A Cheeger-type exponential bound
for the number of triangulated manifolds

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Abstract. In terms of the number of triangles, it is known that there are more than exponentially many triangulations of surfaces, but only exponentially many triangulations of surfaces with bounded genus. In this paper we provide a first geometric extension of this result to higher dimensions. We show that in terms of the number of facets, there are only exponentially many geometric triangulations of space forms with bounded geometry in the sense of Cheeger (curvature and volume bounded below, and diameter bounded above). This establishes a combinatorial version of Cheeger's finiteness theorem.

Further consequences of our work are:

(1) there are exponentially many geometric triangulations of S^d;

(2) there are exponentially many convex triangulations of the d-ball.

1. Introduction

In discrete quantum gravity, one simulates Riemannian structures by considering all possible triangulations of manifolds [30, 6, 33]. The metric is introduced a posteriori, by assigning to each edge a certain length (as long as all triangular inequalities are satisfied). For example, in Weingarten's *dynamical triangulations* model, we simply assign to all edges length 1, and view all triangles as equilateral triangles in the plane [6, 33]. The resulting intrinsic metric is sometimes called "equilateral flat metric", cf. [2].

This model gained popularity due to its simplification power. For example, the *partition function* for quantum gravity, a path integral over all Riemannian metrics, becomes a sum over all possible triangulations with N facets [33]. To

make sure that this sum converges when N tends to infinity, one needs to establish an exponential bound for the number of triangulated d-manifolds with N facets; compare Durhuus and Jonsson [20]. However, already for $d = 2$, this dream is simply impossible: It is known that here are more than exponentially many surfaces with N triangles. For $d = 2$ the problem can be bypassed by restricting the topology, because for fixed g there are only exponentially many triangulations of the genus-g surface, as explained in [6, 31].

In dimension greater than two, however, it is not clear which geometric tools to use to provide exponential cutoffs for the class of triangulations with N simplices. Are there only exponentially many triangulations of S^3, or more? This open problem, first asked in [5], was later put into the spotlight also by Gromov [21, pp. 156–157]. Part of the difficulty is that when $d \geq 3$ many d-spheres cannot be realized as boundaries of $(d + 1)$-polytopes [24, 29], and cannot even be shelled [22]. In fact, we know that shellable spheres are only exponentially many [11].

We tackle the problem from a new perspective. Cheeger's finiteness theorem states that there are only finitely many diffeomorphism types of space forms with "bounded geometry": curvature and volume bounded below, and diameter bounded above. What we achieve is a discrete analogue of Cheeger's theorem, which (roughly speaking) shows that *geometric triangulations* of manifolds with bounded geometry are very few.

Theorem I (Theorem 3.5). *In terms of the number of facets, there are exponentially many geometric triangulations of space forms with bounded geometry* (*and fixed dimension*).

Since every topological triangulation of an orientable surface can be straightened to a geometric one [18, 32], this result is a generalization of the classical exponential bound on the number of triangulated surfaces with bounded genus.

Here is the proof idea. Via Cheeger's bounds on the injectivity radius, we chop any manifold of constant curvature into a finite number of convex pieces of small diameter. Up to performing a couple of barycentric subdivisions, we can assume that each piece is a shellable ball [3], and in particular endo-collapsible [9]. This implies an upper bound for the number of critical faces that a discrete Morse function on the triangulation can have. From here we are able to conclude, using the second author's result that there are only exponentially many triangulations of manifolds with bounded discrete Morse vector [9].

Inspired by Gromov's question, let us now consider the unit sphere S^d with its standard intrinsic metric. Let us agree to call "geometric triangulation" any

tiling of S^d into regions that are convex simplices with respect to the given metric and combinatorially form a simplicial complex. For example, all the boundaries of $(d+1)$-polytopes yield geometric triangulations of S^d, but not all geometric triangulations arise this way. How many geometric triangulations are there?

Once again, by proving that the second derived subdivision of every geometric triangulation is endo-collapsible, we obtain

Theorem II (Theorem 2.9). *There are at most $2^{d^2 \cdot ((d+1)!)^2 \cdot N}$ distinct combinatorial types of geometric triangulations of the standard S^d with N facets.*

Our methods rely, as explained, on convex and metric geometry. Whether *all* triangulations of S^d are exponentially many, remains open. But even if the answer turned out to be negative, Main Theorems I and II provide some support for the hope of discretizing quantum gravity in all dimensions.

Preliminaries

By $\mathbb{R}^d$, $\mathbb{H}^d$ and S^d we denote the euclidean d-space, the hyperbolic d-space, and the unit sphere in $\mathbb{R}^{d+1}$, respectively. A (*euclidean*) *polytope* in $\mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. Similarly, a *hyperbolic polytope* in $\mathbb{H}^d$ is the convex hull of finitely many points of $\mathbb{H}^d$. A *spherical polytope* in S^d is the convex hull of a finite number of points that all belong to some open hemisphere of S^d. Spherical polytopes are in natural one-to-one correspondence with euclidean polytopes, just by taking radial projections; the same is true for hyperbolic polytopes. A *geometric polytopal complex* in $\mathbb{R}^d$ (resp. in S^d or $\mathbb{H}^d$) is a finite collection of polytopes in $\mathbb{R}^d$ (resp. S^d, $\mathbb{H}^d$) such that the intersection of any two polytopes is a face of both. An *intrinsic polytopal complex* is a collection of polytopes that are attached along isometries of their faces (cf. Davis–Moussong [19, Sec. 2]), so that the intersection of any two polytopes is a face of both.

The *face poset* $(C, \subseteq)$ of a polytopal complex C is the set of nonempty faces of C, ordered with respect to inclusion. Two polytopal complexes C, D are *combinatorially equivalent*, denoted by $C \cong D$, if their face posets are isomorphic. Any polytope combinatorially equivalent to the d-simplex, or to the regular unit cube $[0, 1]^d$, shall simply be called a *d-simplex* or a *d-cube*, respectively. A polytopal complex is *simplicial* (resp. *cubical*) if all its faces are simplices (resp. cubes).

The *underlying space* $|C|$ of a polytopal complex C is the topological space obtained by taking the union of its faces. If two complexes are combinatorially

equivalent, their underlying spaces are homeomorphic. If C is simplicial, C is sometimes called a *triangulation* of $|C|$ (and of any topological space homeomorphic to $|C|$). If X is a metric length space, we call *geometric triangulation* any intrinsic simplicial complex C isometric to X such that the simplices of C are convex in the underlying length metric. For example, the boundary of every simplicial $(d + 1)$-polytope yields a geometric triangulation of S^d.
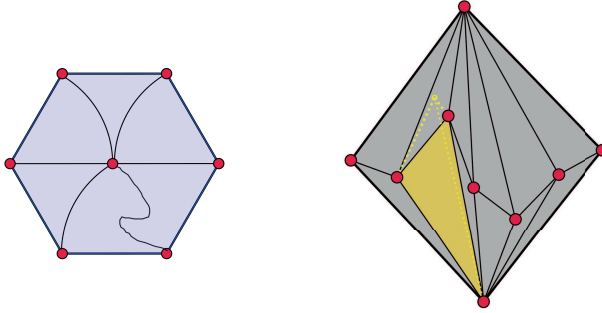

Figure 1. LEFT: A tiling of a disk that is not a geometric triangulation, because the tiles are not convex. RIGHT: Rudin's 3-ball R is a non-shellable subdivision of a convex 3-dimensional polytope with 14 vertices, cf. [35]. Coning off the boundary of R one gets a simplicial complex $\partial(v * R)$ that is a geometric triangulation of S^3, but it is not shellable, hence not polytopal.

A *subdivision* of a polytopal complex C is a polytopal complex C' with the same underlying space of C, such that for every face F' of C' there is some face F of C for which $F' \subset F$. A *derived subdivision* sd C of a polytopal complex C is any subdivision of C obtained by stellarly subdividing at all faces in order of decreasing dimension of the faces of C, cf. [23]. An example of a derived subdivision is the *barycentric subdivision*, which uses as vertices the barycenters of all faces of C.

If C is a polytopal complex, and A is some set, we define the *restriction* $\mathrm{R}(C, A)$ *of C to A* as the inclusion-maximal subcomplex D of C such that $|D|$ lies in A. The *star* of σ in C, denoted by $\mathrm{St}(\sigma, C)$, is the minimal subcomplex of C that contains all faces of C containing σ. The *deletion $C - D$* of a subcomplex D from C is the subcomplex of C given by $\mathrm{R}(C, C \setminus \mathrm{relint}\, D)$. The (*first*) *derived neighborhood* $N(D, C)$ of D in C is the simplicial complex

$$N(D, C) := \bigcup_{\sigma \in \mathrm{sd}\, D} \mathrm{St}(\sigma, \mathrm{sd}\, C).$$

Similarly, for $k > 1$, the k-th derived subdivision of a complex C is recursively defined as sd $C = \mathrm{sd}(\mathrm{sd}^{k-1} C)$. The *$k$-derived neighborhood of D in C*, denoted by $N^k(D, C)$, is the union of $\mathrm{St}(\sigma, \mathrm{sd}^k C)$, with $\sigma \in \mathrm{sd}^k D$.
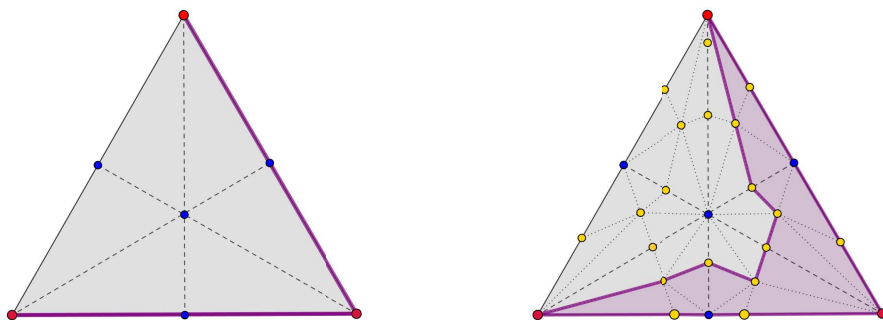
Figure 2. $N^2(D, C)$, where C is a triangle and D the subcomplex formed by two of its edges.

Next comes the geometric definition of the link of a face σ. Intuitively, it is a spherical complex whose face poset is the upper order ideal of σ in the face poset of C. The formal definition is as follows, cf. [4]. Let p be any point of a metric space X. By $\mathrm{T}_p X$ we denote the tangent space of X at p. Let $\mathrm{T}_p^1 X$ be the restriction of $\mathrm{T}_p X$ to unit vectors. If Y is any subspace of X, then $\mathrm{N}_{(p,Y)} X$ denotes the subspace of the tangent space $\mathrm{T}_p X$ spanned by the vectors orthogonal to $\mathrm{T}_p Y$. If p is in the interior of Y, we define $\mathrm{N}_{(p,Y)}^1 X := \mathrm{N}_{(p,Y)} X \cap \mathrm{T}_p^1 Y$. If τ is any face of a polytopal complex C containing a nonempty face σ of C, then the set $\mathrm{N}_{(p,\sigma)}^1 \tau$ of unit tangent vectors in $\mathrm{N}_{(p,\sigma)}^1 |C|$ pointing towards τ forms a spherical polytope $P_p(\tau)$, isometrically embedded in $\mathrm{N}_{(p,\sigma)}^1 |C|$. The family of all polytopes $P_p(\tau)$ in $\mathrm{N}_{(p,\sigma)}^1 |C|$ obtained for all $\tau \supset \sigma$ forms a polytopal complex, called the *link* of C at σ; we will denote it by $\mathrm{Lk}_p(\sigma, C)$. If C is a geometric polytopal complex in $X^d = \mathbb{R}^d$ (or $X^d = S^d$), then $\mathrm{Lk}_p(\sigma, C)$ is naturally realized in $\mathrm{N}_{(p,\sigma)}^1 X^d$. Obviously, $\mathrm{N}_{(p,\sigma)}^1 X^d$ is isometric to a sphere of dimension $d - \dim \sigma - 1$, and will be considered as such. Up to ambient isometry $\mathrm{Lk}_p(\sigma, C)$ and $\mathrm{N}_{(p,\sigma)}^1 \tau$ in $\mathrm{N}_{(p,\sigma)}^1 |C|$ or $\mathrm{N}_{(p,\sigma)}^1 X^d$ do not depend on p; for this reason, p will be omitted in notation whenever possible. By convention, we define $\mathrm{Lk}(\emptyset, C) = C$, and it is the only link that does not come with a natural spherical metric.

If C is a simplicial complex, and σ, τ are faces of C, we denote by $\sigma * \tau$ the minimal face of C containing both σ and τ (if there is one). If σ is a face of C, and τ is a face of $\mathrm{Lk}(\sigma, C)$, then $\sigma * \tau$ is defined as the face of C with $\mathrm{Lk}(\sigma, \sigma * \tau) = \tau$. In both cases, the operation $*$ is called the *join*.

Inside a polytopal complex C, a *free* face σ is a face strictly contained in only one other face of C. An *elementary collapse* is the deletion of a free face σ from a polytopal complex C. We say that C (*elementarily*) *collapses* onto $C - \sigma$,
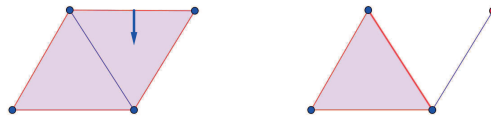
Figure 3. The complex on the left has four free edges (in red). The deletion of any of them is callled an 'elementary collapse'. It yields a complex with two fewer faces and same homotopy type (right). Such complex still has free faces, so the simplification process can be carried out further.

and write $C \searrow_e C - \sigma$. We also say that C *collapses* to a subcomplex C', and write $C \searrow C'$, if C can be reduced to C' by a sequence of elementary collapses. A *collapsible* complex is a complex that collapses onto a single vertex. Collapsibility is a combinatorial property (i.e. it only depends on the combinatorial type), and does not depend on the geometric realization of a polytopal complex.

A *pure* polytopal complex is one where all facets have the same dimension. Elementary collapses may make a pure complex non-pure (Figure 3), and vice versa. A variant of collapsibility where purity is maintained is the following. A pure d-dimensional simplicial complex with N facets is called *shellable* if either $d(N-1) = 0$, or there is an order $F_1, \ldots, F_N$ of its facets such that for each $j \geq 2$ the intersection $F_j \cap \bigcup_{i=1}^{j} F_i$ is a pure $(d-1)$-dimensional subcomplex of ∂F_j. It is easy to see that for pure contractible complexes, shellable implies collapsible. The converse is false, as shown by two triangles glued together at a vertex. We have however the following results:

Theorem 1.1 (Adiprasito and Benedetti [3, 4]). *Let C be a simplicial complex either in $\mathbb{R}^d$ or in $\mathbb{S}^d$. If the underlying space of C is convex, then the second derived subdivision of C is shellable. Moreover, if C is convex in $\mathbb{R}^d$, the first derived subdivision of C is collapsible.*

2. Geometric triangulations

If we want to reach exponential bounds for triangulations of d-manifolds, and d is at least two, we *must* add some geometric or topological assumption. In fact, already for $d = 2$, it is easy to construct $g!$ combinatorially inequivalent triangulations with $14g + 5$ triangles of the genus-g surface, cf. Figure 4. Setting $N = 14g + 5$, clearly $\lfloor \frac{N}{14} \rfloor!$ grows faster than any exponential. This motivates the search for an exponential upper bound to the number of triangulations with extra geometric properties.
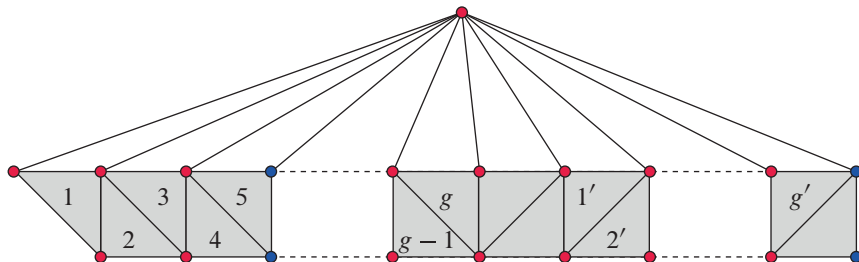
Figure 4. To construct $g!$ triangulations of the genus-g surface, fix a bijection $\pi\colon\{1,\ldots,g\}$ $\to\ \{1,\ldots,g'\}$. Take a strip of $2g+3$ triangles as above, and cone off its boundary to get a 2-sphere with $4g+5$ triangles. Now remove the interiors of the $2g$ triangles $1,\ldots,g,1',\ldots,g'$, and create a genus-g surface by attaching handles between the hole i and the hole $\pi(i)$, for all i. Since every handle can be triangulated using 12 triangles, this yields a triangulation T_π with $14g+5$ triangles of the genus-g surface. By inspecting the link of the highest-degree vertex, from T_π we can recover π, which implies that different permutations yield different triangulations.

Let us recall the notion of *endo-collapsibility*, introduced in [9]. A triangulation C of a d-manifold with non-empty boundary is called *endo-collapsible* if C minus some d-face Σ collapses onto ∂C. A triangulation C of a d-manifold with empty boundary is called *endo-collapsible* if C minus some d-face Σ collapses onto some vertex v. (The choice of Σ and that of v do not matter: if a sphere C is endo-collapsible, then for any face Δ and for any vertex w, one has that C minus Δ collapses onto w.) If C is an endo-collapsible ball with endo-collapsible boundary, then C is collapsible. The converse is false. However, there is a partial converse statement:
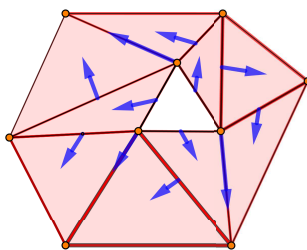

Figure 5. A d-ball is endo-collapsible if it collapses to the boundary after removing the interior of some d-simplex (it does not matter which). In dimension 2, all disks are endo-collapsible. We represent an elementary collapse (σ, Σ) by drawing an arrow between the barycenters of the two faces, with its tail on the lower-dimensional one.

Lemma 2.1 (Benedetti [9, Corollary 3.21]). *Let B be a collapsible triangulation of the d-ball. If* $\operatorname{sd} \operatorname{Lk}(\sigma, B)$ *is endo-collapsible for every face* σ, *then* $\operatorname{sd} B$ *is endo-collapsible.*

This notion is of interest to us for the following exponential upper bound:

Theorem 2.2 (Benedetti and Ziegler [11]). *For fixed d, in terms of the number N of facets, there are at most* $2^{d^2 \cdot N}$ *endo-collapsible triangulations of d-manifolds with N facets.*

Not all triangulations of balls and spheres are endo-collapsible, cf. [11, Theorem 3]. An even stronger fact is that many triangulated spheres are still not endo-collapsible after two barycentric subdivisions [9]. However, Theorem 1.1 and the fact that all shellable manifolds are endocollapsible [9] immediately imply the following:

Proposition 2.3. *Let* $d \geq 2$. *Let C be a geometric triangulation of a convex subset of* S^d *or* $\mathbb{R}^d$. *Then* $\operatorname{sd}^2 C$ *is endo-collapsible.*

Remark 2.4. For geometric triangulations of convex balls in $\mathbb{R}^d$, the bound of Proposition 2.3 can be improved by one: that is, already $\operatorname{sd} C$ is endo-collapsible. This follows from part (C) of the following recent result:

Theorem 2.5 ([4], cf. also [1, Chapter II.2]). *Let C be any convex polytopal d-complex in* S^d. *Let* $\bar{H}_+$ *be a closed hemisphere of* S^d *in general position with respect to C.*

(A) *If* $\partial C \cap \bar{H}_+ = \emptyset$, *then* $N(\operatorname{R}(C, \bar{H}_+), C)$ *is collapsible.*
(B) *If* $\partial C \cap \bar{H}_+$ *is nonempty, and C does not lie in* $\bar{H}_+$, *then* $N(\operatorname{R}(C, \bar{H}_+), C)$ *collapses to the subcomplex* $N(\operatorname{R}(\partial C, \bar{H}_+), \partial C)$.
(C) *If C lies in* $\bar{H}_+$, *there is a facet F of* $\operatorname{sd} \partial C$ *such that* $\operatorname{sd} C$ *collapses to* $\operatorname{sd} \partial C - F$.

Remark 2.6. One may wonder if Proposition 2.3 can be extended to star-shaped d-balls in $\mathbb{R}^d$. The shellability argument above does not extend. However, if C is a star-shaped ball in $\mathbb{R}^d$, the authors in [4, Theorem 3.0.6] showed that $\operatorname{sd}^{d-2} C$ is collapsible; the same proof shows also that the link of any face inside $\operatorname{sd}^{d-2} C$ is collapsible as well. So we are in the position to apply Lemma 2.1. This proves the following:

Proposition 2.7. *Let* $d \geq 2$. *Let C be any star-shaped triangulation of the d-ball in* $\mathbb{R}^d$. *Then* $\operatorname{sd}^{d-1} C$ *is endo-collapsible.*

Using the bound from Theorem 2.2, our next goal is to achieve exponential bounds for the number of convex or star-shaped triangulations. We start with a simple lemma.

Lemma 2.8 (following Bayer [8, Theorem 3]). *Let A and B be two triangulations of the same manifold. If* $\mathrm{sd}\, A = \mathrm{sd}\, B$, *then* $A = B$.

Proof. We make the stronger claim that the barycentric subdivision of any *polytopal* complex T determines T up to duality. The claim immediately implies the conclusion, because if two complexes are dual to one another, only one of them can be simplicial. To prove our claim, we proceed by induction on $\dim T$. Without loss of generality, assume T is connected. Also, assume T is not a single simplex. Following Bayer [8], consider a minimal coloring of the complex $\mathrm{sd}\, T$. Choose a vertex v of $\mathrm{sd}\, T$. Decompose $\mathrm{Lk}(v, \mathrm{sd}\, T)$ minimally into color classes so that $\mathrm{Lk}(v, \mathrm{sd}\, T)$ is a join of the complexes induced by these color classes. If this decomposition is trivial, v corresponds to either a vertex or a facet of T. List all the vertices with trivial decomposition. By the assumption, the induced complex on these vertices is a 2-colorable graph with an edge, with endpoints a and b. If a is a vertex of T, then b must correspond to a facet of T. By inductive assumption we can determine $\mathrm{Lk}(a, T)$ from $\mathrm{Lk}(a, \mathrm{sd}\, T)$; in particular, T is determined by $\mathrm{sd}\, T$. The other (dual) option is if a is a facet; in this case b is a vertex of T and again by induction we can determine $\mathrm{Lk}(b, T)$ from $\mathrm{Lk}(b, \mathrm{sd}\, T)$. □

Theorem 2.9. *In terms of the number N of facets, there are*
- *less than $2^{d^2 \cdot (d+1)! \cdot N}$ geometric triangulations of convex balls in $\mathbb{R}^d$,*
- *less than $2^{d^2 \cdot ((d+1)!)^2 \cdot N}$ geometric triangulations of S^d,*
- *less than $2^{d^2 \cdot ((d+1)!)^{(d-2)} \cdot N}$ star-shaped balls in $\mathbb{R}^d$.*

Proof. We only prove the first item here, the other ones can be proven analogously. Let C be any simplicial subdivision of a convex d-polytope. By Remark 2.4, the derived subdivision $\mathrm{sd}\, C$ is endo-collapsible. Furthermore, if C has N facets, then $\mathrm{sd}\, C$ has $(d+1)! \cdot N$ facets. Hence, by Theorem 2.2, $\mathrm{sd}\, C$ is one of at most $2^{d^2 \cdot (d+1)! \cdot N}$ combinatorial types. Since simplicial complexes with isomorphic derived subdivisions are isomorphic by Lemma 2.8, we conclude that C is one of at most $2^{d^2 \cdot (d+1)! \cdot N}$ combinatorial types. □

3. Triangulated space forms with bounded geometry

In this section, we wish to study *space forms*, which are Riemannian manifolds of constant sectional curvature [17]. We focus on space forms with "bounded geometry", with the goal of establishing an exponential upper bound for the number of triangulations. The following Lemma is well known:

Lemma 3.1. *Let M be a space form of dimension $d \geq 2$. There are* **at least** *exponentially many triangulations of M.*

This motivates the search for an upper bound to the number of such geometric triangulations. We will show that Lemma 3.1 is best possible, in the sense that these triangulations are also *at most* exponentially many (Theorem 3.5).

Our idea is to chop a geometric triangulation of a space form with bounded geometry into a bounded number of endo-collapsible balls. The key for this is given by the following two lemmas: One is Cheeger's bound on the injectivity radius, the other a direct consequence of Toponogov's theorem.

Lemma 3.2 (Cheeger [16]). *Let $-\infty < k < \infty$ and $D, V > 0$. There exists a positive number $\widetilde{C}(k, D, V) > 0$ such that every Riemannian d-manifold with curvature $\geq k$, diameter $\leq D$ and volume $\geq V$, has no closed geodesic of length less than $\widetilde{C}(k, D, V)$.*

By a well-known result of Klingenberg (cf. [26]), the injectivity radius is larger than the minimum of half the length of the shortest closed geodesic and $\sqrt{K}$, where K is the supremum of sectional curvatures on the Riemannian manifold in question. From this we conclude:

Corollary 3.3. *Let $-\infty < k < \infty$ and $D, V > 0$. There exists an integer $C(k, D, V) > 0$ such that every d-dimensional space form with curvature $\geq k$, diameter $\leq D$ and volume $\geq V$ has injectivity radius at least $C(k, D, V)$.*

Finally, we need a lemma to cover a Riemannian manifold by disks.

Lemma 3.4. *Let $-\infty < k < \infty$. Let $D > 0$. Let d be a positive integer. For every $\varepsilon > 0$, there exists a positive integer N_ε such that every Riemannian d-manifold with curvature bounded below by k and diameter at most D can be covered with at most N_ε balls of radius ε.*

Proof. Let X be a Riemannian manifold satisfying the assumptions. Let x be a point of X. Let B_D^d be a ball of radius D in the d-dimensional space of constant

curvature $m = \min\{k, 0\}$. The ball B_D^d has a cover with N_ε balls of diameter ε. Consider the map $\exp_x \widehat{\exp}^{-1}$, where $\widehat{\exp}$ is the exponential map in B_D^d with respect to the center and $\exp_x$ is the exponential map in X with respect to x, cf. [17, Chapter 1, Section 2]. By the assumption, X has curvature bounded below by k. By Toponogov's Theorem, for all $a, b \subset B_D^d$, we have

$$|(\exp_x \widehat{\exp}^{-1})(a)(\exp_x \widehat{\exp}^{-1})(b)| \le |ab|;$$

in other words, $\exp_x \widehat{\exp}^{-1}$ is a non-expansive map. Thus, the images of the N_ε balls that cover B_D^d are contained in N_ε balls of radius at most ε. $\qquad\square$

We are ready for the proof of the main theorem.

Theorem 3.5. *Fix any real number k, positive real numbers D, V, and a positive integer d. In terms of the number N of facets, there are only exponentially many intrinsic simplicial complexes whose underlying spaces are d-dimensional space forms of curvature bounded below by k, of diameter $\le D$ and of volume $\ge V$.*

Proof. Our proof has three parts:

I. we cover a space form X satisfying the constraints above with convex open balls;

II. we count the number of geometric triangulations restricted to each ball;

III. we assemble the triangulated balls together, thus estimating the number of triangulations of X.

Note that convex polyhedra in hyperbolic, spherical and euclidean spaces are related by stereographic projection. Hence, in items (II) and (III), the negatively (resp. positively) curved case is reduced to the zero-curvature case.
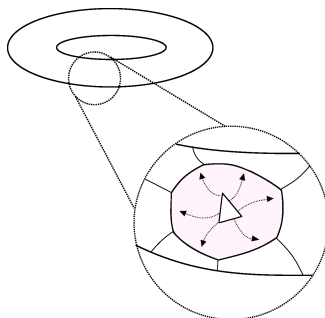

Figure 6. The restriction of the triangulation to each of the B_i ($i \in S$), up to taking two barycentric subdivisions, is shellable. By counting the number of ways in which two of the B_i's can be glued to one another, we determine an upper bound on the number of triangulations.

(I) Let X be a space form of dimension d satisfying the Cheeger constraints (k, D, V). By Lemma 3.4, there exists a set S of $s = s(k, D, V)$ points in such that every point in X lies in distance less than $\varepsilon := \frac{\mathcal{C}(k,D,V)}{4}$ of S. With this choice, any ball of radius ε in X is isometric to a convex ball of radius ε in the unique simply-connected space form of curvature equal to the curvature of X. Let T be a triangulation of X into N simplices. Let $(B_i)_{i \in \{1, \cdots, s\}}$ be the family of open convex balls with radius ε, centered at the points of S.

(II) For any subset $A \subset X$, let V_A denote the vertices of sd T corresponding to faces of T intersecting A. Define T_A to be the subcomplex of sd T induced by V_A.

Let now B_i be one of the convex balls as above. Choose a geometric realization of the derived subdivision of T such that the vertices of V_{B_i} lie in B_i. Consider the complex $T_i' := N(T_{B_i}, \text{sd}\, T)$. Via Theorem 1.1, we obtain that $\text{sd}^3 T_i'$ is shellable, and in particular endo-collapsible, as it can naturally be identified with a triangulation of a hemisphere via stereographic projection. (With some extra work one can show that already $\text{sd}^2 T_i'$ is endo-collapsible. In fact, T_i' is collapsible by Theorem 2.5(A); and since for every face σ of T_i', $\text{sd}^2 \text{Lk}(\sigma, T_i')$ is endo-collapsible by Proposition 2.3, it follows via Lemma 2.1 that $\text{sd}^2 T_i'$ is endo-collapsible.)

Hence, Theorem 2.2 provides a constant κ such that the number of combinatorial types of T_i' is bounded above by $e^{\kappa N}$.

(III) When two endo-collapsible balls are glued by identifying two combinatorially equivalent, endo-collapsible $(d-1)$-subcomplexes of their boundary, the resulting "union" is also endo-collapsible [9]. So since we chopped it into endo-collapsible balls, $\text{sd}^2 T$ is endo-collapsible.

Now, the triangulation $\text{sd}^2 T$ of X is completely determined by

(i) the triangulation of each T_i',
(ii) the triangulation $T_i' \cap T_j'$ and its position in T_i' and T_j'. (This means we have to specify which of the faces of T_i' are faces of $T_i' \cap T_j'$, too.)

As we saw in Part II, we have $e^{\kappa N}$ choices for triangulating each T_i'. Since $T_i' \cap T_j'$ is a disk, it has connected dual graph; hence, if we specify the location of one facet Δ of $T_i' \cap T_j'$ in T_j' and T_i' (including its orientation), this suffices to determine the position of $T_i' \cap T_j'$ in T_i' and T_j'. For this, we have at most $(d+1)!^4 N^2$ possibilities.

In conclusion, the number of geometric triangulations T of d-dimensional space forms with N facets, diameter $\leq D$, volume $\geq v$, and curvature bounded below by k is bounded above by

$$e^{\kappa s N}((d+1)!^2 N)^{s(s-1)}. \qquad \square$$

References

[1] K. A. Adiprasito, *Methods from differential geometry in polytope theory.* Ph.D. thesis. Freie Universität Berlin, Berlin, 2013. arXiv:1403.2657

[2] K. A. Adiprasito and B. Benedetti, Collapsibility of CAT(0) spaces. *Geom. Dedicata* **206** (2020), 181–199. MR 4091542

[3] K. A. Adiprasito and B. Benedetti, Subdivisions, shellability, and collapsibility of products. *Combinatorica* **37** (2017), no. 1, 1–30. MR 3638330 Zbl 1399.52025

[4] K. A. Adiprasito and B. Benedetti, Barycentric subdivisions of convex complexes are collapsible. *Discrete Comput Geom* (2019).
https://doi.org/10.1007/s00454-019-00137-3

[5] J. Ambjørn, B. Durhuus, and T. Jónsson, A model of three-dimensional lattice gravity. *Modern Phys. Lett. A* **7** (1992), no. 18, 1629–1646. MR 1168831 Zbl 1020.83539

[6] J. Ambjørn, B. Durhuus, and T. Jónsson, *Quantum geometry.* A statistical field theory approach. Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1997. MR 1465433 Zbl 0993.82500

[7] F. D. Ancel and C. R. Guilbault, Interiors of compact contractible n-manifolds are hyperbolic ($n \geq 5$). *J. Differential Geom.* **45** (1997), no. 1, 1–32. MR 1443329 Zbl 0877.53030

[8] M. Bayer, Barycentric subdivisions. *Pacific J. Math.* **135** (1988), no. 1, 1–16. MR 0965681 Zbl 0627.57020

[9] B. Benedetti, Discrete Morse theory for manifolds with boundary. *Trans. Amer. Math. Soc.* **364** (2012), no. 12, 6631–6670. MR 2958950 Zbl 1288.57022

[10] B. Benedetti, Smoothing discrete Morse theory. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **16** (2016), no. 2, 335–368. MR 3559605 Zbl 1354.57034

[11] B. Benedetti and G. M. Ziegler, On locally constructible spheres and balls. *Acta Math.* **206** (2011), no. 2, 205–243. MR 2810852 Zbl 1237.57025

[12] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319, Springer-Verlag, Berlin, 1999. MR 1744486 Zbl 0988.53001

[13] D. Burago, Yu. Burago, and S. Ivanov, *A course in metric geometry.* Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, R.I., 2001. MR 1835418 Zbl 0981.51016

[14] K.-U. Bux and S. Witzel, Local convexity in CAT(κ)-spaces. Preprint 2012. arXiv:1211.1871 [math.MG]

[15] R. Charney, *Metric geometry*: *connections with combinatorics.* In L. J. Billera, C. Greene, R. Simion, and R. P. Stanley (eds.), Papers from the Sixth International Conference held in New Brunswick, N.J., May 23–27, 1994. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 24. American Mathematical Society, Providence, R.I., 1996, 55–69. MR 1363506 Zbl 0841.53040

[16] J. Cheeger, Critical points of distance functions and applications to geometry. In P. De Bartolomeis and F. Tricerri (eds.), *Geometric topology*: *recent developments.* Lectures given at the First C.I.M.E. Session on Recent Developments in Geometric Topology and Related Topics held in Montecatini Terme, June 4–12, 1990. Notes in Mathematics, 1504. Springer-Verlag, Berlin, 1991, 1–38. MR 1168042 Zbl 0771.53015

[17] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry.* North-Holland Mathematical Library, 9. North-Holland Publishing Co., Amsterdam and Oxford, and American Elsevier Publishing Co., New York, 1975. MR 0458335 Zbl 0309.53035

[18] Y. Colin de Verdière, Comment rendre géodésique une triangulation d'une surface? *Enseign. Math.* (2) **37** (1991), no. 3-4, 201–212. MR 1151746 Zbl 0753.57009

[19] M. W. Davis and G. Moussong, *Notes on nonpositively curved polyhedra.* In K. Böröczky, Jr., W. Neumann and A. Stipsicz (eds.), *Low dimensional topology.* Papers from the Summer School held in Budapest, 1998 and EMS Summer School No. 1 on Algebraic Geometry held in Eger, 1996. Bolyai Society Mathematical Studies, 8. János Bolyai Mathematical Society, Budapest, 1999, 11–94. MR 1747268 Zbl 0961.53022

[20] B. Durhuus and T. Jonsson, Remarks on the entropy of 3-manifolds. *Nuclear Phys. B* **445** (1995), no. 1, 182–192. MR 1338102 Zbl 0990.57500

[21] M. Gromov, Spaces and questions. *Geom. Funct. Anal.* **2000,** Special Volume, Part I, 118–161. GAFA 2000 (Tel Aviv, 1999). MR 1826251 Zbl 1006.53035

[22] M. Hachimori and G. M. Ziegler, Decompositions of simplicial balls and spheres with knots consisting of few edges. *Math. Z.* **235** (2000), no. 1, 159–171. MR 1785077 Zbl 0965.57022

[23] J. F. P. Hudson, *Piecewise linear topology.* University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees. W. A. Benjamin, New York and Amsterdam, 1969. MR 0248844 Zbl 0189.54507

[24] G. Kalai, Many triangulated spheres. *Discrete Comput. Geom.* **3** (1988), no. 1, 1–14. MR 0918176 Zbl 0631.52009

[25] R. Kirby, *Problems in low–dimensional topology.* http://math.berkeley.edu/~kirby/problems.ps.gz

[26] W. Klingenberg, Contributions to Riemannian geometry in the large. *Ann. of Math.* (2) **69** (1959), 654–666. MR 0105709 Zbl 0133.15003

[27] F. H. Lutz, Examples of $\mathbb{Z}$-acyclic and contractible vertex-homogeneous simplicial complexes. *Discrete Comput. Geom.* **27** (2002), no. 1, 137–154. Geometric combinatorics (San Francisco, CA/Davis, CA, 2000). MR 1871694 Zbl 1011.52007

[28] A. Markov, Unsolvability of certain problems in topology. *Dokl. Akad. Nauk SSSR* **123** (1958) 978–980. In Russian. MR 0103455 Zbl 0090.01202

[29] J. Pfeifle and G. M. Ziegler, Many triangulated 3-spheres. *Math. Ann.* **330** (2004), no. 4, 829–837. MR 2102314 Zbl 1062.52011

[30] T. Regge, General relativity without coordinates. *Nuovo Cimento* (10) **19** (1961), 558–571. MR 0127372

[31] W. T. Tutte, A census of planar triangulations. *Canadian J. Math.* **14** (1962), 21–38. MR 0130841 Zbl 0103.39603

[32] K. Wagner, Bemerkungen zum Vierfarbenproblem. *Jahresbericht der DMV* **46** (1936), 26–32. JFM 62.0656.02 Zbl 0014.18102

[33] D. H. Weingarten, Euclidean quantum gravity on a lattice. *Nuclear Physics B* **210** (1982), 229–245.

[34] J. H. C. Whitehead, Simplicial spaces, nuclei and *m*-groups. *Proc. London Math. Soc.* (2) **45** (1939), no. 4, 243–327. MR 1576810 JFM 65.1443.01 Zbl 0022.40702

[35] R. Wotzlaw, Rudin's non-shellable ball. *Electronic Geometry Model* **2004,** 2004.08.001. http://www.eg-models.de/models/Simplicial_Manifolds/2004.08.001/_preview.html

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