edge labeled A. The latter occurs if and only if α separates ∂Y (thought of as a subset of H).

Motivated by this last observation, we define $|A|_Z$, for arbitrary subsets A, Z of H, to be 1 if $\{A, A^c\}$ separates Z and 0 otherwise. See Figure 5.1.

Remark 5.4. —   Let K be the element of $\mathcal{K}$ represented by $\langle a_1, \ldots, a_N \rangle < F_n$. The marked Stallings graph $(\Gamma \to R_n, id_{R_n})$ representing K can be constructed by taking the core of the graph obtained by starting with the join of loops representing the a_i's and folding; see [Sta83] (or for example [FH99]). A similar statement holds for elements of $X_{\mathcal{K}}$.
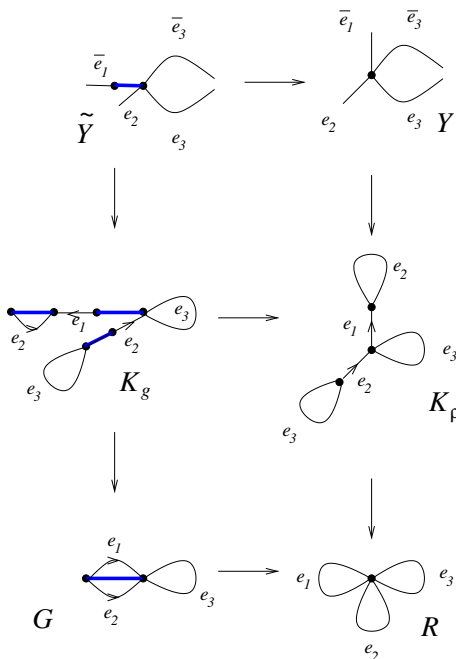
Figure 5.1. The right side illustrates a rose $\rho = (R, r)$, a Stallings graph K_ρ for an element $K \in \mathcal{K}$, and a piece Y with $\partial Y = \{\bar{e}_1, e_2, e_3, \bar{e}_3\}$. (G, g) illustrates a point in $St(\rho)$ with a new (blue) edge corresponding to the ideal edge $E = \{\{\bar{e}_1, \bar{e}_2\}, \{\bar{e}_1, \bar{e}_2\}^c\}$. The partition of ∂Y induced by E is $\{\{\bar{e}_1\}, \{e_2, e_3, \bar{e}_3\}\}$, E separates ∂Y, and $|E|_{\partial Y} = 1$. Cf. [Vog17, Figure 3].

6. Proof of Theorem 1.1

We assume throughout this section that G is trivial. After recalling certain key results from [CV86] and [Vog17] (giving references in [Vog17]), we make the necessary modifications so that they apply in our current context with $\mathcal{K}$ replacing $\mathcal{C}$. Finally, we verify that our contractible subcomplexes form a presentation family and apply Proposition 4.2.

6.1. Contractibility in Case $\mathcal{C}$

Recall that L denotes the subcomplex of the spine of Outer space spanned by graphs with no separating edges. By [Vog17, Proposition 6.1], the spine of Outer space deformation retracts to L.

Fix a sequence $(w_1, \cdots)$ in $\mathcal{C}$ that includes all elements of $\mathcal{C}$ (repeats allowed). In [Vog17], Vogtmann takes the sequence to be without repeats and in non-decreasing reduced word length with respect to a fixed basis, but this is not used in her proof. This flexibility, similar to that in [CV86], will be important for us. For $\rho \in \mathcal{R}$, denote by $\|\rho\|$ the element $(\|\rho\|_{w_1}, \cdots)$ of the ordered abelian group $\mathbb{Z}^{\mathcal{C}}$. Order the set $\mathcal{R}$ of marked roses by $\rho < \rho'$ if $\|\rho\| < \|\rho'\|$.

We record three results from [Vog17] and then prove two corollaries of those results.

PROPOSITION 6.1 ([Vog17, Proposition 5.2]). — *The set $\mathcal{R}$ of roses is well-ordered by $<$.*

Define $L_{<\rho} = \cup_{\rho' < \rho} St(\rho')$

LEMMA 6.2 ([Vog17, Corollary 7.2]). — *If $St(\rho) \cap L_{<\rho}$ is not empty then it contains an element of $\mathcal{R}$ that is obtained from ρ by a Whitehead move.*

PROPOSITION 6.3 ([Vog17, Section 10]). — *$St(\rho) \cap L_{<\rho}$ is either empty or contractible.*

The main result of [Vog17], namely contractibility of L and hence the spine of Outer space, follows easily from Proposition 6.3 and the fact that the spine is connected. We apply that argument here to conclude that the same is true for each $L_{<\rho}$.

COROLLARY 6.4. — *$L_{<\rho}$ is contractible for each non-minimal $\rho \in \mathcal{R}$.*

Proof. — The proof is by (transfinite) induction in two steps. Let $C(\rho)$ be the statement that each component of $L_{<\rho}$ is contractible. This obviously holds for the minimal element of $\mathcal{R}$ so we may assume that ρ is not minimal and that $C(\rho')$ holds for each $\rho' < \rho$. If there is a largest element $\rho' \in L_{<\rho}$ then $C(\rho)$ follows from Proposition 6.3 and the inductive hypothesis. Otherwise, there is an increasing sequence $\rho_1 < \rho_2 < \ldots$ such that $L_{<\rho}$ is the increasing union $\cup_{i=1}^{\infty} L_{\rho_i}$. In this case, $C(\rho)$ follows from the inductive hypothesis.

If the corollary fails, then there exists ρ' such that $L_{\rho'}$ has at least two components, A and B. An induction argument exactly as above shows that A and B are contained in separate components of L_ρ for all $\rho' < \rho$ and hence contained in separate components of L, in contradiction to the fact that L is connected [Vog17, Propositions 4.1 and 6.1]. $\square$

COROLLARY 6.5 (cf. [Whi36b]). — *If $\rho \in \mathcal{R}$ is not minimal then there exists $\rho' \in \mathcal{R}$ such that $\rho' < \rho$ and such that ρ' is obtained from ρ by a Whitehead move.*

Proof. — Let $\hat{\rho}$ be the smallest element of $\mathcal{R}$ that is greater than ρ in the well-ordering. Thus $L_{<\hat{\rho}} = L_{<\rho} \cup St(\rho)$. Corollary 6.4.implies that $L_{<\hat{\rho}}$ is connected and hence that $L_{<\rho} \cap St(\rho)$ is not empty. Lemma 6.2 therefore completes the proof. $\square$

6.2. Contractibility in Case $\mathcal{K}$

We now extend these results so as to apply to $\mathcal{K}$. The only difference between the $\mathcal{C}$ case and $\mathcal{K}$ case is that some pieces in the latter have vertices of valence bigger than two. Many of the arguments in the $\mathcal{C}$ case therefore carry over to the $\mathcal{K}$ setting without change.

Fix a sequence $(K_1, K_2, \cdots)$ in $\mathcal{K}$ that includes every element of $\mathcal{K}$. Update $\|\rho\|$ to now denote the element $(\|\rho\|_{K_1}, \|\rho\|_{K_2}, \cdots)$ of $\mathbb{Z}^{\mathbb{N}}$. We also update $<$ and write $\rho < \rho'$ if $\|\rho\| < \|\rho'\|$.

We have the following analog of Proposition 6.1.

PROPOSITION 6.6. — *$\mathcal{R}$ with the updated $<$ is well-ordered.*

Proof. — With the observation that $w \in \mathcal{C}$ determines an element of $K \in \mathcal{K}$ with $\|\rho\|_w = \|\rho\|_K$, the proof of [Vog17, Proposition 5.2] goes through *mutatis mutandis.* $\qquad\square$

The following proposition is the analog of Proposition 6.3.

PROPOSITION 6.7. — *Let $<$ be the updated well-ordering. Then, for all $\rho \in \mathcal{R}$, $St(\rho) \cap L_{<\rho}$ is either empty or contractible.*

Proof. — The length function $\|\tau\|_w$ is introduced in [Vog17, Section 5]. We have already defined its analog $\|\tau\|_K$. The statement and proofs in [Vog17, Sections 6–8] apply, using this updated norm, without change. In particular, Lemma 6.2 of this paper holds for the updated well-ordering. The star graph Γ_w is introduced in [Vog17, Section 9]. We have already defined its analog Γ_K. No changes are required in [Vog17, Section 10], at the end of which Proposition 6.3 has been shown to follow from a key lemma [Vog17, Lemma 10.1].

Our final modification concerns [Vog17, Lemmas 11.1 and 11.2]. The second part of the former is not true in our current context and the given proof of the latter depends on the former. We state and prove the $\mathcal{K}$-analog of [Vog17, Lemma 11.2] below as Lemma 6.9. With it in place, the proof of [Vog17, Lemma 10.1] (which appears as the proof of [Vog17, Lemmas 11.3]) goes through without further changes and completes the proof of Proposition 6.7. $\qquad\square$

Remark 6.8. — Corollaries 6.4 and 6.5 remain valid with the updated well-ordering because they follow from Propositions 6.3 and 6.1 and Lemma 6.2.

We recall now a definition from [Vog17, Section 11]. Given an arbitrary set $A \subset H$, define $|A| = (|A|_{w_1}, |A|_{w_2}, |A|_{w_3}, \ldots) \in \mathbb{Z}^{\mathcal{C}}$ where $|A|_w$ is the number of edges in the star graph Γ_w with one endpoint in A and the other in A^c. Equivalently, $|A|_w$ is the cardinality of $\{Y \in \mathsf{Y} : \{A, A^c\}$ separates the endpoints of $Y\}$, where $\mathsf{Y} = \{Y\}$ is the set of pieces associated to w in Section 5.

We modify this to fit our context as follows. Given $A \subset H$, $K \in \mathcal{K}$ and the set of pieces $\mathsf{Y} = \{Y\}$ of the Stallings graph K_ρ, define:

$$|A|_{\partial Y} = 1 \text{ if } \{A, A^c\} \text{ separates } \partial Y \text{ and } 0 \text{ otherwise}$$

$$|A|_K = \sum_{Y \in \mathsf{Y}} |A|_{\partial Y} \qquad |A|_{\mathcal{K}} = (|A|_{K_1}, |A|_{K_2}, |A|_{K_3}, \ldots) \in \mathbb{Z}^{\mathcal{K}}$$

The following Lemma is [Vog17, Lemma 11.2] adapted to our current context.

LEMMA 6.9 (cf. [Vog17, Lemma 11.2]). — *Let A and B be subsets of H. Then*

$$|A \cap B|_{\mathcal{K}} + |A \cup B|_{\mathcal{K}} \leqslant |A|_{\mathcal{K}} + |B|_{\mathcal{K}}$$

Proof. — It is enough to prove the following inequality for each $K \in \mathcal{K}$ and each $Y \in \mathsf{Y}$ where Y is the set of pieces in K_ρ:

$$|A \cap B|_{\partial Y} + |A \cup B|_{\partial Y} \leqslant |A|_{\partial Y} + |B|_{\partial Y}$$

We have a partition of H into disjoint subsets $\mathbf{Z} = A \cap B$, $\mathbf{W} = \bar{A} \cap \bar{B}$, $\mathbf{X} = A \cap \bar{B}$, and $\mathbf{Y} = B \cap \bar{A}$. There are cases depending on how ∂Y intersects these subsets. Taking into account that $\partial Y \neq \emptyset$, that A and A^c determine the same partition of H,

and that the inequality in the statement is symmetric in A and B, we are reduced to the following cases.

- ∂Y meets exactly one of $\mathbf{Z}$, $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$: $0 + 0 \leqslant 0 + 0$
- ∂Y meets exactly two of $\mathbf{Z}$, $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$:
  - $\mathbf{Z}$ and $\mathbf{W}$: $1 + 1 \leqslant 1 + 1$
  - $\mathbf{Z}$ and $\mathbf{X}$: $1 + 0 \leqslant 0 + 1$
  - $\mathbf{X}$ and $\mathbf{Y}$: $0 + 0 \leqslant 1 + 1$
- ∂Y meets exactly 3 of $\mathbf{Z}$, $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$:
  - not $\mathbf{X}$: $1 + 1 \leqslant 1 + 1$
  - not $\mathbf{Z}$: $0 + 1 \leqslant 1 + 1$
- ∂Y meets $\mathbf{Z}$, $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$: $1 + 1 \leqslant 1 + 1$. □

As mentioned above, this also finishes the proof of Proposition 6.7.

6.3. Applying Proposition 4.2

Given $\vec{K} = (K_1, \cdots, K_N) \in X_{\mathcal{K}}$, choose an extension $K_1, \cdots, K_N, K_{N+1}, \cdots$ of $\vec{K}$ to an infinite sequence that contains each element of $\mathcal{K}$ and then define a well-ordering $<$ on $\mathcal{R}$ using this infinite sequence as in Section 6.2. If $\rho \in \mathcal{R}$ is not minimal in $<$ then by Remark 6.8 and Corollary 6.5 there exists $\rho' \in \mathcal{R}$ such that $\rho' < \rho$ and such that ρ' differs from ρ by a Whitehead move. Thus, starting with any $\rho \in \mathcal{R}$, we can find $\rho_{min} = (R_{min}, r_{min})$. Define $\vec{z}_{min} = \|\rho_{min}\|_{\vec{K}} = (\|\rho_{min}\|_{K_1}, \ldots, \|\rho_{min}\|_{K_N})$. Let $\mathcal{R}_{\vec{K}}$ be the initial interval of $<$ such that $\rho \in \mathcal{R}_{\vec{K}}$ if and only if $\|\rho\|$ starts with $\vec{z}_{min}$; let $L_{\vec{K}} = \cup_{\rho' \in \mathcal{R}_{\vec{K}}} St(\rho')$. Since $\mathcal{R}_{\vec{K}}$ is an initial interval of $\mathcal{R}$ and since the complement of $\mathcal{R}_{\vec{K}}$ in $\mathcal{R}$ has a first element, Corollary 6.4 implies that $L_{\vec{K}}$ is contractible.

By Proposition 4.2, the following lemma completes the proof of Theorem 1.1.

PROPOSITION 6.10. — $\{L_{\vec{K}}\}$ is a presentation family in L for the action of $\mathsf{Out}(F_n)$ on $X_{\mathcal{K}}$.

Proof. — We check the conditions in Definition 4.1.

We represent elements of $\mathsf{Out}(F_n)$ as homotopy equivalences of R_n or as automorphisms given in terms of their effects on a basis. In the latter case, two such can be composed and are equal iff one composed with the inverse of the other is inner; something that can be checked on a basis. Thus Definition 4.1(1) holds.

We represent elements of $X_{\mathcal{K}}$ as marked Stallings graphs $(\Gamma \to R_n, id_{R_n})$; see Section 5 and especially Proposition 5.2(1). Given $\vec{K}_1, \vec{K}_1 \in X_{\mathcal{K}}$, and $\phi \in \mathsf{Out}(F_n)$, we can compare the Stallings graphs for $\phi(\vec{K}_1)$ and $\vec{K}_2$ using Remark 5.4. Thus Definition 4.1(2) holds.

The action of $\mathsf{Out}(F_n)$ on L preserves the topological types of the vertices of L and so does not invert edges. Thus (3a) holds.

For (3b), suppose that we are given vertices $\tau = (G, g)$ and $\tau' = (G', g')$ of L. Decide if G and G' are cellularly isomorphic. If NO then there does not exist $\theta \in \mathsf{Out}(F_n)$ such that $\tau \cdot \theta = \tau'$. If YES then the finite set of $\theta \in \mathsf{Out}(F_n)$ satisfying $\tau \cdot \theta = \tau'$ can be algorithmically listed. Indeed, such a θ is determined by a homotopy equivalence of the form $(g')^{-1}hg$ where $h : G \to G'$ is a cellular isomorphism. Thus (3b) holds.

For (4b), let $\theta \in \mathsf{Out}(F_n)$ and $\rho \in \mathcal{R}$. By Remark 5.3, $\|\rho\|_{\theta\vec{K}} = \|\rho\theta\|_{\vec{K}} := \|\theta^{-1}\rho\|_{\vec{K}}$. Hence $\rho \in \theta\mathcal{R}_{\vec{K}}$ iff $\rho \in \mathcal{R}_{\theta\vec{K}}$. Thus (4b) holds.

It remains to check (4a). For $S' \subset \mathcal{R}$, let $[S']$ denote the set of equivalence classes of the Stallings graphs $\{\vec{K}_\rho \mid \rho \in S'\}$; see Section 5. We inductively construct a finite set S of $\mathsf{Fix}(\vec{K})$-orbit representatives for $\mathcal{R}_{\vec{K}}$. Start with $S_1 := \{\rho_{min}\}$. Given S_i, let $W(S_i)$ be the subset of $\mathcal{R}_{\vec{K}}$ consisting of elements obtained from elements of S_i by a Whitehead move. If $[W(S_i)] = [S_i]$ then $S := S_i$. Otherwise, $S_{i+1} := S_i \cup W(S_i)$. (We could be more efficient here by only adding new equivalence classes.) This process ends since there are only finitely many equivalence classes with a prescribed volume.

We claim that the union U of the $\mathsf{Fix}(\vec{K})$-orbits of S equals $\mathcal{R}_{\vec{K}}$. Indeed, since $\mathcal{R}_{\vec{K}}$ is $\mathsf{Fix}(\vec{K})$-invariant and $S \subset \mathcal{R}_{\vec{K}}$, we have that $U \subset \mathcal{R}_{\vec{K}}$. For the other inclusion, recall that $\mathcal{R}_{\vec{K}}$ is an initial interval of the total order $<$ on $\mathcal{R}$. Suppose that $\rho \in \mathcal{R}_{\vec{K}} \setminus U$ is minimal. By Remark 6.8 there is $\rho' \in \mathcal{R}_{\vec{K}}$ obtained from ρ by a Whitehead move such that $\rho' < \rho$. So there is $\theta \in \mathsf{Fix}(\vec{K})$ such that $\rho'\theta \in S$. Hence (see Section 3) $\rho\theta$ differs from an element of S by a Whitehead move. By the definition of S, there is $\rho'' \in S$ such that $\vec{K}_{\rho''} = \vec{K}_{\rho\theta}$. By Proposition 5.2(4) $\rho\theta$, hence also ρ, is in the $\mathsf{Fix}(\vec{K})$-orbit of $\rho'' \in S$, a contradiction. It follows that $St(U) = L_{\vec{K}}$. Defining $D_{\vec{K}} = \cup_{\rho \in S} St(\rho)$, we see that (4a) holds. $\qquad\square$

7. From [KV93]

The rest of the paper is devoted to proving our main result, Theorem 1.2. Now the key work is Krstić and Vogtmann's [KV93] and we mimic proofs there. As in the proof of Theorem 1.1, the main work is showing that a certain subcomplex of the spine of Outer space is contractible. As in the proof of Theorem 1.1, we follow the proof of contractibility when $\mathcal{K}$ is replaced by $\mathcal{C}$, making modifications as necessary for our expanded context.

We start by reviewing needed background material; see especially [KV93, Sections 3(B) and 3(C)]. Fix a finite subgroup $\mathsf{G} < \mathsf{Out}(F_n)$ and define the *equivariant spine* to be the subcomplex of the spine of Outer space that is fixed by the action of G. By definition then, the vertices of the equivariant spine are equivalence classes of marked G-graphs $\sigma = (\Gamma, s)$, i.e. Γ is equipped with an action $h : \mathsf{G} \to \mathsf{Aut}(\Gamma)$, such that if $f_x : R_n \to R_n$ represents $x \in \mathsf{G}$ then $h(x) \circ s$ is homotopy equivalent to $s \circ f_x$. Here $\mathsf{Aut}(\Gamma)$ is the (finite) isometry group of Γ where we have identified each edge of Γ with the unit interval. It is required that G inverts no edge of Γ, a requirement that can be achieved by adding midpoints of inverted edges if necessary. Collapsing G-equivariant forests gives the set of vertices of the equivariant spine a partial order that agrees with the one inherited from the spine of Outer space. We view the equivariant spine as the geometric realization of this poset.

An edge of a G-graph Γ is *inessential* if it is contained in every maximal G-invariant forest of Γ. Γ is *essential* if no edge is inessential, all vertices have valence at least two and the two edges terminating at a bivalent vertex are in the same G-orbit. L_G denotes the subcomplex of K_G spanned by essential graphs. L_G is a deformation retract of the equivariant spine [KV93, Proposition 3.3].

The role of roses in this setting is played by *reduced* G-graphs, i.e. vertices $(\Gamma, s) \in L_\mathsf{G}$ without a non-trivial G-forest. The set of reduced G-graphs is denoted $\mathcal{R}_\mathsf{G}$. Every element of L_G is contained in the star of an element of $\mathcal{R}_\mathsf{G}$.

If $f : \Gamma \to \Gamma'$ is a homotopy equivalence of reduced G-graphs of the form $\Gamma \leftarrow G \to \Gamma'$ where G is a G-graph and both maps collapse an edge orbit then we say that $(\Gamma', f \circ s)$ is obtained from (Γ, s) by a *Whitehead move*.

PROPOSITION 7.1 ([KV93, Proposition 5.11]). — *Any two elements of $\mathcal{R}_\mathsf{G}$ are connected by a sequence of Whitehead moves.*

COROLLARY 7.2 ([KV93, Corollary 5.12]). — *L_G is connected.*

Following [KV93, Section 6], we define a norm on $\mathcal{R}_\mathsf{G}$ and a G-star graph. Given $K \in \mathcal{K}$ and $\sigma = (\Gamma, s)$ define

$$\|\sigma\|_{\mathsf{G},K} = \sum_{x \in \mathsf{G}} \|\sigma\|_{xK} = |\mathsf{G}| \cdot \|\sigma\|_K$$

where the volume $\|\sigma\|_K$ is defined in Section 5. As in the proof of Theorem 1.1, we fix a sequence $\vec{K} = (K_1, K_2, \ldots)$ in $\mathcal{K}$ that includes all elements of $\mathcal{K}$. For each $\sigma \in \mathcal{R}_\mathsf{G}$, define

$$\|\sigma\|_\mathsf{G} = (\|\sigma\|_{\mathsf{G},K_1}, \|\sigma\|_{\mathsf{G},K_2}, \ldots) \in \mathbb{Z}^\mathcal{K}$$

and order $\mathcal{R}_\mathsf{G}$ by $\sigma <_\mathsf{G} \sigma'$ if $\|\sigma\|_\mathsf{G} < \|\sigma'\|_\mathsf{G}$.

The Stallings graph K_σ, and its decomposition into pieces $\mathsf{Y} = \{Y_i\}$, is defined as in Section 5 with the role of H being played by the set $E(\Gamma)$ of oriented edges of G. In particular, the elements of ∂Y_i are labeled by distinct elements of $E(\Gamma)$. The G-star graph for $K \in \mathcal{K}$ and $\sigma = (\Gamma, s)$ is defined to be the graph formed by "superimposing" the star graphs of xK for each $x \in \mathsf{G}$ and so has the following description. Begin, as in the non-equivariant case, with a set V of vertices labeled by the elements of $E(G)$. Then, for each $x \in \mathsf{G}$ and $Y \in \mathsf{Y}$, add a copy of xY with $\partial(xY)$ attached to V according to its labels.

Propositions 7.3 and 7.5 below are the analogs of Propositions 6.6 and Lemma 6.2. Their $\mathcal{C}$ versions appear as [KV93, Proposition 6.3 and Lemma 6.5]. The proofs given there carry over to the $\mathcal{K}$ case without modification.

PROPOSITION 7.3. — *The set $\mathcal{R}_\mathsf{G}$ of roses is well-ordered by $<_\mathsf{G}$.*

Define $L_{<_\mathsf{G}}(\sigma) = \cup_{\sigma' < \sigma} St(\sigma)$

LEMMA 7.4. — *If $St(\sigma) \cap L_{<_\mathsf{G}}(\sigma)$ is not empty then it contains an element of $\mathcal{R}$ that is obtained from σ by a Whitehead move.*

We denote by σ_{min} the minimal element of $\mathcal{R}_\mathsf{G}$ with respect to $<_\mathsf{G}$ and by $\mathcal{R}_{\mathsf{G},\vec{K}}$ the subset of $\mathcal{R}_\mathsf{G}$ of σ such that $\|\sigma\|_{\mathsf{G},K_i} = \|\sigma_{min}\|_{\mathsf{G},K_i}$ for $1 \leqslant i \leqslant N$. By $L_{\mathsf{G},\vec{K}}$ denote the union of the stars (in L_G) of the elements of $\mathcal{R}_{\mathsf{G},\vec{K}}$. This notation assumes that we have already specified an element $\alpha \in \mathsf{Hom}(\mathsf{G}, \mathsf{Out}(F_n))$ and $\vec{K} \in X_\mathcal{K}$. If we want to stress the dependence on α and $\vec{K}$ then we will write $\mathcal{R}_{(\alpha,\vec{K})}$ for $\mathcal{R}_{\mathsf{G},\vec{K}}$ and $L_{(\alpha,\vec{K})}$ for $L_{\mathsf{G},\vec{K}}$.

As we did with Theorem 1.1 (see Section 6.3), to prove Theorem 1.2 we will apply Proposition 4.2 – this time:

- $X := \mathsf{Hom}(\mathsf{G}, \mathsf{Out}(F_n)) \times X_{\mathcal{K}}$;
- $G := \mathsf{Out}(F_n)$;
- L is again the subcomplex (still denoted L) of the spine of Outer space consisting of marked graphs with no separating edges; and
- for $x = (\alpha, \vec{K}) \in X$, $L_x := L_{(\alpha, \vec{K})}$.

With only the obvious modifications, which we leave to the reader, the arguments of Section 6.3 reduce the proof of Theorem 1.2 to the following analog of Proposition 6.7. (To verify Definition 4.1 (2), we view an element of $\mathsf{Hom}(\mathsf{G}, \mathsf{Out}(F_n))$ as an action $\mathsf{G} \to \mathsf{Aut}(\Gamma)$ for some marked graph Γ, which is possible by [Cul84, Zim81].)

PROPOSITION 7.5. — $St(\rho) \cap L_{\mathsf{G}, <\rho}$ is either empty or contractible.

The $\mathcal{C}$ version of Proposition 7.5 is proved in [KV93, Sections 7 and 8]. As in the non-equivariant case, most of the arguments extend to our current context without change. The two exceptions are [KV93, Lemmas 7.1 and 7.2], which appear as Lemmas 8.12 and 8.11 in this paper. We state and prove these lemmas in Section 8.3. Once this is done, the arguments in [KV93, Section 8] carry over to complete the proof of Proposition 7.5 and hence the proof of Theorem 1.2.

In summary, we have proved Theorem 1.2 modulo verifying Lemmas 8.12 and 8.11.

8. The Combinatorial Lemmas

In this section we complete the proof of Theorem 1.2 by proving Lemmas 8.12 and 8.11, the analogs of [KV93, Lemmas 7.2 and 7.1].

The following notation is used throughout this section.

Fix $\rho \in \mathcal{R}_{\mathsf{G}}$ and let $E(\Gamma)$ be the set of oriented edges in Γ. The definition of an *ideal edge* α of Γ in the equivariant setting is given in [KV93, Section 5.A]. In particular $\alpha \subset E_v \subset E(\Gamma)$ where v is a vertex of Γ and E_v is the set of oriented edges of Γ terminating at v. We repeatedly use the following property of an ideal edge.

(edge) $$\alpha \cap x\alpha \neq \emptyset, x \in \mathsf{G} \implies x \in \mathsf{Stab}(\alpha)$$

Remark 8.1. — Following Krstić and Vogtmann, let P be the subgroup of $\mathsf{Stab}(v)$ generated by the stabilizers of the oriented edges in $\alpha \subset E_v$. The second condition in their definition of ideal edge is that $\mathsf{G} \cap \alpha = Pe$ for all $e \in \alpha$. From this it follows easily that $P = \mathsf{Stab}(\alpha)$ and that (edge) is satisfied.

For subsets A and Z of $E(\Gamma)$, define $|A|_Z^*$ to be 1 if $\{A, A^c\}$ separates[2] Z and 0 otherwise. The superscript $*$ indicates that we are ignoring the group action. For any subgroup $\mathsf{G}' < \mathsf{G}$, define

$$|A|_{Z, \mathsf{G}'} = \sum_{g \in \mathsf{G}'} |A|_{gZ}^*$$

and

$$|A|_Z = |A|_{Z, \mathsf{G}}$$

---

[2] Equivalently Z meets both A and A^c.

Remark 8.2. — $|gA|_{gZ}^* = |A|_Z^*$ for all $g \in \mathsf{G}$ and all $A, Z \subset E(\Gamma)$. Thus, if $gA = A$ then $|A|_{gZ}^* = |A|_Z^*$. Similarly, $|A|_{Z,\mathsf{G}'} = |A|_{gZ,\mathsf{G}'}$ for all $g \in \mathsf{G}'$. We use these facts repeatedly, usually without comment.

8.1. The First Combinatorial Lemma

The goal of this section is to prove the combinatorial lemma that underlies our analogue (Lemma 8.11) of [KV93, Krstić–Vogtmann's Lemma 7.1].

Notation 8.3. — We assume throughout this subsection that $\alpha, \beta \subset E(\Gamma)$ are ideal edges whose stabilizers $P = \mathsf{Stab}(\alpha)$ and $Q = \mathsf{Stab}(\beta)$ have indices p and q in G, respectively. We also assume such that:

 (1) $\alpha \cap \beta \neq \emptyset$
 (2) $P \leqslant Q$
 (3) $\alpha \cap g\beta \neq \emptyset, g \in \mathsf{G} \implies g \in Q$

Let $\gamma := \alpha \cap \beta$; $\gamma' := Q\gamma = Q\alpha \cap \beta$; $A := \alpha - \gamma$; and $B := \beta \cup Q\alpha = \beta + QA$. (Disjointness of β and QA as well as the final equality follows from the fact that β and β^c are Q-invariant.) In particular,

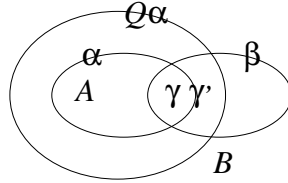 • $Q\alpha$, β, B, B^c are all Q-invariant.

Figure 8.1.

The following lemma is the main result of this subsection.

LEMMA 8.4. — *For any $Z \subset E(\Gamma)$,*

(8.1) $$p|\gamma|_{Z,Q} + q|B|_{Z,Q} \leqslant p|\alpha|_{Z,Q} + q|\beta|_{Z,Q}$$

The proof of Lemma 8.4 occupies the rest of this subsection. We will need the following property of B and Q.

LEMMA 8.5. — $g \in \mathsf{G}, B \cap gB \neq \emptyset \iff g \in Q$.

Proof. — $\Leftarrow$: B is Q-invariant.
$\Rightarrow$: Assuming that $y, g(y) \in B$, we must show that $g \in Q$. Case 1: Assume $y \in \beta$. If $gy \in \beta$ then $g \in Q$ by the (edge) property of β. If $gy \in Q\alpha$ then there is $x \in Q$ such that $x^{-1}gy \in \alpha$ and 8.3 (3) implies that $x^{-1}g \in Q$ and so $g \in Q$. Case 2: Assume $y \in Q\alpha$ and hence there is $x \in Q$ such that $x^{-1}y \in \alpha$. If $gy \in \beta$ then $x^{-1}g^{-1}(gy) \in \alpha$ and 8.3 (3) implies that $x^{-1}g^{-1} \in Q$ and so $g \in Q$. If $gy \in Q\alpha$ then there is $x' \in Q$ such that $x'^{-1}gx(x^{-1}y) = x'^{-1}gy \in \alpha$. By the (edge) property of α, $x'^{-1}gx \in P$ and so $g \in Q$. $\qquad\square$

Notation 8.6. — The disjoint regions in Figure 8.1 are $QA-A$, A, γ, $\gamma'-\gamma$, $\beta-\gamma'$, B^c. Note that all of these regions are P-invariant. We use a sequence of 0s and 1s to indicate which regions contain an element of Z. For example, 011000 denotes that Z meets A and γ and does not meet any other component.

We now prove Lemma 8.4 by a case analysis.

- $Z \subset B^c$: Then $QZ \subset B^c$ and 8.4(8.1) becomes $0 \leqslant 0$.
- Z meets both B and B^c: From $Z \cap B^c \neq \emptyset$ it follows that $|\gamma|^*_{gZ} \leqslant |\alpha|^*_{gZ}$ for all $g \in Q$; in particular, $|\gamma|_{Z,Q} \leqslant |\alpha|_{Z,Q}$. If Z meets β then $|B|^*_{gZ} = |\beta|^*_{gZ} = 1$ for all $g \in Q$. Thus $|B|_{Z,Q} = |\beta|_{Z,Q} = |Q|$ and 8.4(8.1) is satisfied. We may therefore assume that Z misses β and hence meets $Q\alpha$. After replacing Z by gZ for some $g \in Q$, we may assume that Z meets α. Since Z misses β, gZ misses γ for all $g \in Q$ and $|\gamma|_{Z,Q} = 0$. Since gZ meets α for all $g \in P$, $p|\alpha|_{Z,P} = p|P| = |\mathsf{G}|$. 8.4(8.1) therefore follows from $q|B|_{Z,Q} \leqslant q|Q| = |\mathsf{G}|$.
- $Z \subset B$: In this case $|B|_{Z,Q} = 0$. If $|\gamma|_{Z,Q} = 0$ then the left hand side of 8.4(8.1) is 0 and we are done. Hence we may further assume that there exists $x \in Q$ with $|\gamma|^*_{xZ} = 1$. After replacing Z by xZ, we have $|\gamma|_Z = 1$. Thus, Z intersects both γ and γ^c. If Z intersects $\beta-\gamma'$, which is a Q-set in the complement of $Q\alpha$, then gZ intersects α^c for each $g \in Q$. In this case, $|\gamma|^*_{gZ} \leqslant |\alpha|^*_{gZ}$ for all $g \in Q$ and we are done. So we additionally assume that Z doesn't meet $\beta-\gamma'$. Since $Z \subset B$, we have $Z \subset Q\alpha$. To summarize, we are left with proving 8.4(8.1) in the case that $Z \subset Q\alpha$ intersects both γ and γ^c. Using Notation 8.6, Z has the form $\delta_1\delta_21\delta_400$ with $\max(\delta_1,\delta_2,\delta_4) = 1$ where δ_i denotes a 1 or a 0.

The following chart shows the required case analysis. Details are given in Lemma 8.7 and the paragraph that follows it.

	form	$p\lvert\gamma\rvert^*$	$q\lvert\beta \cup Q\alpha\rvert^*$	$p\lvert\alpha\rvert^*$	$q\lvert\beta\rvert^*$
PZ	$\delta_1\delta_21\delta_400$	$\lvert P\rvert p$	0	$\max(\delta_1,\delta_4)\lvert P\rvert p$	$\max(\delta_1,\delta_2)\lvert P\rvert q$
$(Q-P)Z$	$\delta_1'\delta_2'?100$	Δ	0	$\geqslant \Delta$	$\max(\delta_1',\delta_2')\lvert Q-P\rvert q$
column totals: QZ		$\lvert\mathsf{G}\rvert + \Delta$	0	$\max(\delta_1,\delta_4)\lvert\mathsf{G}\rvert + (\geqslant \Delta)$	$\max(\delta_1,\delta_2)\lvert\mathsf{G}\rvert$

Here $\max(\delta_1,\delta_2) = \max(\delta_1',\delta_2')$ and ? denotes an indeterminate 0 or 1 that doesn't factor into the computation. Since Z intersects γ^c, $\max(\delta_1,\delta_2,\delta_4) = 1$, and so 8.4(8.1) is satisfied.

LEMMA 8.7. — *Suppose that $Z \subset Q\alpha$ intersects both γ and γ^c. Define δ_1 [resp. δ_2, δ_4] to be 1 if Z intersects $QA - A$ [resp. A, $\gamma' - \gamma$] and 0 otherwise.*

(a) *For all $g \in Q$, $|B|^*_{gZ} = 0$.*

(b) *For all $g \in Q$, $|\beta|^*_{gZ} = 1 \iff Z$ intersects $QA \iff \max(\delta_1, \delta_2) = 1$.*

(c) *For all $g \in P$: $|\gamma|^*_{gZ} = 1$; and $|\alpha|^*_{gZ} = 1 \iff Z$ intersects $Q\alpha - \alpha \iff \max(\delta_1, \delta_4) = 1$*

(d) *If $g \in Q - P$ then $|\gamma|^*_{gZ} \leqslant |\alpha|^*_{gZ}$*

Proof. — Since B and β are Q-invariant, $|B|^*_{gZ}$ and $|\beta|^*_{gZ}$ are independent of $g \in Q$ and it suffices to verify (a) and (b) with gZ replaced by Z, in which case (a) and (b) follow from the hypotheses of the lemma and the definitions. Item (c) is proved similarly, noting that γ and α are P-invariant. For $g \in Q - P$, the (edge) property of P implies that $gZ \cap \alpha^c \neq \emptyset$ and hence that $|\gamma|^*_{gZ} \leqslant |\alpha|^*_{gZ}$.

Applying Lemma 8.7 and $\max(\delta_1, \delta_2, \delta_4) \geqslant 1$,

$$p|\alpha|_{Z,Q} - p|\gamma|_{Z,Q} + q|\beta|_{Z,Q} - q|B|_{Z,Q}$$

$$\geqslant p|\alpha|_{Z,P} - p|\gamma|_{Z,P} + q|\beta|_{Z,Q} \qquad \text{using (a) and (d)}$$

$$= p|\alpha|_{Z,P} + q|\beta|_{Z,Q} - p|P| \qquad \text{using (c)}$$

$$= p|P| \max(\delta_1, \delta_4) + q|Q| \max(\delta_1, \delta_2) - |\mathsf{G}| \qquad \text{using (b) and (c)}$$

$$= |\mathsf{G}| \max(\delta_1, \delta_4) + |\mathsf{G}| \max(\delta_1, \delta_2) - |\mathsf{G}| \geqslant (\max(\delta_1, \delta_2, \delta_4) - 1)|\mathsf{G}| \geqslant 0$$

This completes the proof in the case that $Z \subset B$ and so also the proof of Lemma 8.4.

$\square$

8.2. The Second Combinatorial Lemma

The goal of this section is to prove the combinatorial lemma that underlies [KV93, Lemma 7.2].

LEMMA 8.8 (cf. [KV93, Lemma 7.2]). — *Suppose that $x \in \mathsf{G}$ and that $\alpha, \beta \subset E(\Gamma)$ are ideal edges whose stabilizers $P = \mathsf{Stab}(\alpha)$ and $Q = \mathsf{Stab}(\beta)$ have indices p and q in G, respectively. Let*

$$\gamma = \alpha \cap Px\beta = P(\alpha \cap x\beta) \qquad \text{and} \qquad \gamma' = \beta \cap Qx^{-1}\alpha = Q\left(\beta \cap x^{-1}\alpha\right)$$

Then for all $Z \subset E(\Gamma)$,

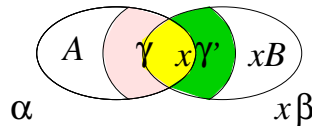$$p|\alpha - \gamma|_Z + q|\beta - \gamma'|_Z \leqslant p|\alpha|_Z + q|\beta|_Z$$

Figure 8.2. The left ellipse represents α and the right $x\beta$. The union of the pink and yellow regions represents γ and the union of the yellow and green $x\gamma'$. $A = \alpha - \gamma$ and $xB = x\beta - x\gamma'$.

Proof. — Letting $A := \alpha - \gamma$ and $B := \beta - \gamma'$, our desired inequality is

$$(8.2) \qquad p|A|_Z + q|xB|_Z \leqslant p|\alpha|_Z + q|\beta|_Z$$

The proof of (8.2) occupies the rest of this subsection. We start with some observations on the action of G on the sets in Figure 8.2.

- $x\gamma' = xQ(\beta \cap x^{-1}\alpha) = Q^x(x\beta \cap \alpha)$ where $Q^x = xQx^{-1} = \mathsf{Stab}(x\beta)$.
- By construction, the six sets $A, \gamma - (\alpha \cap x\beta), \alpha \cap x\beta, x\gamma' - (\alpha \cap x\beta), xB = x\beta - x\gamma'$ and $(\alpha \cup x\beta)^c$ shown in Figure 8.2 are disjoint.
- By definition, α, and so also α^c, is P-invariant; γ, hence also A, is P-invariant. Similarly, $x\beta$, $(x\beta)^c$, $x\gamma'$, and xB are Q^x-invariant.
- $\mathsf{G}(\gamma \cup x\gamma')$ is disjoint from A and xB (which are disjoint from each other). Indeed,
  - $\mathsf{G}\gamma$, A, and xB are pairwise disjoint:
    * $g\gamma \cap A \neq \emptyset \implies g \in P$ (by the (edge) property of α) $\implies g\gamma = \gamma \implies \gamma \cap A \neq \emptyset$, contradiction.
    * $g\gamma \cap xB \neq \emptyset \implies gP(\alpha \cap x\beta) \cap xB \neq \emptyset \implies \exists\, g' \in gP \subset \mathsf{G}, b \in \alpha \cap x\beta$ such that $g'b \in xB$. Thus $g' \in Q^x$ (by the (edge) property of $x\beta$) and so $g'b \in x\gamma'$. This gives that $x(\gamma' \cap B) \neq \emptyset$, which is a contradiction.
  - Symmetrically $\mathsf{G}\gamma' = \mathsf{G}x\gamma'$, A, and xB are pairwise disjoint.

We now use these observations to verify (8.2). If

$$(8.3) \qquad p|A|^*_{gZ} + q|xB|^*_{gZ} \leqslant p|\alpha|^*_{gZ} + q|x\beta|^*_{gZ}$$

holds for all $g \in \mathsf{G}$ then we are done, so assume that (8.3) fails for some $g \in \mathsf{G}$. Thus, either $|A|^*_{gZ} = |\alpha - \gamma|^*_{gZ} > |\alpha|^*_{gZ}$ or $|xB|^*_{gZ} = |x\beta - x\gamma'|^*_{gZ} > |x\beta|^*_{gZ}$. By replacing Z with gZ and reversing the roles of α and β if necessary, we may assume that $|A|^*_Z = |\alpha - \gamma|^*_Z > |\alpha|^*_Z$. It follows that $Z \subset \alpha$ and that Z meets both A and $\gamma = P(\alpha \cap x\beta)$. By replacing Z with gZ for some $g \in P$, we may further assume $Z \subset \alpha$ and Z meets both A and $\alpha \cap x\beta$ and so also intersects $x\beta$.

Notation 8.9. — The disjoint regions in Figure 8.2 are A, $\gamma - x\beta$, $\alpha \cap x\beta$, $x\gamma' - \alpha$, xB, and $(\alpha \cup x\beta)^c$. We use a sequence of 0s and 1s to indicate which regions contain an element of Z. For example, 101000 denotes that Z meets A and $\alpha \cap x\beta$ and does not meet any other component.

With this notation, we are reduced to proving (8.2) assuming that Z has the form 101000 or 111000. The cases are summarized in the following chart, where Δ and Δ' represent nonnegative quantities. We elaborate in Lemma 8.10.

	form	$p|A|^*$	$q|B|^*$	$p|\alpha|^*$	$q|\beta|^*$
$(P \cap Q^x)Z$	101000 or 111000	$|P \cap Q^x|p$	0	0	$|P \cap Q^x|q$
$(P - Q^x)Z$	110000 or 111000	$|P - Q^x|p$	0	0	Δ
$(Q^x - P)Z$	000101	0	0	0	$|Q^x - P|q$
$(\mathsf{G} - (P \cup Q^x))Z$	in α^c and meets $(\alpha \cup x\beta)^c$	0	$\leqslant \Delta'$	0	Δ'
column totals: $\mathsf{G}Z$		$|P|p = |\mathsf{G}|$	Δ'	0	$|\mathsf{G}| + \Delta + \Delta'$

LEMMA 8.10. — *Suppose that $Z \subset \alpha$ intersects both A and $x\beta$.*

(a) *If $g \in (P \cap Q^x)$ then $|A|^*_{gZ} = |x\beta|^*_{gZ} = 1$ and $|\alpha|^*_{gZ} = |xB|^*_{gZ} = 0$.*

(b) *If $g \in (P \cap (Q^x)^c)$ then $|A|^*_{gZ} = 1$ and $|xB|^*_{gZ} = |\alpha|^*_{gZ} = 0$.*

(c) *If $g \in (Q^x \cap P^c)$ then $|A|^*_{gZ} = |xB|^*_{gZ} = |\alpha|^*_{gZ} = 0$ and $|x\beta|^*_{gZ} = 1$.*

(d) *If $g \in \mathsf{G} - (P \cup Q^x)$ then $|A|^*_{gZ} = |\alpha|^*_{gZ} = 0$ and $|xB|^*_{gZ} \leqslant |x\beta|_{gZ}$.*

Proof. — We begin with a pair of preliminary observations: $|A|^*_Z = |x\beta|^*_Z = 1$ because Z intersects both A and $x\beta$, which are disjoint; $|\alpha|^*_Z = |xB|^*_Z = 0$ because $Z \subset \alpha$ and $Z \cap xB = \emptyset$.

If $g \in P \cap Q^x$ then A, xB, α and $x\beta$ are all g-invariant. Item (a) therefore follows from our preliminary observations and Remark 8.2.

If $g \in P \cap (Q^x)^c$ then $|A|^*_{gZ} = |A|^*_Z = 1$ and $|\alpha|^*_{gZ} = |\alpha|^*_Z = 0$ because α and A are g-invariant. From $xB \cap \alpha = \emptyset$ and $g^{-1}(\alpha) = \alpha$, it follows that $g^{-1}(xB) \cap \alpha = \emptyset$ and so $|xB|^*_{gZ} = |g^{-1}xB|^*_Z = 0$. This completes the proof of (b).

If $g \in Q^x \cap P^c$ then $|xB|^*_{gZ} = |xB|^*_Z = 0$ and $|x\beta|^*_{gZ} = |x\beta|^*_Z = 1$ because xB and $x\beta$ are g-invariant. Property (edge) implies that $g\alpha$ is disjoint from α and hence that gZ is disjoint from α and A. It follows that $|A|^*_{gZ} = |\alpha|^*_{gZ} = 0$ completing the proof of (c).

If $g \in \mathsf{G} - (P \cup Q^x)$ then $|A|^*_{gZ} = |\alpha|^*_{gZ} = 0$ as in case (c). Moreover, gZ intersects $g(\alpha \cap x\beta)$ and so intersects the complement of $x\beta$. It follows that $|xB|_{gZ} \leqslant |x\beta|_{gZ}$. $\square$

Equation (8.2), and hence Lemma 8.8, now follows from the following two consequences of Lemma 8.10.

$$p|A|_Z = p|A|_{Z,P} = p|P| = |\mathsf{G}|$$

$$q|x\beta|_Z - q|xB|_Z \geqslant q \sum_{g \in Q^x} \left(|x\beta|^*_{gZ} - |xB|^*_{gZ} \right) = q|Q| = |\mathsf{G}|$$

$\square$

8.3. [KV93, Lemmas 7.1 and 7.2]

Fix two ideal edges α and β of R. Let $P, Q < \mathsf{G}$ denote the respective stabilizers of α and β. The indices of P and Q in G are respectively p and q. Choose double coset representatives $x_1 = 1, x_2, \ldots, x_k$ for $P \backslash \mathsf{G} / Q$, i.e.

$$\mathsf{G} = PQ + Px_2Q + \cdots + Px_kQ$$

where $+$ denotes disjoint union. The intersections $\gamma := \alpha \cap \mathsf{G}\beta$ and $\gamma' := \beta \cap \mathsf{G}\alpha$ decompose as disjoint unions

$$\gamma = \gamma_1 + \cdots + \gamma_k$$
$$\gamma' = \gamma'_1 + \cdots + \gamma'_k$$

where $\gamma_i := \alpha \cap Px_i\beta$ and $\gamma' := \beta \cap Qx_i^{-1}\alpha$.

We assume that at least one γ_i is non-empty; i.e. $G\alpha$ and $G\beta$ *cross* in the notation of [KV93]. After replacing β by some translate $x\beta$, $x \in \mathsf{G}$, we assume that $\gamma_1 = \alpha \cap \beta \neq \emptyset$. If $\gamma_i = \emptyset$ for all $i \neq 1$ then $\mathsf{G}\alpha$ and G_β *cross simply.* Equivalently Notation 8.3 (3) is satisfied.

Recall that we have enumerated the elements $K_1, K_2, \ldots$ of $\mathcal{K}$. Let Y_i be the set of pieces for K_i (see Section 5) and for each $A \subset E(\Gamma)$, define

$$|A|^*_{K_i} := \sum_{Y \in \mathsf{Y}_i} |A|^*_{\partial Y} \qquad |A|_{K_i} := \sum_{x \in \mathsf{G}} |A|^*_{xK_i} \qquad |A| := (|A|_{K_1}, \ldots) \in \mathbb{Z}^{\mathcal{K}}$$

We can now state our versions of [KV93, Lemmas 7.1 and 7.2]. They follow immediately from the definitions and Lemmas 8.4 and 8.8.

LEMMA 8.11 (cf. [KV93, Lemma 7.1]). — *If $\mathsf{G}\alpha$ and $\mathsf{G}\beta$ cross simply, and $P \leqslant Q$, then*

$$p|\alpha \cap \beta| + q|\beta \cup Q\alpha| \leqslant p|\alpha| + q|\beta|$$

LEMMA 8.12 (cf. [KV93, Lemma 7.2]). — *If $\mathsf{G}\alpha$ and $\mathsf{G}\beta$ cross, then for all i,*

$$p|\alpha - \gamma_i| + q\,|\beta - \gamma'_i| \leqslant p|\alpha| + q|\beta|$$

9. Extensions

Let $\vec{K} = (K_1, K_2, \cdots, K_N)$ be a finite sequence of conjugacy classes of finitely generated subgroups of F_n. Less formally, we can (and will!) think of K_i as a subgroup of F_n defined up to conjugation. Let $GMc(\vec{K})$ be the associated generalized McCool group, i.e. the subgroup of $Out(F_n)$ consisting of outer automorphisms that preserve K_i for all i; see Section 2. If H is a subgroup of F_n let $N(H) = \{g \in F_n \mid gHg^{-1} = H\}$ be the normalizer of H in F_n and consider the natural homomorphism $N(H) \to Aut(H)$ that sends $g \in N(H)$ to conjugation by g. When H is noncyclic this homomorphism is injective and we define

$$Out'(H) = Aut(H)/N(H)$$

When H is cyclic we define $Out'(H) = Out(H)$, which is cyclic of order 2 when H is nontrivial. When H is its own normalizer (i.e. $N(H) = H$) then $Out'(H) = Out(H)$.

There is then a well defined restriction homomorphism

$$\Phi : GMc(\vec{K}) \to Out'(K_1) \times Out'(K_2) \times \cdots \times Out'(K_N)$$

Note that the kernel of Φ is the McCool group $Mc(\vec{K})$ (see [GL15] and Section 2). In this section we are interested in the image of Φ, which we denote by $Ext(F_n; \vec{K})$. Thus an element of $Ext(F_n; \vec{K})$ is a tuple $(\theta_1, \cdots, \theta_N)$ of elements of $Out'(K_i)$ that *simultaneously extend* to an outer automorphism of F_n. In [GL21] it was shown that when $N = 1$ and K_1 is a "random" subgroup of F_n then $Ext(F_n; \vec{K})$ is trivial. The main theorem in this section is:

THEOREM 9.1. — *$Ext(F_n; \vec{K})$ is commensurable to a finite product of generalized McCool groups. In particular, it is of type* VF.

Recall that two groups are commensurable if they contain isomorphic finite index subgroups. If there is an epimorphism $G \to H$ with finite kernel, and if G is residually finite, then G and H are commensurable. Finally, recall that $Out(F_n)$ (and hence any of its subgroups) is residually finite [Gro75].

We start by considering a special case.

LEMMA 9.2. — *Suppose F_n admits no free splitting or a splitting over a cyclic subgroup in which every K_i is elliptic. Then the kernel of Φ is finite and thus $Ext(F_n; \vec{K})$ is commensurable with $GMc(\vec{K})$.*

Proof. — If the kernel is infinite, we have a sequence $f_j \in Out(F_n)$ of distinct automorphisms acting as the identity on each K_i. Then after a subsequence the limiting procedure [Pau88, Bes88] yields a nontrivial stable F_n-$\mathbb{R}$-tree T with each K_i elliptic and with all arc stabilizers cyclic (in fact, T belongs to the boundary of Outer space). But then by [BF95, GLP94] F_n has a cyclic splitting with each K_i elliptic, contradiction. □

The next case is that F_n admits splittings over infinite cyclic subgroups with all K_i elliptic, but no such free splittings. In that case we can consider the cyclic JSJ decomposition J of F_n [RS97, DS99, FP06, GL17] relative to $\vec{K}$. Recall that J is a cocompact simplicial F_n-tree in which all K_i are elliptic.

The JSJ decomposition J has a lot of structure: there are two kinds of vertices – rigid and quadratically hanging (QH). The latter correspond to surfaces with boundary, where boundary components represent the incident edge groups or are commensurable with some K_i. The surfaces may be nonorientable, but are required to contain intersecting 2-sided simple closed curves, so e.g. a pair of pants is not allowed. The key universal property that J possesses is the following: if $g \in F_n$ is elliptic in J and not contained in a QH vertex group as a nonboundary element, then g is elliptic in every cyclic splitting of F_n in which all K_i are elliptic. In particular, a rigid vertex group V does not admit any cyclic splittings with ∂V and all K_i contained in V elliptic.

The JSJ tree is usually not unique up to isomorphism, but the set of elliptic elements is independent of the choice and the set of possible JSJ trees forms a contractible deformation space [For02, Cla05, GL07]. All edge stabilizers of J are cyclic, the set of noncyclic vertex stabilizers is independent of the choice of J (maximal noncyclic subgroups of the set of elliptics), and so is the set of commensurability classes of edge stabilizers (intersections of noncyclic point stabilizers).

In particular, for every noncyclic vertex group V, the set ∂V of commensurability classes of edge groups contained in V is well defined. In other words, if f is an outer automorphism of F_n that preserves $\vec{K}$, it may not fix J, but it preserves the conjugacy classes of noncyclic vertex groups (it is allowed to permute them) and commensurability classes of edge groups. It also preserves QH vertex groups and their peripheral subgroups.

LEMMA 9.3. — *Suppose F_n admits splittings over infinite cyclic subgroups with all K_i elliptic, but no such free splittings. Then $Ext(F_n; \vec{K})$ is commensurable with the product of finitely many generalized McCool groups.*

Proof. — There are two kinds of K_i, the ones that are commensurable to edge groups of J or to boundary components of QH vertex groups, and the ones that are not – therefore their representatives fix only vertices.

To fix ideas, we now assume that all K_i are of the latter kind. Then each K_i has a unique noncyclic vertex group (up to conjugacy) $V(K_i)$ that contains it and these vertex groups are rigid. Noncyclic vertex groups are their own normalizer, and

moreover, if $gKg^{-1} < V$ for some $K < V$ and $g \in F_n$ then either $g \in V$ or K is contained in an edge group. Thus K_i gives a unique conjugacy class of subgroups of $V(K_i)$. Let $V_1, V_2, \cdots, V_k$ be the representatives of vertex groups that occur in this way and let $\vec{H}_j$ be the tuple of associated subgroups K_i that occur in V_j. We then claim that the group $Ext(F_n; \vec{K})$ contains a subgroup of finite index isomorphic to

$$Ext\left(V_1; \vec{H}_1'\right) \times \cdots \times Ext\left(V_k; \vec{H}_k'\right)$$

where $\vec{H}_j'$ is obtained from $\vec{H}_j$ by appending V_j-conjugacy classes of cyclic subgroups in ∂V_j, one representative of each commensurability class. Note that some groups in ∂V_j could be conjugate in F_n but not in V_j; however, the number of conjugacy classes up to commensurability in V_j we obtain in this way is still finite and does not exceed the number of V_j-orbits of edges incident to the vertex fixed by V_j. Since V_j does not admit a cyclic splitting with $\vec{H}_j$ and ∂V_j elliptic, Lemma 9.2 finishes the proof. To prove the claim, note that every simultaneous extension permutes the vertex groups, hence fixes the conjugacy classes of the V_j, and it permutes the conjugacy classes in ∂V_j. After taking a suitable finite index subgroup of $Ext(F_n; \vec{K})$, there will be simultaneous extensions that fix ∂V_j and all peripheral subgroups of QH vertex groups. Conversely, if automorphisms can be simultaneously extended to the V_j's fixing ∂V_j, then they can be further extended by the identity to the rest of F_n.

In the general case, when some K_i are commensurable to edge groups or to boundary components of QH vertex groups, we ignore such K_i and we reach the same conclusion as above since our extensions fix all edge groups of J and all peripheral subgroups of QH vertex groups. $\qquad\square$

Proof of Theorem 9.1. — Let $\{G_1, \cdots, G_k\}$ be the smallest free factor system such that each K_i is conjugate into some G_j. This G_j will then be unique since free factors are malnormal. Note that if this free factor system is $\{F_n\}$ then the conclusion follows from Lemma 9.3. After reindexing, $\vec{K}$ yields k tuples $\vec{H}_1, \cdots, \vec{H}_k$ of conjugacy classes of subgroups of $G_1, \cdots, G_k$, and we observe that

$$Ext\left(F_n; \vec{K}\right) = Ext\left(G_1; \vec{H}_1\right) \times \cdots \times Ext\left(G_k; \vec{H}_k\right)$$

Indeed, if a tuple of automorphisms of each $\vec{H}_j$ simultaneously extends to G_j, then these extensions can be combined to an automorphism of $G_1 * \cdots * G_k$ which can then be further extended to F_n since the free product is a free factor of F_n. Conversely, any automorphism of F_n that preserves $\vec{K}$ will also preserve each G_j and the statement follows.

By Lemma 9.3 each $Ext(G_j; \vec{H}_j)$ is commensurable to a product of generalized McCool groups, so the same holds for $Ext(F_n; \vec{K})$. $\qquad\square$

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