The structure of spatial slices
of 3-dimensional causal triangulations

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Abstract. We consider causal 3-dimensional triangulations with the topology of $S^2 \times [0, 1]$ or $D^2 \times [0, 1]$ where S^2 and D^2 are the 2-dimensional sphere and disc, respectively. These triangulations consist of slices and we show that these slices can be mapped bijectively onto a set of certain coloured 2-dimensional cell complexes satisfying simple conditions. The cell complexes arise as the cross section of the individual slices.

Contents

1. Introduction

We investigate in this paper a class of problems that arise in the dynamical triangulation approach to 3-dimensional gravity restricted to the case of so-called *causal triangulations*. For an introduction to the dynamical triangulation approach to discrete quantum gravity we refer to [2] and an account of causal dynamical triangulations can be found in [3]. For the case of 3-dimensional gravity in particular, one may consult [1] and [4].

In [9] we gave a proof that the number $N(V)$ of causal 3-dimensional triangulations homeomorphic to a 3-sphere and consisting of V tetrahedra is exponentially bounded,

$$N(V) \leq C^V,$$

where C is some positive constant. Validity of this bound is crucial in order for the relevant correlation functions to exist, and thereby defining the discretised model (see [2, 3]). The first step of the argument leading to this bound was to decompose the triangulations into slices (which is possible because of the causal structure defined below) and to show that it is sufficient to establish the bound for such causal slices. The number of triangulated causal slices was then shown to be bounded by the number of certain coloured 2-dimensional cell complexes homeomorphic to the 2-sphere. These cell complexes arise as mid-sections of the slices. Finally, bounding the number of those 2-dimensional cell complexes can be done using well-known techniques.

The method of associating a coloured 2-dimensional cell complex with a causal triangulation has been applied earlier by other workers in the field [4, 5] and also used in numerical simulations and in combination with matrix model techniques to extract properties of the model. It has, however, not been established exactly what class of 2-dimensional complexes can occur as mid-sections of a causal slice. As noted in [9], see also [4] Appendix B, some non-trivial constraints have to be imposed on top of its homeomorphism class. It follows that the connection between the model defined in terms of triangulations and the one realised in terms of a specific class of coloured 2-dimensional cell complexes is obscure. It might be the case that the precise class of cell complexes is unimportant in a possible scaling limit of the model, see [5], but this remains to be investigated in detail.

In this paper we provide a complete characterisation of the coloured 2-dimensional cell complexes that correspond to 3-dimensional causal slices. In fact, we shall consider not only the standard notion of causal triangulations homeomorphic to $S^2 \times [0, 1]$, where S^2 denotes the 2-sphere, which in this paper will be referred to as causal sphere-triangulations, but find it useful to generalise the notion of causal triangulation to manifolds homeomorphic to $D^2 \times [0, 1]$, where D^2 is the 2-dimensional disc, and $\partial D^2 \times [0, 1]$ may be viewed as the time-like part of the boundary while the two discs $D^2 \times \{0\}$ and $D^2 \times \{1\}$ form the spatial parts of the boundary. Such triangulations will be called causal disc-triangulations and the corresponding slices will be called causal disc-slices. The coloured cell complexes, defined in Section 3, are in this case homeomorphic to D^2. The main result of this paper, Theorem 1, which is proved in Section 4, states that there is a bijection between the causal disc-slices and the coloured cell complexes. In Section 5 we generalise this result to causal sphere-slices in Theorem 2.

These results might serve as a starting point for an exact enumeration of causal slices by applying well-known techniques for planar surfaces or maps to the coloured cell complexes homeomorphic to the disc. This interesting combinatorial problem is more complicated than those previously considered because of the colouring and the constraints identified in this paper. The result might also be instrumental in finding an appropriate matrix model generating exactly the desired causal slices or as an aid in designing effective numerical algorithms for simulations. These issues are, however, beyond the scope of the present paper.

2. Preliminaries and notation

We will use notation consistent with that of [9]. For the reader's convenience we briefly recall the main conventions, restricting the discussion to the 3-dimensional case. The basic building blocks of our triangulations are tetrahedra or 3-simplices whose vertices have one of two colours: red or blue. Generally we will denote an unoriented simplex with vertices $x_1, \ldots, x_n$ by $(x_1 \ldots x_n)$. If all the vertices in a simplex have the same colour we say that the simplex is monocoloured. This means that if x and y are red vertices and $e = (xy)$ is a 1-simplex then we say that e is red, and a triangle $\Delta = (xyz)$ is red if its vertices (or edges) are red, etc. If a simplex is not monocoloured we say it is two-coloured. It is assumed that all tetrahedra are two-coloured. Thus the tetrahedra come in three types: type (3,1) with three red vertices, type (2,2) with two red vertices and type (1,3) with one red vertex.

We recall that an abstract simplicial complex K is defined by its vertex set K^0, which is assumed to be finite, and a collection of subsets of K^0, called simplices, such that if σ is a simplex and $\sigma' \subset \sigma$, then σ' is also a simplex (see [11]). If a simplex σ contains $p + 1$ vertices we call it a p-simplex. If every simplex in K is contained in some D-simplex we say that D is the dimension of K. Given two abstract simplicial complexes K and L, a bijective map $\psi : K^0 \to L^0$ is called a combinatorial isomorphism if it induces a bijection of the simplices in K and L.

A triangulation is a 3-dimensional simplicial complex which can be viewed as a collection of tetrahedra together with identifications of some pairs of triangles (2-simplices) in the boundaries of the tetrahedra, respecting the colouring, such that any triangle is identified with at most one other triangle. When we identify triangles we also identify all their subsimplices, i.e. their edges (1-simplices) and vertices. This point of view will be used and explained in more detail in Section 4 below. Any pair of identified triangles is called an interior triangle of the triangulation while the other triangles are referred to as boundary triangles. It should be noted that it is implicit in the notion of a simplicial complex that

 (i) two triangles contained in the same tetrahedron cannot be identified,

 (ii) two different triangles in a given tetrahedron cannot be identified with two
      triangles contained in another tetrahedron.

It is common to speak about these two conditions as regularity conditions and
about simplicial complexes as regular triangulations as opposed to singular tri-
angulations, when one or both of these conditions are left out. In this paper we
focus on regular triangulations. Note that the definitions imply that two vertices
are connected by at most one edge and 3 vertices are contained in at most one
triangle.

   One may think of a triangulation either as a purely combinatoric object or
as a topological space embedded in a Euclidean space. In the former case two
triangulations are identified if there is a bijective correspondence between their
vertices respecting the colouring and the pairwise identifications of triangles. In
the latter case two triangulations are identified if there exists a homeomorphism
between them mapping simplices to simplices and thus inducing a combinatorial
isomorphism. It is a fact, explained in e.g. [9], that the two points of view are
equivalent.

   We now introduce the basic objects of study in this paper, the two types of
causal slices that form the building blocks of the general causal triangulations.

Definition 1. A *causal sphere-slice K* is a triangulation fulfilling the following
conditions:

 (i) K is homeomorphic to the cylinder $S^2 \times [0, 1]$;

 (ii) all monocoloured simplices of K belong to the boundary ∂K, such that the
      red ones belong to one boundary component ∂K_{red} and the blue ones belong
      to the other component ∂K_{blue}.

The set of all causal sphere-slices is denoted by $\mathcal{CS}$.

Definition 2. A *causal disc-slice* is a triangulation K fulfilling the following
conditions:

 (i) K is homeomorphic to the 3-dimensional ball B^3;

 (ii) all monocoloured simplices of K belong to the boundary ∂K, such that the
      red ones form a disc D_{red} and the blue ones form a disc D_{blue}, which will be
      called the boundary discs of K.

The set of all causal disc-slices is denoted by $\mathcal{CD}$.

We note that the above definitions imply that all the vertices of a causal slice of either type lie on the boundary. Two-coloured edges are sometimes referred to as *timelike edges*. Two-coloured triangles can be of two types, with two red vertices and one blue or vice versa, and are called *forward directed* and *backwards directed* triangles, respectively.

Relaxing the condition in Definition 2 that D_{red} and D_{blue} are (homeomorphic to) discs to, say, the requirement that they are deformation retracts of discs yields a larger class of causal triangulations, that will not be discussed in detail in this paper (see, however, Section 6 for some further remarks). In the course of the proof of Theorem 2 we shall encounter particular examples of such triangulations and we use the same notation as for causal slices without further comment.

Although the main focus of this paper is on causal slices we introduce for the sake of completeness general causal triangulations in the following definition as a layered union of causal slices. In general these triangulations have interior vertices.

Definition 3. A *causal sphere-triangulation* is a triangulation of the form

$$M = \bigcup_{i=1}^{N} K^i,$$

where each K^i is a causal sphere-slice such that K^i is disjoint from K^j if $i \neq j$ except that $\partial K^i_{\text{blue}} = \partial K^{i+1}_{\text{red}}$ for $i = 1, \ldots, N-1$ as uncoloured 2-dimensional triangulations. The boundary components of M are then $\partial K^1_{\text{red}}$ and $\partial K^N_{\text{blue}}$.

A *causal disc-triangulation* is a triangulation of the form

$$M = \bigcup_{i=1}^{N} K^i,$$

where each K^i is a causal disc-slice (with boundary discs D^i_{red} and D^i_{blue}) such that K^i is disjoint from K^j if $i \neq j$ except that $D^i_{\text{blue}} = D^{i+1}_{\text{red}}$ for $i = 1, \ldots, N-1$ as uncoloured 2-dimensional triangulations. The two discs D^1_{red} and D^N_{blue} are called the red and blue boundary disc of M, respectively.

The following lemma states that, apart from the two boundary discs, the boundary of a causal disc-triangulation consists of a "timelike cylinder," that is a 2-dimensional causal sphere-triangulation, which is defined as in Definitions 1 and 3 with S^2 replaced by S^1.

Lemma 1. *Let K be a causal disc-slice and denote by C the subcomplex of ∂K consisting of two-coloured triangles. Then C is a 2-dimensional causal sphere-slice with boundary $\partial D_{\mathrm{red}} \cup \partial D_{\mathrm{blue}}$, which will be called the side of K.*

More generally, if M is a causal disc-triangulation then the part of the boundary made up of triangles not in the red or blue boundary discs is a 2-dimensional causal sphere-triangulation.

Proof. It is clearly enough to prove the first statement. Choose an orientation of the circle $\partial D_{\mathrm{red}}$ and consider a vertex $v_1 \in \partial D_{\mathrm{red}}$ and its two nearest neighbours v_0 and v_2 in $\partial D_{\mathrm{red}}$ with v_0 preceding v_1 and v_2 succeeding v_1. The edges $(v_0 v_1)$ and $(v_1 v_2)$ are each contained in exactly two triangles of ∂K, one of which is red while the other one contains a blue vertex v_1', resp. v_2', both of which are in $\partial D_{\mathrm{blue}}$. Note that, by the Jordan Curve Theorem, $\partial D_{\mathrm{red}}$ divides ∂K into two discs. It follows that if we denote by $u_1, \ldots, u_k$ the neighbours of v_1 in D_{red} ordered cyclically around v_1 such that $u_1 = v_0$ and $u_k = v_2$, then all other neighbours of v_1 in ∂K are blue and constitute a segment $(w_1^1, w_2^1, \ldots, w_{\ell_1}^1)$ of $\partial D_{\mathrm{blue}}$ with $w_1^1 = v_1'$ and $w_{\ell_1}^1 = v_2'$, see Figure 1. Hence, the triangles

$$(v_0 w_1^1 v_1), (w_1^1 v_1 w_2^1), \quad \ldots, \quad (w_{\ell_1-1}^1 v_1 w_{\ell_1}^1), (w_{\ell_1}^1 v_1 v_2)$$

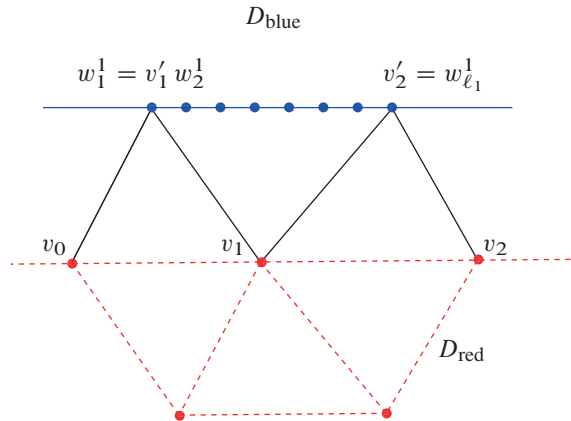make up a segment of a 2-dimensional causal slice.

Figure 1. The sequence of triangles on the side of the causal slice K connecting the red and the blue boundary components.

Repeating this construction with v_0, v_1, v_2 replaced by v_1, v_2, v_3, where v_3 is the successor of v_2 in ∂D_{red}, we obtain a unique continuation of the segment constructed above by

$$(w_1^2 v_2 w_2^2), \quad (w_2^2 v_2 w_3^2), \quad \ldots, \quad (w_{\ell_2-1}^2 v_2 w_{\ell_2}^2), \quad (w_{\ell_2}^2 v_2 v_3),$$

where $w_1^2 = w_{\ell_1}^1$ and $(w_1^2, \ldots, w_{\ell_2}^2)$ is a segment of ∂D_{blue}.

If the vertices of ∂D_{red} are $v_0, \ldots, v_n$ then after n steps we obtain a segment of a causal slice with horizontal edges in $\partial D_{\text{red}} \cup \partial D_{\text{blue}}$ and whose first and last triangle share the vertex v_0. Finally this segment can be completed to a 2-dimensional causal slice C by adjoining the triangles which contain v_0 and have one blue edge (in ∂D_{blue}) and two non-coloured edges. By construction C has boundary $\partial D_{\text{red}} \cup \partial D_{\text{blue}}$ and evidently constitutes all of $\partial K \setminus \text{int}(D_{\text{red}} \cup D_{\text{blue}})$. $\qquad\square$

The next two propositions are elementary and ensure the existence of causal disc-slices and sphere-slices with prescribed boundary discs, respectively boundary components.

Proposition 1. *Given two triangulated discs D_1 and D_2 there exists a causal disc-slice K such that $D_{\text{red}} = D_1$ and $D_{\text{blue}} = D_2$.*

Proof. We give an inductive argument. Suppose first that D_1 and D_2 are triangles Δ_{red} and Δ_{blue}, respectively. In this case, we can choose K to be the prism, depicted in Figure 2, made up of one $(1, 3)$ tetrahedron, one $(3, 1)$ tetrahedron and one $(2, 2)$ tetrahedron.

Now assume K exists for some given D_1 and D_2. If e is an arbitrary edge in $\partial D_1 = \partial D_{\text{red}} \subset \partial K$ one can glue a tetrahedron of type $(3, 1)$ to K along the unique triangle in the side of K which contains e. Thus we obtain a causal disc-slice K' whose red boundary disc has an extra triangle compared to that of K but the blue boundary disc is the same.

Similarly, given two neighbouring edges $e = (x_1 x_2)$ and $e' = (x_2 x_3)$ in ∂D_1, see Figure 3, one can first identify e and e' and then glue a sequence of $(2, 2)$ tetrahedra sharing the identified e and e' and whose blue edges are

$$(y_1 y_2), (y_2 y_3), \quad \ldots, \quad (y_{s-1} y_s)$$

with the notation of Figure 3. The resulting triangulation K'' is a causal disc-slice with red boundary disc D'_{red} obtained from D_{red} by identifying e and e' while D_{blue} is unchanged.
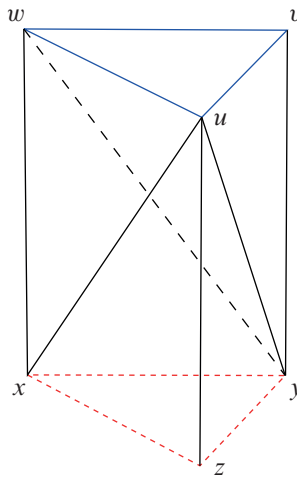
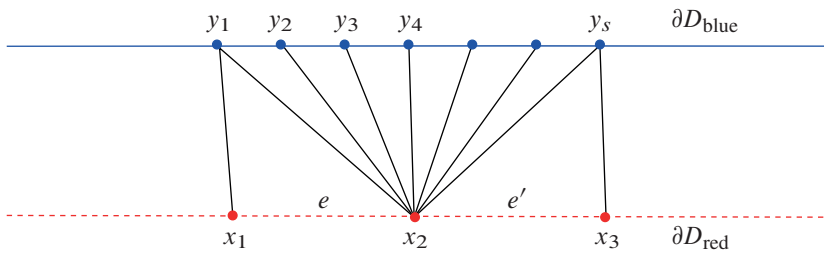Figure 2. A prism made up of 3 tetrahedra $(xyzu)$, $(uvwy)$, and $(xyuw)$.

Figure 3. Triangles in the side of a causal disc slice.

Similar constructions can of course be made with D_{blue} replacing D_{red}. Starting with an arbitrary triangle, it is well known (see e.g. [7] or [6]) that D_1 (and similarly D_2) can be constructed by repeated application of the process of either gluing on a triangle or by identifying two neighbouring boundary edges as described. Hence, the existence of K follows from the preceding discussion by induction. Note that since D_1 and D_2 are regular triangulations so is the causal disc-slice K.   □

Proposition 2. *Given two triangulated* 2*-spheres,* S_1 *and* S_2*, there exists a causal sphere-slice* K *such that* S_1 *is the red boundary of* K *and the blue boundary is* S_2*.*

Proof. We remove one triangle from S_1 and another one from S_2. Then we obtain two triangulated discs D_1 and D_2. By Proposition 1 there exists a causal disc-slice K' with boundary discs $D_{\text{red}} = D_1$ and $D_{\text{blue}} = D_2$. The side of K', C, is a causal

2-dimensional disc-slice whose boundary components are triangles. There are two different possibilities for C, up to combinatorial equivalence, see Figure 4.
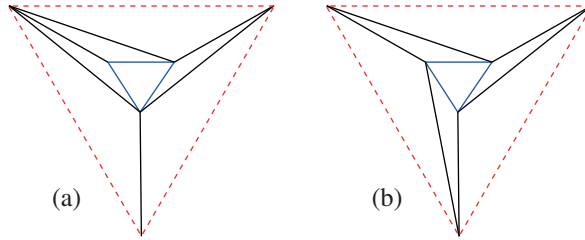

Figure 4. The two possible triangulations of the side of a disc-slice whose red and blue boundary components are both a single triangle.

Note that we may assume that no pair of neighbouring triangles in C belongs to the same tetrahedron. Indeed, if the two triangles are both forward (or both backward) directed, this follows from the fact that otherwise the removed triangle in either S_1 or S_2 would be glued to another triangle along two edges contradicting the regularity of S_1 and S_2. If one backward triangle and one forward triangle in C belong to the same tetrahedron in K' it must be a (2,2)-tetrahedron which can be removed from K' without changing the red and blue boundary discs (only an edge in C gets flipped). Since there are only finitely many (2,2)-tetrahedra in K' the claim follows.

If now C is of type (a) in Figure 4, then we can glue to K' a prism of the form indicated in Figure 2 to obtain K as desired. If C is of type (b) we can glue onto the prism of Figure 2 a (2,2)-tetrahedron to obtain a regular prism whose side C' is of the same type as C and hence can be glued onto K' to obtain K as desired. □

3. Disc-slices and midsections

Given a causal disc-slice K we define its midsection S_K as in [9] for a causal sphere-slice. More explicitly, we view K as being embedded in a Euclidean space and consider a tetrahedron $t = (v_1v_2v_3v_4)$ in K with vertices v_1, v_2, v_3, v_4, i.e. t consists of the points of the form

$$x = s_1v_1 + s_2v_2 + s_3v_3 + s_4v_4, \tag{1}$$

where $s_1, s_2, s_3, s_4 \geq 0$ and $s_1 + s_2 + s_3 + s_4 = 1$. Letting the height $h(x)$ be defined as the sum of the coefficients of red vertices in (1) the 2-cell F corresponding to t is defined as the set of points x in t of height $h(x) = \frac{1}{2}$. If t is of type (3,1) then

F is a triangle whose edges by convention are coloured red; if t is of type (1,3) then F is likewise a triangle whose edges are coloured blue; if t is of type (2,2) then F is a quadrangle each of whose boundary edges are contained in exactly one boundary triangle of t containing a monocoloured edge of t; by convention each edge of F inherits the colour of the corresponding monocoloured edge in t. It is easy to show (see [9] for details) that the 2-cells so obtained define a 2-dimensional cell complex homeomorphic to D^2 with edges coloured red or blue in such a way that triangles are monocoloured while quadrangles are two-coloured with opposite edges of the same colour. Moreover, this cell complex is uniquely defined up to combinatorial isomorphism and is called the *midsection* of K, denoted by S_K.

By construction each vertex a of S_K is contained in a unique edge e_a of K whose endpoints have different colours and vice versa. Similarly, any red (blue) edge of S_K is contained in a unique two-coloured triangle of K which contains a red (blue) edge in ∂K, and vice versa. Finally, each 2-cell of S_K is contained in a unique tetrahedron of K, the tetrahedron being of type (3,1), (2,2) or (1,3) depending on whether the cell is a red triangle, a quadrangle, or a blue triangle, respectively.

In a 2-dimensional coloured cell complex as described we shall use the notation $\langle ab \rangle$ and $\langle abc \rangle$ for edges and triangles with vertices a, b and a, b, c, respectively, whereas a 2-cell with cyclically ordered vertices a, b, c, d, such that $\langle ab \rangle$ and $\langle cd \rangle$ are red edges, will be denoted by $\langle abcd \rangle$. Note that with this convention we have, e.g., $\langle abcd \rangle = \langle dcba \rangle = \langle cdab \rangle$.

By a red, resp. blue, path in S_K we mean a sequence $e_1, \ldots, e_k$ of red, resp. blue, edges such that $e_i = \langle a_i a_{i+1} \rangle$ for each $i = 1, \ldots, k-1$, and some vertices $a_1, \ldots, a_k$. In this case we say that the path connects a_1 and a_k. The path is called simple if either the vertices $a_1, \ldots, a_k$ are all different or if $a_1, \ldots, a_{k-1}$ are different while $a_1 = a_k$, in which case the path is said to be closed.

Remark 1. With the notation just introduced it follows from the definition of S_K that if a and b are two vertices of S_K then the red (blue) endpoints of e_a and e_b are identical if a and b are connected by a blue (red) path. Indeed, if $e = \langle ab \rangle$ is a red edge of S_K, then e is contained in a triangle in K two of whose edges are e_a and e_b sharing a blue vertex. Evidently, the claim follows from this, and similarly if $e = \langle ab \rangle$ is blue.

The converse statement that a and b are connected by a red (blue) path if e_a and e_b share a blue (red) endpoint also holds as a consequence of the proof of Theorem 1 below.

Let us now note that since the monocoloured boundary discs of K are non-empty, it follows that S_K contains at least one triangle of each colour. For the same reason the boundary of S_K must contain edges of both colours. As a consequence, the boundary of S_K consists of a (cyclically ordered) alternating sequence of monocoloured paths which we shall call *boundary arcs*. The following few lemmas describe properties satisfied by any midsection S_K.

Lemma 2. *Two different vertices in a midsection S_K cannot be connected by both a red and a blue path.*

Proof. If two vertices a and b are connected by both a red and a blue path, then the two edges e_a and e_b in K have identical endpoints and hence are identical, which implies $a = b$. □

Since ∂D_{red} is homeomorphic to S^1, there does not exist a sequence of quadrangles $q_1, q_2, \ldots, q_k$ in S_K such that q_i and q_{i+1} share a red edge for $i = 1, 2, \ldots, k - 1$ and the red edges in q_1 and q_k, not shared with q_2 or q_{k-1}, belong to ∂D_{red}. A sequence of quadrangles as we have described will be called a *red path of quadrangles* connecting edges in ∂D_{red}. The corresponding statement with red replaced by blue is of course also true. We define a closed path of quadrangles (red or blue) analogously. Obviously there cannot exist a closed path of quadrangles in S_K because then K would contain a blue or a red edge in its interior. The absence of paths of quadrangles just described follows from the following more restrictive conditions.

Lemma 3. *Let S_K be a midsection, let ρ denote a closed simple red (blue) path in S_K and let μ be a simple red (blue) path connecting two vertices belonging to different blue (red) arcs in the boundary ∂S_K. Then the following hold:*

  (i) *the interior of ρ, i.e. the component of $S_K \setminus \rho$ not containing any boundary edges, contains solely red (blue) edges;*

 (ii) *the two endpoints of μ are the two endpoints of a red (blue) boundary arc.*

Proof. (i) It is sufficient to consider the case when ρ is red. If ρ encloses a blue edge it clearly also encloses a blue triangle. Since the boundary of S_K contains blue edges the exterior of ρ likewise contains a blue triangle. Evidently, all red paths connecting a vertex in an interior blue triangle with a vertex in an exterior blue triangle must intersect ρ. For K this means that D_{blue} consists of two nontrivial subcomplexes sharing a single vertex (namely the common blue endpoint of the edges e_a for $a \in \rho$) which contradicts the fact that D_{blue} is a disc.

(ii) Assume μ is red and connects two vertices a and b belonging to two different blue boundary arcs α_1 and α_2. If the conclusion of ii) does not hold one sees that the common blue endpoint of the edges e_a and e_b in K would separate ∂D_{blue} into two non-trivial parts sharing only this vertex. This contradicts the fact that ∂D_{blue} is a simple closed curve. Obviously, a similar argument applies when μ is blue. $\square$

Lemma 4. *Let $e = \langle ab \rangle$ and $f = \langle a'b' \rangle$ be two disjoint blue (red) edges in the midsection S_K. Suppose a and a' as well as b and b' are connected by a red (blue) path. Then there exists a blue (red) path of quadrangles connecting e and f.*

Proof. It suffices to prove the Lemma for blue edges. Let Δ_e and Δ_f be the two-coloured triangles of K that contain the blue edges e and f, respectively. Then Δ_e and Δ_f share a blue edge (xy) in the blue boundary of K and also have red vertices v_e and v_f in ∂K. Since e and f are disjoint $v_e \neq v_f$. If (xy) belongs to ∂D_{blue}, the star of (xy) consists of a sequence $(xyv_1v_2), (xyv_2v_3), \ldots, (xyv_{k-1}v_k)$ of (2,2)-tetrahedra sharing (xy) and an additional (1,3)-tetrahedron $(xyzv_k)$. We have $v_e, v_f \in \{v_1, \ldots, v_k\}$ and the claim follows since each (2,2)-tetrahedron corresponds to a quadrangle in the midsection. If (xy) is an interior edge in D_{blue} then the first (2,2)-tetrahedron above has to be preceded by a (1,3)-tetrahedron $(xyz'v_1)$ and then the rest of the argument is unchanged. $\square$

Proposition 3. *The midsection S_K of a causal disc-slice K determines K uniquely up to combinatorial equivalence.*

Proof. This follows by arguments identical to those in [9] for causal sphere-slices. $\square$

Definition 4. We let $\mathcal{SD}$ denote the set of all coloured cell complexes S homeomorphic to a disc (with cells as described previously) which have at least one triangle of each colour and satisfy the following conditions.

(α) No pair of different vertices in S are connected by both a red and a blue path.

(β_1) Each closed simple red (blue) path encloses solely red (blue) triangles in its interior.

(β_2) Considering the division of ∂S into red and blue arcs, there is no red (blue) path connecting two vertices belonging to different blue (red) arcs unless they are the two endpoints of a red (blue) arc.

(γ) If $e = \langle ab \rangle$ and $f = \langle cd \rangle$ are two disjoint blue (red) edges in S such that a and c as well as b and d are connected by a red (blue) path, then there exists a blue (red) path of quadrangles connecting e and f.

Remark 2. As mentioned previously, condition (β_1) implies the absence of closed red (blue) paths of quadrangles. Indeed, the outer blue (red) boundary component of such a path would violate (β_1). Similarly, condition (β_2) implies that no two different edges in ∂S can be connected by a red or blue path of quadrangles. Likewise, it follows that two different edges in the same triangle cannot be connected by a path of quadrangles, since it would contradict (α). One can demonstrate by explicit examples that property (γ) does not follow from the first three properties.

It was noted in [4, 5] that the dual graphs that arise in the matrix model formulation of 3-dimensional causal triangulations and correspond to 3-dimensional simplicial manifolds satisfy some extra conditions that are closely related to (α), (β_1) and (γ) above.

Definition 5. We say that a 2-dimensional coloured cell complex satisfies condition (δ) if for any pair of distinct red, resp. blue, triangles it is not possible to join their vertices pairwise by blue, resp. red, paths.

Lemma 5. *Condition (δ) holds for all $S \in \mathcal{SD}$.*

Proof. Given the two triangles, let us assume they are red and that three blue paths exist connecting a to a', b to b', and c to c'. It follows easily from property (α) that the two triangles must be disjoint. Using (γ) there exists a (non-trivial) path ρ_{ab} of quadrangles connecting $\langle ab \rangle$ to $\langle a'b' \rangle$. Similarly, a path ρ_{ac} of quadrangles exists connecting $\langle ac \rangle$ to $\langle a'c' \rangle$. Each of these two paths contains a blue path connecting a to a' which do not intersect each other (although they may touch at some vertices or edges). Hence they define a closed curve, whose interior consists of blue triangles by (β_1). In particular, all edges of those blue triangles as well as all red edges of ρ_{ab} and ρ_{ac} are interior edges of S, and the exterior of the curve contains the two original triangles and also the path ρ_{bc} of quadrangles connecting $\langle bc \rangle$ to $\langle b'c' \rangle$, whose existence again follows from (γ). Similarly, considering the closed blue curves determined by ρ_{ab} and ρ_{bc}, respectively ρ_{bc} and ρ_{ac}, we conclude that all edges are interior in S which contradicts the fact that S is a disc. $\square$

Lemmas 2–5 together with Proposition 3 show that the mapping $\psi \colon K \mapsto S_K$ is a well-defined injective map from the set $\mathcal{CD}$ of causal disc-slices into $\mathcal{SD}$. We now aim to prove the following main result of the present paper.

Theorem 1. *The mapping $\psi \colon K \mapsto S_K$ is bijective from the set $\mathcal{CD}$ of causal disc-slices onto the set $\mathcal{SD}$ of coloured 2-dimensional cell complexes.*

4. Proof of the main result

In this section we prove Theorem 1. The strategy is to show first that from any $S \in \mathcal{SD}$ we can construct a unique simplicial complex. We then proceed to show that this simplicial complex has the topology of a ball and is actually a causal disc-slice. The midsection of this causal slice is the coloured cell complex we started with.

Proof of Theorem 1. Let $S \in \mathcal{SD}$ be given. In order to construct the corresponding $K \equiv K_S \in \mathcal{CD}$ we start by associating to each vertex $a \in S$ a pair of new vertices r_a, b_a which will form the vertex set K_S^0 of K_S with the following identifications:

($\star$) $r_a = r_b$ (resp. $b_a = b_b$) if a and b are connected by a blue (resp. red) path,

where a, b are arbitrary vertices in S. Thus K_S^0 consists of all the vertices r_a, b_a with $a \in S$ subject to the identifications ($\star$). By definition we attach the colour red to the vertex r_a while b_a is coloured blue.

The set K_S^3 of coloured tetrahedra is obtained from the collection of 2-cells of S as follows: for each red triangle $\Delta = \langle abc \rangle$ let t_Δ be the (3,1)-tetrahedron $(r_a r_b r_c b_a)$ where we notice that all 4 vertices are different by (α) and $b_a = b_b = b_c$; similarly, if Δ is a blue triangle, let $t_\Delta = (b_a b_b b_c r_a)$; finally, for the quadrangle $q = \langle abcd \rangle$, we let t_q be the (2,2)-tetrahedron $(r_a r_b b_a b_c)$; again it follows from (α) that the four vertices are different and t_q depends only on the quadrangle q.

Thus to each 2-cell F of S there corresponds a tetrahedron t_F with vertices in K_S^0. This defines an abstract 3-dimensional coloured simplicial complex K_S whose edges and triangles are obtained as sub-simplices of the tetrahedra.

Let us first verify that $F \mapsto t_F$ is bijective between 2-cells in S and tetrahedra in K_S. By definition the mapping is surjective. Consider two 2-cells F and F' such that $t_F = t_{F'}$. Clearly F and F' are both triangles with the same colour or they are both quadrangles.

Suppose $F = \langle abc \rangle$ and $F' = \langle a'b'c' \rangle$ are, say, red triangles. Then $t_F = t_{F'}$ means that $b_a = b_b = b_c = b_{a'} = b_{b'} = b_{c'}$ and $\{r_a, r_b, r_c\} = \{r_{a'}, r_{b'}, r_{c'}\}$. By ($\alpha$) this implies that $\{a, b, c\} = \{a', b', c'\}$ and hence $\langle abc \rangle = \langle a'b'c' \rangle$.

If $F = \langle abcd \rangle$ and $F' = \langle a'b'c'd' \rangle$ are quadrangles then $t_F = t_{F'}$ implies that $\{r_a, r_b\} = \{r_{a'}, r_{b'}\}$ and $\{b_a, b_c\} = \{b_{a'}, b_{c'}\}$. Using ($\alpha$) it is then straightforward to check that $\langle abcd \rangle$ is equal to $\langle a'b'c'd' \rangle$. We have thus established that the tetrahedra of K_S are labelled by the 2-cells of S.

Next let us consider the triangles in K_S. These fall into four disjoint classes:

(i)   Red triangles $(r_a r_b r_c)$ where $\langle abc \rangle$ is a red triangle in S.

(ii)   Blue triangles $(b_a b_b b_c)$ where $\langle abc \rangle$ is a blue triangle in S.

(iii)   Triangles $(r_a r_b b_a)$ where $\langle ab \rangle$ is a red edge in S.

(iv)   Triangles $(b_a b_b r_a)$ where $\langle ab \rangle$ is a blue edge in S.

Using property (δ), which holds by Lemma 5, we see that red triangles in K_S corresponding to different red triangles in S are different. Such triangles are not shared by different tetrahedra and therefore lie in the boundary of K_S. The corresponding statement about blue triangles is also clearly true.

Next consider a triangle $(r_a r_b b_a)$ where $\langle ab \rangle$ is a red edge in S. By definition of t_F it holds that, if $\langle ab \rangle$ belongs to a 2-cell F in S, then $(r_a r_b b_a)$ belongs to the boundary of t_F.

Conversely, suppose the triangle $(r_a r_b b_a)$ belongs to the boundary of a tetrahedron t_F. If F is a red triangle $\langle a'b'c' \rangle$, then $(r_a r_b b_a)$ equals one of the triangles $(r_{a'} r_{b'} b_{a'})$, $(r_{a'} r_{c'} b_{a'})$, $(r_{b'} r_{c'} b_{a'})$. By (α) this implies that $\{a, b\}$ equals one of $\{a', b'\}$, $\{a', c'\}$, $\{b', c'\}$ and hence the edge $\langle ab \rangle$ equals one of the edges $\langle a'b' \rangle$, $\langle a'c' \rangle$, $\langle b'c' \rangle$. Thus, $\langle ab \rangle$ belongs to F. Similarly, if F is a quadrangle $\langle a'b'c'd' \rangle$, it follows that $(r_a r_b b_a)$ equals $(r_{a'} r_{b'} b_{a'})$ or $(r_{a'} r_{b'} b_{c'})$. By (α) this implies that $\langle ab \rangle = \langle a'b' \rangle$ or $\langle ab \rangle = \langle c'd' \rangle$ and hence $\langle ab \rangle$ is an edge in F.

From these observations follows that two tetrahedra t_F and $t_{F'}$ share a triangle $\Delta = (r_a r_b b_a)$ (or $\Delta = (b_a b_b r_a)$) if and only if F and F' share the edge $\langle ab \rangle$. In particular, the interior triangles in K_S are labelled by the interior edges in S. The boundary triangles in K_S are labelled by the boundary edges of S together with the monocoloured triangles in S which label the monocoloured boundary triangles in K_S.

We next consider the edges in K_S. In particular, we want to show that monocoloured edges lie in ∂K_S. Let $(r_a r_b)$ be a red edge in K_S. This means that a, b can be assumed to belong to some 2-cell F in S such that $\langle ab \rangle$ is a red edge in F. If $F = \langle abc \rangle$ is a red triangle it follows from the preceding paragraph that $(r_a r_b)$ belongs to ∂K_S since it belongs to the red triangle $(r_a r_b r_c)$. Alternatively, if $\langle ab \rangle$ belongs to a quadrangle q, then by (β_1) and (β_2) we have that q belongs to a red path of quadrangles connecting either two red triangles or a red triangle and a red boundary edge in S. This shows that $(r_a r_b)$ is an edge in a red triangle in ∂K_S. Applying similar arguments to blue edges in K_S shows that all monocoloured edges lie in the boundary.

The correspondence $a \rightarrow (r_a b_a)$ between vertices in S and the two-coloured edges of K_S is bijective by property (α). Moreover, $(r_a b_a)$ belongs to ∂K_S

if and only if a belongs to ∂S, since it belongs to the two boundary triangles corresponding to the edges in ∂S incident on a.

Finally, consider a red vertex x in K_S. It belongs to some tetrahedron t_F where F is a 2-cell in S. If F is a quadrangle or a red triangle, then x belongs to a red edge and hence, by the preceding paragraph, it belongs to ∂K_S. If F is a blue triangle $\langle abc \rangle$ then $x = r_a = r_b = r_c$. Since we assume that there is a least one triangle of each colour in S we can pick a path in S starting at a and ending at vertex a' in a red triangle $\langle a'b'c' \rangle$. There is a first vertex d in the path which is contained in a red edge and we have $x = r_d$. Hence, x belongs to ∂K_S. An identical argument shows that a blue vertex in K_S is necessarily contained in the boundary. This completes the argument that all monocoloured simplices in K_S belong to the boundary.

The next thing to consider is the structure of ∂K_S. We have seen above that there is a one-to-one correspondence between the boundary edges in S and the two-coloured triangles in ∂K_S. It follows that these triangles form a sequence $\Delta_1, \ldots, \Delta_k$ corresponding to the boundary edges $e_1, \ldots, e_k$ in ∂S ordered cyclically. Setting $e_i = \langle a_i a_{i+1} \rangle$, where $a_1, \ldots, a_k$ are the cyclically ordered vertices of ∂S (with $a_{k+1} = a_1$), we see that Δ_i and Δ_{i+1} share the edge $(r_{a_{i+1}} b_{a_{i+1}})$. By (β_2) the triangles $\Delta_1, \ldots, \Delta_k$ make up a 2-dimensional causal slice C, homeomorphic to the cylinder $S^1 \times [0, 1]$, whose red and blue boundary circles are denoted ∂C_{red} and ∂C_{blue}. At this stage we need the following lemma whose proof we will postpone a little.

Lemma 6. *The simplicial complex K_S is homeomorphic to the 3-ball so its boundary is homeomorphic to the 2-sphere.*

By the above lemma $\partial K_S \setminus C$ consists of two discs D_{red} and D_{blue} whose boundaries are ∂C_{red} and ∂C_{blue}. We claim that D_{red} is made up exactly of the red triangles in ∂K_S and similarly for D_{blue}. Indeed, since D_{red} is a 2-dimensional pseudomanifold any triangle in D_{red} can be edge-connected to $\partial D_{\text{red}} = \partial C_{\text{red}}$ by a sequence of triangles in D_{red}. The triangles in D_{red} are monocoloured and are therefore either red or blue. But a blue triangle cannot be glued to a red triangle so D_{red} consists of red triangles. These must be all the red triangles in ∂K_S since D_{blue} consists solely of blue triangles. We have therefore shown that K_S is a causal disc slice.

Now it is not hard to verify that the midsection of K_S is combinatorially isomorphic to the original midsection S used to construct K_S. We already noted above that the vertices of S are in bijective correspondence with the two-coloured edges in K_S. By the definition of the midsection S' of K_S its vertices are the

midpoints of the two-coloured edges of K_S, and hence can be labelled by the vertices of S. In this way the midsection of t_F clearly gets identified with F and the edge in S' corresponding to an interior triangle $(r_a r_b b_a)$, respectively $(b_a b_b r_a)$, of K_S is the red, respectively blue, edge $\langle ab \rangle$ in S. Hence, the correspondence between vertices induces bijective correspondences between 2-cells and 1-cells as well, and so S and S' are combinatorially isomorphic. This completes the proof of Theorem 1. □

It remains to prove Lemma 6. For this purpose an alternative construction of K_S by a gluing procedure is a useful tool. We begin by explaining this construction.

For each 2-cell F in S and each vertex a of F we define a red vertex r_a^F and a blue vertex b_a^F such that $r_a^F = r_b^F$, resp. $b_a^F = b_b^F$, if $\langle ab \rangle$ is a blue, resp. red, edge in F. In this way four different vertices are defined for each 2-cell F and they in turn define a coloured (abstract) tetrahedron τ_F considered as a simplicial complex including its subsimplices. As before τ_F is of type $(3, 1)$, $(2, 2)$, or $(1, 3)$ depending on whether F is a red triangle, a quadrangle, or a blue triangle, respectively. Without further identifications the tetrahedra so defined are pairwise disjoint and we note that the definition of K_S can be reformulated by stating that K_S is obtained from the collection of tetrahedra τ_F labeled by the 2-cells of S by imposing the identifications of simplices implied by the relations

($\star\star$) $r_a^{F_1} = r_b^{F_2}$ (resp. $b_a^{F_1} = b_b^{F_2}$) if either $a = b$ or a is connected to b by a blue (resp. red) path,

where a, b are arbitrary vertices in S and F_1, F_2 are arbitrary 2-cells in S containing a and b, respectively.

We next show that K_S can equivalently be obtained by applying a suitable gluing procedure to the collection of tetrahedra defined above. Given two 2-cells F_1 and F_2 sharing a red edge $\langle ab \rangle$ we have that τ_{F_1} contains the triangle $(r_a^{F_1} r_b^{F_1} b_a^{F_1})$ and τ_{F_2} contains the triangle $(r_a^{F_2} r_b^{F_2} b_a^{F_2})$. We say that τ_{F_1} is glued to τ_{F_2} along $\langle ab \rangle$ if $r_a^{F_1}$ is identified with $r_a^{F_2}$, $r_b^{F_1}$ is identified with $r_b^{F_2}$, and $b_a^{F_1}$ is identified with $b_a^{F_2}$ in τ_{F_1} and τ_{F_2} and their subsimplices. Similarly, gluing along a blue interior edge in S is defined. In this way, given an interior edge $\langle ab \rangle$ of S, the tetrahedra τ_{F_1} and τ_{F_2} corresponding to the 2-cells F_1 and F_2 sharing $\langle ab \rangle$ can be glued along $\langle ab \rangle$.

We now define the simplicial complex K_S' by imposing the identifications of the simplices in the collection $\{\tau_F\}$ implied by gluing pairs of tetrahedra as described along all interior edges of S. We claim that $K_S = K_S'$. In order to

prove this we need to verify that the identifications of vertices implied by the gluing conditions are identical to those given by (⋆⋆).

First, consider $r_a^{F'}$ and $r_a^{F''}$ where a is a vertex in two different 2-cells F' and F''. Since S is a manifold there exist 2-cells $F_1, F_2, \ldots, F_n$ such that $F_1 = F'$ and $F_n = F''$ and F_i and F_{i+1} share an edge e_i containing a for each $i = 1, \ldots, n$. From the gluing of τ_{F_i} and $\tau_{F_{i+1}}$ along e_i it follows that $r_a^{F_i} = r_a^{F_{i+1}}$ and $b_a^{F_i} = b_a^{F_{i+1}}$ for $i = 1, \ldots, n-1$, and hence we conclude that $r_a^{F'} = r_a^{F''}$ and $b_a^{F'} = b_a^{F''}$.

Next consider two different vertices a and b in S and assume they are connected by a blue path with edges $\langle a_1 a_2 \rangle, \langle a_2 a_3 \rangle, \ldots, \langle a_m a_{m+1} \rangle$, where $a_1 = a$ and $a_{m+1} = b$. Choosing arbitrary 2-cells $F_1, \ldots, F_m$ such that $\langle a_i a_{i+1} \rangle$ belongs to $F_i, i = 1, \ldots, m$, we have that $r_{a_i}^{F_i} = r_{a_{i+1}}^{F_i}$ and by the preceding paragraph $r_{a_{i+1}}^{F_i} = r_{a_{i+1}}^{F_{i+1}}$ for all i. It follows that $r_a^{F_1} = r_b^{F_m}$. Of course, the corresponding result for blue vertices holds if a and b are connected by a red path.

Conversely, it is clear that if a and b are different vertices belonging to 2-cells F' and F'', respectively, and $r_a^{F'} = r_b^{F''}$, then a and b are connected by a blue path. Indeed, there exists a sequence of vertices $b_1, \ldots, b_m$ and corresponding 2-cells $F_1, \ldots, F_m$ such that $a = b_1, F' = F_1$ and $b = b_m, F'' = F_m$ and $r_{b_i}^{F_i}$ is identified with $r_{b_{i+1}}^{F_{i+1}}$ either through a gluing of τ_{F_i} to $\tau_{F_{i+1}}$ along an edge containing b_i, in which case $b_i = b_{i+1}$, or else $F_i = F_{i+1}$ and b_i is connected to b_{i+1} by a blue edge in F_i. Similarly, if $b_a^{F'} = b_b^{F''}$ then a and b are connected by a red path.

This shows that the identifications pertaining to K_S and K_S' are the same and hence $K_S = K_S'$ as claimed. We are now ready to prove Lemma 6.

Proof of Lemma 6. Let $S_1, S_2, \ldots, S_N = S$ be a local construction (see [8, 6]) of the midsection S. This means that S_1 is a coloured 2-cell and S_{n+1} is obtained from S_n by either (i) gluing a coloured 2-cell to S_n along an edge e in ∂S_n or (ii) by identifying two edges e_1 and e_2 in ∂S_n which have the same colour and share a vertex. The existence of such a construction is well known and easy to establish in the 2-dimensional case. For further details we refer to [6, 7]. Evidently, all the S_n's have the topology of a disc and the 2-cells as well as interior edges of S_n can be identified with corresponding 2-cells and interior edges in S.

The correspondence between tetrahedra τ_F in K_S and 2-cells F of S and the local construction of S gives rise to a sequence of coloured simplicial complexes $K_n, n = 1, \ldots, N$, where K_n is defined by gluing the tetrahedra τ_F assigned to the 2-cells in S_n along the interior edges of S_n. In particular, K_1 is a single tetrahedron and $K_N = K_S$, since $S_N = S$ and $K_S = K_S'$ as shown above. Moreover, K_{n+1} is obtained from K_n by gluing a tetrahedron τ_F to a tetrahedron $\tau_{F'}$ in K_n along an

edge $\langle ab \rangle$. In case τ_F is not in K_n already it is clear that the topological class of K_n equals that of K_{n+1}. We shall now argue that the same holds if τ_F and $\tau_{F'}$ both belong to K_n. Since K_1 is homeomorphic to the 3-ball the same will consequently hold for K_S, and hence the proof of the lemma will be completed.

If τ_F and $\tau_{F'}$ belong to K_n and are glued along the edge e in K_n then F and F' belong to S_n, and S_{n+1} is obtained from S_n by identifying two edges e_1 and e_2 in ∂S_n sharing a vertex v, such that the identified edges equal e.
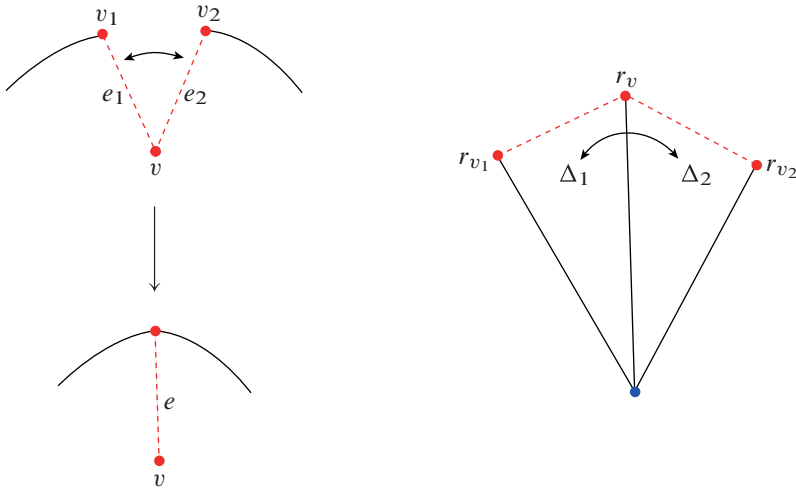
Figure 5. The local construction of the midsection and the corresponding 3-dimensional simplicial complex. Identifying the edges e_1 and e_2 corresponds to identifying triangles Δ_1 and Δ_2.

Let us set $e_1 = \langle vv_1 \rangle$ and $e_2 = \langle vv_2 \rangle$ and assume e is red. Then K_{n+1} is obtained from K_n by identifying the triangles $\Delta_1 = (r_{v_1}^{F_1} r_v^{F_1} b_v^{F_1})$ and $\Delta_2 = (r_{v_2}^{F_2} r_v^{F_2} b_v^{F_2})$ which share the edge $(r_v^{F_1} b_v^{F_1}) = (r_v^{F_2} b_v^{F_2})$ in K_n. This is illustrated on Figure 5. We need to show that no further identifications of simplices are implied.

Additional identifications can only arise if there is a vertex $x \in \partial K_n$ which is a neighbour of both $r_{v_1}^{F_1}$ and $r_{v_2}^{F_2}$ in which case the edges $(xr_{v_1}^{F_1})$ and $(xr_{v_2}^{F_2})$ are identified in the step from K_n to K_{n+1}. We claim that no such x exists except $b_v^{F_1}$ and $r_v^{F_1}$.

First, suppose x is a blue vertex, $x \neq b_v^{F_1}$. Then there are two vertices w_1 and w_2 in the midsection such that $b_{w_1}^{F_1'} = x = b_{w_2}^{F_2'}$ and $r_{w_i}^{F_i'} = r_{v_i}^{F_i}, i = 1, 2$, in K_n for some 2-cells F_1', F_2' in S_n. This implies that w_1 and w_2 are connected by a red

path in S while w_i is connected to v_i by a blue path in S for $i = 1, 2$. When we now take the step from S_n to S_{n+1} and identify e_1 and e_2 the vertices v_1 and v_2 get identified and hence w_1 and w_2 are connected both by a blue path in S and by a red path, see Figure 6. Hence, $w_1 = w_2$, and we get a closed blue path enclosing the red edge e which is impossible by (β_1).
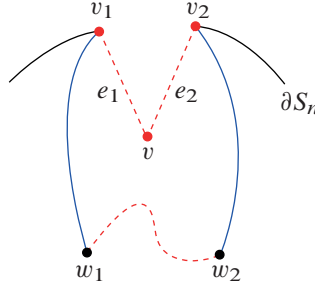

Figure 6. Paths arising in the local construction of the midsection when x is blue.

Now suppose x is a red vertex. Then there are four vertices w_1, w_2, u_1, u_2 in the midsection such that $\langle w_1 w_2 \rangle$ and $\langle u_1 u_2 \rangle$ are red edges and $x = r_{w_2}^{F'} = r_{u_2}^{F''}$, $r_{w_1}^{F'} = r_{v_1}^{F_1}, r_{u_1}^{F''} = r_{v_2}^{F_2}$ for some 2-cells F', F'' in S_n. It follows that w_2 and u_2 are connected by a blue path ρ_1 and similarly that w_1 is connected to v_1 and u_1 is connected to v_2 by a blue path in S, see Figure 7.
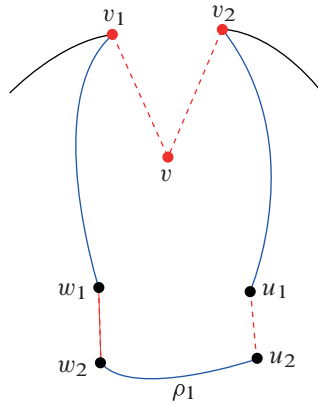

Figure 7. Paths arising in the local construction of the midsection when x is red.

Upon merging e_1 and e_2 we get as above a blue path ρ_2 from w_1 to u_1. It follows by condition (γ) that the edges $\langle w_1 w_2 \rangle$ and $\langle u_1 u_2 \rangle$ are connected by a blue path of quadrangles. This path of quadrangles either lies inside the closed loop made up

of ρ_1, ρ_2 and the edges $\langle w_1 w_2 \rangle$ and $\langle u_1 u_2 \rangle$ or outside it, see Figure 8. In both cases we obtain a closed blue loop containing a red edge in its interior which violates (β_1). This completes the proof of Lemma 6.   □
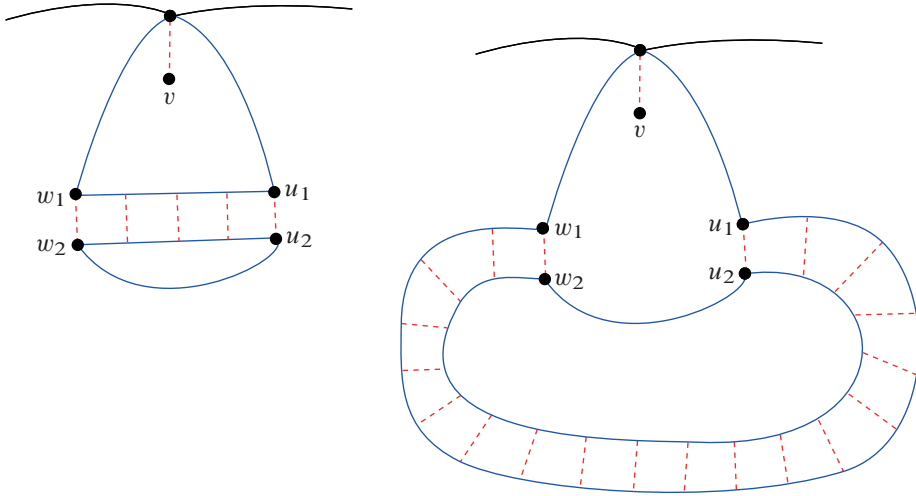

Figure 8. Paths of quadrangles arising in the local construction of the midsection when x is red.

5. Sphere-slices

In this section we generalise Theorem 1 to the case of sphere-slices. It is clear from the proofs of Lemmas 2 and 4 that the midsection of any sphere-slice has properties (α) and (γ), while (β_2) is irrelevant as it refers to the boundary. Furthermore, the proof of Lemma 3 shows that an analogue of (β_1) still holds in the following form:

(β) Each simple closed red (or blue) path divides the midsection into two components, one of which is red (blue).

Since a simplicial decomposition of the 2-sphere requires at least 4 triangles it follows that S must contain at least 4 triangles of each colour. Let ϕ denote the mapping that takes any sphere slice K to its midsection S.

Definition 6. We let $\mathcal{S}$ denote the set of all coloured cell complexes S homeomorphic to the 2-sphere (with cells as described previously) which have at least four triangles of each colour and satisfy the conditions (α), (β), and (γ).

Lemma 7. *Property* (δ) *holds for all* $S \in \mathcal{S}$.

Proof. Applying the same argument as in the proof of Lemma 5 we conclude that if (α), (β), and (γ) hold for a coloured spherical cell complex S and there are two different red triangles in S whose vertices are pairwise connected by blue paths, then there exist three paths of quadrangles connecting the edges of the triangles pairwise and the complement of those paths consists entirely of blue triangles together with the two given red triangles in S. In particular, there can be only two red triangles, which contradicts $S \in \mathcal{S}$. Clearly, a similar argument holds for blue triangles. □

We now state and prove the main result in this section.

Theorem 2. *The map* $\phi \colon K \to S$ *is a bijection from the set* $\mathcal{CS}$ *of causal sphere-slices to the set* $\mathcal{S}$ *of coloured* 2*-dimensional cell complexes.*

Proof. Given $S_0 \in \mathcal{S}$ it is enough to construct a sphere-slice K such that $\phi(K) = S_0$. The idea of the proof is to cut a piece out of S_0 in such a way that we obtain a cell complex S in $\mathcal{CD}$. Then we use Theorem 1 to obtain a disc-slice with the given midsection S. Finally, we fill in the "hole" of the disc-slice to obtain a sphere-slice with midsection S_0. We divide the argument into 5 parts.

(1) Let R be a maximal edge-connected cluster of red triangles in S_0 and let $S = S_0 \setminus \mathrm{int}(R)$. Condition (β) and the Jordan Curve Theorem imply that R and S are (closed) discs. Moreover, since R is maximal, all edges in ∂S are red and contained in quadrangles in S. Pick one of these quadrangles q and let $\Gamma = (q_1, q_2, \dots, q_n)$ be the maximal blue path of quadrangles containing q. Let $\rho = (v_1, v_2, \dots, v_n)$ be the red path in Γ that shares at least one red edge with ∂S, see Figure 9.
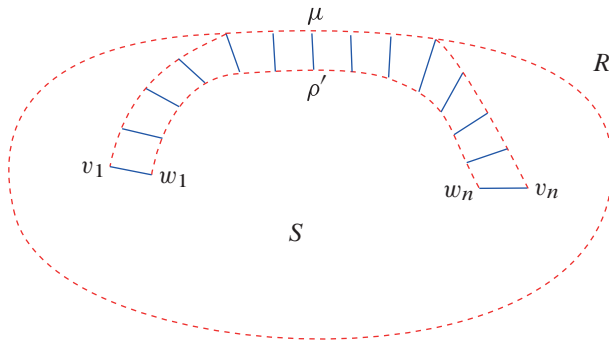

Figure 9. A maximal path of quadrangles in S touching the boundary.

The other red path in Γ, $\rho' = (w_1, w_2, \ldots, w_n)$, shares no vertex with ∂S as a consequence of condition (α). Likewise, ρ and ρ' do not intersect and neither ρ nor ρ' can have multiple vertices, since it would contradict (α). Moreover, if $1 \leq i < j \leq n$ are such that $v_i, v_j \in \partial S$, then $v_k \in \partial S$ for all k between i and j since otherwise R is not maximal due to (β). Hence, Γ intersects ∂S along a curve segment μ, see Figure 9. Note that μ does not contain all of ∂S because this would again contradict condition (α), see Figure 10.
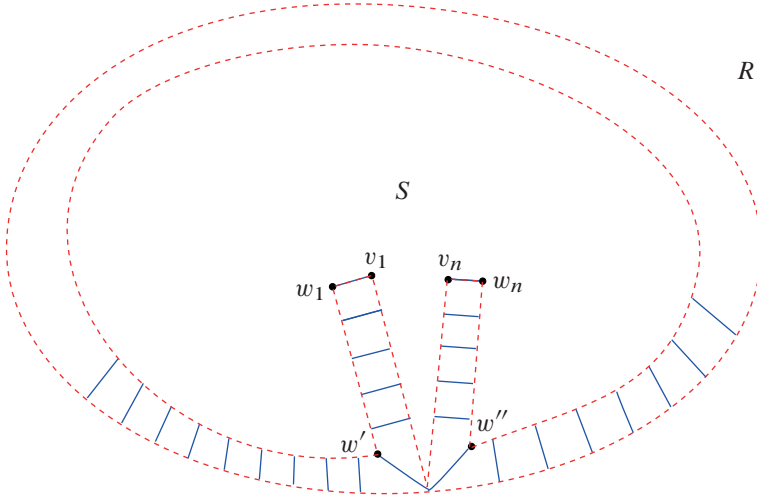

Figure 10. If $\partial S = \mu$ then w' and w'' are connected both by a red and by a blue path.

The above observations show that the boundary of the strip covered by the path of quadrangles Γ, which consists of ρ and ρ' and the two blue edges $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$, is a simple curve. This curve shares exactly a segment $\mu = (v_i, v_{i+1}, \ldots, v_j)$ with ∂S where $1 \leq i < j \leq n$. Hence, removing the strip covered by Γ from S we obtain a new disc S' whose boundary consists of the blue edges $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$ together with the red paths ρ', $\partial S \setminus \mu$, $(v_1, \ldots, v_i)$ and $(v_j, \ldots, v_n)$, see Figure 11.

Now consider the blue triangles Δ_1 and Δ_2 in S' containing $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$, respectively. Denote the third vertex in these triangles by u_1 and u_2, respectively, see Figure 11. It follows from condition (δ) (see Lemma 7) that $u_1 \neq u_2$ and they cannot be connected by a red path. Moreover, u_1 and u_2 are not in $\partial S'$ since this would violate (α). Now, removing Δ_1 and Δ_2 (including edges $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$) from S' we obtain another 2-dimensional cell complex S'' with the topology of a disc and a boundary consisting of the blue paths (v_1, u_1, w_1) and (v_n, u_2, w_n) and the same red paths as $\partial S'$.
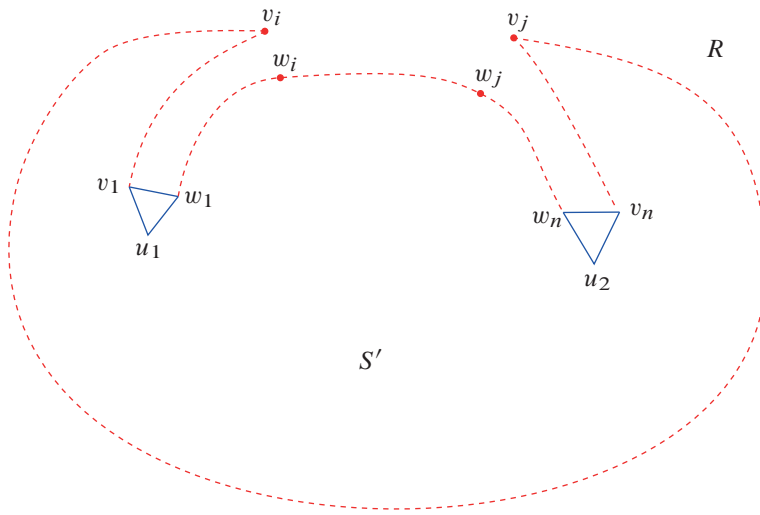
Figure 11. The cell complex S'.

(2) We now claim that S'' belongs to $\mathcal{SD}$. Since paths in S'' are also paths in S_0, conditions (α) and (β_1) are obviously fulfilled. Furthermore, S'' contains at least two blue triangles, and since there exist red edges in $\partial S''$ the existence of at least one red triangle in S'' will follow once (β_2) has been established for S''.

In order to verify (β_2) we recall that u_1 and u_2 cannot be connected by a red path. Connecting u_1 to w_n or v_n or connecting u_2 to w_1 or v_1 by a red path evidently contradicts (α). Similarly, a red path from v_1 to w_n or from w_1 to v_n contradicts (α). This proves (β_2) for S'' as far as red paths are concerned.

Now let η be a blue path in S'' connecting the vertex w_k to a vertex v in the other red arc of $\partial S''$. Viewing η as a path in S_0 and extending it by the edge $\langle w_k v_k \rangle$ we obtain a blue path η' in S_0 connecting v_k to v. If $v \neq v_k$ then condition (α) is violated. If $v = v_k$ then η' is a closed blue path in S_0 with edges of both colours inside and outside unless $k = 1$ or $k = n$. This proves condition (β_2).

In order to verify condition (γ) for S'' let $e_1 = \langle x_1 y_1 \rangle$ and $e_2 = \langle x_2 y_2 \rangle$ be two blue edges in S'' such that x_1 and x_2 as well as y_1 and y_2 are connected by red paths. Then, since (γ) holds for S_0, there is a blue path of quadrangles Λ in S_0 connecting e_1 and e_2. Since the blue boundary edges of S'' are not contained in quadrangles outside S'' the path Λ cannot leave S''. This establishes condition (γ) for the blue edges.

Let now $e_1 = \langle x_1 y_1 \rangle$ and $e_2 = \langle x_2 y_2 \rangle$ be two red edges in S'' such that x_1 and x_2 as well as y_1 and y_2 are connected by blue paths λ' and λ''. By condition (α) the paths λ' and λ'' do not intersect and we can assume without loss of generality that

they have no multiple vertices. Hence, the closed curve consisting of λ', λ'' and the two edges e_1 and e_2 is simple and encloses a disc D in S''. Since condition (γ) holds for S_0 there exists a red path Γ' of quadrangles in S_0 connecting e_1 and e_2 and evidently Γ' is either contained in D or in its exterior. However, the latter case can be excluded as follows. Observe that neither D nor Γ' can intersect R. It follows that one of the closed curves made up of either λ' or λ'' and one of the blue paths in Γ' must separate R (in S_0) from the union of D and Γ' in contradiction with condition (β), see Figure 12. Note, however, that the curve in question is not necessarily simple, since λ' may touch the blue path in Γ' with the same endpoints, and similarly for λ''. Using property (β) one can in this case remove closed parts of the curve so as to obtain a simple closed blue curve with the same separating property. Thus Γ' is contained in D (and hence in S'') and this proves that condition (γ) holds for S''.
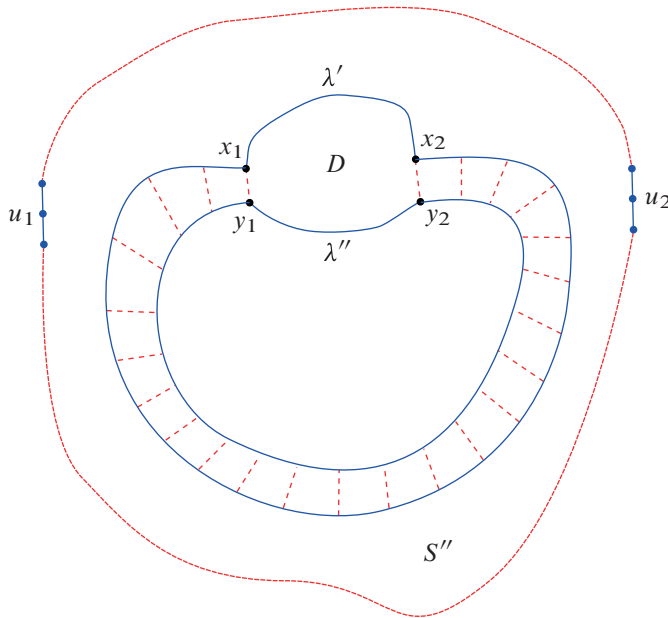

Figure 12. A path of quadrangles outside D connecting the edges e_1 and e_2 and the blue paths λ' and λ'' connecting their endpoints. A priori, parts of the path may be outside S''.

(3) We can now apply Theorem 1 to conclude that there is a unique disc-slice $K'' \in \mathcal{CD}$ whose midsection is S''. In view of the structure of $\partial S''$, the side of K'' consists of two pairs of adjacent backwards directed triangles and two arrays of forward directed triangles corresponding to the two red arcs in $\partial S''$.

We now first construct from K'' a new simplicial complex C by gluing to K'' the tetrahedra corresponding to the quadrangles $q_1, \ldots, q_n$ in Γ and those corresponding to the blue triangles at the ends of Γ. One way to accomplish this is to consider the simplicial complex K_0 whose midsection is Γ with the two blue triangles attached to its ends. Clearly K_0 is a ball and we may think of it as a causal slice whose side consists of two pairs of adjacent backwards directed triangles and two arrays of forward directed triangles of size n whose red edges are pairwise identified and constitute D_{red}. On the other hand, D_{blue} consists of two triangles with one common edge, see Figure 13.
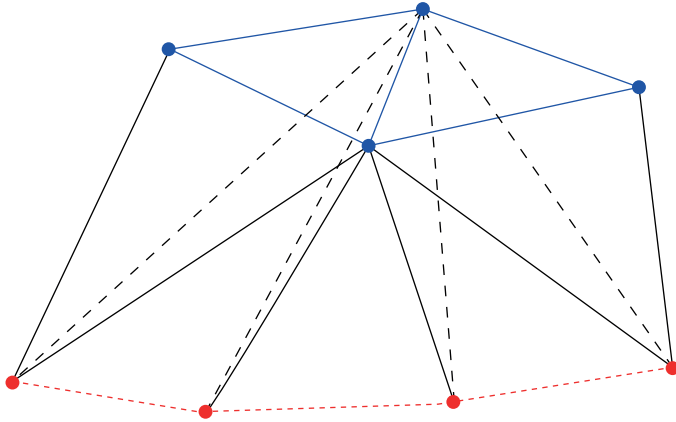

Figure 13. The cell complex K_0 in the case $n = 3$.

Now consider one array of forward directed triangles in ∂K_0 together with the two adjacent backwards directed triangles. These triangles form a disc in ∂K_0 and we can glue K_0 to K'' along this disc and the corresponding disc in $\partial K''$ or, more precisely, the gluing is performed along the edges $\langle u_1 w_1 \rangle, \langle w_1 w_2 \rangle, \ldots, \langle w_n u_2 \rangle$, see Figure 11. Since both K_0 and K'' are simplicial balls, so is the resulting complex $K'' \# K_0$. Its Euler characteristic is therefore -1.

Note also that $K'' \# K_0$ has a side consisting of two pairs of adjacent backwards directed triangles and two arrays of forward directed triangles, one of which has length n. The blue edges in the backwards directed triangles form a closed curve. We now identify the two backwards directed triangles in each pair, i.e. we glue tetrahedra in $K'' \# K_0$ along the edges $\langle u_1 v_1 \rangle$ and $\langle u_2 v_n \rangle$. Doing this, the number of triangles in the simplicial complex decreases by 2, the number of edges decreases by 4 and the number of vertices decreases by 1 and we obtain a new simplicial complex C' with Euler characteristic -2.

The side of C' is a closed circuit of forward directed triangles sharing a single blue vertex. We now obtain the simplicial complex C by gluing tetrahedra in C' along the egdes $\langle v_k v_{k+1} \rangle$ in the midsection, for $k = 1, \ldots, i - 1$ and $k = j, \ldots, n - 1$. It is easy to check that the Euler characteristic does not change with these identifications so

$$\chi(C) = -2.$$

By construction, the midsection of C is $S_0 \setminus R$.

(4) Now return to the disc R that was removed from S_0 in the first step. Let $\bar{R}$ be the cone over R, i.e. $\bar{R}$ consists of the tetrahedra obtained by adding a common blue vertex to all the triangles in R or alternatively, by gluing the tetrahedra corresponding to triangles in R along interior edges in R. We can now glue $\bar{R}$ to C along their sides, i.e. along the edges in $\partial S = \partial R$. The resulting simplicial complex K has Euler characteristic

$$\chi(K) = \chi(C) + \chi(\bar{R}) + \chi_2(\overline{\partial R}) = -2,$$

since $\chi(\bar{R}) = -1$, and $\chi_2(\overline{\partial R}) = 1$ is the Euler characteristic of the cone $\overline{\partial R}$ over ∂R, which is in fact a disc.

(5) Since K'' is a disc-slice it is evident from the preceding construction that ∂K consists of two triangulated 2-spheres, one red and one blue. The midsection of K is by construction S_0. By gluing on cones over the two boundary components of K we obtain a simplicial complex $\tilde{K}$ with Euler characteristic 0. Since $\tilde{K}$ is clearly a pseudomanifold we conclude (see [12], p. 216) that $\tilde{K}$ is also a manifold. In fact, $\tilde{K}$ is simply connected since any closed curve can be deformed to a closed curve on the midsection which is simply connected. Hence, $\tilde{K}$ is a 3-sphere by [10] and K is a cylinder as desired.[1] This completes the proof of Theorem 2. □

6. Discussion

A few remarks pertaining to extensions and variations of the present work are in order. As already indicated, the requirement in Definition 2 that D_{red} and D_{blue} be discs is unnecessarily restrictive. Thus, replacing (ii) of Definition 2 by

(ii′) *all monocoloured simplices of K belong to the boundary ∂K, such that the red (resp. blue) ones form a connected and simply connected subsimplex D_{red} (resp. D_{blue}) of ∂K,*

---

[1] This can presumably be proven by lesser means but the argument does not seem to be entirely trivial and we will not elaborate on it here.

leads to a convenient class of causal triangulations. Let us call this class $\mathcal{CD}'$. In particular, any coloured tetrahedron belongs to $\mathcal{CD}'$ (but not to $\mathcal{CD}$) and so does the complex depicted in Figure 13.

By inspection of the proofs given above it is seen that condition (β_2) is used in the construction of a causal disc slice from a cell complex in $\mathcal{CD}$ solely to ensure that ∂D_{red} and ∂D_{blue} are simple curves. It follows that dropping (β_2) one obtains a one-to-one correspondence between $\mathcal{CD}'$ and the set $\mathcal{SD}'$ of coloured cell complexes homeomorphic to a disc and fulfilling conditions (α), (β_1), and (γ).

On a different note, one may observe that condition (γ) is used partly to ensure the validity of condition (δ) (see Lemma 5) and otherwise only in the final part of the proof of Lemma 6 ensuring that the complex K is a simplicial ball. It is natural to consider replacing property (γ) by property (δ) or dropping both of them. This would require stepping outside the category of simplicial complexes and adopt a different setting encompassing singular triangulations.

Finally, one might also envisage characterising causal triangulations in higher dimensions, which evidently would lead to more involved higher dimensional coloured cell complexes of which little is known at present. In [9] it is explained how the midsections of 4-dimensional causal triangulations are made up of tetrahedra and prisms which replace the triangles and quadrangles which make up the midsections considered in this paper.

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