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A notion analogous to the discriminant for transcendental elements in certain extensions of local fields

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Abstract – Let $(K, |\cdot|)$ be a local field. In this paper we define an invariant analogous to the discriminant over K for certain transcendental elements over K.

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1. Introduction

Let $(K, |\cdot|)$ be a local field. Let $\bar{K}$ be a separable algebraic closure of K and let $\tilde{\bar{K}} = \Omega$ be the completion of $\bar{K}$ with respect to the unique extension of $|\cdot|$ also denoted $|\cdot|$ such that $K \subseteq \bar{K} \subseteq \Omega = \tilde{\bar{K}}$. The unique extension by continuity of $|\cdot|$ to Ω will also be denoted by $|\cdot|$. In particular we may consider $K = \mathbf{Q}_p$ and $|\cdot| = |\cdot|_p$ the $p-$adic absolute value normalised such that $|p| = \frac{1}{p}$. We also denote by $|\cdot|_p$ the unique extension of $|\cdot|_p$ to $\bar{\mathbf{Q}}_p$ and further by continuity to $\mathbf{C}_p$, the completion of $\bar{\mathbf{Q}}_p$ with respect to $|\cdot|_p$, usually called the p-adic complex field. Generalizations of all of these may be found in [2].

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In this paper we define an invariant analogous to the discriminant over K for transcendental elements of $\tilde{\bar{K}} = \Omega$ over K.

2. Background material

By a local field $(K, |\cdot|)$ we understand a complete field with respect to a discrete absolute value $|\cdot|$ with a finite residue field. For example, if char $K = 0$ then K is isomorphic to a finite extension of the p-adic field $\mathbf{Q}_p$ and $|\cdot|$ is the unique extension to K of the p-adic absolute value $\bar{\mathbf{Q}}_p$, normalized such that $|p| = \frac{1}{p}$. It is well known (see [2]) that $|\cdot|$ uniquelly extends to a fixed algebraic closure $\bar{K}$ of $K$and further by continuity to $\tilde{\bar{K}} = \Omega$, the completion of $\bar{K}$ with respect to $|\cdot|$.

Since any $\sigma \in \mathrm{Gal}(\bar{K}/K)$ is an isometry with respect to $|\cdot|$, it follows that σ uniquelly extends to a continuous K-automorphism of Ω. Let us denote $G_K = \mathrm{Gal}_{\mathrm{cont}}(\Omega/K) \simeq \mathrm{Gal}(\bar{K}/K)$. Let $T \in \Omega$. Let $C_K := \{\sigma(T), \sigma \in G_K\}$ be the orbit of T with respect to the action of G_K on Ω. Recall from [3], Theorem 3.5, that if T is transcendental over K then C_K is an infinite compact subset of Ω.

In [4] at page 29 it is associated a chain to each $T \in \Omega$ defined in terms of the distances from T to its conjugates over K, that is $\{\sigma(T), \sigma \in G_K = \mathrm{Gal}_{\mathrm{cont}}(\Omega/K) \cong \mathrm{Gal}(\bar{K}/K)\}$. The orbit of T with respect to the action of the group G_K is $C_K(T) := \{\sigma(T), \sigma \in G_K\}$. $C_K(T)$ is always a compact set and it is a finite set if and only if $T \in \bar{K}$. Let us denote $B[T, \epsilon] = \{\beta \in \Omega, |\beta - T| \leq \epsilon\}$.

Let $N(K, T, \epsilon)$ the number of disjoint such closed balls of radius ϵ covering $C_K(T)$. The function

$$\begin{aligned} (0, \infty) &\longrightarrow \mathbf{N} - \{0\}, \\ \epsilon &\longmapsto N(K, T, \epsilon), \end{aligned}$$

is a decreasing step function. It is bounded if and only if $T \in \bar{K}$. Its image is an increasing sequence $1 = N_1 < N_2 < \cdots$ whch is infinite if and only if $T \in \Omega - \bar{K}$.

Let $\epsilon_j = \inf\{\epsilon > 0, N(K, T, \epsilon) = N_j\}$. Since each $\sigma \in G_K$ is an isometry, each of the N_j balls of radius ϵ_j covering $C_K(T)$ is covered by the same number of balls of radius ϵ_{j+1} which intersect $C_K(T)$. It follows that $N_j | N_{j+1}$, for all j. If $T \in \Omega - \bar{K}$ we obtain the infinite chain

$$\mathcal{N}_K(T) = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \cdots \\ N_1 & N_2 & \cdots \end{pmatrix}.$$

Note that $\epsilon_1 = \sup\{|T - \sigma(T)|, \sigma \in G_K\}$ is the diameter of C_K and $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \cdots$ and $\lim_{n \to \infty} \epsilon_n = 0$. The sequence $(\epsilon_n)_{n \geq 1}$ is said to be *the fundamental* *sequence* associated to T. For $T = \alpha \in \overline{K}$ one obtains a finite chain

$$\mathcal{N}_K(\alpha) = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{l'_K}(\alpha) \\ N_1 & N_2 & \cdots & N_{l'_K}(\alpha) \end{pmatrix},$$

where $\epsilon_{l'_K}(\alpha) = 0$ and $N_{l'_K}(\alpha) = \deg_K(\alpha)$.

3. Main result

Let $B_1^m, B_2^m, \ldots, B_{N_m}^m$, be a partition of $C_K(T)$ with balls of radius ϵ_m, that is $B_1^m = \{z \in C_K(T), |z - T| \le \epsilon_m\}$. Any $\sigma \in G_K$ permutes $B_1^m, B_2^m, \ldots, B_{N_m}^m$, and if we denote $H_m = \{\sigma \in G_K, \sigma(B_1^m) = B_1^m\} \subseteq G_K$ we have $N_m = [G_K : H_m]$. Let $T_i \in B_i^m$ arbitrarily fixed with $T_1 = T$. Then $|T - T_i| > \epsilon_m$ for all $i \ge 2$. For example, if R_m is a set of representatives for $(G_K/H_m)_{\text{left}}$ then we can pick $T_i = \sigma_i T, \sigma_i \in R_m$. Let us notice that the number $x_m := \prod_{i=2}^{N_m} |T - T_i|$ does not depend on the choice of $T_i \in B_i^m$, it depends on the choice of T (in fact, of $C_K(T)$) only. Indeed, for $T'_i \in B_i^m$ we have $|T - T'_i| = |T - T_i + T_i - T'_i| = |T - T_i|$ since $|T_i - T'_i| < |T - T_i|$.

THEOREM 3.1. *For m_0 large enough we have $x_{m+1}^{\frac{1}{N_{m+1}}} < x_m^{\frac{1}{N_m}}$ for all $m \ge m_0$.*

PROOF. Since $N_m | N_{m+1}$ we have $N_{m+1} = N_m \cdot E_m$ that is each ball B_i^m of diameter (radius) ϵ_m has a partition consisting of E_m balls B_{ij}^{m+1} of diameter ϵ_{m+1} satisfying $E_m = [H_m : H_{m+1}]$.

Now let $i \ge 2$, $T_i \in B_i^m = \bigcup_{j=1}^{E_m} B_{ij}^{m+1}$. From each B_{ij}^{m+1} we pick a conjugate T_{ij} of $T = T_1$ with $T_{11} = T_1 = T$. Then $|T - T_i| = |T - T_{ij}|$ for all $1 \le j \le E_m$ thus $|T - T_i|^{E_m} = \prod_{j=1}^{E_m} |T - T_{ij}|$ therefore $|T - T_i|^{\frac{1}{N_m}} = \prod_{j=1}^{E_m} |T - T_{ij}|^{\frac{1}{N_{m+1}}}$ since $N_{m+1} = N_m E_m$.

From $\epsilon_m \to 0$ it follows that $\epsilon_m < 1$ for $m \ge m_0$, m_0 large enough. Thus $|T - T_{1j}| = |T_1 - T_{1j}| \le \epsilon_m < 1$ for all $j \ge 2$ therefore $1 > \prod_{j=2}^{E_m} |T - T_{1j}|$.

Finally,

$$x_m^{\frac{1}{N_m}} = \prod_{i=2}^{N_m} |T - T_i|^{\frac{1}{N_m}} = \prod_{i=2}^{N_m} \prod_{j=1}^{E_m} |T - T_{ij}|^{\frac{1}{N_{m+1}}}$$

$$> \prod_{i=2}^{N_m} \prod_{j=1}^{E_m} |T - T_{ij}|^{\frac{1}{N_{m+1}}} \cdot \prod_{j=2}^{E_m} |T - T_{1j}|^{\frac{1}{N_{m+1}}} = x_{m+1}^{\frac{1}{N_{m+1}}}. \qquad \square$$

COROLLARY 3.2. *The sequence $(x_m^{\frac{1}{N_m}})_m$ converges and its limit is an invariant of T (and of $C_K(T)$).*

Let us denote

$$\Delta_K(T) := \lim_{m\to\infty} x_m^{\frac{1}{N_m}}$$

Then $\Delta_K(T)$ can be considered an analogous of the discriminant of $T = \alpha \in \bar{K}$ according to the following observation: $x_m^{N_m} = \prod_{i\neq j} |T_j - T_i|$ since we can take $T_i = \sigma_i T$, for a suitable $\sigma_i \in R_m$ and the automorphisms of G_K are also isometries with respect to $|\cdot|$.

Now we give a method for computing the numerical invariants $\Delta_K(T)$ in certain cases. For this we make use of the following results from [3]:

Let K be a complete valued field of rank one valuation v. Let $\bar{K}$ be a fixed algebraic closure such that $K \subset \bar{K}$ is a countably generated extension and let Ω be the completion of $\bar{K}$ with respect to the unique prolongation (also denoted by v) of the valuation v to $\bar{K}$. we denote by $|.|$ the absolute value associated to v on Ω. (that is $|x| = c^{v(x)}$ with $c \in (0,1)$). From [3] there exists a one-to-one correspondence between the set of closed subfields L satisfying $K \subseteq L \subseteq \Omega$ and the subfields l satisfying $K \subseteq l \subseteq \bar{K}$ and $L = \tilde{l}$ (the closure with respect to $|.|$) and $l = L \cap \bar{K}$.

Let us recall that $T \in \Omega$ is said to be a generic element for the closed subfield L if $L = \widetilde{K(T)}$. It is proved that if $\Omega = \mathbf{C}_p$ endowed with the usual p-adic absolute value and $T \in \mathbf{C}_p - \bar{\mathbf{Q}}_p$ then $\widetilde{\mathbf{Q}_p(T)} = \tilde{\mathbf{Q}}_p[T]$ (see [1]).

Theorem 2 from [3] implies the following:

Let $L \subset \Omega$ be a closed subfield such that $K \subseteq L$ is a transcedental extension. There is an element $T \in \Omega$ such that $L = \widetilde{K(T)}$. Such an element T can be obtained as follows: let $l = L \cap \bar{K}$. We construct a sequence $(\alpha_n)_{n\geq 0}$, $\alpha_n \in l$ satisfying:

(1) for all n we have $|\alpha_{n+1} - \alpha_n| < min\{|\sigma(a_n) - a_n|, \sigma \in G_K, \sigma(\alpha_n) \neq \alpha_n\}$;

(2) $|\alpha_{n+1} - \alpha_n| \to 0$;

(3) $\bigcup_n K(\alpha_n) = l$.

Let $d_n = [K(\alpha_n) : K] = \deg_K \alpha_n$ and let $T = \lim_n \alpha_n \in \Omega - \bar{K}$. From (1) it follows that in each of the R_n balls of radius $\epsilon_n = |T - \alpha_n| = |\alpha_{n+1} - \alpha_n|$ which gives a partition of the orbit of T with respect to the action of G_K there exists exactly one conjugate of α_n over K. According to the notation of the previous paragraph it follows that $\prod_{\sigma_i\in R_n} |T - \sigma_i T| = \prod_{\sigma_i \in R_n} |\alpha_n - \sigma_i(\alpha_n)|$ and $|R_n| = d_n$. Since each $\sigma \in G_K$ is an isometry it follows that $x_n^{d_n} = \prod_{\sigma_i\in R_n, \sigma_i\neq\sigma_j} |\sigma_i(\alpha_n) - \sigma_j(\alpha_n)| = \Delta_K(\alpha_n)$. In conclusion:

Theorem 3.3. *Let L be a closed subfield of Ω. Then there exists a generic element T for L, that is $L = \widetilde{K(T)}$ satisfying $T = \lim_n \alpha_n$, $\alpha_n \in l = L \cap \bar{K}$ and $\Delta_K(T) = \lim_n \mathrm{disc}_K(\alpha_n)^{\frac{1}{d_n(d_n-1)}}$ where $d_n = \deg_K(\alpha_n)$.*

Now let $K = \mathbf{Q}_p$ thus $\tilde{\bar{K}} = \tilde{\bar{\mathbf{Q}}}_p = \mathbf{C}_p$ (the completion with respect to $|\cdot|_p$). Let $T, U \in \mathbf{C}_p - \bar{\mathbf{Q}}_p$ satisfying $|T|_p \le 1, |U|_p \le 1$ and $\tilde{\mathbf{Z}}_p[T] = \tilde{\mathbf{Z}}_p[U]$. We want to prove that $\Delta_{\mathbf{Q}_p}(T) = \Delta_{\mathbf{Q}_p}(U)$.

From $\tilde{\mathbf{Z}}_p[T] = \tilde{\mathbf{Z}}_p[U]$ it follows that there exist sequences of polynomials $P_n, R_n \in \mathbf{Z}_p[X]$ such that $U = \lim_{n\to\infty} P_n(T)$ and $T = \lim_{n\to\infty} R_n(U)$. For each $\sigma \in G = \mathrm{Gal}_{\mathrm{cont}}(\mathbf{C}_p/\mathbf{Q}_p)$ and for each positive integer n we have $|P_n(T) - \sigma P_n(T)| = |P_n(T) - P_n(\sigma T)| \le |T - \sigma T|$ since $P_n \in \mathbf{Z}_p[X]$. Thus $|U - \sigma U| = |\lim_{n\to\infty} P_n(T) - \sigma \lim_{n\to\infty} P_n(T)| = |\lim_{n\to\infty}(P_n(T) - \sigma P_n(T))| \le |T - \sigma T|$ for all $\sigma \in G$. By symmetry, we have that $|T - \sigma T| \le |U - \sigma U|$, for all $\sigma \in G$ thus $|T - \sigma T| = |U - \sigma U|$, for all $\sigma \in G$.

Let $F : C_{\mathbf{Q}_p}(T) \to C_{\mathbf{Q}_p}(U)$ defined as follows: $F(x) = \lim_{n\to\infty} P_n(x)$, for all $x = \sigma T \in C_{\mathbf{Q}_p}(T)$. Its inverse is $F^{-1}\colon C_{\mathbf{Q}_p}(U) \to C_{\mathbf{Q}_p}(T)$, $F^{-1}(y) = \lim_{n\to\infty} R_n(y)$, $\forall y = \sigma U \in C_{\mathbf{Q}_p}(U)$. For each ball B of radius ϵ in $C_{\mathbf{Q}_p}(T)$ its image $F(B)$ is a ball of radius ϵ in $C_{\mathbf{Q}_p}(U)$. Therefore T and U have the same fundamental associated sequence $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_m > \cdots$ with $\epsilon_m \to 0$. For each $m \ge 1$ we consider the unique partition $B_1^{(m)}, B_2^{(m)}, \ldots, B_{N_m}^{(m)}$ with ϵ_m-radius closed balls of $C_{\mathbf{Q}_p}(T)$. It follows that $F(B_1^{(m)}), F(B_2^{(m)}), \ldots, F(B_{N_m}^{(m)})$ is the unique partition with ϵ_m-radius closed balls of $C_{\mathbf{Q}_p}(U)$.

As above, let H_m be both the G-stabilizer of $B_1^{(m)}$ and the G-stabilizer of $F(B_1^{(m)})$. Let $R_m = \{\sigma_i\}_{i=1,..N_m}$ be a complete system of representatives of $(G/H_m)_{\mathrm{left}}$. Since $|T - \sigma T| = |U - \sigma U|$, for all $\sigma \in G$, it follows that

$$x_m(T) = \prod_{i=2}^{N_m} |T - \sigma_i T| = \prod_{i=2}^{N_m} |U - \sigma_i U| = x_m(U)$$

thus

$$\Delta_{\mathbf{Q}_p(T)} = \lim_{m\to\infty} x_m(T)^{1/N_m} = \lim_{m\to\infty} x_m(U)^{1/N_m} = \Delta_{\mathbf{Q}_p(U)}$$

We proved the following

Theorem 3.4. *Let $T, U \in \mathbf{C}_p - \bar{\mathbf{Q}}_p$ satisfying $|T|_p \le 1, |U|_p \le 1$ and $\tilde{\mathbf{Z}}_p[T] = \tilde{\mathbf{Z}}_p[U]$. Then $\Delta_{\mathbf{Q}_p}(T) = \Delta_{\mathbf{Q}_p}(U)$ is an invariant of the closed ring $\tilde{\mathbf{Z}}_p[T]$.*

Remark 3.5. The above theorem is the transcedental analogue for the following well-known result (see [6]). Let T, U be algebraic over $\mathbf{Q}_p$ satisfying $|T|_p \le 1$, $|U|_p \le 1$ and $\mathbf{Z}_p[T] = \mathbf{Z}_p[U]$. Then $\mathrm{disc}_{\mathbf{Q}_p(T)} = \mathrm{disc}_{\mathbf{Q}_p(U)}$

4. A notable example

In this section we give an example of a non-zero $\Delta_K(T)$.

First we recall notions and properties from [1].

For $K = \mathbf{Q}_p$ and $\Omega = \mathbf{C}_p$ the absolute value $|\cdot| = |\cdot|_p$ is associated to the p-adic valuation v via $|x|_p = (\frac{1}{p})^{v(x)}$ for all $x \in \mathbf{C}_p$. According to [1], Proposition 2.2, p. 135, for all $T \in \mathbf{C}_p$, T transcendental over $\mathbf{Q}_p$, there exists a so called *distinguished* sequence $(\alpha_n)_{n\geq 0}$ with $\alpha_n \in \bar{Q}_p$ such that $T = \lim_n \alpha_n$, where $\alpha_0 \in \mathbf{Q}_p$ and $|T - \alpha_0|_p = min\{|T - \alpha|_p, \alpha \in \mathbf{Q}_p\}$. The *distinguished* sequence $(\alpha_n)_{n\geq 0}$ satisfies the following conditions:

(1) $D_n = \deg \alpha_n > D_m = \deg \alpha_m$ for all $m < n$, futhermore D_m divides D_n;

(2) $|T - \alpha_n|_p < |T - \alpha_{n-1}|_p$;

(3) if $\gamma \in \bar{\mathbf{Q}}_p$ and $\deg \gamma < \deg \alpha_n$, then $|T - \gamma|_p \leq |T - \alpha_{n-1}|_p$.

We also quote from [1] the following: if we denote f_n the minimal polynomial of α_n over $\mathbf{Q}_p$, $n \geq 0$; and if we also denote $\gamma_n = v_p(f_n(\alpha_{n+1}))$ we have $\gamma_n > \gamma_{n-1}$ and $\frac{\gamma_n}{D_n} > \frac{\gamma_{n-1}}{D_{n-1}}$ for all $n \geq 1$. Thus there exists $l(T) = \lim_n \frac{\gamma_n}{D_n} \in \mathbf{R}_+ \cup \{+\infty\}$. Also recall that the numbers D_n, $|T - \alpha_n|$, $|\gamma_n|$ depend on T only; they do not depend on the distinguished sequence associated to T. The above statements are equivalent to $|f_n(T)|_p^{\frac{1}{D_n}} < |f_{n-1}(T)|_p^{\frac{1}{D_{n-1}}}$ since $|f_n(T)|_p = \left(\frac{1}{p}\right)^{\gamma_n}$, thus there exists $(\frac{1}{p})^{l(T)} = \lim_n |f_n(T)|_p^{\frac{1}{D_n}} \in [0, +\infty)$.

Now, in order to construct the example, we need the lemma below.

Let $T \in \mathbf{C}_p - \bar{\mathbf{Q}}_p$ and let $(\alpha_m)_{m\geq 0}$ be a distinguished sequence converging to T. Put $\overline{\epsilon_m} := |T - \alpha_m|_p$. Let $\overline{B_1^m}, \overline{B_2^m}, \dots, \overline{B_{\overline{N_m}}^m}$ be closed balls in $\mathbf{C}_p$ of radius $\overline{\epsilon_m}$ such that their intersections to the orbit $C_{\mathbf{Q}_p}(T)$ give a partition of $C_{\mathbf{Q}_p}(T)$. For this it suffices that the balls are conjugated and they give a partition of the orbit of α_m. More precisely, $\overline{B_1^m} = \{x \in \mathbf{C}_p, |T - x| \leq \overline{\epsilon_m}\}$ and $\overline{B_i^m} = \sigma_i \overline{B_1^m}$ for some $\sigma_i \in G_{\mathbf{Q}_p}$. Since each ball $\overline{B_i^m}$ contains the same number F_m of conjugates of α_m we have $\deg \alpha_m = D_m = \overline{N_m} F_m$. Let $T_i \in \overline{B_i^m} \cap C_{\mathbf{Q}_p}(T)$, $i \geq 2$ and let $\overline{x_m} := \prod_{i=2}^{\overline{N_m}} |T - T_i|$. Note that the product which defines $\overline{x_m}$ does not depend on the choice of T_i. Let us denote by $\epsilon_{m'}$ the radius of the ball $\overline{B_i^{m'}} = \overline{B_i^m} \cap C_{\mathbf{Q}_p}(T)$. Notice that $\epsilon_{m'}$ is a term of the fundamental sequence associated to T. We have that $\epsilon_{m'} \leq \epsilon_m$. The balls $\overline{B_i^{m'}}$, $i = 1, \overline{N_m}$ are conjugated and they give a partition of the orbit $C_{\mathbf{Q}_p}(T)$ and we also have $N_{m'} = \overline{N_m}$ and $x_{m'} = \overline{x_m}$, with x_m and N_m defined as in Theorem1. From $\overline{\epsilon_m} \to 0$ it follows that $\epsilon_{m'} \to 0$ thus $N_{m'} \to \infty$. Therefore the sequence $(\overline{x_m}^{\frac{1}{\overline{N_m}}})_{m\geq 0}$ is a subsequence of $(x_m^{\frac{1}{N_m}})_{m\geq 0}$ studied in Theorem 3.1.

LEMMA 4.1. *For $|T|_p \leq 1$ and $m \geq 1$ we have* $\overline{x_m}^{\frac{1}{\overline{N_m}}} > |f_m(T)|_p^{\frac{1}{D_m}}$.

PROOF. Since $D_m = \overline{N_m} F_m$ we have to prove that $\overline{x_m}^{F_m} > |f_m(T)|_p$. We have $\overline{x_m} := \prod_{i=2}^{\overline{N_m}} |T - T_i|$, $T_i \in \overline{B_i^m}$ and $|f_m(T)|_p = \prod_{s=1}^{D_m} |T - T_{\sigma_s(\alpha_m)}|$, where $\{\sigma_s(\alpha_m), s = 1..D_m\}$ are the conjugates of α_m with $\alpha_m = \sigma_1(\alpha_m)$. Let us denote $\{\sigma_{i_s}(\alpha_m), s = 1, \ldots, F_m\}$ those conjugates of α_m belonging to the ball $\overline{B_i^m}$. Then for all $i \geq 2$ we have $|T - T_i|_p > |T_i - \sigma_{i_s}(\alpha_m)|_p$ thus $|T - T_i|_p = |T - \sigma_{i_s}(\alpha_m)|_p$. Therefore

$$|T - T_i|_p^{F_m} = \prod_{s-1}^{F_m} |T - \sigma_{i_s}(\alpha_m)|_p.$$

On the other hand in the ball B_1^m there is at least one conjugate of α_m (for example α_m itself) and for each of these conjugates we have $|T - \sigma_{1_s}(\alpha_m)|_p < 1$. Therefore

$$\prod_{s=1}^{F_m} |T - \sigma_{1_s}(\alpha_m)|_p < 1$$

and by multiplying the above two products we obtain $\overline{x_m}^{F_m} > |f_m(T)|_p$. □

Taking into account that the sequence $(\overline{x_m}^{\frac{1}{\overline{N_m}}})_{m\geq 0}$ is a subsequence of $(x_m^{\frac{1}{N_m}})_{m\geq 0}$ studied in Theorem 3.1 it follows by applying the above lemma that

$$\Delta_{\mathbf{Q}_p}(T) \geq \lim_n |f_n(T)|_p^{\frac{1}{D_m}} = \left(\frac{1}{p}\right)^{l(T)} \geq 0$$

REMARK 4.2. In [1] page 142, using an argument from [5], it is given an example of an element $T \in \mathbf{C}_2 - \mathbf{Q}_2$ with $l(T) = 2$. Therefore, for that T, we have $\Delta_{\mathbf{Q}_2}(T) \geq (\frac{1}{2})^2 > 0$.

A thoroughly study of how $\Delta_{\mathbf{Q}_p}(T)$ and $\left(\frac{1}{p}\right)^{l(T)}$ are related to each other is yet to be done.

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