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AN INTEGRATED INVENTORY MODEL INVOLVING DISCRETE SETUP COST REDUCTION, VARIABLE SAFETY FACTOR, SELLING PRICE DEPENDENT DEMAND, AND INVESTMENT

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Abstract. This paper develops a sustainable integrated inventory model for maximizing profit with a controllable lead time, discrete setup cost reduction, and consideration of environmental issues. Contrary to the available literature, this paper considers a discrete setup cost for the vendor, thus making the integrated model sustainable. The customer's demand is assumed to be selling-price dependent to increase the number of sales, and the lead time demand follows a Poisson distribution. The integrated model is used to optimized the total shipment number, volume of shipments, safety factor, investments, selling-price, and probability of moving between the *"in-control"* to *"out-of-control"* states. An algorithm is developed to obtain the numerical results. Numerical examples and sensitivity analyses are given to illustrate the model.

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1. Introduction

In the real world each business industry wants more profit with a smaller production cost. Customer satisfaction is another big issue for any industry, and for that the quality of the product should also be improved. A huge amount of demand for any product results is more profit for the companies and more shortages may also occur due to the large demand. As a result, shortages are a great issue for customer satisfaction, for which safety stocks may provide an answer. A small investment can increases the quality of a product such as through brand image which is beneficial to any industry such approaches can also improve the total profit. Until now a basic economic production quantity (EPQ) models have been more efficient for integrated models by considering the setup cost as constant (see Ref. [32]). One such model Sarkar *et al.* [32] was developed by considering partial backlogging as where Cárdenas-Barrón [6] investigated two types of backorder costs for economic order quantity (EOQ) and EPQ inventory models which were solved geometrically and algebraically. The setup cost for any production system can be reduced in the inventory model first discussed by Porteus [23]. He showed that the

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setup cost is not always constant, and that some continuous investment can be used to reduce the setup cost. In this direction Wee *et al.* [40] developed an alternative EPQ model, by considering planned backorders for a single-stage production process with reworking. Quality improvement of products has always had a great impact in for any industry, and based on this, Ouyang *et al.* [15] introduced quality improvements with a reduced setup cost in an integrated inventory model, where he developed two models, a model with a normal distributed lead time demand and another model without any distribution for the lead time demand. For both cases, they proved that continuous investment to reduce the setup cost and improve the quality are much more effective than a constant setup cost or fixed quality level. They proved that such a model improved the costs with respect to a constant backorder rate, which was extended by Sarkar and Moon [29] by introducing a lead time dependent variable backorder rate. Similar to Ouyang *et al.* [15], Sarkar and Moon [29] proved that by implementing continuous investments, setup cost reduction and quality improvement can both be easily achieved. In the same direction, many researchers developed various different models. Pal *et al.* ([19, 20]) incorporated strategies for two-echelon supply chains with improvement of quality where, the cost for production was decreasing. They also considered non-linear demand for their models [20]. Annadurai and Uthaykumar [2] discussed setup cost reduction for an integrated inventory model with lost sales reduction. The length of the credit period for the seller in an integrated inventory model was considered by Abad and Jaggi [1], and they also set unit price. In this model they further considered the end demand as price sensitive. A Single Setup Multi Delivery (SSMD) policy that was more effective for a two-echelon supply chain model was developed by Sarkar and Majumder [28]. They also discussed setup cost reduction in their model. Sarkar *et al.* [37] reduced the setup cost for a two-echelon supply chain model and in this model they compare two delivery strategies SSSD and SSMD with improve quality of the product. Sarkar and Mahapatra [27] extended the model of Annadurai and Uthaykumar [2] by considering a fuzzy demand, demonstrating the efficiency of setup cost reduction and quality improvement. In this direction Shah and Gor [38] developed an integrated economic lot-size model, where the input was random. In this model the buyer's cost increased due to the random input but the total cost was optimized in the presence of trade credit. Considering stock-out conditions, Pal and Adhikari [16] introduced an inventory model, where the production was imperfect and price sensitive. Taleizadeh *et al.* [39] expanded an EOQ model by including partial backorder and a special pricing. A new approach with two cycles in the two-echelon supply chain model was introduced by Pal *et al.* [18] , where production was imperfect and demand was price dependent. In this model, they considered two cycles. In the first cycle, all good quality products are sold at the original price, and in the second cycle those products are instead sold at some discount. The setup cost for an inventory model can be reduced by considering service level constraints (see Ref. [34]). An integrated inventory model was extended by Sarkar *et al.* [35] *via* continuous setup cost reduction, unequal shipment sizes and on SSMD policy, where carbon emission costs are also considered to maintain the environmental effects. More recently Pal *et al.* [21] developed an integrated supply chain model where the demand depends on the price and credit period. In this model they considered two rivaling retailers and a common buyer. However the, above mentioned authors did not consider discrete setup cost reduction, where realistically improvement that always continuous may not be needed, sometimes less or more investment is needed. Based on this strategy Huang *et al.* [11] introduced discrete setup cost investment to reduce the setup cost in a basic model. The proposed model utilizes this idea of discrete investment to reduce the setup cost in an improved integrated inventory model.

It is quite natural that an SSMD policy is always convergent over an SSSD (single-setup-single-delivery) policy, when the buyer's holding cost is more than the vendor's holding cost (Goyal [9]). There is a trade-off between the increased transportation costs of the vendor and the reduced holding costs of the buyer. Based on this policy Kim and Ha [10] implemented an SSMD policy with an improved cost saving policy when the buyer's holding cost is more than that of the vendor. Yang and Wee [41] extended this concept *via* a three stage inspection process in an improved integrated inventory model. Sarkar *et al.* [36] introduced the concept of fixed and variable carbon emission costs, where they extended an SSMD policy by implementing a trade-off between the buyer's reduced holding cost and the vendor's increased carbon emission cost. Sarkar *et al.* [36] extended Ben-Daya's [5] model with a fixed and variable transportation cost and carbon emission cost under an extended SSMD policy. The proposed model uses this same idea for the sustainable integrated inventory model.

The lead time is the delay between initiation and implementation of the process. In reality, lead time reduction is important to the manufacturing system. Chang and Lo [7] considered inventory models with continuous and discrete lead times, backorders and lost sales. They suggested several policies with continuous and discrete lead times in order to improve upon traditional policies. Johansen and Thorstenson [12] developed an optimal and approximate (Q, r) inventory including lost sales and a gamma-distributed lead time, considering a poisson process for the order quantity and reorder point. Bijvank and Johansen [4] investigated a periodic review lost-sales inventory model with compound poisson demand and constant lead times of various lengths. They developed limited base-stock policies under several situations. Pang and Chen [22] introduced the coordinated pricing and inventory control with batch production, as well as Erlang and Poisson distributed demand, to determine the optimal inventory-pricing policy and maximize the total profit. Many researchers have assumed the ordering cost to be constant but this is not accurate. Recently Kim and Sarkar [14] developed a multi-stage cleaner production where the ordering cost was dependent on the lead time thus improving the quality of the product. Transportation in any production system has a large impact. Considering this, Deyi and Xiaoqian [8] optimised a transportation strategy using a stochastic programming.

Constant demand is a business service that helps customers find new customers and penetrate new markets. This can optimize the inside sales and marketing activities to achieve high quality sales. Variable demand can predict and quantify changes that are caused by transport conditions for the demand. Since a high selling price negatively affect how likely clients are to buy products or services, assuming the demand to be selling-price-dependent is more realistic. Karaoz *et al.* [13] considered an EOQ model with price and time dependent demand under the influence of complement and substitute product's selling-prices. The finite replenishment inventory model was developed by considering that the demand was sensitive to change in the time and selling-price.

Sana [24] introduced the price-sensitive demand for perishable items in an inventory model. The finite time deterministic EOQ model was considered in this model, where the of demand decreased with selling-price. Trade credits and price-discounts are a very good policy in any production system and based on this Sarkar *et al.* [31] investigated an EOQ model, where replenishment was finite and then optimised the profit. The demand for any inventory system is not always constant, and may depend on time, selling price, inventory etc., Sarkar and Sarkar [30] established an inventory model where the demand was inventory dependent and an algorithm was developed to maximize the profit. To maximize the vendor's profit, the optimal ordering quantity and selling-price were optimized *via* an analytical procedure. Pal *et al.* [17] developed a multi-item inventory model, where the demand was sensitive to the selling-price and price-break. The objective of the model proposed here is to determine the optimal order quantities and selling-prices to maximize the profit of the vendor. An integrated model with developed lead time and production rate was discussed by Azadeh and Paknafs [3]. Sarkar *et al.* [33] developed an EMQ model with price and time dependent demand under the effects of reliability and inflation.

Zhang *et al.* [42] considered a deterministic inventory with partial backordering and correlated demand resulting from cross-selling. In this study, the cross-selling effects were presented in fractional contexts of different products. Due to cross-selling, a two-item inventory model was proposed where the demand for minor items was related to that for major items. An optimal policy was also considered in this paper. Many of the previous works considered linear backorders, whereas this paper considers the basic concepts of analytic geometry and algebra. The optimal lot size and backorders are both linear and fixed backorders costs can be determine from the proposed method. Sarkar [26] developed a two-echelon supply chain model with backordering and three-stage inspection using an algebraic procedure. Sarkar [25] developed an integrated inventory model with discount offers from the vendor to buyer and a variable backorder rate, which was solved *via* an algebraic approach. In that study, the author considered a selling price dependent demand with shortages in an EOQ model, but a variable safety factor with a reduced setup cost and improved product quality that considers the effects on the environment has not yet been considered.

Major contributions of various author(s) are given in Table 1.

This paper is arranged as follows: the problem definition, notation, and assumptions are developed in Section 2. The mathematical model is formulated in Section 3, and the solution methodology is described in Section 4. In Section 5 the numerical examples are illustrated and sensitivity analysis is shown in Section 6. Some managerial insights are given in next section. Finally, conclusions are given in Section 8.

Table 1. Author(s) contributions table.

Author(s)	Setup cost reduction	Poisson distributed lead time	Variable safety factor	Selling price dependent demand	Shortage	EOQ
Porteus [23]	√					
Ouyang *et al.* [15]	√					√
Sarkar and Moon [29]						√
Sarkar *et al.* [34]	√					√
Huang *et al.* [11]		√			√	√
Cheng and Lo [7]			√			
Pang and Chen [22]		√				
Sana [24]				√	√	√
Pal *et al.* [17]				√		√
Sarkar *et al.* [33]			√	√	√	
Zhang *et al.* [42]					√	√
Cárdenas-Barrón [6]					√	√
This model	√	√	√	√	√	√

2. Problem definition, notation, and assumptions

In this section the problem definition is discussed along with notation and assumptions.

2.1. Problem definition

The aim of the study is to construct a sustainable integrated inventory model by simultaneously optimizing the total shipment number, volume of shipments, safety factor, investments, selling-price, and the probabilities of moving between the *"in-control"* to *"out-of-control"* states ultimately optimizing the profit. An effort is made to reduce the setup cost of the vendor. Thus, a single-setup multi-delivery (SSMD) policy is implemented for transportation of the products from the vendor to the buyer, where there is a trade-up between buyer's holding cost and the vendor's transportation as well as carbon emission costs. As the demand during the lead time follows a Poisson distribution, a lead time crashing cost is used to reduce the lead time. Thus, a sustainable integrated inventory model will continue forever if the integrated profit is optimized subject to optimum values of the decision variable.

2.2. Notation

The following notation and assumptions are used for this model.

2.2.1. Index

- i component for lead time with minimum duration ($i = 1, 2, \ldots, n$)
- j component for lead time with normal duration ($j = 1, 2, \ldots, n$)

2.2.2. Decision variables

- B each shipment's volume (unit)
- k safety factor (\$/week)
- L lead time (weeks)
- P selling-price (\$/unit)
- θ probability that the production process will change to the *out-of-control* state

2.2.3. Parameters

n	total shipment per lot, a positive integer
Q	lot size (units)
A	investment for reducing setup cost S ($)
Y_i	ith demand quantity, $i = 1, 2, \dots, n$ (units)
D	buyer's demand rate (units)
R	vendor's production rate, $R > D$ (units/ time)
T	cycle time, $T = Q/D = nB/D$ (time)
S_0	initial setup cost per production run ($/setup)
S	setup cost, which is necessarily a decreasing function of A, with $S(A) = S_0 e^{-rA}$, where r is a parameter ($/unit)
h	inventory holding cost per unit per unit time ($/unit/unit time)
C_v	vendor's unit production cost per unit ($/unit)
C_b	buyer's unit purchase cost per unit ($/unit purchased)
π	unit shortage cost per unit ($/unit shortage)
φ	standard normal probability density function
α	annual fractional cost for the capital investment ($/invetment)
β	price elasticity parameter
Φ	standard normal cumulative distribution function
$N(L)$	consumer's total number during lead time L
$X(L)$	total amount purchased by time L
$C(L)$	total crashing cost is related to the lead time ($/week)
u_i	ith component of lead time with u_i as minimum duration (days)
v_i	ith component of lead time with v_i as normal duration (days)
O_c	ordering cost for buyer ($/unit)
C_{fcb}	fixed carbon emission cost of consumer ($/shipment)
C_{vcb}	variable carbon emission cost of consumer ($/unit)
C_{fcv}	fixed carbon emission cost of vendor ($/shipment)
Y_v	variable carbon emission cost of vendor ($/unit)
F	fixed transportation cost ($/shipment)
V	variable transportation cost ($/unit)

2.3. Assumptions

The following assumptions are considered when developing this model.

1. This paper assumes a single-vendor single-buyer integrated-inventory model with single types of products.
2. In the literature, many production models consider a constant setup cost, but it is possible to reduce the setup cost using a continuous investment function (see Ref. [29]). However if the setup cost is reduced during every setup, a continuous investment is not needed. Thus, a discrete investment function is used to reduce the setup cost (see Ref. [11]) specifically, $S(A_i) = S_0 e^{-rA}$, where $i = 0, 1, \dots, n$ and $A_0 = 0$.
3. The total number of consumers during the lead time $[N(L)]$ follows a poisson distribution with mean λL. The Y_i are independent and identically normally distributed with mean μ and standard deviation σ, and, then $\{X(L) = \sum_{(i=1)}^{N(L)} Y_i, L \geq 0\}$ is a compound poisson process (Huang *et al.* [11]).
4. The reorder point is equal to the sum of the expected demand during the lead time and the safety stock, *i.e.*, $RP = DL + k\sigma\sqrt{L}$.
5. In a long-run system, the process changes to the *out-of-control* state from the *in-control* state, and as a result, defective items are produced, which need to be improved *via* an investment function.

6. Demand is assumed to depend on the selling-price of products as $D = \alpha P^{-\beta}$.
7. Shortages are permitted and fully backordered.
8. The lead time L has n mutually independent components and to reduce the lead time each component has a different crashing cost. Let v_i be the normal duration of the ith component and u_i be the minimum duration where the crashing costs per unit time m_i satisfy $m_1 \leq m_2 \leq \ldots \leq m_n$. Suppose L_i is the length of the lead time for which components $1, 2, 3, \ldots, i$ crash to their minimum duration, and let $L_0 = \sum_{j=1}^{n} v_j$. Then $L_i = L_0 - \sum_{j=1}^{i}(v_j - u_j)$ and the crashing cost $C(L)$ can be written as $C(L) = m_i(L_i - L) + \sum_{j=1}^{i}(v_j - u_j)$ for $i = 1, 2, \ldots, n$.
9. The model follows a single-setup multi-delivery (SSMD) policy for transportation. The reason for using SSMD is that the total supply chain has a reduced cost since the holding cost of the buyer is larger than the holding cost of the vendor. Therefore, the vendor produces items in a single-lot but delivers using a in multi delivery system. Due to the increasing number of deliveries, the transportation cost increases. Thus, there is a trade-off between the buyer's holding cost and the increased transportation cost. This model considers fixed and variable transportation costs as well as carbon emission costs to make the model realistic.

3. Mathematical model

The derivation of the integrated inventory model is presented in this section. The inventory level is depicted in Figures 1 and 2 for the buyer and vendor respectively. Based on the conclusions of Sarkar [26], cooperative integrated inventory models result in more profit than non-cooperative integrated models, thus the proposed model considers a cooperative integrated inventory model. For this purpose, one needs to calculate the profit for the buyer and vendor as follows.

3.1. Buyer's profit

The vendor produced items according to a finite production rate $(R > D)$ at a single step for the buyer's order lot size Q, and deliver a total of n items with a shipment size B.

Thus, on the vendor's side, the setup cost and the inventory cost are reduced, because the lot is shipped to the buyer right after manufacturing the lot size B.

According to assumption 3, the demand during the lead time is

$$E(X(L)) = E\left(X_1 + X_2 + \cdots + X_{N(L)}\right) E(X_i) = \lambda L u$$

and the variance is

$$\begin{aligned} Var(X) &= [E(X_i)]^2 \, Var(N(L)) + E((L))Var(X_i) \\ &= u^2 \lambda L + \lambda L \sigma^2 \\ &= \lambda L(u^2 + \sigma^2). \end{aligned}$$

Hence, $RP = \lambda u L + k\Sigma\sqrt{\lambda L}$ is the reader point, where $\Sigma^2 = u^2 + \sigma^2$. According to Ouyang's *et al.* [15] model the shortage per replenishment cycle is $C(r) = \Sigma\sqrt{\lambda L}\Psi(k)$, where, $\Psi(k) = \phi(k) - k[1 - \Phi(k)]$ $\left(\frac{B}{2}\right) + k\Sigma\sqrt{\lambda L}$ is the on-hand inventory average for the buyer. Therefore, $hC_B\left(\left(\frac{B}{2}\right) + k\Sigma\sqrt{\lambda L}\right)$ is the holding cost per year.

3.1.1. Ordering cost

In any vendor-buyer production model, the first thing the buyer does is order products from the vendor. To place any order with the vendor, there must be some associted cost called the ordering cost. The ordering cost for the buyer in this integrated model is as follows

$$\frac{O_c \alpha P^{-\beta}}{B}.$$

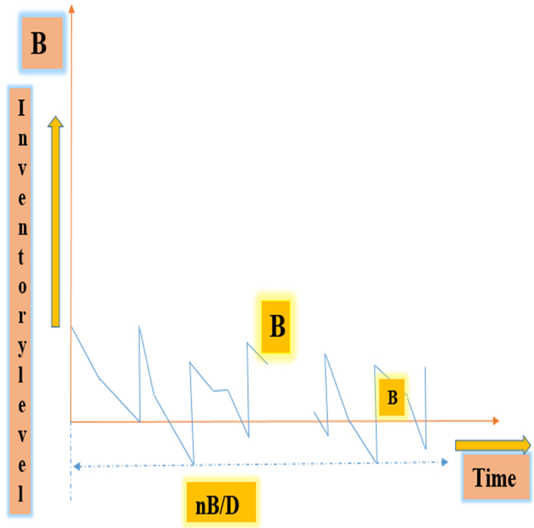

FIGURE 1. Inventory level for the buyer.

3.1.2. Holding cost

After placing the order, one must consider, where the ordered products to be stored, for that a place or a store or a warehouse is needed which may be own or may be rented. There is always some cost associated with the warehouse is needed even if it is owned, and if it is rented then direct cash is needed. So the average inventory for the buyer is as follows

$$\left[\frac{B}{2} + k\varSigma\sqrt{\lambda L}\right]$$

and the predicted holding cost for the buyer is

$$hC_B\left[\frac{B}{2} + k\varSigma\sqrt{\lambda L}\right].$$

3.1.3. Shortage cost

When the customer demand is greater than the production rate, then a shortage must occurs due to the lack of products, which indirectly creates a bad image for that product's company and directly affects the integrated profit. In this model the shortage per replenish cycle is given by $C(r) = \varSigma\sqrt{\lambda L}\varPsi(k)$, where, $\varPsi(k) = \phi(k) - k[1 - \varPhi(k)]$. Thus, the total shortage cost is given by

$$\frac{\pi\alpha P^{-\beta}}{B}C(r).$$

3.1.4. Lead time crashing cost

The time gap between placing an order and receiving the product in hand is called the lead time. Many researchers consider the lead time to be negligible but in real situations this does not happen, there must be some lead time and anyone want to reduced this time gap. To reduce the lead time, a crashing cost is used. The

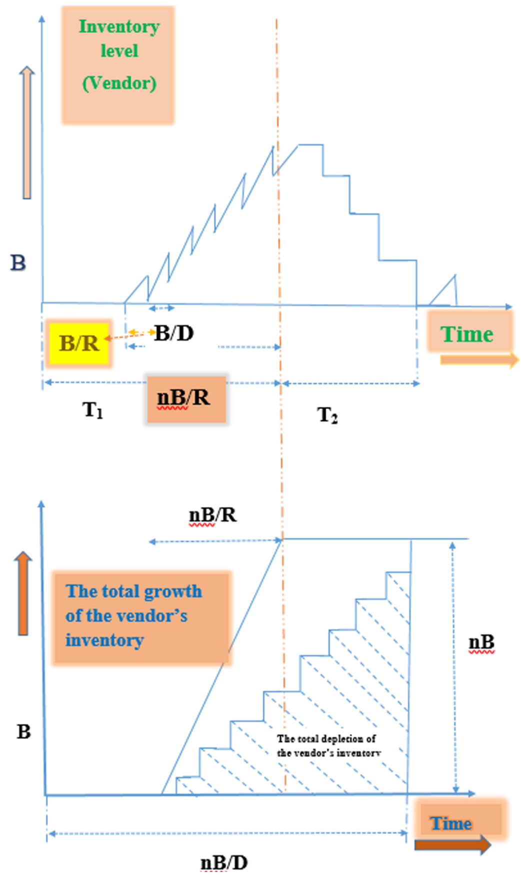

FIGURE 2. Inventory position for the vendor.

cost for lead time crashing is given by

$$\frac{D}{Q}C(L).$$

Therefore, the total cost for the buyer is

$$\begin{aligned} TC_B(B,k,L,P) &= hC_B\left(\frac{B}{2}+k\Sigma\sqrt{\lambda L}\right)+\frac{\pi\alpha P^{-\beta}}{B}C(r)+\frac{D}{Q}C(L)+\frac{O_c\alpha P^{-\beta}}{B} \\ &= hC_B\left(\frac{B}{2}+k\Sigma\sqrt{\lambda L}\right)+\frac{\pi\alpha P^{-\beta}}{B}C(r)+\frac{D}{nB}C(L)+\frac{O_c\alpha P^{-\beta}}{B}. \end{aligned}$$

The revenue for the buyer is $= (C_v - C_B)\alpha P^{-\beta}$ thus, buyer's profit function is

$$\begin{aligned} \text{Profit}_B(B,k,L,P) &= (C_V - C_B)\alpha P^{-\beta} - TC_B(B,k,L,P) \\ &= (C_V - C_B)\alpha P^{-\beta} - \left[hC_B\left(\frac{B}{2}+k\Sigma\sqrt{\lambda L}\right)+\frac{\pi\alpha P^{-\beta}}{B}C(r)+\frac{D}{Q}C(L)+\frac{O_c\alpha P^{-\beta}}{B}\right]. \quad (3.1) \end{aligned}$$

3.2. Vendor's profit

Figure 2 shows the accumulation and depletion process of the vendor's inventory, according to Ha and Kim [10] model. To calculate the vendor's profit, the model has to calculate all costs of the vendor. Thus, the costs related to the vendor's profit are calculated as follows

3.2.1. Setup cost

To produce any product, company must need some setup cost for the whole production system. A basic production system has always a fixed setup cost. But here an attempt of discrete investment is used to reduce the initial setup cost. Therefore, the combined investment and setup cost for the vendor is obtain as follows

$$S_0 e^{-rA} + A.$$

3.2.2. Transportation cost

In any production system, transportation is one of the most important things to transport the product from vendor's warehouse to buyer's warehouse. Due to the SSMD policy, the number of transports increases and thus, both fixed and variable carbon emission are considered for quality improvement.

The transportation cost is $nF + VnB$.

3.2.3. Investment

The amount of defective items are $\frac{SDQ\theta}{2}$, and the quality improvement is given by

$$b \ln\left(\frac{\theta_0}{\theta}\right).$$

3.2.4. Carbon emission cost (CC_v)

Due to transportation, a carbon effects on the environment are implied, for that reason, two types of carbon emission costs for the vendor are considered here, the fixed carbon emission cost for the vendor is $C_{vcv}n$ and the variable cost with respect to the number of received shipments is $(C_{fcv}n)$. Thus, the total carbon emission cost is

$$CC_v = C_{fcv}n + C_{vcv}Q = C_{fcv}n + C_{vcv}Bn.$$

3.2.5. Holding cost for the vendor

To produce the product and store them a production house and a warehouse are required, for that some holding cost is also needed. According to Figure 2, the average inventory of the vendor is

$$\frac{\text{[bold area-shaded]}}{nB/\alpha P^{-\beta}} = \frac{\left[\left(n\alpha P^{-\beta}\left(\frac{B}{R} + (n-1)\frac{B}{\alpha P^{-\beta}}\right) - \frac{nB\left(\frac{nB}{R}\right)}{2}\right) - T[B + 2B + \cdots + (n-1)B]\right]}{n\alpha P^{-\beta}/B}$$
$$= \left[\frac{B}{2} + \frac{(n-2)B}{2}\left(1 - \frac{\alpha P^{-\beta}}{R}\right)\right].$$

Therefore, the total holding cost for the vendor is $= \left[\frac{B}{2} + \frac{(n-2)B}{2}\left(1 - \frac{\alpha P^{-\beta}}{R}\right)\right] C_v h.$

The annual total cost for the vendor is

$$\begin{aligned} TC_V(B,k,P,\theta) &= \frac{S_0 e^{-rA}}{nB} + \frac{B}{2}\left[1+(n-2)\left(1-\frac{\alpha P^{-\beta}}{R}\right)\right] C_V h + \frac{\alpha P^{-\beta} A}{nB} + \frac{SDQ\theta}{2} + b\ln\frac{\theta_0}{\theta} \\ &+ \frac{(nF+VnB)D}{nB} + \frac{(C_{fcv}n + C_{vcv}Bn)D}{nB} \\ &= \frac{S_0 e^{-rA}}{nB} + \frac{B}{2}\left[1+(n-2)\left(1-\frac{\alpha P^{-\beta}}{R}\right)\right] C_V h + \frac{\alpha P^{-\beta} A}{nB} + \frac{nSDB\theta}{2} + b\ln\frac{\theta_0}{\theta} \\ &+ \frac{FD}{B} + VD + \frac{DC_{fcv}}{B} + C_{vcv}D. \end{aligned}$$

The revenue of vendor is $(P - C_v)\alpha P^{-\beta}$.

Then, the vendor's profit function is

$$\begin{aligned} \text{Profit}_V(B,k,P,\theta) &= (P-C_V)\alpha P^{-\beta} - TC_V(B,k,P,\theta) \\ &= (P-C_V)\alpha P^{-\beta} - \left[\frac{S_0 e^{-rA}}{nB} + \frac{B}{2}\left[1+(n-2)\left(1-\frac{\alpha P^{-\beta}}{R}\right)\right] C_V h \right. \\ &+ \frac{\alpha P^{-\beta} A}{nB} + \frac{nBSD\theta}{2} + b\ln\frac{\theta_0}{\theta} + \frac{FD}{B} + VD \\ &\left. + \frac{DC_{fcv}}{B} + C_{vcv}D\right]. \end{aligned} \tag{3.2}$$

Here, equations (3.1) and (3.2) providing the buyer's annual total profit and the vendor's annual total profit respectively. Thus, the joint total expected annual profit is given by

$$\begin{aligned} \text{Total profit}(B,L,k,P,\theta) &= (P-C_B)\alpha P^{-\beta} - TC(B,k,P,\theta) \\ &= (P-C_B)\alpha P^{-\beta} - \frac{\alpha P^{-\beta}}{B}\left[\frac{S_0 e^{-rA}}{n} + \frac{A}{n} + O_c + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right] \\ &- \frac{B}{2}h\left\{1+(n-2)\left(1-\frac{\alpha P^{-\beta}}{R}\right)\right\} C_V - \frac{Bh}{2}C_B - hC_B k\Sigma\sqrt{\lambda L} \\ &- \frac{nBSD\theta}{2} - b\ln\left(\frac{\theta_0}{\theta}\right) - \frac{D}{nB}C(L) - \frac{FD}{B} - VD - \frac{DC_{fcv}}{B} - C_{vcv}D. \end{aligned} \tag{3.3}$$

4. Solution methodology

The aim now is to maximize the total profit. To maximize the expected total profit with respect to given restrictions, one can first ignored the restrictions and solve the non-linear problem by calculating all the partial derivatives of the total profit function with respect to the decision variables. Then after applying these restrictions, the values are obtained as follows

$$\begin{aligned} \frac{\partial TP(B,k,L,P,\theta)}{\partial B} &= \frac{\alpha P^{-\beta}}{B^2}\left[\frac{S_0 e^{-rA}}{n} + \frac{A}{n} + O_c + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right] \\ &- \frac{1}{2}h\left[\left\{1+(n-2)\left(1-\frac{\alpha P^{-\beta}}{R}\right)\right\} C_V + C_B\right] + \frac{nSD\theta}{2} + \frac{1}{B^2}(FD + DC_{fcv}) \\ \frac{\partial TP(B,L,k,P,\theta)}{\partial L} &= -\frac{\alpha P^{\beta}}{2B}\pi\Sigma\sqrt{\frac{\lambda}{L}}\psi(k) - \frac{hC_B}{2}k\Sigma\sqrt{\frac{\lambda}{L}} - \frac{D}{Q}m_i \\ \frac{\partial TP(B,L,k,P,\theta)}{\partial k} &= -\frac{\alpha P^{-\beta}}{B}\pi\Sigma\sqrt{\lambda L}\left(\Phi(k)-1\right) - hC_B\Sigma\sqrt{\lambda L} \end{aligned}$$

$$\frac{\partial TP(B,L,k,P,\theta)}{\partial P} = \alpha P^{-\beta} - \alpha\beta P^{-\beta} + \alpha\beta P^{-\beta-1}(C_B + V + C_{vcv}) + \alpha\beta P^{-\beta-1}\Big(\frac{1}{B}\Big(\frac{S_0 e^{-rA}}{n} + \frac{A}{n} + O_c$$
$$+F + C_{fcv} + C(L)/n + \pi\Sigma\sqrt{\lambda L}\Psi(k)\Big) - \frac{B}{2}h(n-2)\frac{C_V}{R}\Big)$$
$$\frac{\partial TP(B,L,k,P,\theta)}{\partial \theta} = -\frac{nBSD}{2} + \frac{b}{\theta}.$$

For given $B,\ P,\ k$, and θ, $TP(B,L,k,P,\theta)$ is a convex function in L for $L>0$, because:

$$\frac{\partial^2 TP(B,L,k,P,\theta)}{\partial L^2} = \frac{\alpha P^{\beta}}{2B}\pi\Sigma\sqrt{\frac{\lambda}{L^3}}\psi(k) + \frac{hC_B}{2}k\Sigma\sqrt{\frac{\lambda}{L^3}} > 0.$$

If one regards the values of B,k,P, and θ, as constant, then the function $\partial TP(B,L,k,P,\theta)$ is convex with respect to L. Thus, for constant values of L,k,P, and θ, the maximum expected profit can be obtained from the end points of $[L_i, L_{i-1}]$. Therefore, one can obtain the values of B,k,P, and θ, and for a given $n \in [L_i, L_{i-1}]$. Now equating the first order partial derivatives to zero, one can obtain

$$B_i^* = \sqrt{\frac{\alpha P^{-\beta}\left[\frac{S_0 e^{-rA}}{n} + \frac{A}{n} + O_c + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right] + DF + DC_{fcv}}{\frac{1}{2}hC_V\left[\left\{1 + (n-2)\left(1 - \frac{\alpha P^{-\beta}}{R}\right)\right\} + C_B\right] + \frac{nSD\theta}{2}}} \tag{4.1}$$

$$\phi(K_i^*) = 1 - \frac{hC_B B}{\pi\alpha P^{-\beta}} \tag{4.2}$$

$$P^* = \frac{\left(1 - \frac{1}{\beta}\right)}{\frac{1}{B}\left(\frac{S_0 e^{-rA}}{n} + \frac{A}{n} + O_c + F + C_{fcv} + C(L)/n + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right) - \frac{B}{2}h(n-2)\frac{C_V}{R}} \tag{4.3}$$

$$\theta^* = \frac{2b}{SDBn}. \tag{4.4}$$

As, it is a non-linear program, one can construct a lemma in order to obtain the optimum value.

Lemma 4.1. *For a given $L \in [L_i, L_{i-1}]$, the Hessian matrix for $TP(B,L,k,P,\theta)$ is always at the optimal values $(B_i^*, k_i^*, P^*, \theta^*)$.*

Proof. See Appendix A. □

Solution algorithm

Step 1. Set $n=1$, and set all values for the lot size, vendor's production rate, shipment's number, setup cost, holding cost, vendor's production cost, buyer's purchase cost, and ordering cost.

Step 2. For each L_i, $i = 1,2,\ldots,n$ repeat Step (2.2), (2.3), and (2.4) until there are no changes in $(P^*, B^*, k^*, \theta^*)$

Step 2.1. Start with $P_i = P_0$.

Step 2.2. Find k_i by substituting P_i and the result from Step 2.1 into equation (4.2).

Step 2.3. Substitute k_i and P_i into equation (4.1) to evaluate B_i.

Step 2.4. Find P_{i+1} by employing the result of Step 2.1

Step 3. For each $L \in [L_i, L_{i-1}]$, find A using equation (4.3)

Step 4. Find the relation between P and B using L and equation (4.3)

Step 5. Utilize equation (3.3) in order to find the corresponding expected total profit.

Step 6. Find $TP(B,L,k,P,\theta) = Max[TP(B^*, L^*, k^*, P^*, \theta^*)]$ for every $L \in [L_i, L_{i-1}]$, then this is the maximum total profit of this model, and $(B^*, L^*, P^*, k^*, \theta^*)$ is the optimal solution.

5. Numerical example

This paper utilizes the same data as in Huang's *et al.* [11] and Sarkar and Moon's [29] models. The suggested analytic solution process is used to solve the numerical example.

Contemplate an inventory system with the following characteristics (Tab. 2).

Table 2. Parametric value.

Parameter	Value	Parameter	Value
β	1.5	b	400
F	0.2 (\$/shipment)	π	5 (/unit shortage)
C_V	20 (\$/unit)	S_0	1000 (\$/setup)
C_B	25 (\$/unit)	R	3200 (units/year)
O_c	200 (\$/order)	λ	2
σ	7 (units/week)	h	0.2 (\$/unit/time)
θ_0	0.0002	r	0.01
C_{vcv}	0.1 (\$/unit)	C_{fcv}	0.2 (\$/unit)
		α	15

With an added cost of investment, the setup cost can be reduced.

The unit crashing cost is taken as in Sarkar and Moon's [29] model (Tab. 3).

Table 3. Lead time data.

Lead time component i	Normal duration v_i (days)	Minimum duration u_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

The profit is shown in Table 4.

Table 4. Profit for the integrated inventory model.

L (weeks)	B (units)	K (\$/week)	P (\$/unit)	θ	Profit (\$/week)
3*	**10.1917***	**0.4764***	**0.8405***	**0.000001***	**1802.0635***
4	10.4660	0.5008	0.7999	0.000001	1734.7783
6	9.9281	0.5784	0.7403	0.000001	1622.6877

The bold data show the optimal results for this model. One can see that the optimal integrated profit is 1802.0635\$, which is more beneficial from the previous integrated models. Thus, by using an investment the setup cost was reduced and the profit was maximized with an improved product quality, where the lead time is 3, the optimized shipment volume is 10.19, the selling-price is 0.8405(\$/*unit*), and the probability that the production process changes to an *"out-of-control"* state is near about 0, which is also a valuable finding in this model.

A sensitivity analysis is provided below to demonstrate the effects on the profit by changing the parameter values.

6. Sensitivity analysis

The combined profit increase is $CPI = \frac{Change\ of\ profit}{orginal\ profit\ TP} \times 100\%$.
The symbol "–" indicates there is no effect on the profit.
Table 5 shows the effect on the profit when the parameters C_b, O_c, C_{fcv}, C_{vcv}, F, π, and C_b are varied. The effect of varying the parameters is shown in Figure 3.
In Table 5

- One can easily find that the buyer's unit purchase cost *i.e.*, C_b is more sensitive, meaning that small changes in C_b have a great impact on the total profit. If one increase the buyer's purchasing cost a small amount, the total profit decreases simultaneously.
- The buyer ordering cost O_c has also a large impact on total profit meaning small increase or decrease in the ordering cost affects the total profit very strongly.

Table 5. Percentage change in profit.

Rate of change (in %)		$E[TP]$	CPI (%)	Rate of change (in %)		$E[TP]$	CPI (%)
	30	757.60	−57.96		30	1346.00	−25.31
	20	1110.10	−38.40		20	1493.78	−17.11
	10	1457.63	−19.12		10	1645.48	−8.69
C_b	0	1802.06	–	O_c	0	1802.06	–
	−10	2146.60	19.12		−10	1964.74	9.03
	−20	2495.50	38.48		−20	2135.01	18.48
	−30	2846.81	57.97		−30	2314.64	28.44
	30	1807.78	0.32		30	1832.06	1.66
	20	1805.88	0.21		20	1822.06	1.11
	10	1803.97	0.11		10	1812.06	0.55
C_{fcv}	0	1802.06	–	C_{vcv}	0	1802.06	–
	−10	1800.15	−0.11		−10	1792.06	−0.55
	−20	1798.23	−0.21		−20	1782.06	−1.11
	−30	1796.31	−0.32		−30	1772.06	−1.66
	30	1807.78	0.32		30	1554.31	−13.75
	20	1805.88	0.21		20	1634.55	−9.30
	10	1803.97	0.11		10	1716.99	−4.72
F	0	1802.06	–	π	0	1802.06	–
	−10	1800.15	−0.11		−10	1890.32	4.90
	−20	1798.23	−0.21		−20	1982.61	10.02
	−30	1796.31	−0.32		−30	2080.33	15.44

Rate of change (in %)		$E[TP]$	CPI(%)
	30	1694.11	−5.99
	20	1725.55	−4.25
	10	1761.46	−2.25
C_v	0	1802.06	–
	−10	1847.64	2.53
	−20	1898.56	5.35
	−30	1955.28	8.50

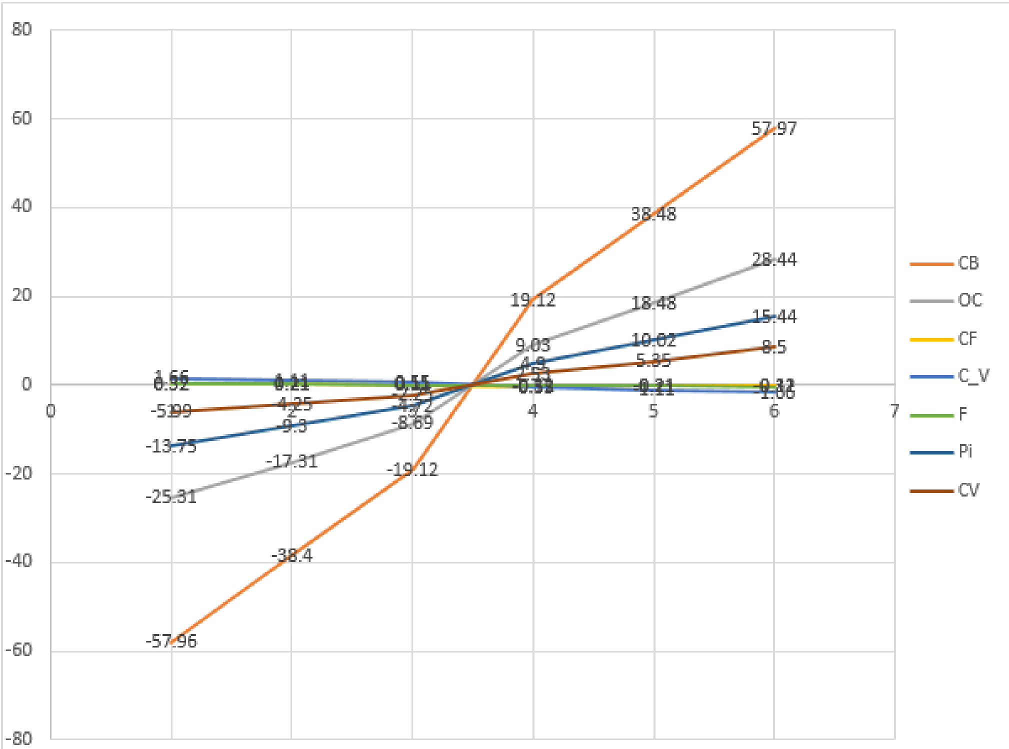

FIGURE 3. Effect of changes in the parametric value.

- Shortage is also a vital factor for any inventory model. In Table 5 one can show that the unit shortage cost has a vital impact on the total profit. Specifically, the total profit is decreases with small increases in the unit shortage cost.
- The vendor unit production cost *i.e.*, C_v has a comparatively smaller impact on the total profit. That is, there is not much change in the total profit when the production cost is changed.
- Variable carbon emission costs for the vendor have little impact on the total profit, whereas, fixed carbon emission costs for the vendor have an almost negligible impact on the total profit.
- Fixed transportation costs is also do not too much affect on the total profit.

7. MANAGERIAL INSIGHTS

- This is a integrated inventory model where the total profit is maximized with the optimized values of shipment size, lead time period, safety factor, selling price and probability that the production process will change to the *"out-of-control"* state.
- This model developed an integrated inventory model in the presence of environmental issues, thus this model is very effective in today's earth-conscious environment.
- One can also reduce the setup cost for production which is more beneficial to most industries. Not only that, but also discussed the improved quality of the product by some initial investment, which has also a large impact on today's business world.
- Shortages, can create a bad image for any business industry. In this model the safety factor is also calculated according to the lead time period, which maintains the brand image as well as profits.

8. Conclusions

This paper extended an integrated inventory model with setup cost reduction by discrete investment considering environmental impacts. Due to the SSMD policy, the variable and fixed transportation costs and carbon emission cost were taken into consideration. Two investments were used to reduce the setup cost and to improve the quality to make a sustainable integrated inventory model. The model was solved analytically. One lemma was used to show the global maximum profit of the model. An improved algorithm was developed to obtain the numerical results. The study demonstrated that this model reduced the initial setup cost and improved the initial quality. The model considered continuous investment for quality improvement, which is a limitation of this model. However, discrete investment can be used for setup cost reduction, as well as improving the quality of products. This model can extended by considering two types of warehouse for vendor and buyer. This model can be extended by considering unreliable vendor/retailier. This model is developed without considering the inspection policy, incorporate of inspection is one of the more realistic research in this direction. And if vendor produced single type of assembled item with backorder where backorder are follows different types of distribution that is a very effective research in this direction.

Appendix A

Proof of Lemma 4.1. This paper computes the Hessian matrix at the optimal values for a given $L \in [L_i, L_{i-1}]$ as follows:

$$|H(TP)| = \begin{vmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} & \frac{\partial^2 TP(.)}{\partial B \partial P} & \frac{\partial^2 TP(.)}{\partial B \partial P} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k \partial P} & \frac{\partial^2 TP(.)}{\partial k \partial P} \\ \frac{\partial^2 TP(.)}{\partial P \partial B} & \frac{\partial^2 TP(.)}{\partial P \partial k} & \frac{\partial^2 TP(.)}{\partial P^2} & \frac{\partial^2 TP(.)}{\partial P \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta \partial B} & \frac{\partial^2 TP(.)}{\partial \theta \partial k} & \frac{\partial^2 TP(.)}{\partial P \partial \theta} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{vmatrix}$$

where $TP(.) = TP(n, B, k, \theta, P)$. The second order partial derivatives at the optimal values are

$$\begin{aligned}
\frac{\partial^2 TP(.)}{\partial B^2} &= -2\frac{\alpha P^{-\beta}}{B^3}\left[\frac{S_0 e^{-rA}}{n} + A + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right] \\
\frac{\partial^2 TP(.)}{\partial k^2} &= -\frac{\alpha P^{-\beta}}{B}\pi\Sigma\sqrt{\lambda L}\Phi(k) \\
\frac{\partial^2 TP(.)}{\partial P^2} &= -2\alpha\beta P^{-\beta-1} + (P - C_B)\alpha\beta(\beta+1)P^{-\beta-2} \\
&\quad -\frac{\alpha\beta(\beta+1)P^{-\beta-1}}{B}\left(\frac{S_0 e^{-rA}}{n} + A + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right) + \frac{BhC_V\alpha\beta(\beta+1)P^{-\beta-2}}{2R} \\
\frac{\partial^2 TP(.)}{\partial B \partial k} &= \frac{\alpha P^{-\beta}}{B^2}\pi\Sigma\sqrt{\lambda L}(\Phi(k) - 1) \\
\frac{\partial^2 TP(.)}{\partial B \partial P} &= \frac{\alpha(-\beta)P^{-\beta-1}}{B}\left(\frac{S_0 e^{-rA}}{n} + A + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right) - \frac{1}{2}h(n-2)C_V\left(\frac{\alpha\beta P^{-\beta-1}}{R}\right) \\
\frac{\partial^2 TP(.)}{\partial k \partial P} &= \frac{\alpha\beta P^{-\beta-1}}{B}\pi\Sigma\sqrt{\lambda L}(\Phi(k) - 1).
\end{aligned}$$

The first principal minor at the optimal values is

$$\det(H_{11}) = \det\left(\frac{\partial^2 TP(.)}{\partial B^2}\right) = -2\frac{\alpha P^{-\beta}}{B^3}\left[\frac{S_0 e^{-rA}}{n} + A + \pi\Sigma\sqrt{\lambda L}\Psi(k)\right] < 0.$$

The first principal minor is smaller than zero since all the terms are positive:

The second principal minor of $H(TP)$ is

$$\det(H_{22}) = \det\begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} \end{pmatrix}$$
$$= 2\frac{\alpha P^{-\beta}}{B^3}\left[\frac{S_0 e^{-rA}}{n} + A + \pi \Sigma \sqrt{\lambda L}\Psi(k)\right] \times \frac{\alpha P^{-\beta}}{B}\pi \Sigma \sqrt{\lambda L}\Phi(k) - \left[\frac{\alpha\beta P^{-\beta}}{B^2}\pi\Sigma\sqrt{\lambda L}(\Phi(k)-1)\right]^2$$
$$= \left(\frac{\alpha\beta P^{-\beta}}{B^2}\right)^2\left[\frac{S_0 e^{-rA}}{\pi n} + \frac{A}{\pi} + \Sigma\sqrt{\lambda L}\Psi(k)\right] \times \left(\Sigma\sqrt{\lambda L}\Phi(k)\right)\left(\Sigma\sqrt{\lambda L}(1-\Phi(k))\right)^2 > 0.$$

That is because $\Psi(k) = \phi(k) - k\,[1-\Phi(k)] > 1-\Phi(k)$ and $\phi(k) > 1-\Phi(k)$.

The third principal minor of $H(TP)$ is

$$\det(H_{33}) = \det\begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} & \frac{\partial^2 TP(.)}{\partial B \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta \partial B} & \frac{\partial^2 TP(.)}{\partial \theta \partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{pmatrix}$$
$$= \frac{\partial^2 TP(.)}{\partial B \partial \theta} \times \det\begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} \end{pmatrix} + \frac{\partial^2 TP(.)}{\partial \theta^2} \times \det(H_{22})$$
$$= -\left(\frac{\partial^2 TP(.)}{\partial B \partial \theta}\right)^2 \times \left(\frac{\partial^2 TP(.)}{\partial k^2}\right)^2 + \frac{\partial^2 TP(.)}{\partial \theta^2} \times \det(H_{22}) < 0$$

since, $\frac{\partial^2 TP(.)}{\partial \theta^2} < 0$ and $\det(H_{22}) > 0$.

The forth principal minor of $H(TP)$ is

$$\det(H_{44}) = \begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} & \frac{\partial^2 TP(.)}{\partial B \partial \theta} & \frac{\partial^2 TP(.)}{\partial B \partial P} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k \partial \theta} & \frac{\partial^2 TP(.)}{\partial k \partial P} \\ \frac{\partial^2 TP(.)}{\partial \theta \partial B} & \frac{\partial^2 TP(.)}{\partial \theta \partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} & \frac{\partial^2 TP(.)}{\partial \theta \partial P} \\ \frac{\partial^2 TP(.)}{\partial P \partial B} & \frac{\partial^2 TP(.)}{\partial P \partial k} & \frac{\partial^2 TP(.)}{\partial P \partial \theta} & \frac{\partial^2 TP(.)}{\partial P^2} \end{pmatrix}$$
$$= \frac{\partial^2 TP(.)}{\partial B \partial P} \times \det\begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} & \frac{\partial^2 TP(.)}{\partial B \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta \partial B} & \frac{\partial^2 TP(.)}{\partial \theta \partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{pmatrix} - \frac{\partial^2 TP(.)}{\partial k \partial P}$$
$$\times \det\begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} & \frac{\partial^2 TP(.)}{\partial B \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta \partial B} & \frac{\partial^2 TP(.)}{\partial \theta \partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{pmatrix} + \frac{\partial^2 TP(.)}{\partial \theta \partial P}$$
$$\times \det\begin{pmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B \partial k} & \frac{\partial^2 TP(.)}{\partial B \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k \partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k \partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta \partial B} & \frac{\partial^2 TP(.)}{\partial \theta \partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{pmatrix} - \frac{\partial^2 TP(.)}{\partial P^2} \times \det(H_{33}).$$

Now,

$$\begin{vmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B\partial k} & \frac{\partial^2 TP(.)}{\partial B\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k\partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta\partial B} & \frac{\partial^2 TP(.)}{\partial \theta\partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{vmatrix} = \frac{\partial^2 TP(.)}{\partial B^2}\left(\frac{\partial^2 TP(.)}{\partial k^2}.\frac{\partial^2 TP(.)}{\partial \theta^2} - \frac{\partial^2 TP(.)}{\partial k\partial \theta}.\frac{\partial^2 TP(.)}{\partial \theta\partial k}\right)$$
$$-\frac{\partial^2 TP(.)}{\partial B\partial k}\left(\frac{\partial^2 TP(.)}{\partial k\partial B}.\frac{\partial^2 TP(.)}{\partial \theta^2} - \frac{\partial^2 TP(.)}{\partial \theta\partial B}.\frac{\partial^2 TP(.)}{\partial \theta\partial K}\right)$$
$$+\frac{\partial^2 TP(.)}{\partial B\partial \theta}\left(\frac{\partial^2 TP(.)}{\partial k\partial B}.\frac{\partial^2 TP(.)}{\partial \theta\partial k} - \frac{\partial^2 TP(.)}{\partial \theta\partial B}.\frac{\partial^2 TP(.)}{\partial K^2}\right)$$
$$> 0.$$

which is true because

$$\frac{\partial^2 TP(.)}{\partial B^2}\left(\frac{\partial^2 TP(.)}{\partial k^2}.\frac{\partial^2 TP(.)}{\partial \theta^2} - \frac{\partial^2 TP(.)}{\partial k\partial \theta}.\frac{\partial^2 TP(.)}{\partial \theta\partial k}\right)$$
$$< \frac{\partial^2 TP(.)}{\partial B\partial k}\left(\frac{\partial^2 TP(.)}{\partial k\partial B}.\frac{\partial^2 TP(.)}{\partial \theta^2} - \frac{\partial^2 TP(.)}{\partial \theta\partial B}.\frac{\partial^2 TP(.)}{\partial \theta\partial K}\right) + \frac{\partial^2 TP(.)}{\partial B\partial \theta}\left(\frac{\partial^2 TP(.)}{\partial k\partial B}.\frac{\partial^2 TP(.)}{\partial \theta\partial k} - \frac{\partial^2 TP(.)}{\partial \theta\partial B}.\frac{\partial^2 TP(.)}{\partial K^2}\right).$$

Thus, one can easily show that

$$\frac{\partial^2 TP(.)}{\partial B\partial P}\begin{vmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B\partial k} & \frac{\partial^2 TP(.)}{\partial B\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k\partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta\partial B} & \frac{\partial^2 TP(.)}{\partial \theta\partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{vmatrix} > 0.$$

Similarly it can easily be shown that

$$\frac{\partial^2 TP(.)}{\partial k\partial P}\begin{vmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B\partial k} & \frac{\partial^2 TP(.)}{\partial B\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k\partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta\partial B} & \frac{\partial^2 TP(.)}{\partial \theta\partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{vmatrix} < 0$$

and

$$\frac{\partial^2 TP(.)}{\partial \theta\partial P}\det\begin{vmatrix} \frac{\partial^2 TP(.)}{\partial B^2} & \frac{\partial^2 TP(.)}{\partial B\partial k} & \frac{\partial^2 TP(.)}{\partial B\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial k\partial B} & \frac{\partial^2 TP(.)}{\partial k^2} & \frac{\partial^2 TP(.)}{\partial k\partial \theta} \\ \frac{\partial^2 TP(.)}{\partial \theta\partial B} & \frac{\partial^2 TP(.)}{\partial \theta\partial k} & \frac{\partial^2 TP(.)}{\partial \theta^2} \end{vmatrix} > 0$$

and, $\frac{\partial^2 TP(.)}{\partial P^2} \times \det(H_{33}) > 0$
Thus, finally one can conclude that, the fourth principal minor is greater than zero. □

References

[1] P.L. Abad and C.K. Jaggi, A joint approach for setting unit price and the length of the credit period for a seller when end demand is price sensitive. *Int. J. Prod. Econ.* **83** (2003) 115–122.

[2] K. Annadurai and R. Uthayakumar, Reducing lost-sales rate in (T, R, L) inventory model with controllable lead time. *Appl. Math. Mod.* **34** (2010) 3465–3477.

[3] A. Azadeh and B. Paknafs, Integrated simulation modeling of business, maintenance and production systems for concurrent improvement of lead time, cost and production rate. *Ind. Eng. Manag. Syst.* **15** (2017) 403–431.

[4] M. Bijvank and S.G. Johansen, Periodic review lost-sales inventory models with compound poisson demand and constant lead times of any length. *Eur. J. Oper. Res.* **220** (2012) 106–114.

[5] M. Ben-Daya, R. As'ad and M. Seliaman, An integrated production inventory model with raw material replenishment considerations in a three layer supply chain. *Int. J. Prod. Econ.* **143** (2013) 53–61.

[6] L.E. Cárdenas-Barrón, The derivation of EOQ/EPQ inventory models with two backorders costs using analytic geometry and algebra. *Appl. Math. Mod.* **35** (2011) 2394–2407.

[7] C.T. Chang and T.Y. Lo, On the inventory model with continuous and discrete lead time, backorders and lost sales. *Appl. Math. Mod.* **33** (2009) 2196–2206.

[8] M. Deyi and Z. Xiaoqian, Stochastic programming for the optimization of transportation-inventory strategy. *Ind. Eng. Manag. Syst.* **16** (2017) 44–51.

[9] S.K. Goyal, A joint economic-lot-size model for purchaser and vendor: a comment. *Decis. Sci.* **19** (1988) 236–241.

[10] D. Ha and S.L. Kim, Implementation of JIT purchasing: an integrated approach. *Prod. Plann. Cont.* **8** (1997) 152–157.

[11] C.K. Huang, T.L. Cheng, T.C. Kao and S.K. Goyal, An integrated inventory model involving manufacturing setup cost reduction in compound poisson process. *Int. J. Prod. Res.* **49** (2011) 1219–1228.

[12] S.G. Johansen and A. Thorstenson, Optimal and approximatem(Q, r) inventory policies with lost sales and gamma-distributed lead time. *Int. J. Prod. Econ.* **30** (1993) 179–194.

[13] M. Karaöz, A. Erçlu and A. Sütçü, An EOQ model with price and time dependent demand under the influence of complement and substitute product's selling-prices. *J. Alanya Fac. Bus./Alanya Isletme Fakültesi Dergisi* **3** (2011) 21–32.

[14] M. Kim and B. Sarkar, Multi-stage cleaner production process with quality improvement and lead time dependent ordering cost. *J. Clean. Prod.* **144** (2017) 572–590.

[15] L.Y. Ouyang, C.K. Chen and H.C. Chang, Quality improvement, setup cost and lead-time reductions in lot size reorder point models with an imperfect production process. *Comput. Oper. Res.* **29** (2002) 1701–1717.

[16] B. Pal and S. Adhikari, Price-sensitive imperfect production inventory model with exponential partial backlogging. To appear in: *Int. J. Syst. Sci. Oper. Logist.* (2017). DOI: 10.1080/23302674.2017.1323133

[17] B. Pal, S.S. Sana and K. Chaudhuri, Multi-item EOQ model while demand is sales price and price break sensitive. *Econ. Model.* **29** (2012) 2283–2288.

[18] B. Pal, S.S. Sana and K.S. Chaudhuri, Joint pricing and ordering policy for two-echelon imperfect production inventory model with two cycles. *Int. J. Prod. Econ.* **155** (2014), 229–238.

[19] B. Pal, S.S. Sana and K.S. Chaudhuri, Two-echelon manufacturer–retailer supply chain strategies with price, quality, and promotional effort sensitive demand. *Int. Trans. Oper. Res.* **22** (2015), 1071–1095.

[20] B. Pal, S.S. Sana, K.S. Chaudhuri, Coordination contracts for competitive two-echelon supply chain with price and promotional effort sensitive non-linear demand. *Int. J. Syst. Sci. Oper. Logist.* **2** (2015) 113–124.

[21] B. Pal, S.S. Sana and K. Chaudhuri, Two-echelon competitive integrated supply chain model with price and credit period dependent demand. *Int. J. Syst. Sci.* **47** (2016) 995–1007.

[22] Z. Pang and F.Y. Chen, Coordinated pricing and inventory control with batch production and Erlang lead times. *Prob. Eng. Inf. Sci.* **28** (2014) 529–563.

[23] E.L. Porteus, Optimal lot sizing, process quality improvement and setup cost reduction. *Oper. Res.* **34** (1986) 137–144.

[24] S.S. Sana, Price-sensitive demand for perishable items – an EOQ model. *Appl. Math. Mod.* **217** (2011) 6248–6259.

[25] B. Sarkar, A production-inventory model with probabilistic deterioration in two-echelon supply chain management. *Appl. Math. Mod.* **37** (2013) 3138–3151.

[26] B. Sarkar, Supply chain coordination with variable backorder, inspections, and discount policy for fixed lifetime products. *Math. Prob. Eng.* **2016** (2016) 6318737.

[27] B. Sarkar and A.S. Mahapatra, Periodic review fuzzy inventory model with variable lead time and fuzzy demand. *Int. Trans. Oper. Res.* **24** (2017) 1197–1227.

[28] B. Sarkar and A. Majumder, Integrated vendor–buyer supply chain model with vendor's setup cost reduction. *Appl. Math. Comput.* **224** (2013) 362–371.

[29] B. Sarkar and I. Moon, Improved quality, setup cost reduction, and variable backorder costs in an imperfect production process. *Int. J. Prod. Econ.* **155** (2014) 204–213.

[30] B. Sarkar and S. Sarkar, Variable deterioration and demand – an inventory model. *Econ. Mod.* **31** (2013) 548–556.

[31] B. Sarkar, S.S. Sana and K.S. Chaudhuri, An inventory model with finite replenishment rate, trade credit policy and price-discount offer. *J. Ind. Eng.* **2013** (2013) 672504.

[32] B. Sarkar, L.E. Cárdenas-Barrón, M. Sarkar and M.L. Singgih, An economic production quantity model with random defective rate, rework process and backorders for a single stage production system. *J. Manuf. Syst.* **33** (2014) 423–435.

[33] B. Sarkar, P. Mandal and S. Sarkar, An EMQ model with price and time dependent demand under the effect of reliability and inflation. *Appl. Math. Comput.* **231** (2014) 414–421.

[34] B. Sarkar, K.S. Chaudhuri and I. Moon, Manufacturing setup cost reduction and quality improvement for the distribution free continuous-review inventory model with a service level constraint. *J. Manuf. Syst.* **34** (2015) 74–82.

[35] B. Sarkar, S. Saren, D. Sinha and S. Hur, Effect of unequal lot sizes, variable setup cost, and carbon emission cost in a supply chain model. *Math. Prob. Eng.* **2015** (2015) 469486.

[36] B. Sarkar, B. Ganguly, M. Sarkar and S. Pareek, Effect of variable transportation and carbon emission in a three-echelon supply chain model. *Trans. Res. E: Logist. Transp. Rev.* **91** (2016) 112–128.

[37] B. Sarkar, A. Majumder, M. Sarkar, B.K. Dey and G. Roy, Two-echelon supply chain model with manufacturing quality improvement and setup cost reduction. *J. Ind. Manag. Opt.* **13** (2017) 1085–1104.

[38] N.H. Shah and A.S. Gor, An integrated economic lot-size model for vendor–buyer inventory system when input is random. *Math. Comp. Mod.* **49** (2009) 1326–1330.

[39] A.A. Taleizadeh, D.W. Pentico, M. Aryanezhad and S.M. Ghoreyshi, An economic order quantity model with partial backordering and a special sale price. *Eur. J. Oper. Res.* **221** (2012) 571–583.

[40] H.M. Wee, W.T. Wang and L.E. Cárdenas-Barrón, An alternative EPQ model with rework process at a single-stage manufacturing system with planned backorders. *Comput. Ind. Eng.* **64** (2013) 748–755.

[41] P.C. Yang and H.M. Wee, A single-vendor and multiple-buyers production–inventory policy for a deteriorating item. *Eur. J. Oper. Res.* **143** (2002) 570–581.

[42] R.Q. Zhang, I. Kaku and Y.Y. Xiao, Deterministic EOQ with partial backordering and correlated demand caused by cross-selling. *Eur. J. Oper. Res.* **210** (2011) 537–551.