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Divisibility and duo-rings

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Abstract – This paper investigates the projective dimension of the maximal right ring of quotients $Q^r(R)$ of a right non-singular ring R. Our discussion addresses the question under which conditions $\mathrm{pd}(Q) \leq 1$ guarantees that the module Q/R is a direct sum of countably generated modules extending Matlis' Theorem for integral domains to a non-commutative setting.

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Contents

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1. Introduction

There are various ways to extend the concept of divisibility from integral domains to arbitrary rings. A right R-module D is *divisible in the classical sense* if $Dc = D$ for every regular element $c \in R$. E. Matlis extended upon this concept and called a module D *h-divisible* if it is an epimorphic image of an injective module [18]. On the other hand, we can generalize using homological properties and define D to be *divisible* if $\mathrm{Ext}^1_R(R/rR, D) = 0$ for every $r \in R$. The question of when these various notions coincide for integral domains has been investigated by several authors, and a summary of their results can be found in [10]. The non-commutative case was addressed by one of the authors in [1], and is considered further in Section 2 of this paper.

It is always the case that h-divisibility implies classic divisibility, but the converse fails in general [1]. If R is a semi-prime right Goldie-ring, then a non-singular module D is divisible if and only if it is divisible in the classical sense if and only if it is injective [1, Corollary 4.5]. Here, a ring R is a *right Goldie-ring* if it has finite right Goldie-dimension and satisfies the ascending chain condition on right annihilators. A ring R has *finite right Goldie-dimension* if every direct sum of nonzero right ideals of R contains only finitely many direct summands. A semi-prime right and left Goldie-ring R has a semi-simple Artinian classical right and left ring of quotients $Q = Q^r = Q^l$, which is also its right and left maximal ring of quotients [12].

In [10, Theorem VII.2.8], Fuchs and Salce show that all three notions of divisibility coincide for countable integral domains (see also [18]). This does not hold true if R is a non-commutative domain (see [2, sections 4 and 5]). However, it will hold if R is a semi-prime right and left Goldie p.p.-ring for which the maximal ring of quotients Q is countably generated as a right R-module [1, Theorem 5.5]. Moreover, questions concerning divisibility are closely related to the projective dimension of Q. An integral domain R with $\mathrm{pd}_R(Q) \leq 1$ is called a *Matlis domain*. E. Matlis [18], S. B. Lee [17], and L. Fuchs and L. Salce [10, Chapter VII, Theorem 2.8] characterize Matlis domains by showing that the following three conditions are equivalent for an integral domain R:

a) R is a Matlis domain;

b) divisible R-modules are h-divisible;

c) Q/R is a direct sum of countably generated (divisible) submodules.

Furthermore, L. Fuchs and S. B. Lee show in [9, Theorem 6.4] that a commutative ring R is a Matlis ring if and only if Q/R is a direct sum of countably presented modules if and only if divisible R-modules are h-divisible. It is the main focus of this paper to investigate whether the equivalence of the above conditions extends to a non-commutative setting. We want to remind the reader of the following result from [1]:

Theorem 1.1 ([1, Theorem 5.2]). *Let R be a semi-prime right and left Goldie-ring. If Q/R is a direct sum of countably generated submodules, then* $\mathrm{pd}_R(Q) \leq 1$.

We begin our discussion in Section 2 by focusing on the various notions of divisibility and related concepts. Our results will establish that c) $\Longrightarrow$ b) and b) $\Longrightarrow$ a) remain valid for semi-prime right and left Goldie rings (Theorem 2.2). However, we give an example that a) $\Longrightarrow$ c) may fail in the non-commutative setting (Theorem 2.4). Therefore, the remaining part of this paper will focus on establishing a non-commutative setting in which a) $\Longrightarrow$ c) is valid (sections 3, 4, and 5). In the course of our discussion, we extend several of Kaplansky's Change of Rings Lemmas to a non-commutative setting (Section 3). We will obtain a direct sum decomposition as in c) via a transfinite induction, at the core of which is a Step-Lemma (Theorem 4.4) similar to the one used in applications of set-theoretic methods to groups and modules [11].

2. Divisibility and projective dimension

We want to remind the reader that

$$Z(M) = \{x \in M : xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$$

denotes the *singular submodule of M*. Moreover, a right R-module A has *projective dimension* $\leq n$, denoted $\mathrm{pd}_R(A) \leq n$, if there exists a finite projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in which $P_0, \ldots, P_n$ are projective.

A ring R is a *right p.p.-ring* if aR is projective for all $a \in R$. Equivalently, R is right p.p. if and only if right annihilators of elements are generated by an idempotent. If R is a right and left Goldie-ring, then right p.p.-rings are also left p.p.-rings [21]. In this case, we simply call R a p.p.-ring. Clearly, every ring without zero-divisors is a p.p.-ring.

Proposition 2.1 ([1, Corollary 4.6a]). *If R is a semi-prime right and left Goldie p.p.-ring, then the class of divisible modules coincides with the class of modules which are divisible in the classical sense.*

A right R-module M is *weakly cotorsion* if $\operatorname{Ext}^1_R(Q^r, M) = 0$.

Theorem 2.2. *Consider the following conditions for a semi-prime right and left Goldie-ring R with classical right and left ring of quotients Q, and let $K = Q/R$:*

a) $K_R \cong \bigoplus_I A_i/R$ *where each A_i is a subring of Q such that $(A_i)_R$ is countably generated.*

b) K_R *is a direct sum of countably generated submodules*;

c) *every divisible module is h-divisible*;

d) *all divisible modules are weakly cotorsion*;

e) $Z(D)$ *is a direct summand of D whenever D is divisible*;

f) $\operatorname{pd}_R(Q/R) \leq 1$.

Then a) $\Longrightarrow$ b) $\Longrightarrow$ c). *Furthermore, if R is a p.p.-ring in addition, then we get* c) $\Longrightarrow$ d) $\Longrightarrow$ e) $\Longrightarrow$ f). *Theorem 2.4 will show that* f) $\Longrightarrow$ a) *may fail even if R is a semi-prime right and left Goldie p.p.-ring.*

Proof. Since a) $\Longrightarrow$ b) is obvious, we turn to b) $\Longrightarrow$ c). Let D be a divisible right R-module, and consider $a \in Z(D)$. Since R is a semi-prime right and left Goldie-ring, there is a regular element s_0 of R such that $as_0 = 0$.

Write $K(R) = \bigoplus_I K_i/R$ where each K_i is a countably generated submodule of Q_R containing R. Since R is a semi-prime right and left Goldie-ring, Q is the classical right and left ring of quotients of R. Therefore, every element of Q can be written as $c^{-1}r$ for some regular element c of R. Hence, if U is a countably generated submodule of Q_R, then we can find regular elements $\{c_n | n < \omega\}$ of R such that $U \subseteq \Sigma_{n<\omega} c_n^{-1} R$. At the same time, there is a countable subset J of I such that $U \subseteq \Sigma_{j \in J} K_j$. Using a standard back and forth argument beginning with U, we can find a countable subset $\{d_n | n < \omega\}$ of regular elements of R such that $V = (\Sigma_{n<\omega} d_n^{-1} R)/R$ is a direct summand of Q/R. In fact, V will be a direct sum of countably many of the K_i.

Applying the construction from the last paragraph to $U = s_0^{-1}R$, we select regular elements $\{s_1, s_2, \ldots\}$ of R such that $E = (\Sigma_{n<\omega} s_n^{-1} R)/R$ is a direct summand of Q/R. Inductively, we show that we can find regular elements t_n of R with $t_0 = s_0$ such that $Rt_{n+1} \subseteq Rt_n$ for all $n < \omega$ and $\Sigma_{n<\omega} s_n^{-1} R \subseteq \bigcup_{n<\omega} t_n^{-1} R$. Assume that we have already constructed $t_0, \ldots, t_n$ with the desired properties such that $s_0^{-1}, \ldots, s_n^{-1} \in t_n^{-1} R$. Since R is a semi-prime right and left Goldie ring, Rt_n and Rs_{n+1} are essential left ideals of R because t_n and s_{n+1} are regular.

Thus, $Rt_n \cap Rs_{n+1}$ is an essential left ideal, and it contains a regular element t_{n+1} of R. We can find $r_{n+1}, t \in R$ such that $t_{n+1} = ts_{n+1} = r_{n+1}t_n$. Observe that r_{n+1} has to be left regular since t_{n+1} is regular. Since R is a semi-prime right and left Goldie-ring, r_{n+1} is regular. Inside Q, we obtain $s_{n+1}^{-1} = t_{n+1}^{-1}t$ and $t_n^{-1} = t_{n+1}^{-1}r_{n+1}$. Therefore, $s_{n+1}^{-1}, t_n^{-1} \in t_{n+1}^{-1}R$. In particular,

$$R \subseteq t_0^{-1}R \subseteq t_1^{-1} \subseteq \cdots \subseteq t_n^{-1}R \subseteq \cdots.$$

Then, $V = \bigcup_{n<\omega} t_n^{-1}R$ contains R, and $E \subseteq V/R$.

To show that $Z(D)$ is h-divisible, we let $a_0 = a$ and $r_0 = s_0$. Select $\{a_n \in D\colon n < \omega\}$ such that $a_{n+1}r_{n+1} = a_n$ for $n < \omega$ where $t_{n+1} = r_{n+1}t_n$ as in the last paragraph. Since $t_n^{-1}R$ is a free right R-module, setting $\alpha_n(t_n^{-1}) = a_n$ defines a map $\alpha_n\colon t_n^{-1}R \to D$. Moreover,

$$\alpha_{n+1}(t_n^{-1}) = \alpha_{n+1}(t_{n+1}^{-1})r_{n+1} = a_{n+1}r_{n+1} = a_n = \alpha_n(t_n^{-1}).$$

Therefore, $\alpha_{n+1}|t_n^{-1}R = \alpha_n$. Moreover, $\alpha_0(1) = \alpha_0(t_0^{-1}s_0) = a_0s_0 = 0$ yields $\alpha_n(R) = 0$ for all $n < \omega$. Thus, the α_n induce a map $\alpha\colon V/R \to D$ with $\alpha(t_0^{-1} + R) = a$. However, $t_0^{-1} + R = s_0^{-1} + R \in E$. Consequently, a is contained in the image of $\alpha|E\colon E \to D$. Since E is a direct summand of Q/R, we obtain a map $\beta\colon K(R) = Q_R/R \to D$ such that $a \in im\ \beta$. Because Q/R is singular, $\beta(Q/R) \subseteq Z(D)$. In particular, $Z(D)$ is an epimorphic image of a direct sum of copies of $K(R)$. But then, $Z(D)$ is an image of copies of Q_R, and hence h-divisible. By Part b) of Theorem 1.1, $Z(D)$ is divisible and weakly cotorsion. Since R is a semi-prime right and left Goldie-ring, every non-singular module, which is divisible in the classical sense, is actually a Q-module, and hence injective. This holds in particular for $D/Z(D)$, and so $\mathrm{Ext}^1_R(D/Z(D), Z(D)) = 0$. This shows that $D \cong Z(D) \oplus D/Z(D)$ is h-divisible.

From this point on, we will assume that R is a p.p.-ring in addition to being a semi-prime right and left Goldie-ring.

c) $\Longrightarrow$ d). Let D be a divisible module. By c), D is h-divisible. Hence, $Z(D)$ is a direct summand of D by Theorem 4.1 of [1]. Moreover, all divisible modules are divisible in the classical sense and vice-versa by Proposition 2.1. Combining these two observations yields that all divisible modules are weakly cotorsion by Part b) of [1, Corollary 4.6].

d) $\Longrightarrow$ e). Since all divisible modules are divisible in the classical sense and vice-versa by Proposition 2.1, the fact that every divisible module D is weakly cotorsion yields that $Z(D)$ is a direct summand of D by Part b) of [1, Corollary 4.6].

Finally, e) $\Longrightarrow$ f) follows directly from Proposition 2.1 and [1, Proposition 5.1]. □

The equivalence of a) and b) was discussed in [1, Proposition 5.3]. A decomposition $Q/R = A/R \oplus B/R$, where A and B are submodules of Q_R containing R, has the additional property that A and B are subrings of Q exactly if A and B are also submodules of ${}_RQ$.

Although h-divisible modules are divisible in the classical sense, they need not be divisible.

Proposition 2.3. [1, Corollary 4.2] *The following are equivalent for a right non-singular ring R of finite right Goldie-dimension*:

a) *R is a right p.p.-ring*;

b) *every h-divisible right R-module is divisible.*

Therefore, it is not surprising that p.p-rings entered the discussion in Theorem 2.2. Moreover, the ring $M_2(\mathbb{Z}[x])$ is an example of a ring for which not all h-divisible modules are divisible [2].

The next result shows that f) $\implies$ a) may fail, even if R is a semi-prime right and left Goldie p.p.-ring, by constructing a right hereditary ring R for which $(Q/R)_R$ is not the direct sum of countably generated submodules A_i/R where each A_i is a subring of Q. Since R is right hereditary, $\mathrm{pd}_R(Q_R) \leq 1$.

We want to remind the reader that a ring R is a *right duo ring* if $Ra \subseteq aR$ for every $a \in R$, and it is a *a duo ring* if it is both a right and left duo ring. It is easy to see that Mr is a submodule of M for all right R-modules M if and only if R is a left duo ring.

Theorem 2.4. *Let R be a right Noetherian, right chain domain whose lattice of right ideals is inversely order isomorphic to an ordinal σ of uncountable cardinality. Then, R is a right hereditary right duo ring with classical right ring of quotients Q such that $(Q/R)_R$ is not the direct sum of countably generated submodules A_i/R where each A_i is a subring of Q.*

Proof. Bessenrodt, Brungs, and Törner show in [6, Lemmas 1.4, 3.2] that R is a right duo ring. Hence, every right ideal of R is two-sided. Moreover, R is a right hereditary ring since every right ideal of R is principal [6, Lemma 3.1], and R has a classical right ring of quotients Q since every right Noetherian domain is a right Ore domain.

We first show that ${}_RQ$ is not countably generated. If it were, then we could find $\{c_n \in R : n < \omega\}$ such that $Q = \sum_{n<\omega} Rc_n^{-1}$. We consider the right ideals c_nR of R , and observe that $\bigcap_{n<\omega} c_nR \neq 0$ since σ is of uncountable cardinality. We pick a non-zero $d \in \bigcap_{n<\omega} c_nR$, and write $d = c_nr_n$ for all $n < \omega$. In particular, we have $qd \in R$ for all $q \in Q$. Specifically, $c^{-1}d \in R$ for all $0 \neq c \in R$. Thus, $d \in \bigcap_{c\neq 0} cR$. In particular, $0 \neq d^2$ and $d^2R \subseteq dR \subseteq \bigcap_{c\neq 0} cR \subseteq d^2R$, and we can find $r \in R$ such that $d = d^2r$. Since R has no zero-divisors, $1 = dr$. Hence, $d \notin J(R)$ and d is a unit, from whence it follows $R = Qd = Q$, a contradiction. Thus, ${}_RQ$ is not countably generated.

Now assume $(Q/R)_R \cong \bigoplus_I A_i/R$ for some index set I, where A_i/R is countably generated and A_i is a subring of Q containing R. By the discussion following Theorem 2.2, each A_i is a two-sided submodule of Q. Pick a countable subset $J_0 \subseteq I$, and write $\sum_{J_0} A_j = \sum_{n<\omega}(r_nc_n^{-1})R$. Then, $r_nc_n^{-1} \in \sum_m Rc_m^{-1}$. However, Rc_m^{-1} is also an R-submodule of Q_R. To see this, let $r \in R$ and pick $s \in R$ such that $rc_m = c_ms$. This is possible since a right Noetherian, right chain ring is right duo by [7]. Then $c_m^{-1}r = sc_m^{-1}$, and thus $\sum_{J_0} A_j \subseteq \sum_m Rc_m^{-1}$. Since ${}_RQ$ is not countably generated, we may assume that this inclusion is proper. Otherwise, we can add Rd^{-1} to the sum on the right-hand side, and proceed with $\sum_m Rc_m^{-1} + Rd^{-1}$ such that $d^{-1} \notin \sum_m Rc_m^{-1}$.

We can find a countable subset J_1 of I such that $J_0 \subseteq J_1$ and $c_m^{-1} \in \sum_{J_1} A_j$. Since each A_j is two-sided,

$$\sum_{J_0} A_j \subsetneq \sum_m Rc_m^{-1} \subseteq \sum_{J_1} A_j.$$

Inductively, we obtain an ascending chain $J_0 \subseteq J_1 \subseteq \cdots$ of countable subsets of I and a countable family $\{d_n : n < \omega\} \subseteq R$ such that $J = \bigcup_{n<\omega} J_n$ is a countable subset of I with $\sum_J A_j = \sum_{n<\omega} Rd_n^{-1}$. If ${}_RQ \neq \sum_{n<\omega} Rd_n^{-1}$, then there exists $0 \neq c \in R$ such that $c^{-1} \notin \sum_{n<\omega} Rd_n^{-1}$. Since R is a right chain ring, either $cR \subseteq d_nR$ or $d_nR \subseteq cR$. If the latter occurs, then $d_n = ct_n$ for some $t_n \in R$ and $c^{-1} = t_nd_n^{-1}$, a contradiction. Thus, $c = d_ns_n$ for some $s_n \in R$ and $d_n^{-1} = s_nc^{-1}$. It readily follows that $\sum_{n<\omega} Rd_n^{-1} \subseteq Rc^{-1}$.

However, $R \subseteq Rc^{-1}$, so that

$$\sum_J A_j \subseteq \sum_{n<\omega} Rd_n^{-1} \subseteq Rc^{-1}$$

implies $\bigoplus_J A_j/R \subseteq Rc^{-1}/R$. Thus,

$$Rc^{-1}/R = \bigoplus_J (A_j/R) \oplus U/R$$

for some $R \subseteq U \subseteq Rc^{-1}$ since $\left(\bigoplus_J A_j\right)/R$ is a direct summand of Q/R. Observe that $Q/R = \bigoplus_I A_i/R$ is a decomposition of both $(Q/R)_R$ and ${}_R(Q/R)$ since A_i is a two-sided submodule for each $i \in I$. Moreover, the module $\bigoplus_J (A_j/R)$ is not finitely generated since $\sum_{J_n} A_j \subsetneq \sum_{J_{n+1}} A_j$ for every $n < \omega$, and we obtain a contradiction. Therefore, $Q = \sum_{n<\omega} Rd_n^{-1}$, contradicting the fact that $_RQ$ is not countably generated. Thus, $(Q/R)_R$ is not the direct sum of countably generated submodules A_i/R where each A_i is a subring of Q. □

3. Duo rings and projective dimension

Kaplansky's Change of Rings Lemmas investigate the relationship between the projective dimensions of modules over the commutative rings R and R/sR, where $s \in R$ is a non-zero divisor. If one wants to attempt to extend them to a non-commutative setting, some obvious restrictions need to be imposed on s to avoid obvious counter-examples. In particular, $Rs = sR$ has to be satisfied. The proof of the next result carries over directly from the commutative setting and is therefore omitted.

Lemma 3.1. *Let R be a duo ring, and suppose that $s \in R$ is regular. If M is a right R-module such that $xs \neq 0$ for every $0 \neq x \in M$, then* $\mathrm{pd}_{R/sR}(M/Ms) \leq \mathrm{pd}_R(M)$.

We next consider two other versions of Kaplansky's Change of Rings Lemmas, namely [20, Proposition 8.39] and [10, Lemma VI.2.11]. In contrast to the last result, these proofs fail to carry over to duo rings because they rely on the fact that right multiplication by s is a right R-module homomorphism. More precisely, either proof considers a right R-module M with $Ms = 0$ and a free resolution $0 \to K \to F \to M \to 0$, and shows that the induced map $F/K \to Fs/Ks$ defined by $x + K \to xs + Ks$ is an isomorphism of R-modules. Unfortunately, this map is an R-morphism only if s is a central element of R [19, Theorem 9.33]. To prove the theorem in the case that s is not central, a different approach is needed which we base on

Lemma 3.2 ([19, Theorem 9.32]). *If $\varphi\colon R \to R^*$ is a ring homomorphism and A^* is a right R^*-module, then* $\mathrm{pd}_R(A^*) \leq \mathrm{pd}_{R*}(A^*) + \mathrm{pd}_R(R^*)$.

If R is a ring and $\sigma\colon R \to R$ is an automorphism of rings, then every right R-module M carries another R-module structure induced by σ: for $x \in M$ and $r \in R$, define $x * r = x\sigma(r)$. Let M^* denote the R-module M with the structure induced by σ. Since $1 * r = 1\sigma(r)$, we have that R^* is a free right R-module. Hence, $\mathrm{pd}_R(R^*) = 0$ and $\mathrm{pd}_R(M) = \mathrm{pd}_R(M^*) \leq \mathrm{pd}_{R^*}(M^*)$ by Lemma 3.2. Since σ is an isomorphism, we can use σ^{-1} to get the reverse inequality. Therefore, $\mathrm{pd}_R(M) = \mathrm{pd}_{R^*}(M^*)$.

It is easy to see that the regular elements in a duo ring R satisfy the right and left Ore condition. Thus, R has a classical right and left ring of quotients Q.

Proposition 3.3. *Let R be a duo ring with classical right and left ring of quotients Q, and let $0 \neq s \in R$ be regular. If $\sigma\colon R \to R$ is the automorphism defined by $\sigma(r) = s^{-1}rs$, then the map $\bar{\sigma}\colon R/sR \to R/sR$ defined by $\bar{\sigma}(r+sR) = \sigma(r) + sR$ is an automorphism of R/sR.*

Proof. Observe that s is a unit of Q. For $r \in R$, we can select $r' \in R$ such $rs = sr'$ since R is duo. Computing in Q, we obtain $s^{-1}rs = s^{-1}sr' \in R$. Hence, $\sigma\colon R \to R$. It is easy to see that σ is one-to-one and a morphism of rings. Moreover, if $t \in R$, then we can find $t' \in R$ with $t's = st$ since R is duo. Then $\sigma(t') = s^{-1}t's = S^{-1}st = t$, and σ is an isomorphism of rings.

If $r' = r+st$ for some $t \in R$, then $s^{-1}r's = s^{-1}rs+s^{-1}sts = s^{-1}rs+ts$. Since $sR = Rs$, we have $\bar{\sigma}(r'+sR) = \bar{\sigma}(r+sR)$, and hence $\bar{\sigma}$ is well-defined. It is easily seen that $\bar{\sigma}$ is an epimorphism and an R-map. To see that $\bar{\sigma}$ is a monomorphism, suppose that $\bar{\sigma}(r+sR) = 0$. The duo condition yields $s^{-1}rs = ts$ for some $t \in R$. Hence $t = s^{-1}r \in Q$, and $r = st \in sR$. Therefore, $\bar{\sigma}$ is an automorphism of R/sR. □

A right R/sR-module U can be viewed as a right R-module with $Us = 0$. Moreover, using the maps σ and $\bar{\sigma}$ from Proposition 3.3, we have

$$u * (r + sR) = u\bar{\sigma}(r + sR) = u(\sigma(r) + sR) = u\sigma(r) = u \times r$$

where $*$ and $\times$ denote the module structures induced by $\bar{\sigma}$ and σ, respectively.

Theorem 3.4. *Let R be a right and left duo ring, and let $0 \neq s \in R$ be regular. If M is a right R/sR-module such that $\operatorname{pd}_{R/sR}(M) = n < \infty$, then $\operatorname{pd}_R(M) = n + 1$.*

Proof. To begin our induction, let $\operatorname{pd}_{R/sR}(M) = 1$. Observe that Lemma 3.2 yields $\operatorname{pd}_R(M) \leq 2$. We assume $\operatorname{pd}_R(M) \leq 1$, and consider an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ of right R-modules with P_0 and P_1 projective. Applying the functor $\_ \otimes_R R/sR$ induces the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(M, R/sR) \longrightarrow P_1 \otimes_R R/sR \longrightarrow P_0 \otimes_R R/sR \longrightarrow M \otimes_R R/sR \longrightarrow 0$$

of right R/sR-modules.

However, $M \otimes_R R/sR \cong M$ since M is an R/sR-module. Furthermore, since $P_i \otimes_R R/sR$ is a projective R/sR-module for $i = 0, 1$, and $\operatorname{pd}_{R/sR}(M) = 1$, we have that $\operatorname{Tor}_1^R(M, R/sR)$ is a projective R/sR-module.

Now, the sequence $0 \to sR \xrightarrow{\iota} R \to R/sR \to 0$ is an exact sequence of R-R-bimodules where $\iota\colon sR \to R$ is the inclusion map. We consider the induced sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(M, R/sR) \xrightarrow{\partial} M \otimes_R sR \xrightarrow{\iota^*} M \otimes_R R.$$

Computing in $M \otimes_R R$, we have $x \otimes st = xs \otimes t = 0$ since M is a right R-module satisfying $Ms = 0$. Thus, $im\ \iota^* = 0$, and ∂ is an isomorphism. Consequently, $A = M \otimes_R sR \cong \operatorname{Tor}_1^R(M, R/sR)$ as an R-module, and hence as an R/sR-module. Therefore, A is a projective R/sR-module.

Let A^* denote the R-module A with the module structure induced by $\bar{\sigma}$ as defined in Proposition 3.3. For $x \otimes ts \in A$, we have

$$(x \otimes ts) * r = x \otimes tss^{-1}rs = x \otimes trs.$$

However, $\lambda\colon A^* \to M$ defined by $\lambda(x \otimes ts) = xt$ is an isomorphism of R-modules, and hence also of R/sR-modules. As previously shown, Lemma 3.2 implies that A and A^* have the same projective dimension as both R and R/sR-modules since σ and $\bar{\sigma}$ are automorphisms of R and R/sR, respectively. Thus, we have a contradiction since this leads to

$$1 = \operatorname{pd}_{R/sR}(M) = \operatorname{pd}_{R/sR}(A^*) = \operatorname{pd}_{R/sR}(A) = \operatorname{pd}_{R/sR}(\operatorname{Tor}_1^R(M, R/sR)) = 0.$$

Therefore, $\operatorname{pd}_R(M) > 1$ and $\operatorname{pd}_R(M) = 2$.

For the induction step, assume that $\operatorname{pd}_R(M) = n$ whenever $\operatorname{pd}_{R/sR}(M) = n - 1$. Suppose $\operatorname{pd}_{R/sR}(M) = n$. If $\operatorname{pd}_R(M) \le n$, then there exists an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ of right R-modules with P_i projective for $i = 0, 1, \ldots, n$. As before, this induces the exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Tor}_1^R(M, R/sR) \longrightarrow P_n \otimes_R R/sR \longrightarrow \cdots \\ \longrightarrow P_0 \otimes_R R/sR \longrightarrow M \otimes_R R/sR \longrightarrow 0 \end{aligned}$$

of right R/sR-modules. Since $\operatorname{pd}_{R/sR}(M) = n$ and each $P_i \otimes_R R/sR$ is projective as a right R/sR-module, we have $\operatorname{pd}_{R/sR}(\operatorname{Tor}_1^R(M, R/sR)) = n - 1$. By the inductive hypothesis, $\operatorname{pd}_R(\operatorname{Tor}_1^R(M, R/sR)) = n$. Hence, $\operatorname{pd}_R(A) = n$, which leads to a contradiction since

$$n = \operatorname{pd}_R(A) = \operatorname{pd}_{R/sR}(A) = \operatorname{pd}_{R/sR}(\operatorname{Tor}_1^R(M, R/sR)) = n - 1.$$

Therefore, $\operatorname{pd}_R(M) > n$ and the claim follows using Lemma 3.2 once more. □

4. Normal submonoids and prime ideals

If Q is the classical right and left ring of quotients of R, then the ring of quotients R_T with respect to a right and left Ore-set $T \subseteq R^\times$ can naturally be viewed as a subring of Q, and we will identify R_T with this subring. In the following, we are particularly interested in subrings of Q arising as localizations at completely prime ideals, where an ideal P of R is *completely prime* if $xy \in P$ implies that $x \in P$ or $y \in P$ for every $x, y \in R$. If R is a duo ring, then every prime ideal is completely prime, and the localization at $R\backslash P$, denoted R_P, can be obtained as in the commutative setting. However, although $R \setminus P$ is multiplicatively closed, it may still contain zero divisors, so that R_P cannot always be embedded into Q. To avoid this additional complexity, we assume that R does not contain any zero divisors. Furthermore, Brungs showed in [7] that the localization R_P of a duo ring R at a prime ideal P need not be duo. However, R_P is a duo ring if it satisfies the ascending chain condition for principal right and left ideals.

A submonoid T of a monoid S is *normal*, denoted $T \lhd S$, if $sT = Ts$ for every $s \in S$.

Proposition 4.1. *Let R be a duo ring without zero-divisors with classical ring of quotients Q. If T is a normal submonoid of $R^\times$ and P is a prime ideal of R, then*

a) *$R_T R_P = \{rt^{-1}sx^{-1} \mid r, s \in R, t \in T, x \in R \setminus P\}$ is a subring of Q and the same holds for $R_P R_T$;*

b) *$(R_T)_P = (R_P)_T = R_T R_P = R_P R_T \subseteq Q$.*

Proof. a) Let $0 \neq r \in R$ and $t \in T$. Since R is a duo ring, we can find $s \in R$ such that $rt = ts$. Thus, T is a right Ore-subset of R.

Moreover, to see that $R_T R_P$ is a subring of Q, consider $u_1, u_2 \in R_T$ and $v_1, v_2 \in R_P$. We may find $r_1, r_2, s_1, s_2 \in R$, $t \in T$ and $x \in R \setminus P$ such that $u_i = r_i t^{-1}$ and $v_i = s_i x^{-1}$ or $i = 1, 2$. Since T is normal in $R^\times$, we can find $t_1, t_2 \in T$ such that $s_i t_i = t s_i$ so that $t^{-1} s_i = s_i t_i^{-1}$. Since T is a right and left Ore-set, there exists $r_1', r_2' \in R$ and $t_3 \in T$ with $t_i^{-1} = r_i' t_3^{-1}$. Thus,

$$\begin{aligned}(u_1 + u_2)(v_1 + v_2) &= [(r_1 + r_2)t^{-1}s_1 + (r_1 + r_2)t^{-1}s_2]x^{-1} \\ &= [(r_1 + r_2)s_1t_1^{-1} + (r_1 + r_2)s_2t_2^{-1}]x^{-1} \\ &= [(r_1 + r_2)s_1r_1' + (r_1 + r_2)s_2r_2']t_3^{-1}x^{-1} \in R_T R_P.\end{aligned}$$

A similar argument shows that $R_T R_P$ is multiplicatively closed. The case $R_P R_T$ is treated similarly.

b) We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R\otimes_R R_P & \xrightarrow{\alpha} & R_T\otimes_R R_P & \xrightarrow{\beta} & (R_T/R)\otimes_R R_P & \longrightarrow & 0 \\ & & \wr\downarrow\phi & & \downarrow\psi & & & & \\ 0 & \longrightarrow & R_P & \xrightarrow{\subseteq} & R_T R_P & & & & \end{array}$$

in which ϕ and ψ are the multiplication maps. Clearly, ψ is onto. For $r, s \in R$, $t \in T$ and $x \in R \setminus P$, select $s' \in R$ such that $st = ts'$ using the fact that R is duo. Then

$$[rt^{-1} \otimes sx^{-1}]xt = rt^{-1} \otimes st = rt^{-1} \otimes ts' = rs' \otimes 1,$$

which shows that $(R_T/R) \otimes_R R_P$ is a singular right R-module.

If $y \in \ker\ \psi$, then there is a regular element $c \in R$ such that $yc = \alpha(y')$. Then $0 = \psi(yc) = \psi\alpha(y') = \phi(y')$ and thus $yc = 0$. Observe that $R_T \otimes_R R_P$ is non-singular since R_T is a flat module. Hence, $yc = 0$ implies $y = 0$. Therefore, $(R_T)_P = R_T R_P$.

For $r, s \in R$, $t \in T$ and $x \in R \setminus P$, select $s', r' \in R$ such that $st = ts'$ and $rx = xr'$ using the fact that R is a duo ring. Since T is normal in $R^\times$, there are $t', t'' \in T$ with $xt = t'x$ and $tx = xt''$. Then

$$rt^{-1}sx^{-1} = rs't^{-1}x^{-1} = rs'(xt)^{-1} = rs'(t'x)^{-1} = rs'x^{-1}t'^{-1} \in R_P R_T,$$

and $R_T R_P \subseteq R_P R_T$. Moreover,

$$sx^{-1}rt^{-1} = sr'x^{-1}t{-}1 = sr'(tx)^{-1} = sr'(xt'')^{-1} = sr't''^{-1}x^{-1} \in R_T R_P,$$

and $R_P R_T \subseteq R_T R_P$. □

Lemma 4.2. *Let R be a duo ring without zero-divisors. If $T \lhd S$ are normal submonoids of $R^\times$ such that $\mathrm{pd}_R(R_S) \leq 1$ and $\mathrm{pd}_R(R_S/R_T) \leq 1$, and if P is a prime ideal of R with $T \cap P \neq 0$, then R_T/sR_T is projective as a left R/sR-module for any $s \in S$.*

Proof. Clearly, R_T is a left R-module in view of $r(at^{-1}) = (ra)t^{-1} \in R_T$ for any $r \in R$ and any $at^{-1} \in R_T$. Since R is a duo ring, $sR = Rs$. Thus, for any $r \in R$, we can find $r_1 \in R$ such that $rs = sr_1$. Hence, $r(sat^{-1}) = sr_1(at^{-1})$, and sR_T is a submodule of ${}_R R_T$. Since R is duo, sR is a two-sided ideal of R, and we can view R_T/sR_T as a left R/sR-module.

Consider the exact sequence

$$0 \longrightarrow R_T \longrightarrow R_S \longrightarrow R_S/R_T \longrightarrow 0.$$

By assumption, $\mathrm{pd}_R(R_S) \le 1$ and $\mathrm{pd}_R(R_S/R_T) \le 1$. If $\mathrm{pd}_R(R_T) > 1$, then $\mathrm{pd}_R(R_T) > \mathrm{pd}_R(R_S)$ and hence $\mathrm{pd}_R(R_S/R_T) = \mathrm{pd}_R(R_T) + 1 > 1$ by [20, Exercise 8.5]. However, this is a contradiction, and thus $\mathrm{pd}_R(R_T) \le 1$. Consequently, $\mathrm{pd}_{R/sR}(R_T/sR_T) \le \mathrm{pd}_R(R_T) \le 1$ by Lemma 3.1. Now, consider the exact sequence

$$0 \longrightarrow R_T/sR_T \longrightarrow R_S/sR_T \longrightarrow R_S/R_T \longrightarrow 0$$

of left R-modules. Since R_S is s-divisible, $sR_S = R_S$ and

$$R_S/R_T \cong sR_S/sR_T = R_S/sR_T.$$

Thus, $\mathrm{pd}_R(R_S/R_T) = \mathrm{pd}_R(R_S/sR_T)$. Therefore,

$$\mathrm{pd}_R(R_T/sR_T) < \mathrm{pd}_R(R_S/R_T) = \mathrm{pd}_R(R_S/sR_T) \le 1$$

using standard properties of the projective dimension. However, Theorem 3.4 shows that if $\mathrm{pd}_{R/Rs}(R_T/sR_T) = 1$ then $\mathrm{pd}_R(R_T/sR_T)$ must be 2, which is a contradiction. Therefore, $\mathrm{pd}_{R/sR}(R_T/sR_T) = 0$, and R_T/sR_T is projective as a left R/sR-module. □

Lemma 4.3. *Let R be a duo ring without zero-divisors. If $T \lhd S$ are normal submonoids of $R^\times$ such that $\mathrm{pd}_R(R_S) \le 1$ and $\mathrm{pd}_R(R_S/R_T) \le 1$, and if P is a prime ideal of R with $T \cap P \ne 0$, then $(R_T)_P = (R_S)_P$ and R_T/R is S-divisible.*

Proof. By Proposition 4.1, $(R_T)_P = R_T R_P$ and $(R_S)_P = R_S R_P$. Observe that R_P is a local ring since P is completely prime. Hence, R_P/sR_P is local too. Moreover, since R is a duo ring and T is a normal submonoid of $R^\times$, we can view $(R_T)_P$ as a left R_P-module. To see this, take $(at^{-1})m^{-1} \in (R_T)_P$ and $bn^{-1} \in R_P$ where $n, m \in R \setminus P$. The duo condition provides $a_1 \in R$ such that $an = na_1$. Since T is normal, we can find $t_1 \in T$ such that $tn = nt_1$. Thus,

$$bn^{-1}(at^{-1}m^{-1}) = ba_1n^{-1}t^{-1}m^{-1} = (ba_1t_1^{-1})(mn)^{-1} \in (R_T)_P.$$

Since localization at P is an exact functor, $(R_T)_P/s(R_T)_P$ is projective as a left $(R_P)/s(R_P)$-module by what was shown in the preceding paragraph. Since projective modules over local rings are free (see for example [20, Theoem 4.58] and the note after it), $(R_T)_P/s(R_T)_P$ is a free $(R_P)/s(R_P)$-module.

Now assume $(R_T)_P/s(R_T)_P \ne 0$, and consider $t \in T \cap P \ne \emptyset$. Suppose t is a unit of R_P. Then there exists $rm^{-1} \in R_P$ such that $trm^{-1} = 1$. However, this leads to a contradiction since it implies that $t^{-1} = rm^{-1} \in R_P$ and hence $t \in R\setminus P$. Furthermore, if $(au^{-1})m^{-1} \in (R_T)_P$, then the duo condition provides $a_1 \in R$ such that

$$au^{-1}m^{-1} = tt^{-1}au^{-1}m^{-1} = ta_1(ut)^{-1}m^{-1} \in t(R_T)_P.$$

Hence, $t(R_T)_P = (R_T)_P$.

Since $(R_T)_P/s(R_T)_P$ is a free $(R_P)/s(R_P)$-module, there exists some index set I such that $(R_T)_P/s(R_T)_P \cong \bigoplus_I (R_P)/s(R_P)$. Moreover, since $(R_T)_P$ is divisible by t, it must also be the case that

$$\bigoplus_I (R_P)/s(R_P) = t\big[\bigoplus_I (R_P)/s(R_P)\big].$$

However, this implies $R_P/sR_P = t(R_P/sR_P)$. But $t \in PR_P$, which is a contradiction since t is not a unit in R_P. Therefore, given any $s \in S$, we have $(R_T)_P/s(R_T)_P = 0$ and hence $(R_T)_P = s(R_T)_P$. That is, $s^{-1}(R_T)_P = (R_T)_P$ for any $s \in S$.

Now, to see that $(R_P)_S \subseteq (R_T)_P$, take $(rm^{-1})u^{-1} \in (R_P)_S$. Since S is a normal submonoid of $R^\times$ and m is regular, there exists $u_1 \in S$ such that $um = mu_1$, and hence $m^{-1}u^{-1} = u_1^{-1}m^{-1}$. Moreover, we can use the duo condition to find $r_1 \in R$ such that $u_1 r = r_1 u_1$, from whence it follows $ru_1^{-1} = u_1^{-1}r_1$. Observe also that $r_1 m^{-1} \in (R_T)_P$ since $r_1 \in R \subseteq R_T$ and $m \in R\backslash P$. Therefore, $(R_P)_S \subseteq (R_T)_P$ since

$$(rm^{-1})u^{-1} = u_1^{-1}r_1m^{-1} \in u_1^{-1}(R_T)_P = (R_T)_P.$$

It is easily seen that $(R_T)_P \subseteq (R_P)_S$ since $xT = Tx$ for every $x \in R^\times$ and $T \subseteq S$. For if $rt^{-1}m^{-1} \in (R_T)_P$, then there exists $t_1 \in T \subseteq S$ such that

$$rt^{-1}m^{-1} = rm^{-1}t_1^{-1} \in (R_P)_S.$$

Thus, $(R_T)_P = (R_P)_S$. By Proposition 4.1, $(R_P)_S = (R_S)_P$. Therefore, we have $(R_T)_P = (R_P)_S = (R_S)_P$, and it readily follows from the S-divibility of R_S that $(R_T/R)_P = (R_T)_P/R_P = (R_S)_P/R_P$ is S-divisible. Consequently, R_T/R is S-divisible. □

Theorem 4.4 (Step-Lemma). *Let R be a duo ring without zero-divisors. If $T \triangleleft S$ are normal submonoids of $R^\times$ such that* $\mathrm{pd}_R(R_S) \leq 1$ *and* $\mathrm{pd}_R(R_S/R_T) \leq 1$, *then R_T/R is a direct summand of R_S/R.*

Proof. As a first step, we show that $(R_T/R)_P$ is S-divisible for all prime ideals P of R. Since R is a duo ring, P is completely prime, and $R\backslash P$ is multiplicatively closed. If $T \cap P = \emptyset$, then $T \subseteq R\backslash P$, and so $(R_T/R)_P = 0$.

Now, assume $T \cap P \neq \emptyset$. It follows from Lemma 4.3 that $(R_T/R)_P$ is S-divisible and $(R_T)_P = (R_S)_P$. Moreover, R_T/sR_T is projective as a left R/sR-module by Lemma 4.2.

Suppose $s \in S$. By the S-divisibility of R_T/R, we have $s(R_T/R) = R_T/R$, and hence $sR_T + R = R_T$. Furthermore, R_T/sR_T is projective as a left R/sR-module. Hence,

$$R/(R \cap sR_T) \cong (sR_T + R)/sR_T = R_T/sR_T$$

is projective as a left R/sR-module. The epimorphism $\pi\colon R/sR \to R/(R \cap sR_T)$ defined by $\pi(r + sR) = r + (R \cap sR_T)$ induces the exact sequence

$$0 \longrightarrow (R \cap sR_T)/sR \longrightarrow R/sR \longrightarrow R/(R \cap sR_T) \longrightarrow 0$$

which splits since $R/(R \cap sR_T)$ is projective as a R/sR-module. However, left multiplication by s induces isomorphisms

$$s^{-1}R/R \cong R/sR$$

and

$$(s^{-1}R \cap R_T)/R \cong (R \cap sR_T)/sR$$

of right R-modules. Hence

$$[s^{-1}R/R]/[(s^{-1}R \cap R_T)/R] \cong s^{-1}R/[(s^{-1}R \cap R_T)] \cong R/(R \cap sR_T)$$

is a projective R/sR-module. Thus,

$$[s^{-1}R/R] = [(s^{-1}R \cap R_T)/R] \oplus C/R$$

for some submodule C of $s^{-1}R$ containing R. Observe that $C/R \cong R_T/sR_T$. Using the notation of Fuchs and Salce, let $B = \bigcap_{P \in \mathcal{W}}(R_P \cap R_S)$ where $\mathcal{W}$ is the set of maximal ideals P with $T \cap P \neq \emptyset$. By Lemma 4.3, $(R_T)_P = (R_S)_P$ in the case that $T \cap P \neq \emptyset$. Hence,

$$\begin{aligned}(C/R)_P &\cong (R_T/sR_T)_P = (R_T)_P/s(R_T)_P\\ &= (R_S)_P/s(R_S)_P = (R_S)_P/(R_S)_P = 0\end{aligned}$$

from which we obtain $C_P = R_P$. Since $C \subseteq R_S$ and $(s^{-1}R \cap R_T)/R \subseteq R_T/R$, we have $s^{-1}R/R \leq R_T/R + B/R$ for every $s \in S$. Thus

$$R_S/R = R_T/R + B/R.$$

It remains to be seen that $(R_T/R) \cap (B/R) = 0$. Once this is established, we have shown that

$$R_S/R = (R_T/R) \oplus (B/R).$$

Again using the notation of Fuchs and Salce, let $A = \bigcap_{P \in \mathcal{V}}(R_P \cap R_S)$, where $\mathcal{V}$ is the set of maximal ideals with $T \cap P = \emptyset$. Since R_T is clearly contained in A and $R_T \cap B \leq A \cap B$, it suffices to show that $A \cap B = R$. It is easily seen that $R \subseteq A \cap B$. For if $x \in R$, then $x \in R_T$ for any submonoid T of $R^\times$. Hence, $x \in R_P \cap R_S$ for every maximal ideal P and thus $x \in A \cap B$.

To see that $A \cap B \subseteq R$, it suffices to show that

$$R = \left[\bigcap_{P \in \text{m-Spec}} R_P\right] \cap R_S$$

where m-Spec is the set of all maximal ideals of R. Let $x = us^{-1} \in R_S \backslash R$ and consider the right ideal $I_x = \{r \in R : xr \in R\}$. Note that $I_x \neq \{0\}$ since $xs = us^{-1}s = u \in R$ yields $s \in I_x$. Moreover, I_x is a proper right ideal since $1 \notin I_x$. Hence, it follows that there exists a maximal right ideal P containing I_x. Since R is duo, P is a two-sided ideal. If $x \in R_P$, then $x = rm^{-1}$ for some $r \in R$ and $m \in R \backslash P$. However, $xm = r \in R$ implies that $m \in I_x \subseteq P$, which is a contradiction. Thus, given $x \in R_S \backslash R$, there exists some maximal ideal P of R such that $x \notin R_P$. Hence, $x \in R$ whenever $x \in R_P$ for every maximal ideal P of R. Therefore, $R = \left[\bigcap_{P \in \text{m-Spec}} R_P\right] \cap R_S$ and $A \cap B = R$. □

5. Pre-Matlis duo domains and tight systems

We now turn to obtaining the desired direct sum decomposition of Q/R. For a right R-module M, a set $\mathcal{S} = \{M_i : i \in I\}$ of submodules of M is called a $G(\aleph_0)$*-family* if the following are satisfied:

i) $0, M \in \mathcal{S}$;

ii) $\mathcal{S}$ is closed under unions of chains;

iii) for every $M_i \in \mathcal{S}$ and every countable subset X of M, there exists $M_j \in \mathcal{S}$ such that $M_i \leq M_j$, $X \subseteq M_j$ and M_j/M_i is countably generated.

A submodule N of a right R-module M is called *tight* if $\text{pd}_R(M/N) \leq \text{pd}_R(M)$. For a right R-module M with $\text{pd}_R(M) \leq 1$, a family $\mathcal{T} = \{M_i : i \in I\}$ of tight submodules of M is called a *tight system* if it is a $G(\aleph_0)$-family such that $\text{pd}_R(M_j/M_i) \leq \text{pd}_R(M) \leq 1$ whenever $M_i, M_j \in \mathcal{T}$ with $M_i \subseteq M_j$. The following result ensures the existence of a tight system in our setting in the case that $\text{pd}_R(M) \leq 1$. The proof is similar to the integral domain case found in [10, Proposition 5.1] and is therefore omitted.

Lemma 5.1 ([10]). *Let R be a semi-prime right and left Goldie-ring and M a right R-module. If $\text{pd}_R(M) \leq 1$, then M admits a tight system.*

Once we have an appropriate $G(\aleph_0)$-family of tight submodules, we will use the following lemma to extract a well-ordered ascending chain of direct summands.

LEMMA 5.2 ([15, Lemma 7.2]). *Let R be a ring and let M be a right R-module. Let $\mathcal{U}$ be a family of submodules of M, and take $\mathcal{U}_0$ to be a subset of $\mathcal{U}$. Assume that for a suitable ordinal β there exists a chain $\{M_\gamma\}_{\gamma\leq\beta}$ such that*

i) *for every $\gamma < \beta$, $M_{\gamma+1} = M_\gamma \oplus U_\gamma$ for some $U_\gamma \in \mathcal{U}_0$;*

ii) *$M_0 = 0$, and $M_\gamma = \bigcup_{\nu<\gamma} M_\nu$ for every limit ordinal $\gamma \leq \beta$, and $M = M_\beta$.*

Then, $M = \bigoplus_{\gamma<\beta} U_\gamma$ is a direct sum of modules with $U_\gamma \in \mathcal{U}_0$ for every $\gamma < \beta$.

The monoid of regular elements $R^\times$ has a *κ-filtration* if it is the union of a smooth well-ordered ascending chain

$$\{1\} = T_0 \leq T_1 \leq \cdots \leq T_\alpha \leq \cdots \leq T_\kappa = R^\times$$

of submonoids. We want to remind the reader that submonoids of $R^\times$ are right and left Ore-sets if R is a duo ring.

One of the main difficulties encountered in our discussion is that, in the noncommutative setting, $R^\times$ does not necessarily have κ-filtrations with the same properties as those in integral domains. In particular, if we consider a submonoid T of $R^\times$ and a countable subset S of $R^\times$, then it is not guaranteed that the localization at the submonoid generated by T and S is countably generated over the localization at T. For instance, Theorem 2.4 provides an example of a ring for which $R^\times$ does not have a desired filtration.

To overcome these difficulties, we introduce a notion similar to the Third Axiom of Countability introduced by P. Griffith and P. Hill in [14]. A monoid T satisfies the *third axiom of countability* if there exists a family $\mathcal{C} = \{T_i : i \in I\}$ of submonoids of T such that

i) $1 \in \mathcal{C}$;

ii) $\mathcal{C}$ is closed under unions of chains;

iii) if $i \in I$ and $X \subseteq T$ is countable, then there exists $i_0 \in I$ such that $T_i, X \subseteq T_{i_0}$ and T_{i_0} is countably generated over T_i.

We refer to the family $\mathcal{C}$ as an *Axiom* III family of T.

DEFINITION 5.3. A ring R is a *pre-Matlis ring* if $R^\times$ is the union of a smooth chain

$$\{1\} = T_0 \leq T_1 \leq \cdots \leq T_\alpha \leq \cdots \leq T_\kappa = R^\times$$

of submonoids with the following properties:

(i) $T_\alpha \lhd R^\times$ for every $\alpha < \kappa$;

(ii) if $\alpha < \kappa$ and $X \subseteq R^\times$ is countable, then there exists $\beta < \kappa$ such that $T_\alpha, X \subseteq T_\beta$ and T_β is countably generated over T_α.

We consider an example from Bessenrodt, Brungs, and Törner in [6] of a ring whose monoid of regular elements has the desired filtration of normal submonoids. For an ordered group $(G, \leq)$ with identity e, let $G^+ = \{g \in G : e \leq g\}$ denote the positive cone of G. For a division algebra K, consider the collection of power series of the form $a = \sum_{g \in G} g a_g$, with $a_g \in K$. Define the support of a to be $\mathrm{supp}(a) = \{g \in G : a_g \neq 0\}$, and refer to a as a generalized power series if $\mathrm{supp}(a)$ is a well-ordered subset of G. If $ag = ga$ for every $a \in K$ and $g \in G$, then the set of all generalized power series, denoted $K[[G]]$, is a ring with normal power series addition and multiplication. Moreover, $K[[G]]$ is a division ring and [6, Proposition 1.24] shows that $K[[G^+]]$ is a duo chain domain with quotient ring $K[[G]]$.

Theorem 5.4. *Let $(G, \leq)$ be an ordered group which has an Axiom III family of normal subgroups, and let $R = K[[G^+]]$. Then R is a pre-Matlis domain.*

Proof. Suppose G has an Axiom III family $C = \{N_\alpha : \alpha < \kappa\}$ of normal subgroups. Since $G^+ \cap N_\alpha$ is a normal subroup of G^+ for each $\alpha < \kappa$, it is easily seen that $C' = \{G^+ \cap N_\alpha : \alpha < \kappa\}$ is an Axiom III family of G^+.

i) $\{e\} = G^+ \cap \{e\} \in C'$ since $\{e\} \in C$.

ii) If $\{G^+ \cap N_\beta\}_{\beta<\gamma}$ is a chain in C', then $\{N_\beta\}_{\beta<\gamma}$ is a chain in C. Hence, $\bigcup_{\beta<\gamma} N_\beta \in C$, from whence it follows $G^+ \cap \left(\bigcup_{\beta<\gamma} N_\beta\right) \in C'$.

iii) Let $\alpha < \kappa$ and let $X \subseteq G^+ \subseteq G$ be countable. Since C is an Axiom III family, there exists $\beta < \kappa$ such that $N_\alpha, X \subseteq N_\beta$ and N_β is countably generated over N_α. Therefore, $G^+ \cap N_\alpha, X \subseteq G^+ \cap N_\beta$ and $G^+ \cap N_\beta$ is countably generated over $G^+ \cap N_\alpha$.

For each $\alpha < \kappa$, define $T_\alpha = K[[G^+ \cap N_\alpha]] \backslash \{0\}$ to be the set of all non-zero generalized power series $\sum g a_g$ over $G^+ \cap N_\alpha$ and K. By [6, Proposition 1.24], we obtain that $K[[G^+ \cap N_\alpha]]$ is a duo ring, and hence $rT_\alpha = T_\alpha r$ for every $r \in R^\times$. By extending property iii) of the Axiom III family of G^+ to $\{T_\alpha\}_{\alpha<\kappa}$, we obtain that the second condition of our filtration is satisfied. Therefore, $K[[G^+]]$ is a pre-Matlis domain. □

We are now ready for our main result, which extends the characterization of Matlis domains to duo rings not containing zero-divisors. For a semi-prime right and left Goldie-ring R with classical right and left ring of quotients Q, let $K = Q/R$.

Theorem 5.5. *The following conditions are equivalent if R is a right and left duo pre-Matlis domain*:

a) $K_R \cong \bigoplus_I [A_i/R]$ *where each* A_i *is a subring of* Q *such that* $(A_i)_R$ *is countably generated*;

b) *every divisible module is h-divisible*;

c) $\mathrm{pd}_R(Q/R) \leq 1$.

Proof. By Theorem 2.2, it remains to show c) $\Longrightarrow$ a). Suppose $\mathrm{pd}_R(Q/R) = 1$, and assume that R has the desired filtration

$$\{1\} = T_0 \leq T_1 \leq \cdots \leq T_\alpha \leq \cdots \leq T_\kappa = R^\times.$$

Let $\mathcal{U} = \{R_{T_\alpha}/R \colon \alpha \leq \kappa\}$. Observe that for each $\alpha < \kappa$, R_{T_α}/R is a submodule of Q/R. We show that $\mathcal{U}$ is a $G(\aleph_0)$-family of Q/R. Clearly, condition i) is satisfied since $\{0\} = R_{\{1\}}/R \in \mathcal{U}$ and $Q/R = R_{R^\times}/R \in \mathcal{U}$. Moreover, $\mathcal{U}$ is closed under unions of chains since $\{T_\alpha\}_{\alpha \leq \kappa}$ forms a smooth chain and includes $R^\times = \bigcup_{\alpha<\kappa} T_\alpha$.

To see that condition iii) is satisfied, take $R_{T_\alpha}/R \in \mathcal{U}$ and let

$$X = \{r_j s_j^{-1} + R \colon r_j, s_j \in R \text{ with } s_j \text{ regular}, j < \omega\}$$

be a countable subset of Q/R. Using condition ii) of the filtration, there exists $\beta < \kappa$ such that $T_\alpha \subseteq T_\beta$, $\{s_j \colon j < \omega\} \subseteq T_\beta$, and T_β is countably generated over T_α. Hence, $R_{T_\alpha}/R, X \subseteq R_{T_\beta}/R$ and there exists a countable subset $S_\alpha \subseteq T_\beta$ such that $T_\beta = S_\alpha T_\alpha = T_\alpha S_\alpha$. Thus, if $t \in T_\beta$, there exists $s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n} \in S_\alpha$ and $t_{\alpha_1}, t_{\alpha_2}, \ldots, t_{\alpha_n} \in T_\alpha$ such that

$$t = s_{\alpha_1} t_{\alpha_1} s_{\alpha_2} t_{\alpha_2} \ldots s_{\alpha_n} t_{\alpha_n}.$$

Then if $rt^{-1} + R_{T_\alpha} \in R_{T_\beta}/R_{T_\alpha}$, we have

$$rt^{-1} = rt_{\alpha_n}^{-1} s_{\alpha_n}^{-1} \ldots t_{\alpha_2}^{-1} s_{\alpha_2}^{-1} t_{\alpha_1}^{-1} s_{\alpha_1}^{-1}.$$

Therefore,

$$(R_{T_\beta}/R)/(R_{T_\alpha}/R) \cong R_{T_\beta}/R_{T_\alpha}$$

is countably generated by $\{s^{-1} \colon s \in S_\alpha \backslash T_\alpha\}$ and $\mathcal{U}$ is a $G(\aleph_0)$-family of Q/R.

It follows from Lemma 5.1 that Q/R admits a tight system $\mathcal{T}$. It is clear that $\mathcal{T}$ is also a $G(\aleph_0)$-family of Q/R, and it is easily seen that $\mathcal{U} \cap \mathcal{T}$ is a $G(\aleph_0)$-family of tight submodules of Q/R of the form R_{T_α}/R for $\alpha < \kappa$. Thus, given any $R_{T_\alpha}/R \in \mathcal{U} \cap \mathcal{T}$,

$$\mathrm{pd}_R(Q/R_{T_\alpha}) = \mathrm{pd}_R((Q/R)/(R_{T_\alpha}/R)) \leq \mathrm{pd}_R(Q/R) \leq 1.$$

Theorem 4.4 yields that R_{T_α}/R is a direct summand of Q/R for every $\alpha < \kappa$. Since $R^\times = \bigcup_{\alpha<\kappa} T_\alpha$, we have $Q/R = \bigcup_{\alpha<\kappa} R_{T_\alpha}/R$. Moreover, the smooth filtration ensures that

$$R_{T_\beta}/R = \bigcup_{\gamma<\beta} R_{T_\gamma}/R \in \mathcal{U} \cap \mathcal{T},$$

and hence there exists $\beta \leq \kappa$ and a continuous well-ordered ascending chain $\{R_{T_\gamma}/R \colon \gamma < \beta\} \subseteq \mathcal{U} \cap \mathcal{T}$ of submodules of Q/R such that R_{T_γ}/R is a direct summand of Q/R and $R_{T_{\gamma+1}}/R_{T_\gamma}$ is countably generated. Hence, $Q/R = \bigoplus_{\gamma<\beta} A_\gamma/R$ where each A_γ is countably generated. Finally, since R is right and left duo and R_{T_γ} is a subring of Q for each γ, we have that each A_γ is a two-sided submodule of Q. □

Theorem 2.4 showed that, without some additional filtration properties, implication f) $\Longrightarrow$ a) of Theorem 2.2 may fail if Q^r is not countably generated. However, we can find the following filtration of countable submonoids of $R^\times$ if $(Q/R)_R$ is generated by $\aleph_1$-many elements.

Corollary 5.6. *Suppose R is a semi-prime right and left Goldie-ring such that $(Q/R)_R$ is a direct sum of $\aleph_1$ many countable modules, then there exists a smooth ascending chain $T_0 \leq T_1 \leq \cdots \leq T_\alpha \leq \cdots$, $\alpha < \aleph_1$, of countable submonoids of $R^\times$ such that $R^\times = \bigcup_{\alpha<\aleph_1} T_\alpha$.*

Proof. Let $T_0 = \{1\}$ and let $T_\sigma = \bigcup_{\beta<\sigma} T_\beta$ for each limit ordinal $\sigma < \aleph_1$. Note that each T_σ is countable as the countable union of a countable set. Let $\alpha < \aleph_1$ and suppose that for each $\beta \leq \alpha$, T_β has been defined so that R_{T_β}/R is a direct sum of countably many A_ν/R. Then

$$R_{T_\alpha}/R = \bigoplus_{I_\alpha}[A_\nu/R]$$

is a direct summand of Q/R for some countable set I_α. If $R_{T_\alpha} = Q$, then we are done. Otherwise, there exists $\mu < \aleph_1$ with $A_\mu \not\subseteq R_{T_\alpha}$. Let $A_\mu = \langle r_n t_n^{-1} \colon n < \omega\rangle$

and define $T_\alpha^1 = \langle T_\alpha, t_n : n < \omega\rangle$. Observe that T_α^1 is countable since it is countably generated by countable sets. Since

$$R_{T_\alpha^1}/R \subseteq Q/R = \bigoplus_{\nu<\aleph_1} [A_\nu/R],$$

we can find a countable subset $I_\alpha^1 \supseteq I_\alpha$ such that $R_{T_\alpha^1}/R \subseteq \bigoplus_{I_\alpha^1}[A_\nu/R]$.

If $R_{T_\alpha^1} = Q$, then we are done. Otherwise, there exists $\mu_2 < \aleph_1$ with $A_{\mu_2} \not\subseteq R_{T_\alpha^1}$. As before, let $A_{\mu_2} = \langle r_n t_{n,2}^{-1} : n < \omega\rangle$ and define $T_\alpha^2 = \langle T_\alpha^1, t_{n,2} : n < \omega\rangle$. Then, T_α^2 is countable and we can find a countable subset $I_\alpha^2 \supseteq I_\alpha^1$ such that $R_{T_\alpha^2}/R \subseteq \bigoplus_{I_\alpha^2} A_\nu/R$. Note that

$$R_{T_\alpha^1}/R \subseteq \bigoplus_{I_\alpha^1} A_\nu/R \subseteq R_{T_\alpha^2}/R \subseteq \bigoplus_{I_\alpha^2} A_\nu/R.$$

Continue this process to find

$$I_\alpha \subseteq I_\alpha^1 \subseteq I_\alpha^2 \subseteq \cdots \subseteq I_\alpha^n \subseteq \cdots$$

and

$$T_\alpha \subseteq T_\alpha^1 \subseteq T_\alpha^2 \subseteq \cdots \subseteq T_\alpha^n \subseteq \cdots$$

satisfying

$$R_{T_\alpha^n}/R \subseteq \bigoplus_{I_\alpha^n}[A_\nu/R] \subseteq R_{T_\alpha^{n+1}}/R \subseteq \bigoplus_{I_\alpha^{n+1}}[A_\nu/R].$$

Let $T_{\alpha+1} = \bigcup_{n<\omega} T_\alpha^n$ and let $I = \bigcup_{n<\omega} I_\alpha^n$. Observe that both $T_{\alpha+1}$ and I are countable since each T_α^n and each I_α^n are countable. If $rt^{-1} + R \in R_{T_{\alpha+1}}/R$, then $t \in T_\alpha^n$ for some $n < \omega$. Hence, $rt^{-1} + R \in \bigoplus_{I_\alpha^n}[A_\nu/R] \subseteq \bigoplus_I [A_\nu/R]$ and so $R_{T_{\alpha+1}}/R \subseteq \bigoplus_I [A_\nu/R]$. On the other hand, if

$$x \in \bigoplus_I [A_\nu/R] = \bigcup_n \bigoplus_{I_\alpha^n}[A_\nu/R],$$

then $x \in \bigoplus_{I_\alpha^n}[A_\nu/R]$ for some $n < \omega$, and thus $x \in R_{T_\alpha}^{n+1}/R \subseteq R_{T_{\alpha+1}}/R$. Hence,

$$R_{T_{\alpha+1}}/R = \bigoplus_I [A_\nu/R]$$

is a direct summand of Q/R. Therefore, T_α is defined for every $\alpha < \aleph_1$ and

$$T_0 \leq T_1 \leq \cdots \leq T_\alpha \leq \ldots$$

with $\alpha < \aleph_1$ is a smooth ascending chain of countable submonoids of $R^\times$ such that $R^\times = \bigcup_{\alpha<\aleph_1} T_\alpha$. ☐

References

[1] U. Albrecht, *Right Utumi p.p.-rings,* Rend. Semin. Mat. Univ. Padova 124 (2010), pp. 25–42.

[2] U. Albrecht – J. Dauns – L. Fuchs, *Torsion-freeness and non-singularity over right p.p.-rings,* J. Algebra 285 (2005), no. 1, pp. 98–119.

[3] U. Albrecht – J. Trilifaj, *Cotilting classes of torsion-free modules,* J. Algebra Appl. 5 (2006), no. 6, pp. 747–763.

[4] R. Baer, *Abelian groups without elements of finite order,* Duke Math. J. 3 (1937), no. 1, pp. 68–122.

[5] S. Bazzoni – P. Eklof – J. Trlifaj, *Tilting cotorsion pairs,* Bull. London Math. Soc. 37 (5) (2005), pp. 683-696.

[6] C. Bessenrodt – H. H. Brungs – G. Törner, *Right chain rings,* Part 1, Schriftenreihe des Fachbereichs Mathematik der Universität Duisburg, Heft 74, 1985.

[7] H. H. Brungs, *Three questions on duo rings,* Pacific J. Math. 58 (1975), no. 2, pp. 345–349.

[8] E. Enochs – O. Jenda, *Relative homological algebra,* De Gruyter Expositions in Mathematics, 30. Walter de Gruyter & Co., Berlin, 2000.

[9] L. Fuchs – S. B. Lee, *On modules over commutative rings,* J. Aust. Math. Soc. 103 (2017), no. 3, 341–356.

[10] L. Fuchs – L. Salce, *Modules over non-Noetherian domains,* Mathematical Surveys and Monographs 84, American Mathematical Society, Providence, R.I., 2001.

[11] R. Göbel – J. Trlifaj, *Approximations and endomorphism algebras of modules,* De Gruyter Expositions in Mathematics 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2006.

[12] K. Goodearl, *Ring theory,* Nonsingular rings and modules, Pure and Applied Mathematics 33, Marcel Dekker, New York and Basel, 1976.

[13] A. Hattori, *A foundation of torsion theory for modules over general rings,* Nagoya Math. J. 17 (1960), pp. 147–158.

[14] P. Hill, *The third axiom of countability for Abelian groups,* Proc. Amer. Math. Soc. 82 (1981), no. 3, pp. 347–350.

[15] L. A. Hügel – D. Herbera – J. Trlifaj, *Divisible modules and localization,* J. Algebra 294 (2005), no. 2, pp. 519–551.

[16] J. Jans, *Rings and homology,* Holt, Rinehart, and Winston, New York, 1964.

[17] S. B. Lee, *On divisible modules over domains,* Arch. Math. (Basel) 53 (1989), no. 3, pp, 259–262.

[18] E. Matlis, *Divisible modules,* Proc. Amer. Math. Soc. 11 (1960), pp. 385–391.

[19] J. Rotman, *An introduction to homological algebra,* 1st edition, Pure and Applied Mathematics 85, Academic Press, New York and London, 1979.

[20] J. Rotman, *An introduction to homological algebra,* 2nd edition, Universitext, Springer, New York, 2009.

[21] B. Stenström, *Ring of quotients,* An introduction to methods of ring theory, Die Grundlehren der Mathematischen Wissenschaften 217, Springer-Verlag, Berlin etc., 1975.

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