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CONVEX GREY OPTIMIZATION

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Abstract. Many optimization problems are formulated from a real scenario involving incomplete information due to uncertainty in reality. The uncertainties can be expressed with appropriate probability distributions or fuzzy numbers with a membership function, if enough information can be accessed for the construction of either the probability density function or the membership of the fuzzy numbers. However, in some cases there may not be enough information for that and grey numbers need to be used. A grey number is an interval number to represent the value of a quantity. Its exact value or the likelihood is not known but the maximum and/or the minimum possible values are. Applications in space exploration, robotics and engineering can be mentioned which involves such a scenario. An optimization problem is called a grey optimization problem if it involves a grey number in the objective function and/or constraint set. Unlike its wide applications, not much research is done in the field. Hence, in this paper, a convex grey optimization problem will be discussed. It will be shown that an optimal solution for a convex grey optimization problem is a grey number where the lower and upper limit are computed by solving the problem in an optimistic and pessimistic way. The optimistic way is when the decision maker counts the grey numbers as decision variables and optimize the objective function for all the decision variables whereas the pessimistic way is solving a minimax or maximin problem over the decision variables and over the grey numbers.

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1. Introduction

Decision making is an old practice of humans. Everyday, we deal with different decision making problems in our day to day activities. Deciding what to eat and which route to take to go from one place to another are good examples. In these cases, since there are a limited number of options a decision can easily be made based on the outcome of each options. The increase in the options makes the problem challenging and a systematic solution approach needs to be devised. The application of these problems is not limited in our daily activities but exists in a complex and challenging applications in different studies including management [32], medicine [3], politics [26], engineering [28], transportation studies [37] and agriculture [34].

Once a decision making problem is formulated as a mathematical optimization problem, different solution methods can be used as a solution procedure. These solution methods can depend on the behavior of the problem. Basically, there are two categories of optimization methods, namely deterministic and non-deterministic methods

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[41]. Deterministic solution methods are optimization methods where a mathematical or statistical arguments are used to arrive to the exact optimal solution, however, most non-deterministic algorithms uses different approaches to approximate the optimal solution, this includes metaheuristic algorithms [41].

Decision making problems can also be categorized into categories based on the amount of information known in the problem formulation stage. If the complete information is known then the problem is called deterministic problem otherwise non-deterministic. Most real problems are non-deterministic where complete information is not available. A problem with partial information is called stochastic problem if the partial information is sufficiently enough to construct a probability distribution function to represent the non-deterministic values [6]. Similarly, it is called a fuzzy problem if the values of the problem which lacks complete information has sufficiently enough information to construct a fuzzy membership function so that the non-deterministic values can be represented by a fuzzy number [30]. A stochastic programming or fuzzy optimization methods can be used in these cases. In some cases, there may not be enough information to construct either the probability distribution function or the fuzzy membership function to represent the non-deterministic values of a model. Rather, a simple interval values without probability distribution functions or without membership functions will be available [4], these problems are called grey optimization problems. The former cases has been extensively studied but a limited research has been reported on grey optimization, unlike its wide applications [2, 5, 25, 33, 35, 36, 38–40]. The applications of grey theory can be found in a wide range of disciplines, including economic forecasting, waste management, project management and scheduling [12, 14, 42] (interested readers on the applications can refer [18,23,45]). Even though different methods and approaches are used to deal with grey decision making problems, there is a lack of mathematical analysis and discussion. Recent reviews and solution methods highlighted the importance of further analysis on the existing methods which will give insight for further study and introduction of effective solution methods [8]. Hence, this paper discusses the convex grey optimization problems and its corresponding solution method. Possible extensions and future works are also discussed, hence this paper can be used as a foundation where other researches can be based.

The paper is organized as follows. In the next section a literature review will be presented followed by a discussion on grey number system in Section 3. Section 4 discusses convex grey optimization problems with numerical examples given in Section 5. A discussion with possible future works is given in Section 6. Finally, a conclusion will be given in Section 7.

2. Previous works

Grey system theory was proposed by Julong Deng in early 1980s. It was a new methodology focusing on studies involving missing or limited information [17, 22]. It is an interdisciplinary concept where its application can be used in different disciplines and also bridges the gap between a social and natural science [18].

Grey system theory has been used in different applications. It has been used in economic and currency exchange predictions [7, 13]. It has also been used towards airlines network design with incomplete information [9]. In [15], the growth of Japanese Larch has been formulated and forecasted using grey system theory, specically G(1,1). To predict the number of women committing suicide has been studied as a case study using grey system theory [27]. Many other applications including applications in agriculture, ecology, medicine, history, geography, geology, military, sports, traffic study, environmental science and material science are given in [18].

Unlike its applications, not much study has been done in the analysis and the solution procedures of grey optimization. Most of the research conducted focuses on prediction, to aid a deterministic solution approach by providing preference information and pre optimization tuning and problem formulation with the exception of a number of studies on grey optimization. Grey linear programming is among the main focus of researchers in grey programming [10, 11, 16, 31].

In [16], an optimization problem with grey coefficients in the objective function is discussed. The problem is converted to a multiobjective optimization problem based on the decision makers preference to the values of the grey numbers. However, in [11], rather than converting the problem to multiobjective optimization, multiple simplex methods runs are used from different combination of values of the grey number both in the objective function and the constraint set. A detailed discussion and analysis for the linear case is given in [21]. In addition to the linear grey programming, Liu and Lin [21], discussed, non linear programming as well. They used single values from the interval for each grey number and solve the resulting deterministic problem.

For a generalized grey optimization problem, Zheng and Lewis [46] used a weighted sum of the objective function as well as the constraint function for different sample points representing the grey number from the interval. Hence, it will use the converted problem from grey to white. Since, the approach depends on the values of the selected sample points and the conversion may not be unique.

In [29], linear problems involving grey coefficients have been dealt by what they called a two stage method. In the method they use the end points of the grey numbers and try to find the upper limit and lower limit of the objective function. The method they used is only applicable for linear cases and applications. This paper generalized their result for any convex grey optimization problem.

Recent advances in the solution methods of grey optimization problem can be found in [8]. It is shown that the solution methods proposed are trying to find a single solution. However, this paper is devoted to finding a range of solutions for the convex grey optimization problem.

3. Grey numbers

Definition 3.1. A grey number is a number with clear upper and/or lower boundaries but which has an unknown position within the boundaries [29].

A grey number α can be represented by $\alpha^{\pm}$ with a lower and upper bound of α^- and α^+, respectively, with $\alpha^- < \alpha^+$. If $\alpha^- = \alpha^+$, then the number is no more a grey number but a crisp or white number. A grey number can be represented as follows:

$$\alpha^{\pm} = [\alpha^-, \alpha^+] = \{t \in \Re : \alpha^- < t < \alpha^+\}. \tag{3.1}$$

Definition 3.2. A kernel of a grey number is the central value of the grey number. A kernel, $\hat{\alpha}$, of a continuous grey number $\alpha^{\pm}$ is expressed using equation (3.2):

$$\hat{\alpha} = \frac{\alpha^- + \alpha^+}{2}\cdot \tag{3.2}$$

A grey number is called discrete if it takes discrete values in the interval. For a discrete grey number which takes n values in an interval, say $\alpha_1, \alpha_2, \ldots, \alpha_n$, the kernel is given by [24]:

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} \alpha_i}{n}\cdot \tag{3.3}$$

Definition 3.3. The basic operations on grey numbers, $\alpha^{\pm}$ and $\beta^{\pm}$, are dened as follows [11]:

(1) $\alpha^{\pm} + \beta^{\pm} = [\alpha^- + \beta^-, \alpha^+ + \beta^+]$
(2) $\alpha^{\pm} - \beta^{\pm} = [\alpha^- - \beta^+, \alpha^+ - \beta^-]$
(3) $\alpha^{\pm} \times \beta^{\pm} = [\min\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\}, \max\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\}]$
(4) $\frac{\alpha^{\pm}}{\beta^{\pm}} = [\min\{\frac{\alpha^-}{\beta^-}, \frac{\alpha^-}{\beta^+}, \frac{\alpha^+}{\beta^-}, \frac{\alpha^+}{\beta^+}\}, \max\{\frac{\alpha^-}{\beta^-}, \frac{\alpha^-}{\beta^+}, \frac{\alpha^+}{\beta^-}, \frac{\alpha^+}{\beta^+}\}]$, (provided that $\beta^-, \beta^+ \neq 0$)
(5) ${\alpha^{\pm}}^{-1} = [\frac{1}{\alpha^+}, \frac{1}{\alpha^-}]$, (provided that $\alpha^-, \alpha^+ \neq 0$)
(6) $t\alpha^{\pm} = [\min\{t\alpha^-, t\alpha^+\}, \max\{t\alpha^-, t\alpha^+\}]$, for a real number t.

Definition 3.4. Consider two grey numbers $\alpha^{\pm}$ and $\beta^{\pm}$, based on [43]

(1) They are called non-overlapping if $\alpha^- > \beta^+$ or $\beta^- > \alpha^+$
(2) They are called partially overlapping if $\beta^- \leq \alpha^- \leq \beta^+ < \alpha^+$ or $\alpha^- \leq \beta^- \leq \alpha^+ < \beta^+$
(3) They are called completely overlapping if $\alpha^- \leq \beta^- < \beta^+ \leq \alpha^+$ or $\beta^- \leq \alpha^- < \alpha^+ \leq \beta^+$.

Order relation of an interval number system can be useful for grey numbers as well [19].

Definition 3.5. For two grey numbers $\alpha^{\pm}$ and $\beta^{\pm}$

(1) $\alpha^{\pm} = \beta^{\pm}$ if and only if $\alpha^{-} = \beta^{-}$ and $\alpha^{+} = \beta^{+}$
(2) $\alpha^{\pm} < \beta^{\pm}$ if and only if $\alpha^{+} < \beta^{-}$
(3) $\alpha^{\pm} \leq \beta^{\pm}$ if and only if $\alpha^{-} \leq \beta^{-}$ and $\alpha^{+} \leq \beta^{+}$.

4. Convex grey optimization

4.1. Convex optimization

Definition 4.1. A given set of points S is said to be convex if and only if for any two points in the set, the line joining them is entirely in the set.

In other words, all the points on the line joining two members of a convex set are also a member of the set. That is, if x_1 and x_2 in S, then $tx_1 + (1-t)x_2 \in S$ for $t \in \Re$ and $0 \leq t \leq 1$. Consequently, the linear combination of members of the set is in the set, *i.e.* $\sum_{i=1}^{n} \lambda_i x_i \in S$ for $x_i \in S, \forall i$, for a real number $\lambda_i \in [0,1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. $\sum_{i=1}^{n} \lambda_i x_i$ is called the convex combination of $x_i's$.

Definition 4.2. A real valued function, *i.e.* $\Re^n \mapsto \Re$, is said to be a convex function if the line joining any two points on the graph of $f(x)$ is above or on the graph of $f(x)$.

That is, $f(x)$ is convex if and only if for x_1 and x_2 in the domain of $f(x)$, $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$. If the inequality is strict then it is called strictly convex. If the Hessian matrix of $f(x)$ is positive semidenite then it is convex. In other word, if the eigenvalues of the hessian matrix is non-negative then the function is convex.

Convex optimization problem is an optimization problem where the objective function as well as the feasible region or the constraint set are convex. For a convex optimization problem, if a local solution exists then it is a global solution. In addition, the set of minimum solution is convex. In addition, if the objective function is strictly convex then the optimal solution is unique. It is applied in different areas including in finance, signal processing, wireless networks and trust topology design [20]. Due to its application in wide range of areas, different solution algorithms are proposed and used including subgradient method, cutting plane method and interior method [1].

4.2. Convex grey optimization

A grey optimization problem is an optimization problem which involves grey numbers. A general single objective grey optimization problem can be expressed as given in equation (4.1).

$$\begin{aligned} &\min\ f(x, a^{\pm}) \\ &\text{s.t.}\ \ x \in S = \{x \in \Re^n : G(x, b^{\pm}) \leq 0, \\ &\qquad\qquad\qquad x_{\min} \leq x_i \leq x_{\max}\} \end{aligned} \tag{4.1}$$

where $a^{\pm}$ represents the grey coefficient values of $a_i^{\pm}$'s in the objective function and $b^{\pm}$ represents the grey coefficient values of $b_i^{\pm}$'s in the constraint set S.

Definition 4.3. A grey optimization problem, given in equation (4.1), is convex if and only if

(1) The feasible region S is convex for all values of b in the interval $[b^{-}, b^{+}]$.
(2) The objective function is convex for any value of a in the interval $[a^{-}, a^{+}]$, *i.e.* $f(tx_1 + (1-t)x_2, a) \leq t(f(x_1, a) + (1-t)f(x_2, a), \forall a_i \in [a_i^{-}, a_i^{+}]$ and $x_1, x_2 \in S$.

The optimal solution for a grey optimization problem (GOP) depends on the values of the grey numbers. For different values of the grey numbers, the optimal solution of the problem can be different. It is possible to consider the grey numbers as bounded variables by the minimum and maximum values of the grey number. Hence, the problem can be dealt in the following two ways.

(a) *Optimistic way.* The optimistic way of solving the problem is by assuming that the white value of a grey number will be favorable to the optimization problem. In this case, the decision maker can consider the grey numbers as variables and reformulate the problem as given in equation (4.2).

$$\begin{aligned} &\min\, f_o(x,a,b) \\ &\text{s.t.}\;\; (x,a,b) \in \bar{S} = \{(x,a,b) \in \Re^{n\times n_a\times n_b} : G(x,b) \leq 0, \\ &\qquad x_{\min} \leq x_i \leq x_{\max}, a^- \leq a_i \leq a^+, b^- \leq b_i \leq b^+\}. \end{aligned} \tag{4.2}$$

The solution resulting from this formulation is an ideal solution which is the best possible solution that can be achieved. Let it be represented by (x^o, a^o, b^o).

(b) *Pessimistic way.* The pessimistic way is when trying to minimize the objective function under worst scenario. That is a minimax problem as given in equation (4.3).

$$\begin{aligned} &\min_x \max_{(a,b)}\, f_p(x,a,b) \\ &\text{s.t.}\qquad (x,a,b) \in \bar{S} = \{(x,a,b) \in \Re^{n\times n_a\times n_b} : G(x,b) \leq 0, \\ &\qquad\quad x_{\min} \leq x_i \leq x_{\max}, a^- \leq a_i \leq a^+, b^- \leq b_i \leq b^+\}. \end{aligned} \tag{4.3}$$

This can also be considered as a bi-objective optimization problem and can be expressed using equation (4.4)

$$\begin{aligned} &\min_x\, f(x,a,b) \\ &\text{s.t.}\;\; (x,a,b) \in \bar{S} \\ &\qquad \max_{(a,b)} f(x,a,b) \\ &\qquad \text{s.t.}\qquad (x,a,b) \in \bar{S}. \end{aligned} \tag{4.4}$$

Let the resulting solution from this solution be called pessimistic solution and represent by (x^p, a^p, b^p).

Theorem 4.4. *The value of the objective function for a fixed value of the grey numbers is bounded by the objective functions of the optimistic and pessimistic values. That is,* $f(x^o, a^o, b^o) \leq f(x|_{(a,b)}, a, b) \leq f(x^p, a^p, b^p)$ *where* $x|_{(a,b)}$ *is the minimizer of* f *for fixed values of* a *and* b*.*

Proof. Since the optimistic solution is the ideal solution, $f(x^o, a^o, b^o) \leq f(x, a, b)$, for any x, a and b. That includes $x|_{(a,b)}$ for any a and b.

Hence, we only need to show $f(x|_{(a,b)}, a, b) \leq f(x^p, a^p, b^p)$. Suppose it is not true, *i.e.* $f(x|_{(a,b)}, a, b) > f(x^p, a^p, b^p)$.

But that is in contradiction with minimax solution being (x^p, a^p, b^p).

Hence, $f(x^o, a^o, b^o) \leq f(x|_{(a,b)}, a, b) \leq f(x^p, a^p, b^p)$. □

Hence, the resulting value for the objective function can be expressed by a grey number $f^{\pm} = [f^-, f^+]$, where $f^- = f(x^o, a^o, b^o)$ and $f^+ = f(x^p, a^p, b^p)$.

Theorem 4.5. *For a convex grey optimization problem, the solution is a grey number given by* $x^{*\pm}$ *where* $x_i^{*-} = \min\{x_i^o, x_i^p\}$ *and* $x_i^{*+} = \max\{x_i^o, x_i^p\}$*.*

Proof. Since x^o is the ideal solution and using theorem 1, we have $f(x^o) \leq f(x|_{(a,b)}) \leq f(x^p)$. There are two cases to consider.

(1) **Case 1:** The sign of $f(x^o)$ and $f(x^p)$ are the same, *i.e.* consider both are positive. If both are negative we can multiply the system by negative in order to make both positive.
There is a sufficiently large $t \leq 1$ such that $f(x|_{(a,b)}) \leq tf(x^p)$.
$\Rightarrow f(x|_{(a,b)}) \leq tf(x^p) + (1-t)f(x^o)$
Hence, $x|_{(a,b)}$ is in between x^o and x^p. (Since f is convex)

(2) **Case 2:** The sign $f(x^o)$ and $f(x^p)$ are different. Here there are two cases where $x|_{(a,b)}$ is either positive or not. If it is negative by multiplying the system by negative one it is possible to convert to a form where $x|_{(a,b)} \leq 0$. Hence let us consider the case where $f(x|_{(a,b)}) \leq 0$ and $f(x^p) \geq 0$.
There exist a sufficiently large t with $0 \leq t \leq 1$ for which $f(x^o) \leq f(x|_{(a,b)}) + (t-1)f(x^o)$. Since both $f(x^o)$ and $f(x|_{(a,b)})$ are not positive we have $f(x^o) \leq f(x|_{(a,b)}) + (t-1)f(x^o) \leq 0 \leq tf(x^p)$
$\Rightarrow f(x|_{(a,b)}) \leq tf(x^p) + (1-t)f(x^o)$
Hence $x|_{(a,b)}$ is in between x^o and x^p.
Hence, the optimal solution for fixed (a, b) is in between the optimistic and pessimistic value provided that the grey optimization problem is convex.

□

5. Numerical examples

Even though the proposed theorems are proved in the previous section, for the sake of completeness some examples are given below to elaborate the idea.

Example 5.1. The first example is taken from [46]. It is given in equation (5.1).

$$\min\ f(x) = x_1^2 + (1+a^{\pm})x_2^2 - \frac{1}{4}(a^{\pm})^2 x_1 - 4x_2 + 1 \tag{5.1}$$

for $a = [a^-, a^+] = [0, 4]$.

The hessian matrix of the objective function is $H = \begin{pmatrix} 2 & 0 \\ 0 & 2+2a \end{pmatrix}$ and it is positive definite. Hence the optimization problem given is convex.

The equivalent optimistic model can be given by:

$$\begin{aligned} &\min_{(x,a)} f_o(x,a) = x_1^2 + (1+a)x_2^2 - \tfrac{1}{4}(a)^2 x_1 - 4x_2 + 1 \\ &\text{s.t.}\quad x \in \Re^2, \\ &\qquad\ \ 0 \leq a \leq 4. \end{aligned} \tag{5.2}$$

The optimal solution for the optimistic problem is $x = (0.4735, 0.6788)$ with $a = 1.9463$. Hence, the optimistic functional value is -0.5819.

The pessimistic equivalent problem is given by:

$$\begin{aligned} &\min_{(x)} g_1(x,a) = x_1^2 + (1+a)x_2^2 - \tfrac{1}{4}(a)^2 x_1 - 4x_2 + 1 \\ &\text{s.t.}\quad \text{a solves} \\ &\qquad \max_{a} g_2(x,a) = x_1^2 + (1+a)x_2^2 - \tfrac{1}{4}(a)^2 x_1 - 4x_2 + 1 \\ &\qquad x \in \Re^2 \\ &\qquad 0 \leq a \leq 4. \end{aligned} \tag{5.3}$$

By solving the leaders problem, *i.e.* the minimization problem, the parametric solution with parameter a is $x = (\frac{a^2}{8}, \frac{2}{a+1})$. Hence, the followers, *i.e.* the maximization, problems objective function can be reconstructed by substituting the value of x from the parametric solution. Hence, the resulting problem becomes:

$$\begin{aligned} &\max_{(a)} \tfrac{-a^5 - a^4 + 64a - 192}{64a + 64} \\ &\text{s.t.}\quad a \in [0, 4]. \end{aligned} \tag{5.4}$$

The maximizer of the above problem is $a = 1.9463$. Hence the pessimistic solution is given by $x^p = (0.4735, 0.6788)$ with a functional value of -0.5819.

Hence, the optimal value of the objective function is $f^{\pm} = [-0.5819, -0.5819]$ with optimal grey solution of $x^{*\pm}$. where $x^{*-} = [0.4735, 6788]$ and $x^{*+} = [0.4735, 0.6788]$.

Example 5.2. The second problem is given as follows:

$$\begin{aligned} \min_{(x,a)}\ & f(x,a) = x_1^2 - a_1^{\pm} x_2^2 + a_2^{\pm} x_3^2 + x_1 x_3 + a_3^{\pm} \\ \text{s.t.}\ & x \in \Re^3 \\ & x_i \geq 0 \end{aligned} \tag{5.5}$$

where $[a_1^-, a_1^+] = [-5, -2]$, $[a_2^-, a_2^+] = [2.2, 4.2]$ and $[a_3^-, a_3^+] = [-1, 2]$.

The eigenvalues of the hessian matrix of $f = x_1^2 - a_1 x_2^2 + a_2 x_3^2 + x_1 x_3 + a_3$ are $\lambda_1 = -2a_1$, $\lambda_2 = a_2 + 1 + \sqrt{1 + (a_2 - 1)^2}$ and $\lambda_3 = a_2 + 1 - \sqrt{1 + (a_2 - 1)^2}$. The values are real. Furthermore, $\lambda_3 > 0$ if and only if $a_2 > \frac{1}{4}$. Hence, the problem is a convex problem with positive eigenvalues.

The optimistic corresponding problem is given as:

$$\begin{aligned} \min_{(x,a)}\ & f(x,a) = x_1^2 - a_1 x_2^2 + a_2 x_3^2 + x_1 x_3 + a_3 \\ \text{s.t.}\ & x \in \Re^3 \\ & x_i \geq 0 \\ & -5 \leq a_1 \leq -2 \\ & 2.2 \leq a_2 \leq 4.2 \\ & -1 \leq a_3 \leq 1. \end{aligned} \tag{5.6}$$

The corresponding solution is $(x_1, x_2, x_3, a_1, a_2, a_3) = (0, 0, 0, a_1, a_2, 1)$ with a functional value of -1. The solution is valid for any values of a_1 and a_2.

Since the problem is separable for variable a_3 and the other variables, *i.e.* $f(x,a) = g(x, a_1, a_2) + h(a_3)$, the pessimistic problem can be expressed by:

$$\begin{aligned} \min_{x}\ & g_1 = x_1^2 - a_1 x_2^2 + a_2 x_3^2 + x_1 x_3 + a_3 \\ \text{s.t.}\ & \text{a solves} \\ & \max_{a}\ g_2 = a_3 \\ & x \in \Re^2 \\ & -5 \leq a_1 \leq -2 \\ & 2.2 \leq a_2 \leq 4.2 \\ & -1 \leq a_3 \leq 1. \end{aligned} \tag{5.7}$$

The corresponding solution will be $(x_1, x_2, x_3, a_1, a_2, a_3) = (0, 0, 0, a_1, a_2, 1)$ with objective function value of 1.

Hence, $f^{\pm} = [-1, 1]$ with the optimal solution of the problem being $x_i^{*\pm} = [0, 0] = 0$

Example 5.3. The third problem is given as follows:

$$\begin{aligned} \min_{x}\ & f(x) = 2x_3 - x_1 x_3 - x_1 x_2 - a_2^{\pm} x_2 - x_1 \mathrm{e}^{a_1^{\pm} x_2} \\ \text{s.t.}\ & x \in \Re^3 \\ & x_1 \leq 0 \end{aligned} \tag{5.8}$$

where $x_1 > 0, [a_1^-, a_1^+] = [1, 5]$ and $[a_2^-, a_2^+] = [-4, -1]$.

$H = \begin{pmatrix} 0 & -1 - a_1 \mathrm{e}^{a_1 x_2} & -1 \\ -1 - a_1 \mathrm{e}^{a_1 x_2} & -a_1^2 x_1 \mathrm{e}^{a_1 x_2} & 0 \\ -1 & 0 & 0 \end{pmatrix}$ is the hessian matrix of the objective function f is and is positive definite. Hence it is a convex problem in the feasible region, *i.e.* for all possible values of a_i's.

The optimistic corresponding problem is given as:

$$\begin{array}{ll} \min\limits_{(x,a)} & f_o = 2x_3 - x_1x_3 - x_1x_2 - a_2x_2 - x_1 \mathrm{e}^{a_1x_2} \\ \text{s.t.} & x \in \Re^3 \\ & x_1 > 0 \\ & 1 \leq a_1 \leq 5 \\ & -4 \leq a_2 \leq -1 \end{array} \tag{5.9}$$

The corresponding solution is $(x_1, x_2, x_3, a_1, a_2) = (2, 0, -1, a_1, -2 - a_1)$ for $1 \leq a_1 \leq 2$ with a functional value of -2.

The pessimistic version of the problem can be formulated by finding the parametric solution of the problem in terms of a_i's. Hence, the solution becomes, $(x_1, x_2, x_3) = (2, \frac{1}{a_1}\ln(\frac{-2-a_2}{2a_1}), \frac{2+a_2}{2a_1} - \frac{1}{a_1}\ln(\frac{-2-a_2}{2a_1}))$. Note that a new restriction which is $-2 - a_2 > 0$ needs to be added, *i.e.* $a_2 \leq -2.0001$. By substituting these values the pessimistic problem can be given by:

$$\begin{array}{ll} \min\limits_{x} & g_1(x,a) = 2x_3 - x_1x_3 - x_1x_2 - a_2x_2 - x_1\mathrm{e}^{a_1x_2} \\ \text{s.t.} & \text{a solves} \\ & \max\limits_{a} g_2(a) = \frac{2+a_2}{a_1}(1 - \ln(\frac{-2-a_2}{2a_1})) \\ & x \in \Re^3 \\ & x_1 > 0 \\ & 1 \leq a_1 \leq 5 \\ & -4 \leq a_2 \leq -2.0001. \end{array} \tag{5.10}$$

Solving the follower's problem will give a solution of $(a_1, a_2) = (5, -2.0001)$. Hence the pessimistic solution becomes (2, −2.3026, 2.3026, 5, −2.0001) with objective function value of −0.0003.

Hence, $f^{\pm} = [-2, 0.0003]$ with the optimal solution of the problem being $x^{*+} = [2, 0, 2.3026]$ and $x^{*-} = [2, -2.3026, 1]$.

Example 5.4. The fourth problem is given as follows:

$$\begin{array}{ll} \min\limits_{x} & f(x) = x_1x_3 - a_1^{\pm}\sin(x_2) + a_2^{\pm}x_3^2 - x_1^2 - a^{\pm}b^{\pm} + x_1 \\ \text{s.t.} & x \in \Re^3 \end{array} \tag{5.11}$$

where $[a_1^-, a_1^+] = [-5, -1]$ and $[a_2^-, a_2^+] = [0, 2]$.

$H = \begin{pmatrix} -2 & 0 & 1 \\ 0 & a_1\sin(x_2) & 0 \\ 1 & 0 & 2a_2 \end{pmatrix}$ is the hessian matrix of the objective function f and is positive definite under the given constraint set. Hence it is a convex problem in the feasible region, *i.e.* for all possible values of a_i's.

The optimistic corresponding problem is given as:

$$\begin{array}{ll} \min\limits_{x,a} & f(x,a) = x_1x_3 - a_1^{\pm}\sin(x_2) + a_2^{\pm}x_3^2 - x_1^2 - a^{\pm}b^{\pm} + x_1 \\ \text{s.t.} & x \in \Re^3 \\ & a_1 \in [-5, -1] \\ & a_2 \in [0, 2]. \end{array} \tag{5.12}$$

The corresponding solution is $(x_1, x_2, x_3) = (\frac{2}{3}, \frac{\pi}{2}, \frac{1}{3})$ for $a = (\frac{1}{9}, -1)$'s, with a functional value of 0.3333.

The pessimistic version of the problem can be formulated by finding the parametric solution of the problem in terms of a_i's. Hence, the solution becomes, $(x_1, x_2, x_3) = (\frac{2a_2}{1+4a_2}, \frac{\pi}{2}, -\frac{1}{1+4a_2})$. By substituting these values the pessimistic problem can be given by:

$$\begin{array}{ll}\min\limits_{x} & g_1(x,a) = x_1x_3 - a_1\sin(x_2) + a_2x_3^2 - x_1^2 - ab + x_1 \\ \text{s.t.} & \text{a solves} \\ & \max\limits_{a} g_2(a) = \frac{a_2}{4a_2+1} - a_1(a_2+1) \\ & x \in \Re^3 \\ & a_1 \in [-5,-1] \\ & a_2 \in [0,2]. \end{array} \tag{5.13}$$

Solving the follower's problem will give a solution of $(a_1, a_2) = (-5, 2)$. Hence the pessimistic solution becomes $(\frac{4}{9}, \frac{\pi}{2}, -\frac{1}{9}, -5, 2)$ with objective function value of 5.0124.

Hence, $f^{\pm} = [0.3333, 5.0124]$ with the optimal solution of the problem being $x^{*+} = [0.6667, \frac{\pi}{2}, 0.3333]$ and $x^{*-} = [0.4444, \frac{\pi}{2}, -0.1111]$.

6. Discussion

The study of grey numbers and system has been introduced in early 1980's. With the wide applications of the theory, it has been used in different applications. However, grey optimization is not explored enough. Recent studies showed that the study conducted in this regard is very limited. Studying different categories of grey optimization will help to solve problems from different grey optimization categories and may also help to develop usable dynamic algorithms for generalized grey optimization problem.

In this paper, a convex grey optimization problem is discussed along with corresponding reformulation of the problem into two equivalent deterministic problems, called the optimistic and pessimistic versions. Solution methods proposed for deterministic problems can be used to produce a grey solution for the original grey problem by solving these two deterministic problems. Numerical examples are given to demonstrate the idea.

The proposed approach and discussion is based on convex grey optimization problem. Hence, one possible future work is to relax this assumption and the study of non-convex problems. Research questions in this regard can be; will the result hold if only either the objective function or the feasible region is convex? if both are not convex?

There are not well developed benchmark problems for grey optimization. Therefore, another possible future work is developing benchmark problems which can be used to test the effectiveness of different approaches.

In this paper, a pure grey optimization problem is discussed. That is the uncertain values are all represented by a grey number. In some cases it is possible to find a mixed problems of grey-fuzzy or grey-stochastic problems, where some of the uncertain values are grey whereas the others are fuzzy or stochastic. How to deal with this cases is another issue for possible future works.

Dealing with applications where the uncertain or level of the missing information reduces through time is another interesting future work. Possibly, a dynamic way of adjustment to the problem as well as the solution approach can be studied. With the new information a grey optimization problem can be grey-fuzzy or grey-stochastic problem. Studying how that affect the solution found in the original grey optimization problem can be studied further.

Testing proposed approaches, finding their strength and weakness using real problems as well as mathematical benchmark problems is another possible future work. This can be very useful not only for testing the algorithms but also to determine which methods to use in different cases.

7. Conclusion

A grey optimization problem is an optimization problem where some of the coefficients are grey numbers. The paper discussed two approaches to deal with the problem, namely, the optimistic and pessimistic approach. The optimistic approach is when the grey number is also considered as a decision variable and the problem is solved in one level. Here, the decision maker assumes optimistically that the best values for the grey numbers can be used to compute the optimal solution. The second model is a pessimistic model. Here the decision maker tries to minimize the objective function by assuming worst values for the grey numbers. Hence, a minimax model (or a bi-level programming approach) is used to deal with the pessimistic model. It is also proved that for a convex grey optimization problem the optimal solution is bounded by the optimistic and pessimistic solutions, so does the objective function values. Four problems, one taken from a literature, are used to demonstrate the proposed method.

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