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AN OPTIMAL INVENTORY MODEL WITH INTERACTION OF LOT SIZE, PRODUCTION RATE AND LEAD-TIME IN A FUZZY BACK-ORDER SYSTEM

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Abstract. This paper examines the decision-making about the interaction of lot size, production rate and lead time between a vendor and a buyer with the consideration of trade credit and fuzzy back-order rate. We assume that the lead time demand is distribution free and the back-order rate is triangular fuzzy number. An economic model is design to determine the optimal lot-size, production rate and lead time while minimizing system total cost. A minimax approach is applied to tackle the model and designed an iterative algorithm to obtain the optimal strategy. Numerical example and sensitivity analyses are given to demonstrate the performance of the proposed methodology and to highlight the differences between crisp and the fuzzy cases. This paper provides optimal decision support tools for managers in the form of mathematical model that improve operational, tactical, and strategic decision making in the fuzzy system. This paper aims to raise the awareness of managers with regard to realistic inventory problems.

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1. Introduction

Coordination among different business entities such as buyer, vendor, producer, etc. are an important way to gain today's competitive advantages. It involves synchronization of different efforts or actions of the various units of an organization to provide the requisite amount, timing, quality and sequence of efforts so that the planned objectives may be achieved with minimum conflict. In the retailing industry, WalMart and Proctor and Gamble received substantial collaboration benefits by implementing collaborative planning, and replenishment, a business model that intends to help supply chain members to collaborate in both tactical and strategic levels. Accordingly Yang and Wee [44] stated that the integrated policy results in an impressive cost reduction when it is compared with the independent decisions made by the vendor and the buyer. Recently several researchers such as Hoque [13], Kebing *et al.* [18] and Fernandes *et al.* [6], Bibhas *et al.* [3] and Jun-Yeon *et al.* [16], Ali *et al.* [1], Hong-Fwu and Wen-Kai [12] addressed integrated inventory models under various environment.

Trade credit is a powerful tool to improve sales and profits in an industry in real life business *via* share marketing. In developing countries, vendor providing credit to their buyers is an important form of financing for

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business and particularly role of trade credit is immense where growth of financial institutions is less compared to developed nations. The volume of trade credit in aggregate represents 17.8% of total assets for United States firms, 22% for United Kingdoms firms, and more than 25% for countries such as Germany, France and Italy. Hence, trade credit is one of the most important sources of short-term external financing for firms in a wide range of industries and economies in today's business transactions. In view of that trade credit inventory problem has been studied many times in the literatures such as Sarmah *et al.* [33], Thangam and Uthayakumar [41], Kreng and Tan [20], Mahata [23], Zeng *et al.* [45], Pramanik *et al.* [29] and Salem *et al.* [30] with less attention being given to probabilistic continuous review inventory system; however, there have been few number of works in this area in recent years like Wu [43], Salameh *et al.* [31] and Huang [14].

Make-to-order is a production approach where products are not built until a confirmed order for products is received. Similarly, in make-to-stock, products are manufactured based on demand forecasts. As the accuracy of the forecasts will prevent excess inventory and opportunity loss due to stock-out, the issues are very important in real life problems. But how do we obtain the forecast demands accurately? Thus, these two approaches are almost near to each other. In addition, the make-to-stock production system is applicable for managing most of standard products. That is, reserved on-hand stock is immediately available to meet demand when an order is received. In this case, the production process may start immediately or later when customers place their orders. Clearly, it is infeasible to utilize such a production policy for custom-made products. In view of this we will study on the make-to-order approach only.

Cost and operation of inventory depends a great deal on what happens to demand when the system is out of stock. In view of this Montgomery *et al.* [24] addressed continuous review inventory problem with a mixture of backorders and lost sales. Recently the authors Thangam and Uthayakumar [40] and Taleizadeh *et al.* [38], Javad *et al.* [15] and San-Jose *et al.* [32] addressed back-ordering inventory system with the assumption that the fraction of excess demand backordered (or lost) is a fixed constant. In the real situation, when stockout occurs many potential factors such as properties of products and/or image of selling shop may affect customers' wills of backorders. In other words, the amount of lost demand caused by stockout probably has a little disturbance due to various uncertainties [28]. If we express the fuzzy backorder rate as the neighborhood of the fixed backorder (or lost sales) rate, then it will more match with the real situation.

Fuzzy sets representing linguistic concepts such as low, medium, high, *etc.*, are employed to define states of a variable. The membership function of a fuzzy set posses a quantity meaning and may be viewed as a fuzzy number provided they satisfy certain conditions. The application of fuzzy set concepts on economic order quantity (EOQ) inventory models have been proposed by many authors like Chang *et al.* [4], Ouyang and Chang [28], Lin [22], Taleizadeh *et al.* [39], Wang *et al.* [42] and Sonia *et al.* [37], Amalesh *et al.* [2]. Ouyang and Chang [28] presented a more extensive EOQ model to modify Moon and Choi's [26] model by fuzzifying the lost sales rate and to solve the new inventory model in the fuzzy sense. Lin [22] developed a periodic review inventory model involving fuzzy expected demand short and fuzzy backorder rate.

Lead time and production rate are essential factors in any supply chain system. In stochastic inventory models, lead time and production rate are often viewed as a prescribed constant or a random variable that is not subject to control and both are independent. In many practical situations, lead times can be controlled by paying additional investment and also it depends on the production rate. Liao and Shyu [21] introduced notion of the crashing cost into stochastic inventory model, in which lead time can be controlled by additional investment. Many researchers have developed various analytical inventory models to extend Liao and Shyu's model. In this connection, recently, Sofiene *et al.* [34] derived joint integrated production-maintenance policy with production plan smoothing through production rate control. Chi *et al.* [5] addressed an integrated production-inventory model for deteriorating items with consideration of optimal production rate and deterioration during delivery.

The aforementioned lead time/or production rate inventory models are based on the following two unrealistic assumptions: (i) lead time is independent of the production process and ordering policy, and has no relationships with production capacity and order size; (ii) lead time is controlled only by one side of a supply chain. Kim and Benton [19] challenge the unrealistic assumption (i) on lead time. They established a linear relationship between lead time and lot size. After that, Hariga [10], Hariga [11], Moon and Cha [25] and Noblesse *et al.* [27]

TABLE 1. Comparison with the literature.

Author (s)	Integrated inventory model	Lot size and/or production rate dependent Lead time	Trade credit policy	Fuzzy back-order rate
Kim and Benton [19]		✓		
Chang *et al.* [4]				✓
Hariga [10]		✓		
Ouyang and Chang [28]				✓
Sarmah *et al.* [33]	✓		✓	
Lin [22]				✓
Song *et al.* [36]	✓	✓		
Kreng and Tan [20]			✓	
Glock [9]	✓	✓		
Song *et al.* [35]	✓	✓		
Taleizadeh *et al.* [39]				✓
Noblesse *et al.* [27]		✓		
Bibhas *et al.* [3]	✓		✓	
Kumari and Pakkala 2016	✓		✓	
Javad *et al.* [15]	✓			✓
Salem *et al.* [30]	✓		✓	
Pramanik *et al.* [29]	✓		✓	
Sofiene *et al.* [34]	✓			
This paper	✓	✓	✓	✓

incorporated this lead time/lot size relation in the classical stochastic continuous review (Q, r) model, however, these papers have been developed from the buyer's point of view by neglecting seller's decisions. Recently, Song *et al.* [35, 36] relaxed the common assumptions on lead time (i) and (ii) with the aim to authorize for a more practical analysis of production-inventory management of a supply chain, and to extend Moon and Cha's [25] inventory model from one-side viewpoint to an interactive decision-making Stackelberg game model from both two-sides' viewpoint in a supply chain. Glock [9] also framed lead time reduction strategies in a single-vendor-single-buyer integrated inventory model with lot size-dependent lead times and stochastic demand.

Today many complex multi-stage manufacturing companies have high levels of work-in-process because of queueing delays at raw materials (or shortage of raw materials) from outside retailers and consequently long manufacturing lead times. These delays are directly related to production lot-sizes. Also crisp data are inadequate to accurate the lost sales rate since human judgements are often vague and decision makers cannot estimate their lost sales rate with an exact numerical value due to the uncertainties of customer demand and the raw material arrivals. In view of the above said scenarios we considered the replenishment lead time (*i.e.*, there is no order lead-time, no delivery lead-time etc.) which is dependent on both order size and production rate as well as triangular fuzzy numbers are used to express the accurate lost sales ratings of decision makers.

From the literature, till to date, none of the authors framed the decision-making based on the above scenario. The contribution of the proposed paper intends to fill this remarkable gap in the inventory literature. A comparison of our paper with the literature is provided in Table 1. This paper designs an optimal inventory strategy for a two-echelon supply chain system with the consideration of the following realistic assumptions: (i) lead time is dependent upon the production process and ordering policy, (ii) the vendor provides a trade credit period to the buyer and (iii) the lead time demand is distribution free and the back-order (or lost sales) rate is triangular fuzzy number. A mathematical model and an algorithm are designed to obtain the optimal strategy for the inventory system. Numerical example and sensitivity analyses are given to demonstrate the novelty of the proposed approach and to highlight the differences between proposed study and existing studies.

The subsequent sections have been organized as follows: Section 2 outlines the notations, assumptions and preliminary concepts that have been used for model building purposes. In Section 3, the model developing methodology was formulated. A numerical example illustrates the proposed methodology in Section 4. Section 5 demonstrates the sensitivity analysis and some managerial implications of the model. Finally some concluding remarks have been made in Section 6.

2. Notations, assumptions and preliminaries

We list the following notations, assumptions and preliminaries which will be used throughout the context.

2.1. Notations

Q	Order lot-size, a decision variable of the buyer.
P	Production rate, that is production quantity per year, a decision variable of the vendor.
k	Safety factor, a decision variable of the buyer.
L	Replenishment lead time.
P_0	Regular production quantity per year.
P_1	Maximum production quantity per year.
D	Average demand per year.
A	buyer's ordering cost per order.
S	Setup cost for the vendor.
p	Unit purchase cost paid by the buyer.
s	Unit selling price by the buyer.
C_v	Additional cost per unit product for increasing production rate.
h_b	Cost of holding a unit product per year for the buyer.
h_v	Cost of holding a unit product per year for the vendor.
π_0	Marginal profit per unit.
β	Fraction of the shortage that will be backordered, $0 \leqslant \beta \leqslant 1$.
I_d	Interest rate of deposit for the buyer per year.
I_c	Interest charge to be paid per $ in stock to the bank per year.
I_v	Interest rate for calculating company's opportunity interest loss due to the delay payment per year.
t_c	The length of the trade credit period, in years.
σ	Standard variance of demand per year.
r	Reorder point.
s_s	Safety stock.
X	Demand during lead time, a stochastic variable.

2.2. Assumptions

(1) The buyer and the vendor belong to different corporate entities and are enthusiastic to have the collaboration inventory system. Thus, both members agree to minimize the integrated expected annual total cost in the integrated strategy.

(2) The information about the form of cumulative distribution function (cdf) F of lead time demand X is unknown, while only the mean DL and standard deviation $\sigma\sqrt{L}$ are known.

(3) buyer uses a continuous review inventory policy and the order quantity Q is placed whenever inventory level falls to the reorder point r. The safety stock is established based on the criterion of service level per replenishment cycle. The safety factor k is determined by:

$$1 - F(k) = \int_k^\infty f(x)\mathrm{d}x.$$

(4) The reorder point r = expected demand during the lead time (DL) + the safety stock (s_s), and $s_s = k \times$ (standard deviation of lead time demand), *i.e.*

$$r = DL + k\sigma\sqrt{L}. \tag{2.1}$$

(5) The shortages occur when $X > r$, therefore the buyer's expected shortages at the end of the cycle time is given by

$$E(X-r)^+ = \int_r^\infty (x-r)f(x)\mathrm{d}x.$$

(6) There is no order lead-time (for processing the order placed), no delivery lead-time *etc.* The replenishment lead time is considered which is dependent on both order size and production rate, *i.e.*,

$$L = \frac{Q}{P} \tag{2.2}$$

here, it is assumed that the lead time is in direct proportion to the buyer's ordering size and in inverse proportion to the vendor's production rate [35].

(7) The vendor offers a certain trade credit period, t_c, to attract the buyer to cooperate in the integrated strategy. Thus, the buyer is not necessary to pay immediately after receiving the product. Besides, the credit period t_c is less than the reorder interval, which means that the credit period cannot be longer than the time at which another order is placed. This is in agreement with the usual practice.

(8) The buyer deposits the sale income in a bank with annual interest rate I_d before the payment is due. At the payment time, the buyer pays off the purchased products' cost to the vendor. The buyer has a loan from a bank for the unpaid purchase cost of unsold units. During the period of delayed payment, the vendor has an opportunity interest loss with annual rate I_v where $I_v = I_d$.

Note: For the fuzzy back-order (or lost sales) rate in multi-echelon integrated inventory model based on the Centroid or center of gravity method, all pertinent definitions of fuzzy sets are given in Appendix A.

Remark 2.1. Centroid or center of gravity method obtains the center of area (x^*) occupied by the fuzzy sets. It is given by the following formula $x^* = \frac{\int x\mu(x)\mathrm{d}x}{\int \mu(x)\mathrm{d}x}$ where $\mu(x)$ is membership function [8].

3. Model formulation

In this study a continuous review integrated inventory policy is adopted for a two-stage supply chain with interactive decisions on lead time between a vendor and a buyer. In the proposed scenario, replenishment lead time is interactively determined by both the two sides' decisions. For the vendor, once an order is placed from the buyer (here, vendor manufactures product according to make-to-order mode. Hence, the buyer's order size is also his production lot-size), he starts producing product and his inventory rises until the production finishes. As soon as the buyer's order size is completed by the vendor, it is delivered to the buyer and he receives it immediately (zero delivery time). This is a reasonable assumption as long as the delivery time is shorter than the manufacturing time. In addition, the vendor offers a certain trade credit period to attract the buyer to cooperate in the integrated strategy. Hence, the buyer deposits the sale income in a bank with annual interest rate before the payment is due. The decision objectives of the vendor and buyer are to minimize their joint expected total cost of the system. The inventory profile for both the vendor (or vendor) and the buyer (buyer) is depicted in Figure 1.

In this study, the integrated expected annual total cost of the supply chain $C(Q,P,k)$ is sum of buyer's expected total cost, $C_b(Q,k)$, and vendor's expected total cost, $C_v(Q,P)$.

The buyer places an order of Q units, therefore for expected cycle time of Q/D, the expected ordering cost per unit time can be given by AD/Q. Buyer's expected net inventory level just before arrival of a procurement quantity Q is only the safety stock $s_s = r - DL$. The buyer's expected net inventory level immediately after arrival of a procurement Q is $Q + s_s$ units. Hence, the buyer's expected inventory over the cycle is $\frac{Q}{2} + k\sigma\sqrt{L}$.

Let t_c be the credit period and let h_b be the unit stock-holding cost per unit time excluding interest charges for stock financing. The expected inventory over the cycle is $\frac{Q}{2} + k\sigma\sqrt{L}$, where $\frac{Q}{2}$ is the expected cycle stock. Hence, the buyer's holding cost for the cycle stock per unit time is $\frac{h_bQ}{2}$.

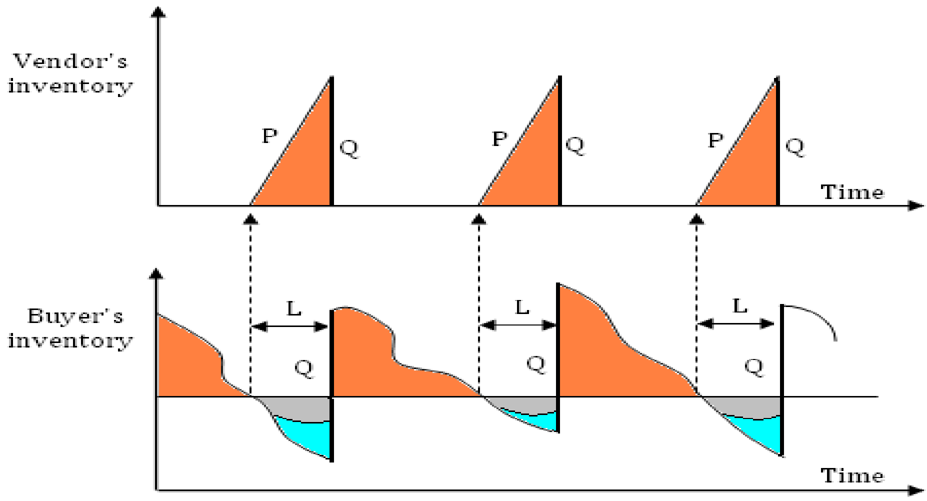

FIGURE 1. The inventory pattern for the vendor and the buyer.

On the other hand, the safety stock $k\sigma\sqrt{L}$ which is held throughout the cycle. The vendor charges interest at rate I_c for this portion of the stock and the buyer must pay the corresponding holding cost as per the trade credit policy. Therefore, buyer's safety stock cost is the sum of the holding cost and interest charged: $(h_b + pI_c)k\sigma\sqrt{L}$.

The credit period, t_c, which is assumed less than the reorder interval. Note that this assumption is reasonable, because the payments of previous order should be made before another order is placed. Therefore, when the buyer's permissible delay period expires on or before all inventories are depleted completely, the buyer can sell the items and earn interest with the rate of I_d until the end of the credit period t_c. Hence, the buyer's interest earned per unit time is $\frac{sI_dD}{Q}\int_0^{t_c} Dt\mathrm{d}t = \frac{D^2t_c^2sI_d}{2Q}$. In addition, the expected shortage, $E(X-r)^+$, is partially backordered in the previous cycle and is partially cleared in the beginning of the current cycle. Therefore, the buyer earns interest of $\frac{Dst_cI_d}{Q}\beta E(X-r)^+$ per unit time during the credit period. On the other hand, the lost sales $(1-\beta)E(X-r)^+$ which is held throughout the cycle. Therefore, the vendor charges interest at rate I_c for this portion of the stock and the buyer must pay the corresponding holding cost as well as the buyer incurs the marginal profit of the lost sales. Hence, the buyer's total cost for the shortage stock per unit time is the sum of the holding cost, interest charged and marginal profit: $(1-\beta)E(X-r)^+\left(\frac{D\pi_0}{Q} + h_b + pI_c\right)$.

Conversely, the buyer still has $(Q - Dt_c)$ unsold units at the end of the permissible delay period. The vendor charges interest for this portion of the stock. However, the buyer has a loan from a bank for unpaid purchase costs for unsold units, at the common interest rate of I_c. Therefore, the opportunity interest cost per cycle time for unsold units is $\frac{pI_cD}{Q}\int_{t_c}^{\frac{Q}{D}}(Q-Dt)\mathrm{d}t = \frac{(Q-Dt_c)^2pI_c}{2Q}$.

Accordingly, the expected annual total cost per unit time for the buyer comprised of ordering cost, holding cost, safety stock cost, opportunity interest cost, shortage cost and interest earned, is expressed by

$$C_b(Q,k) = \frac{D}{Q}A + \frac{h_bQ}{2} + (h_b + pI_c)k\sigma\sqrt{L} + (1-\beta)E(X-r)^+\left[\frac{D\pi_0}{Q} + h_b + pI_c\right] + \frac{(Q-Dt_c)^2pI_c}{2Q} - \frac{D^2t_c^2sI_d}{2Q} - \frac{Dst_cI_d}{Q}\beta E(X-r)^+. \tag{3.1}$$

For the vendor, his expected annual total cost can be given by $C_v(Q,P)$ = setup cost+holding cost+productivity improvement investment + opportunity interest loss.

Here, vendor's annual setup cost per year can be given by $\frac{D}{Q}S$; his annual holding cost given by Song *et al.* [35] is $\frac{Q}{2}\frac{D}{P}h_v$. The productivity improvement investment given by Moon and Cha [25] is $\frac{D}{Q}(P-P_0)LC_v = \left(1-\frac{P_0}{P}\right)DC_v$. The expected opportunity interest loss per unit time for the vendor is $I_v p t_c D$.

Therefore, the vendor's expected annual total cost is given by

$$C_v(Q,P) = \frac{D}{Q}S + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + I_v p t_c D. \tag{3.2}$$

From the view of the whole supply chain, the integrated expected annual total cost is the sum of the buyer's and the vendor's total annual cost. That is,

$$\begin{aligned} C(Q,P,k) = {} & \frac{D}{Q}\left\{A + S - \beta s t_c I_d E(X-r)^+\right\} + (h_b + pI_c)\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) \\ & + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + (1-\beta)E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right]. \end{aligned} \tag{3.3}$$

Now we attempt to modify model (3.3) by fuzzifying the backorder rate (or equivalently, fuzzifying the lost sales rate). For convenience, we first let $\delta \equiv 1-\beta$ denote the lost sales rate. Therefore, for any $Q > 0$, $P > 0$ and $k > 0$, we may remodify the expected annual total cost function (as expressed in Eq. (3.3)) as follows:

$$\begin{aligned} C_{(Q,P,k)}(\delta) \equiv {} & C(Q,P,k) \\ = {} & \frac{D}{Q}\left\{A + S - \beta s t_c I_d E(X-r)^+\right\} + (h_b + pI_c)\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) \\ & + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + \delta E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right]. \end{aligned} \tag{3.4}$$

It should be noted that in the above model, the lost sales rate δ during the planning horizon is assumed a fixed constant. However, when the inventory planning is completed, due to various uncertainties the lost sales rate in practical problem may be not equal to δ but just close to it. Therefore, it is difficult for the decision maker to determine a fixed value δ for the lost sales rate in the planning horizon. On the contrary, it is easier to set the lost sales rate in the interval $[\delta-\Delta_1, \delta+\Delta_2]$, where $0 < \Delta_1 < \delta$, $0 < \Delta_2$ and Δ_1, Δ_2 are determined by the decision-maker. To find the corresponding fuzzy set with this interval $[\delta-\Delta_1, \delta+\Delta_2]$, we take any value $\dot{\delta}$ from this interval and then compare it with δ of crisp value. If $\dot{\delta} = \delta$, then we define the error $|\dot{\delta} - \delta| = 0$. In the fuzzy sense, we can use the term confidence level instead of error.

When the error is zero the confidence level will be the largest, and we set it to be 1. If $\dot{\delta}$ is located in $[\delta-\Delta_1, \delta]$ or $[\delta, \delta+\Delta_2]$ the farther the value $\dot{\delta}$ deviates from δ, the larger of the error $|\dot{\delta} - \delta|$, and hence, the smaller the confidence level. When $\dot{\delta} = \delta - \Delta_1$ and $\dot{\delta} = \delta + \Delta_2$, the error $|\dot{\delta} - \delta|$ will attain to the largest, and the confidence level will be the smallest and we set it be zero. Therefore, corresponding to the interval $[\delta-\Delta_1, \delta+\Delta_2]$ we set the triangular fuzzy number, $\tilde{\delta} = (\delta-\Delta_1, \delta, \delta+\Delta_2)$, $0 < \Delta_1 < \delta$, $0 < \Delta_2$.

The membership grade of δ in $\tilde{\delta}$ is 1. The farther the point in $[\delta-\Delta_1, \delta+\Delta_2]$ is from both sides of δ, the less the membership grade is. The membership grade shares the same property with the confidence level. If we make a correspondence between membership grade and confidence level, it is reasonable to set a fuzzy number in $\tilde{\delta} = (\delta-\Delta_1, \delta, \delta+\Delta_2)$ corresponding to the interval $[\delta-\Delta_1, \delta+\Delta_2]$. Further, here we describe the membership function of $\tilde{\delta}$ similar to [28] as follows:

$$\mu_{\tilde{\delta}}(x) = \begin{cases} \frac{x-\delta+\Delta_1}{\Delta_1}, \text{ if } & \delta-\Delta_1 \le x \le \delta \\ \frac{\delta+\Delta_2-x}{\Delta_2}, \text{ if } & \delta \le x \le \delta+\Delta_2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

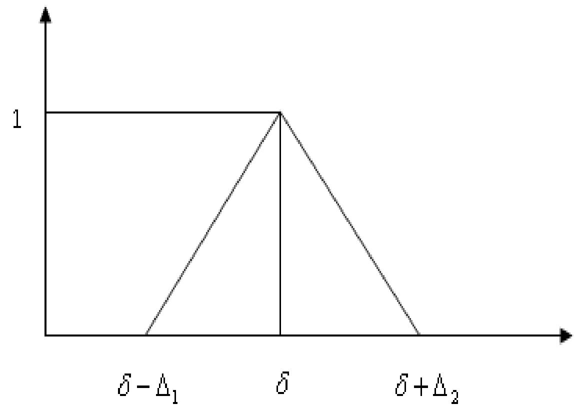

FIGURE 2. Triangular fuzzy number $\tilde{\delta}$ [28].

Then, according to [28], the centroid for $\mu_{\tilde{\delta}}(x)$ is given by

$$\delta^* = \frac{\int_{-\infty}^{\infty} x\mu_{\tilde{\delta}}(x)\mathrm{d}x}{\int_{-\infty}^{\infty} \mu_{\tilde{\delta}}(x)\mathrm{d}x} = \delta + \frac{1}{3}(\Delta_2 - \Delta_1).$$

We regard this value as the estimate of lost sales rate in the fuzzy sense.

For any $Q > 0$, $P > 0$ and $k > 0$, we let $C_{(Q,P,k)}(x) = y$ (>0). By extension principle [17], the membership function of the fuzzy cost $C_{(Q,P,k)}(\tilde{\delta})$ similar to [28] is given by

$$\mu_{C_{(Q,P,k)}(\tilde{\delta})}(y) = \begin{cases} \sup_{x \in C^{-1}_{(Q,P,k)}(y)} \mu_{\tilde{\delta}}(x), & \text{if } \ C^{-1}_{(Q,P,k)}(y) \neq \emptyset \\ 0, & \text{if } \ C^{-1}_{(Q,P,k)}(y) = \emptyset. \end{cases} \tag{3.6}$$

From $C_{(Q,P,k)}(x) = y$ and equation (3.4), we get

$$\begin{cases} \frac{D}{Q}\left\{A + S - \beta s t_c I_d E(X-r)^+\right\} + (h_b + pI_c)\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) \\ + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + \left(1 - \frac{P_0}{P}\right)DC_v + xE(X-r)^+\left[\left\{h_b + pI_c + \frac{\pi_0 D}{Q}\right\}\right] = y. \end{cases} \tag{3.7}$$

Hence,

$$x = \frac{yQ - \left\{WD + QW_0 + \frac{D^2t^2}{2}(pI_c - sI_d) + QDt_c p(I_v - I_c) + \frac{Q^2}{2}\frac{D}{P}h_v + Q\left(1 - \frac{P_0}{P}\right)DC_v\right\}}{E(X-r)^+\left[Q(h_b + pI_c) + \pi_0 D\right]} \tag{3.8}$$

where $W = A + S - \beta s t_c I_d E(X-r)^+$ and $W_0 = (h_b + pI_c)\left(\frac{Q}{2} + k\sigma\sqrt{L}\right)$.

Therefore, from equations (3.5) and (3.8), the membership function of $C_{(Q,P,k)}(\tilde{\delta})$ can be written as

$$\mu_{C_{(Q,P,k)}(\tilde{\delta})}(y) = \begin{cases} \frac{yQ - \left\{WD + QW_0 + \frac{D^2t^2}{2}(pI_c - sI_d) + QDt_c p(I_v - I_c) + \frac{Q^2}{2}\frac{D}{P}h_v + Q\left(1 - \frac{P_0}{P}\right)DC_v\right\}}{E(X-r)^+[Q(h_b + pI_c) + \pi_0 D]\Delta_1} - \frac{\delta - \Delta_1}{\Delta_1}, & y_1 \leq y \leq y_2 \\ \frac{\delta + \Delta_2}{\Delta_2} + \frac{\left\{WD + QW_0 + \frac{D^2t^2}{2}(pI_c - sI_d) + QDt_c p(I_v - I_c) + \frac{Q^2}{2}\frac{D}{P}h_v + Q\left(1 - \frac{P_0}{P}\right)DC_v\right\} - yQ}{E(X-r)^+[Q(h_b + pI_c) + \pi_0 D]\Delta_2}, & y_2 \leq y \leq y_3 \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

where

$$y_1 = \frac{WD}{Q} + W_0 + \frac{D^2t^2}{2Q}(pI_c - sI_d) + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + DC_v\left(1 - \frac{P_0}{P}\right) + (\delta - \Delta_1)E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right],$$

$$y_2 = \frac{WD}{Q} + W_0 + \frac{D^2t^2}{2Q}(pI_c - sI_d) + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + DC_v\left(1 - \frac{P_0}{P}\right) + \delta E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right],$$

$$y_3 = \frac{WD}{Q} + W_0 + \frac{D^2t^2}{2Q}(pI_c - sI_d) + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + DC_v\left(1 - \frac{P_0}{P}\right) + (\delta + \Delta_2)E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right].$$

The pictorial of the membership function of $C_{(Q,P,k)}(\tilde{\delta})$ is shown in Figure 3. Now we derive the centroid of $\mu_{C_{(Q,P,k)}(\tilde{\delta})}(y)$ as follows:

$$\begin{aligned} IETC(Q,P,k) &= \frac{\int_{-\infty}^{\infty} y\mu_{C_{(Q,P,k)}(\tilde{\delta})}(y)\mathrm{d}y}{\int_{-\infty}^{\infty} \mu_{C_{(Q,P,k)}(\tilde{\delta})}(y)\mathrm{d}y} = \frac{1}{3}(y_1 + y_2 + y_3) \\ &= C(Q,P,k) + \frac{(\Delta_2 - \Delta_1)}{3}E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right]. \end{aligned}$$

Thus, we obtain the following property.

Property 3.1. For any $Q > 0$, $P > 0$ and $k > 0$, the estimate of the integrated expected annual total inventory cost in the fuzzy sense is

$$\begin{aligned} IETC(Q,P,k) = {} & \frac{D}{Q}\left\{A + S - \beta s t_c I_d E(X-r)^+\right\} + (h_b + pI_c)\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) \\ & + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + \left(1 - \frac{P_0}{P}\right)DC_v + \delta E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right] \\ & + \frac{(\Delta_2 - \Delta_1)}{3}E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right]. \end{aligned} \tag{3.10}$$

Remark 3.2. If we let $Z = \frac{(\Delta_2 - \Delta_1)}{3.C(Q,P,k)}E(X-r)^+\left(h_b + pI_c + \frac{\pi_0 D}{Q}\right)$, then from equation (3.10) we obtain $\frac{IETC(Q,P,k) - C(Q,P,k)}{C(Q,P,k)} \times 100\% = Z \times 100\%$, which implies $[IETC(Q,P,k) - C(Q,P,k)] \times 100\% = Z \times C(Q,P,k) \times 100\%$.

Note 1. If $\Delta_1 = \Delta_2$, then Figure 2 is an isosceles triangle and equation (3.10) reduces to $IETC(Q,P,k) = C(Q,P,k)$, this implies that the fuzzy case becomes the crisp case; that is, the fixed lost sales rate inventory model is a special case of our new fuzzy lost sales rate inventory model.

Note 2. If $\Delta_1 < \Delta_2$, then the triangle in Figure 2 is skewed to the right. In this case, $IETC(Q,P,k) > C(Q,P,k)$ and the increment of $IETC(Q,P,k)$ is $Z\%$ of $C(Q,P,k)$.

Note 3. If $\Delta_1 > \Delta_2$, then the triangle in Figure 2 is skewed to the left. In this case, $IETC(Q,P,k) < C(Q,P,k)$ and the decrement of $IETC(Q,P,k)$ is $|Z|\%$ of $C(Q,P,k)$.

Now, the objective of this study is to minimize the integrated expected annual total cost in the fuzzy sense (as expressed in Eq. (3.10)) by simultaneously selecting lot-size Q, production rate P and safety factor k. Thus

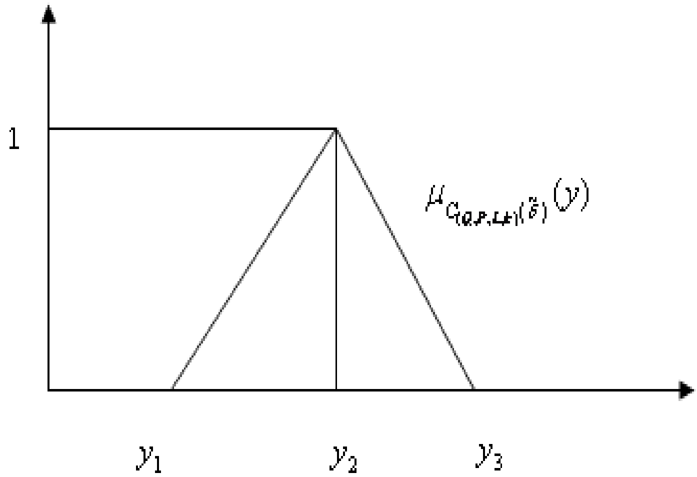

FIGURE 3. Triangular fuzzy number $C_{(Q,P,k)}(\tilde{\delta})$ [28].

the problem of integrated inventory system with trade credit financing and fuzzy backorder rate when lead time depends on both lot-size and production rate can be mathematically formulated as the following model:

$$\begin{aligned}\text{Min}_{Q,P,k}\text{Max}_{F\in\Omega} IETC(Q,P,k) &= \frac{D}{Q}\left\{A+S-\beta st_c I_d E(X-r)^+\right\} + (h_b+pI_c)\left(\frac{Q}{2}+k\sigma\sqrt{L}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d)\\ &+ Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + \delta E(X-r)^+\left[h_b + pI_c + \frac{\pi_0 D}{Q}\right]\\ &+ \frac{(\Delta_2-\Delta_1)}{3}E(X-r)^+\left[h_b+pI_c+\frac{\pi_0 D}{Q}\right]. \end{aligned} \quad (3.11)$$

Subject to $P_1 \geq P \geq P_0$, $Q > 0$, $k > 0$.

Now, we try to use minimax approach similar to Song *et al.* [35] to solve this distribution-free problem. The minimax approach is to find the most unfavorable cdf F in Ω (which is a class of F with mean DL and standard deviation $\sigma\sqrt{L}$) for each (Q,P,k) and then to minimize the expected annual total cost over (Q,P,k). In order to minimize (3.11) we need the following lemma from [7]:

Lemma 3.3. *For any $F \in \Omega$,*

$$E(X-r)^+ \leq \frac{1}{2}\left(\sqrt{\sigma^2 L + (r-DL)^2} - (r-DL)\right). \quad (3.12)$$

Moreover, the upper bound equation (3.12) is tight.

Proof. Proof of Lemma 3.3 is given in [7] so we skip the proof details. □

Now, substituting equation (2.1) into (3.12), we can get

$$E(X-r)^+ \leq \frac{1}{2}\sigma\sqrt{L}\left(\sqrt{1+k^2} - k\right). \quad (3.13)$$

For convenience, letting $\Psi(k) = \frac{1}{2}\left(\sqrt{1+k^2}-k\right)$, we have

$$E(X-r)^+ \leq \sigma\sqrt{L}\,\Psi(k). \quad (3.14)$$

Using equation (3.14), we can transform equation (3.11) denoting $\varPi = \varPi(Q,P,k) = \mathrm{Max}_{F\in\Omega} IETC(Q,P,k)$ to

$$\begin{aligned}\varPi = {} & \frac{D}{Q}\left\{A+S-\beta st_c I_d\sigma\sqrt{L}\,\Psi(k)\right\} + (h_b+pI_c)\left(\frac{Q}{2}+k\sigma\sqrt{L}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) \\ & + Dt_c p(I_v - I_c) + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + \delta\sigma\sqrt{L}\,\Psi(k)\left[h_b+pI_c+\frac{\pi_0 D}{Q}\right] \\ & + \frac{(\varDelta_2-\varDelta_1)}{3}\sigma\sqrt{L}\,\Psi(k)\left[h_b+pI_c+\frac{\pi_0 D}{Q}\right]. \end{aligned} \tag{3.15}$$

Subject to $P_1 \geq P \geq P_0$, $Q > 0$, $k > 0$.

Now, inserting equation (2.1) into (3.15), we obtain

$$\begin{aligned}\varPi = {} & \frac{D}{Q}\left\{A+S-\beta st_c I_d\sigma\sqrt{\frac{Q}{P}}\,\Psi(k)\right\} + (h_b+pI_c)\left(\frac{Q}{2}+k\sigma\sqrt{\frac{Q}{P}}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) + Dt_c p(I_v - I_c) \\ & + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + \sigma\sqrt{\frac{Q}{P}}\,\Psi(k)\left(h_b+pI_c+\frac{\pi_0 D}{Q}\right)\left[\delta+\frac{(\varDelta_2-\varDelta_1)}{3}\right]. \end{aligned} \tag{3.16}$$

Subject to $P_1 \geq P \geq P_0$, $Q > 0$, $k > 0$.

Thus, the objective of this study (as expressed in Eq. (3.11)) is equivalent to

$$\begin{aligned}\mathrm{Min}_{Q,P,k}\ \varPi = {} & \frac{D}{Q}\left\{A+S-\beta st_c I_d\sigma\sqrt{\frac{Q}{P}}\,\Psi(k)\right\} + (h_b+pI_c)\left(\frac{Q}{2}+k\sigma\sqrt{\frac{Q}{P}}\right) + \frac{D^2t^2}{2Q}(pI_c - sI_d) + Dt_c p(I_v - I_c) \\ & + \frac{Q}{2}\frac{D}{P}h_v + \left(1-\frac{P_0}{P}\right)DC_v + \sigma\sqrt{\frac{Q}{P}}\,\Psi(k)\left(h_b+pI_c+\frac{\pi_0 D}{Q}\right)\left[\delta+\frac{(\varDelta_2-\varDelta_1)}{3}\right]. \end{aligned} \tag{3.17}$$

Subject to $P_1 \geq P \geq P_0$, $Q > 0$, $k > 0$.

In order to solve the above problem, some analysis need to be done firstly similar to Song *et al.* [35], For convenience, we temporarily ignore the constraints of the problem.

Lemma 3.4. *For fixed P and k, $\varPi$ is convex in Q and the optimal solution to minimize $\varPi$ occurs at the unique value which satisfies that $\frac{\partial \varPi}{\partial Q} = 0$.*

Proof. Taking the first partial derivative of $\varPi$ using equation (3.17) with respect to Q, we obtain

$$\frac{\partial \varPi}{\partial Q} = \frac{H}{2} + \frac{Dh_v}{2P} + Q^{-1/2}\left(\frac{Hk\sigma}{2\sqrt{P}} + \frac{H\zeta}{2\sqrt{P}} - \frac{D\left[\pi_0\zeta - \beta st_c I_d\sigma\Psi(k)\right]}{2\sqrt{P}Q} - \frac{D(A+S)}{Q^{3/2}} - \frac{D^2t_c^2(pI_c - sI_d)}{2Q^{3/2}}\right)$$

where $H = h_b + pI_c$ and $\zeta = \sigma\Psi(k)\left(\delta + \frac{\varDelta_2-\varDelta_1}{3}\right)$.

Denote

$$\begin{aligned} G(Q) &= Q^{-1/2}\left(\frac{Hk\sigma}{2\sqrt{P}} + \frac{H\zeta}{2\sqrt{P}} - \frac{D\left[\pi_0\zeta - \beta st_c I_d\sigma\Psi(k)\right]}{2\sqrt{P}Q} - \frac{D(A+S)}{Q^{3/2}} - \frac{D^2t_c^2(pI_c - sI_d)}{2Q^{3/2}}\right), \\ g_1(Q) &= Q^{-1/2}, \end{aligned} \tag{3.18}$$

and

$$g_2(Q) = \left(\frac{Hk\sigma}{2\sqrt{P}} + \frac{H\zeta}{2\sqrt{P}} - \frac{D\left[\pi_0\zeta - \beta st_c I_d\sigma\Psi(k)\right]}{2\sqrt{P}Q} - \frac{D(A+S)}{Q^{3/2}} - \frac{D^2t_c^2(pI_c - sI_d)}{2Q^{3/2}}\right). \tag{3.19}$$

Then we have $G(Q) = g_1(Q)g_2(Q)$ when $\frac{H(k\sigma+\zeta)}{2\sqrt{P}} > \frac{D}{2\sqrt{P}}\left\{-\frac{\left[\pi_0\zeta-\beta st_c I_d\sigma\Psi(k)\right]}{Q} - \frac{2\sqrt{P}(A+S)}{Q^{3/2}} - \frac{Dt_c^2(pI_c-sI_d)\sqrt{P}}{Q^{3/2}}\right\}$, and

$$\frac{\partial \Pi}{\partial Q} = \frac{HP + Dh_v}{2P} + G(Q).$$

It is obvious that $g_2(Q)$ is an increasing function of Q for $Q > 0$.

Since $\lim_{Q\to 0^+} g_2(Q) \to -\infty$ and $\lim_{Q\to\infty} g_2(Q) = \frac{H(k\sigma+\zeta)}{2\sqrt{P}} > 0$, there exists a unique $Q = Q_{01}$ which satisfies $g_2(Q_{01}) = 0$ and $G(Q_{01}) = 0$.

Based on equation (3.18), we can observe that for $Q > 0, g_1(Q) > 0$ and $g_1(Q)$ is a decreasing function of Q. Then, for any $Q_1 \in (0, Q_{01}), Q_2 \in (0, Q_{01})$ and $Q_1 < Q_2$, we know that $g_1(Q_1) > g_1(Q_2) > 0$, and $0 > g_2(Q_1) > g_2(Q_2)$.

Hence, $g_1(Q_1)g_2(Q_1) < g_1(Q_2)g_2(Q_2) < 0$, *i.e.*, $G(Q_1) < G(Q_2) < 0$.

This shows that $G(Q)$ is an increasing function of Q on the interval $(0, Q_{01})$. Thus, $\frac{\partial \Pi}{\partial Q} = \frac{HP+Dh_v}{2P} + G(Q)$ is also an increasing function of Q on the interval $(0, Q_{01})$.

For

$$\lim_{Q\to 0^+} \frac{\partial \Pi}{\partial Q} = -\infty, \left.\frac{\partial \Pi}{\partial Q}\right|_{Q=Q_{01}} = \frac{HP + Dh_v}{2P} > 0$$

and

$$\frac{\partial \Pi}{\partial Q} \text{ is an increasing function of } Q \text{ on the interval } (0, Q_{01}),$$

therefore there exists a unique solution to the equation $\frac{\partial \Pi}{\partial Q} = 0$ on the interval $(0, Q_{01})$.

Let Q_{02} denote the solution to the equation $\frac{\partial \Pi}{\partial Q} = 0$ on the interval $(0, Q_{01})$, we observe that

(i) for $0 < Q < Q_{02}, \frac{\partial \Pi}{\partial Q} < 0$;
(ii) for $Q_{02} < Q < Q_{01}, \frac{\partial \Pi}{\partial Q} > 0$;
(iii) for $Q > Q_{01}, \frac{\partial \Pi}{\partial Q} > 0$. This is because $g_1(Q) > 0, g_2(Q) > 0$.

Hence, $G(Q) = g_1(Q)g_2(Q) > 0$ and $\frac{\partial \Pi}{\partial Q} = \frac{HP+Dh_v}{2P} + G(Q) > 0$. This means that $\frac{\partial \Pi}{\partial Q} < 0$ for $0 < Q < Q_{02}$ and $\frac{\partial \Pi}{\partial Q} > 0$ for $Q > Q_{02}$.

Thus, Π is convex in Q and the minimum value of Π occurs at the unique value Q_{02}.

Now, setting $\frac{\partial \Pi}{\partial Q}$ equal to 0 and solving for Q, we obtain

$$Q = \sqrt{\frac{P^{1/2}\left[2D(A+S)P^{1/2} + D^2t_c^2(pI_c - sI_d)P^{1/2} + \zeta\pi_0 DQ^{1/2} - D\beta st_cI_d\sigma\Psi(k)Q^{1/2} - Hk\sigma Q^{3/2} - HQ^{3/2}\zeta\ \right]}{HP + Dh_v}}. \tag{3.20}$$

□

Lemma 3.5. *For fixed Q and P, Π is convex in k and the optimal solution to minimize Π occurs at the unique value which satisfies that* $\frac{\partial \Pi}{\partial k} = 0$.

Proof. Taking first and second partial derivative of Π using equation (3.17) with respect to k, we have

$$\frac{\partial \Pi}{\partial k} = \frac{H\sigma Q^{1/2}}{P^{1/2}} + \frac{\sigma Q^{1/2}}{2P^{1/2}}\left(\frac{k}{\sqrt{1+k^2}} - 1\right)\left(H + \frac{\pi_0 D}{Q}\right)\left\{\delta + \frac{\Delta_2 - \Delta_1}{3}\right\} - \frac{D\beta st_cI_d\sigma}{2P^{1/2}Q^{1/2}}\left(\frac{k}{\sqrt{1+k^2}} - 1\right) \tag{3.21}$$

and

$$\frac{\partial^2 \Pi}{\partial k^2} = \frac{D\beta st_cI_d\sigma}{2P^{1/2}Q^{1/2}}\left(\frac{k^2}{(1+k^2)^{3/2}} - \frac{1}{\sqrt{1+k^2}}\right) + \frac{\sigma Q^{1/2}}{2P^{1/2}}\left(\frac{1}{\sqrt{1+k^2}} - \frac{k^2}{(1+k^2)^{3/2}}\right)\left(H + \frac{\pi D}{Q}\right)\left\{\delta + \frac{\Delta_2 - \Delta_1}{3}\right\} > 0. \tag{3.22}$$

From equation (3.22), we can observe that $\varPi(Q,P,k)$ is convex in k.

Now, setting $\frac{\partial \varPi}{\partial k}$ equal to 0 and solving for k, we obtain

$$k=\sqrt{\frac{1}{\frac{4H}{\left\{\left(H+\frac{\pi_0 D}{Q}\right)\left(\delta+\frac{\varDelta_2-\varDelta_1}{3}\right)-\frac{D\beta st_c I_d}{Q}\right\}}\left(1-\frac{H}{\left\{\left(H+\frac{\pi_0 D}{Q}\right)\left(\delta+\frac{\varDelta_2-\varDelta_1}{3}\right)-\frac{D\beta st_c I_d}{Q}\right\}}\right)}-1}\cdot \tag{3.23}$$

□

Lemma 3.6. *For fixed Q and k, $\varPi$ is strictly increasing or decreasing or a concave function when $P_1 \geq P \geq P_0$. Therefore, the optimal production rate P to minimize $\varPi$ is either P_0 or P_1.*

Proof. Taking first partial derivative of $\varPi$ using equation (3.17) with respect to P, we can obtain

$$\frac{\partial \varPi}{\partial P}=\frac{D\beta st_c I_d\sigma\varPsi(k)}{2Q^{1/2}P^{3/2}}+\frac{DC_vP_0}{P^2}-\frac{Hk\sigma Q^{1/2}}{2P^{3/2}}-\frac{QDh_v}{2P^2}-\frac{Q^{1/2}\sigma\varPsi(k)}{2P^{3/2}}\left(H+\frac{\pi D}{Q}\right)\left\{\delta+\frac{\varDelta_2-\varDelta_1}{3}\right\}. \tag{3.24}$$

For fixed Q and k, without loss of generality, equation (3.24) can be re-written as:

$$\frac{\partial \varPi}{\partial P}=\frac{1}{2P^2}\left(\lambda-\eta Q^{1/2}\sqrt{P}\right) \tag{3.25}$$

where

$$\lambda=\frac{D\beta st_c I_d\sigma\varPsi(k)P^{1/2}}{Q^{1/2}}+2DC_vP_0$$

and

$$\eta=H\sigma k+\frac{Q^{1/2}Dh_v}{P^{1/2}}+\sigma\varPsi(k)\left(H+\frac{\pi D}{Q}\right)\left\{\delta+\frac{\varDelta_2-\varDelta_1}{3}\right\}.$$

Based in equation (3.25), we can observe that the sign of $\frac{\partial \varPi}{\partial P}$ is determined by the value $\left(\lambda-\eta Q^{1/2}\sqrt{P}\right)$, and it is obvious that $\lambda>0$. When $P_1 \geq P \geq P_0$, the maximum value of $\left(\lambda-\eta Q^{1/2}\sqrt{P}\right)$ is $\left(\lambda-\eta Q^{1/2}\sqrt{P_0}\right)$ and the minimum value of $\left(\lambda-\eta Q^{1/2}\sqrt{P}\right)$ is $\left(\lambda-\eta Q^{1/2}\sqrt{P_1}\right)$. Since the trend of $\varPi$ depends on the sign of $\frac{\partial \varPi}{\partial P}$, we discuss the signs of $\frac{\partial \varPi}{\partial P}$ in three cases:

Case 1. If $\left(\lambda-\eta Q^{1/2}\sqrt{P_0}\right)<0$, then $\frac{\partial \varPi}{\partial P}<0$ for all $P\in[P_0,P_1]$. Hence, $\varPi$ is a strictly decreasing function of P on the interval $[P_0,P_1]$.

In this case, P_1 is the optimal production rate for minimizing $\varPi$.

Case 2. If $\left(\lambda-\eta Q^{1/2}\sqrt{P_1}\right)>0$, then $\frac{\partial \varPi}{\partial P}>0$ for all $P\in[P_0,P_1]$. Hence, $\varPi$ is a strictly increasing function of P on the interval $[P_0,P_1]$.

In this case, P_0 is the optimal production rate for minimizing $\varPi$.

Case 3. If $\left(\lambda-\eta Q^{1/2}\sqrt{P_0}\right)\geq 0$ and $\left(\lambda-\eta Q^{1/2}\sqrt{P_1}\right)\leq 0$, then the case shows that $\varPi$ is a concave function on the interval $[P_0,P_1]$.

In this case, the optimal production rate that minimize $\varPi$ for fixed Q and k can be selected as either P_0 or P_1 by comparing $\varPi(Q,P_0,k)$ and $\varPi(Q,P_1,k)$.

Therefore, the optimal production rate is always P_0 or P_1. □

As analyzed above, there is no closed-form solution to problem (3.11). Hence, iterative algorithm is used to search for the optimal solutions. In terms of Lemma 3.5, if substituting a given Q into the right-hand side of equation (3.23), then a unique k can be obtained by equation (3.23). Then, using the present values of Q and k,

TABLE 2. Solving process of the model.

Δ_1	Δ_2	Iteration	Q	k	P
0.2	0.2	0	1277.3	2.6570	109 500
		1	1278.5	2.4572	109 500
		2	1278.5	2.4560	109 500
		3	1278.5	2.4560	109 500
0.1	0.4	0	1276.3	2.9361	109 500
		1	1277.2	2.7204	109 500
		2	1277.2	2.7194	109 500
		3	1277.2	2.7194	109 500
0.4	0.1	0	1278.6	2.3450	109 500
		1	1280.2	2.1627	109 500
		2	1280.2	2.1612	109 500
		3	1280.2	2.1612	109 500

a unique value of P can be determined according to Lemma 3.6, *i.e.*, $P = \text{argmin}_{\text{P}}\{\Pi(Q, P_0, k), \Pi(Q, P_1, k)\}$. Next, in terms of Lemma 3.4, if substituting the present values of (Q, P, k) into the right-hand side of equation (3.20), then a new unique value of Q can be obtained by equation (3.20). With the new value of Q, repeat the above mentioned process. Once no change occurs in the values (Q, P, k), the optimal solution to problem (3.11) can be got. Combining the above finding and Lemmas, we propose the following iterative algorithm similar to Song *et al.* [35] for solving the problem.

Algorithm

Step 1. Set an error tolerance, $\varepsilon > 0$ (small positive number), go to step 2.

Step 2. Set $Q_0 = \sqrt{\frac{2D(A+S)}{H}}$. Compute k by substituting Q_0 for Q in equation (3.23). Next, find P using Lemma 3.6, that is, $P = \text{argmin}_{\text{P}}\{\Pi(Q, P_0, k), \Pi(Q, P_1, k)\}$.
Step 3. Compute Q by substituting the present value of k, Q_0 and P for k, Q and P respectively in equation (3.20).
Step 4. If $|Q - Q_0| < \varepsilon$, then output the Q, P and k and END the iterations; else let $Q_0 = Q$, go to Step 2.

4. Numerical example

In this section, a numerical example is given to illustrate the above solution procedure, and to highlight the differences between crisp and the fuzzy cases. The solutions to this example is obtained by using the computer MatLab software. We consider the values of the following parameters which are used in Song *et al.* [35]: $D = 36\,500$ units/year, $\sigma = 955$ units/year, $A = \$4000$/order, $S = \$5000$/setup, $h_b = \$500$/unit/year, $h_v = \$100$/unit/year, $\pi_0 = \$1000$/unit, $P_0 = 73\,000$ units/year, $P_1 = 109\,500$ units/year and $C_v = \$5$/unit. For measuring lead time in day units, it is assumed that there are 365 days in a year.

In addition, for trade credit and fuzzy backorder integrated inventory model, we take $p = \$600$/unit, $s = \$800$, $I_d = 0.02$, $I_v = 0.02$, $I_c = 0.06$ and $\beta = 0.5$. Here, we consider three cases: $(\Delta_1, \Delta_2) = (0.2, 0.2)$, $(\Delta_1, \Delta_2) = (0.1, 0.4)$ and $(\Delta_1, \Delta_2) = (0.4, 0.1)$. Further, we solve each case for lost sale rate $\delta = 0.5$. Given error tolerance $\varepsilon = 0.01$, the solving process for these cases is summarized in Table 2.

From Table 2, when $\Delta_1 = \Delta_2 = 0.2$ (in this situation, the fuzzy case becomes the crisp case), by comparing $IETC(Q^*, P^*, k^*)$, we obtain the optimal solution $(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) = (1278.5, 109\,500, 2.4560)$ and the corresponding minimum expected total annual cost $IETC(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) = \$969\,530$. Moreover, when $\Delta_1 = 0.1$ and $\Delta_2 = 0.4$, *i.e.*, the fuzzy number $\tilde{\delta} = (0.4, 0.5, 0.9)$, we have $(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) = (1277.2, 109\,500, 2.7194)$ and $IETC(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) = \$998\,550$. Note that since $C(Q, P, k) = \$969\,530$ is the corresponding minimum expected total annual cost in the

TABLE 3. Effects of D on optimal solution.

% of change in D	(Δ_1, Δ_2)	Q	k	P	L	Π
+50	(0.2, 0.2)	1649.7	2.6666	109 500	5.4990	1 256 300
	(0.1, 0.4)	1648.4	2.9478	109 500	5.4947	1 291 600
	(0.4, 0.1)	1618.5	2.3788	73 000	8.0925	1 211 700
			$R_Z = 2.81\%$		$\tilde{R}_Z = 3.55\%$	
+25	(0.2, 0.2)	1468.7	2.5723	109 500	4.8957	1 115 500
	(0.1, 0.4)	1467.4	2.8455	109 500	4.8913	1 147 900
	(0.4, 0.1)	1445.8	2.2884	73 000	7.2290	1 078 500
			$R_Z = 2.90\%$		$\tilde{R}_Z = 3.32\%$	
–25	(0.2, 0.2)	1075.2	2.3052	109 500	3.5840	814 940
	(0.1, 0.4)	1073.9	2.5560	109 500	3.5797	840 150
	(0.4, 0.1)	1076.8	2.0241	109 500	3.5893	786 690
			$R_Z = 3.09\%$		$\tilde{R}_Z = 3.47\%$	
–50	(0.2, 0.2)	850.3	2.0941	109 500	2.8343	645 040
	(0.1, 0.4)	849.0	2.3277	109 500	2.8300	665 730
	(0.4, 0.1)	852.0	1.8316	109 500	2.8400	621 820
			$R_Z = 3.21\%$		$\tilde{R}_Z = 3.60\%$	

crisp case, and hence the absolute relative variation in the fuzzy sense for the minimum expected total cost is

$$R_Z = \frac{\left|IETC(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) - C(Q,P,k)\right|}{C(Q,P,k)} \times 100\% = \frac{|998\,550 - 969\,530|}{969\,530} \times 100\% = 2.99\%.$$

Similarly, for the case $\Delta_1 = 0.4$ and $\Delta_2 = 0.1$, *i.e.*, the fuzzy number $\tilde{\delta} = (0.1, 0.5, 0.6)$, we have $(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) = (1280.2, 109500, 2.1612)$ and $IETC(Q_{\tilde{\delta}}, P_{\tilde{\delta}}, k_{\tilde{\delta}}) = \$937\,050$, and the absolute relative variation in the fuzzy sense for the minimum expected total annual cost is

$$\tilde{R}_Z = \frac{|937\,050 - 969\,530|}{969\,530} \times 100\% = 3.35\%.$$

5. Effects of parameters and managerial implications

5.1. Effects of parameters

To further illustrate the model and algorithm, we now study the effects of parameters $D, A, S, C_v, h_b, \sigma, s$ and π_0. The sensitivity analysis is performed by changing the parameters of $D, A, S, C_v, h_b, \sigma, s$ and π_0 by +50%, +25%, −25% and –50%. The effects of parameters are shown in Tables 2 to 9. On the other hand, we summarize the computational results for different $t_c = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 in Table 10.

TABLE 4. Effects of A on optimal solution.

% of change in A	$(\varDelta_1, \varDelta_2)$	Q	k	P	L	$\varPi$
+50	(0.2, 0.2)	1375.9	2.3584	109 500	4.5863	1 024 500
	(0.1, 0.4)	1374.5	2.6134	109 500	4.5817	1 053 600
	(0.4, 0.1)	1377.7	2.0727	109 500	4.5923	991 970
			$R_Z = 2.84\%$		$\tilde{R}_Z = 3.18\%$	
+25	(0.2, 0.2)	1328.1	2.4050	109 500	4.4270	997 540
	(0.1, 0.4)	1326.7	2.6641	109 500	4.4223	1 026 600
	(0.4, 0.1)	1329.9	2.1149	109 500	4.4330	965 010
			$R_Z = 2.91\%$		$\tilde{R}_Z = 3.26\%$	
–25	(0.2, 0.2)	1227.0	2.5121	109 500	4.0900	940 400
	(0.1, 0.4)	1225.7	2.7803	109 500	4.0857	969 380
	(0.4, 0.1)	1228.6	2.2121	109 500	4.0953	907 950
			$R_Z = 3.08\%$		$\tilde{R}_Z = 3.45\%$	
–50	(0.2, 0.2)	1173.1	2.5745	109 500	3.9103	909 980
	(0.1, 0.4)	1171.9	2.8484	109 500	3.9063	938 930
	(0.4, 0.1)	1174.7	2.2686	109 500	3.9157	877 580
			$R_Z = 3.18\%$		$\tilde{R}_Z = 3.56\%$	

TABLE 5. Effects of S on optimal solution.

% of change in S	$(\varDelta_1, \varDelta_2)$	Q	k	P	L	$\varPi$
+50	(0.2, 0.2)	1399.2	2.3365	109 500	4.6640	1 037 700
	(0.1, 0.4)	1397.7	2.5898	109 500	4.6590	1 066 800
	(0.4, 0.1)	1401.0	2.0528	109 500	4.6700	1 005 100
			$R_Z = 2.80\%$		$\tilde{R}_Z = 3.14\%$	
+25	(0.2, 0.2)	1340.2	2.3930	109 500	4.4673	1 004 400
	(0.1, 0.4)	1338.8	2.6510	109 500	4.4627	1 033 400
	(0.4, 0.1)	1342.0	2.1040	109 500	4.4733	971 840
			$R_Z = 2.89\%$		$\tilde{R}_Z = 3.24\%$	
–25	(0.2, 0.2)	1213.7	2.5271	109 500	4.0457	932 920
	(0.1, 0.4)	1212.5	2.7965	109 500	4.0417	961 890
	(0.4, 0.1)	1215.3	2.2257	109 500	4.0510	900 480
			$R_Z = 3.11\%$		$\tilde{R}_Z = 3.48\%$	
–50	(0.2, 0.2)	1145.3	2.6084	109 500	3.8177	894 240
	(0.1, 0.4)	1144.1	2.8850	109 500	3.8137	923 170
	(0.4, 0.1)	1146.8	2.2993	109 500	3.8227	861 850
			$R_Z = 3.20\%$		$\tilde{R}_Z = 3.62\%$	

5.2. Results analysis

In this section, we investigate the proposed model based on the numerical results and sensitivity analyses.

(i) Table 2 shows that the optimal solution for the given data is obtained after three iterations. We observe that the computational effort and time are small to find the optimal solutions of the given data using the proposed algorithm.

(ii) Tables 3 to 5 show that when the parameters D, A and S decrease, the absolute relative variation in the fuzzy sense for the minimum expected total annual cost increases. On the other hand, when σ and s decrease, the absolute relative variation in the fuzzy sense for the minimum expected total annual cost also decreases (see Tabs. 8 and 9).

(iii) From Tables 2 to 11, it is interesting to note that when $(\Delta_1, \Delta_2) = (0.4, 0.1)$, the order lot-size Q is higher than $(\Delta_1, \Delta_2) = (0.2, 0.2)$ and $(\Delta_1, \Delta_2) = (0.1, 0.4)$, but the $IETC(Q, P, k)$ is lower than others.

TABLE 6. Effects of C_v on optimal solution.

% of change in C_v	(Δ_1, Δ_2)	Q	k	P	L	Π
+50	(0.2, 0.2)	1261.2	2.4744	73 000	6.3060	980 240
	(0.1, 0.4)	1259.6	2.7398	73 000	6.2980	1 015 800
	(0.4, 0.1)	1263.2	2.1776	73 000	6.3160	940 460
			$R_Z = 3.62\%$		$\tilde{R}_Z = 4.06\%$	
+25	(0.2, 0.2)	1261.2	2.4744	73 000	6.3060	980 240
	(0.1, 0.4)	1259.6	2.7398	73 000	6.2980	1 015 800
	(0.4, 0.1)	1263.2	2.1776	73 000	6.3160	940 460
			$R_Z = 3.62\%$		$\tilde{R}_Z = 4.06\%$	
–25	(0.2, 0.2)	1278.5	2.4560	109 500	4.2617	954 330
	(0.1, 0.4)	1277.2	2.7194	109 500	4.2573	983 340
	(0.4, 0.1)	1280.2	2.1612	109 500	4.2673	921 840
			$R_Z = 3.04\%$		$\tilde{R}_Z = 3.40\%$	
–50	(0.2, 0.2)	1278.5	2.4560	109 500	4.2617	9391 20
	(0.1, 0.4)	1277.2	2.7194	109 500	4.2573	968 130
	(0.4, 0.1)	1280.2	2.1612	109 500	4.2673	906 630
			$R_Z = 3.09\%$		$\tilde{R}_Z = 3.46\%$	

TABLE 7. Effects of h_b on optimal solution.

% of change in h_b	(Δ_1, Δ_2)	Q	k	P	L	Π
+50	(0.2, 0.2)	1066.7	2.1954	109 500	3.5557	1 170 700
	(0.1, 0.4)	1065.4	2.4370	109 500	3.5513	1 206 100
	(0.4, 0.1)	1068.5	1.9241	109 500	3.5617	1 131 100
			$R_Z = 3.02\%$		$\tilde{R}_Z = 3.38\%$	
+25	(0.2, 0.2)	1158.3	2.3098	109 500	3.8610	1 074 800
	(0.1, 0.4)	1156.9	2.5610	109 500	3.8563	1 107 100
	(0.4, 0.1)	1160.0	2.0284	109 500	3.8667	1 038 600
			$R_Z = 3.00\%$		$\tilde{R}_Z = 3.37\%$	
–25	(0.2, 0.2)	1446.5	2.6543	109 500	4.8217	851 610
	(0.1, 0.4)	1445.2	2.9346	109 500	4.8173	876 930
	(0.4, 0.1)	1422.9	2.3656	73 000	7.1145	821 500
			$R_Z = 2.97\%$		$\tilde{R}_Z = 3.54\%$	
–50	(0.2, 0.2)	1663.1	2.9931	73 000	8.3155	712 830
	(0.1, 0.4)	1704.0	3.2596	109 500	5.6800	736 000
	(0.4, 0.1)	1664.9	2.6474	73 000	8.3245	684 040
			$R_Z = 3.25\%$		$\tilde{R}_Z = 4.04\%$	

(iv) Table 11 shows that when trade credit period t_c increases, the lead time L and integrated expected total annual cost $IETC(Q, P, k)$ also increase without affecting production rate P.

(v) From Table 11, we can observe that if the trade credit period t_c increases, then the absolute relative variation in the fuzzy sense for the minimum expected total annual cost increase.

(vi) The results of Tables 2–11 highlighted the differences between crisp and the fuzzy cases.

TABLE 8. Effects of σ on optimal solution.

% of change in σ	(Δ_1, Δ_2)	Q	k	P	L	Π
+50	(0.2, 0.2)	1281.0	2.4533	109 500	4.2700	1 105 100
	(0.1, 0.4)	1279.0	2.7173	109 500	4.2633	1 148 700
	(0.4, 0.1)	1283.6	2.1603	109 500	4.2787	1 056 400
			$R_Z = 3.95\%$		$\tilde{R}_Z = 4.41\%$	
+25	(0.2, 0.2)	1279.8	2.4546	109 500	4.2660	1 037 300
	(0.1, 0.4)	1278.1	2.7183	109 500	4.2603	1 073 600
	(0.4, 0.1)	1281.9	2.1596	109 500	4.2730	996 720
			$R_Z = 3.50\%$		$\tilde{R}_Z = 3.91\%$	
–25	(0.2, 0.2)	1259.7	2.4761	73 000	6.2985	897 190
	(0.1, 0.4)	1276.3	2.7204	109 500	4.2543	923 490
	(0.4, 0.1)	1261.2	2.1796	73 000	6.3060	867 360
			$R_Z = 2.93\%$		$\tilde{R}_Z = 3.32\%$	
–50	(0.2, 0.2)	1258.2	2.4777	73 000	6.2910	814 130
	(0.1, 0.4)	1257.5	2.7422	73 000	6.2875	831 890
	(0.4, 0.1)	1259.2	2.1815	73 000	6.2960	794 250
			$R_Z = 2.18\%$		$\tilde{R}_Z = 2.44\%$	

TABLE 9. Effects of s on optimal solution.

% of change in s	(Δ_1, Δ_2)	Q	k	P	L	Π
+50	(0.2, 0.2)	1202.6	2.5387	109 500	4.0087	920 800
	(0.1, 0.4)	1201.4	2.8094	109 500	4.0047	950 290
	(0.4, 0.1)	1204.2	2.2359	109 500	4.0140	887 660
			$R_Z = 3.20\%$		$\tilde{R}_Z = 3.60\%$	
+25	(0.2, 0.2)	1241.2	2.4957	109 500	4.1373	945 500
	(0.1, 0.4)	1239.9	2.7626	109 500	4.1330	974 750
	(0.4, 0.1)	1242.8	2.1971	109 500	4.1427	912 680
			$R_Z = 3.09\%$		$\tilde{R}_Z = 3.47\%$	
–25	(0.2, 0.2)	1314.9	2.4188	109 500	4.3830	992 970
	(0.1, 0.4)	1313.5	2.6790	109 500	4.3783	1 021 700
	(0.4, 0.1)	1316.6	2.1276	109 500	4.3886	960 810
			$R_Z = 2.89\%$		$\tilde{R}_Z = 3.24\%$	
–50	(0.2, 0.2)	1350.2	2.3842	109 500	4.5007	1 015 900
	(0.1, 0.4)	1348.8	2.6413	109 500	4.4960	1 044 400
	(0.4, 0.1)	1352.0	2.0963	109 500	4.5067	984 020
			$R_Z = 2.81\%$		$\tilde{R}_Z = 3.14\%$	

TABLE 10. Effects of π_0 on optimal solution.

% of change in π_0	$(\mathit{\Delta}_1, \mathit{\Delta}_2)$	Q	k	P	L	$\mathit{\Pi}$
+50	(0.2, 0.2)	1277.6	3.0602	109 500	4.2587	1 036 300
	(0.1, 0.4)	1276.5	3.3760	109 500	4.2550	1 071 100
	(0.4, 0.1)	1279.0	2.7081	109 500	4.2633	997 530
			$R_Z = 3.36\%$		$\tilde{R}_Z = 3.74\%$	
+25	(0.2, 0.2)	1278.0	2.7745	109 500	4.2600	1 004 800
	(0.1, 0.4)	1276.8	3.0653	109 500	4.2560	1 036 800
	(0.4, 0.1)	1279.5	2.4498	109 500	4.2650	968 970
			$R_Z = 3.18\%$		$\tilde{R}_Z = 3.57\%$	
–25	(0.2, 0.2)	1279.3	2.0899	109 500	4.2643	929 040
	(0.1, 0.4)	1277.8	2.3228	109 500	4.2593	954 700
	(0.4, 0.1)	1281.3	1.8280	109 500	4.2710	900 190
			$R_Z = 2.76\%$		$\tilde{R}_Z = 3.11\%$	
–50	(0.2, 0.2)	1263.7	1.6588	73 000	6.3185	870 420
	(0.1, 0.4)	1261.4	1.8585	73 000	6.3070	897 150
	(0.4, 0.1)	1266.6	1.4325	73 000	6.3330	840 090
			$R_Z = 3.07\%$		$\tilde{R}_Z = 3.48\%$	

5.3. Managerial implications

Today's high competitive business environment, decision-making is very important as there are consequences to making the wrong decision. When managers are making decisions on behalf of the company, it is important that they weigh their options because poor choices can result in legal, financial or brand issues. To make better decisions, most managers start by defining the problem. Defining the problem removes distractions that are irrelevant to the decision. Once they have a clear understanding the problem, they can determine alternate ways of approaching the problem and the problem must first be defined for proper framing. The same decision process managers use works for day-to-day scenarios as well. However, due to uncertain environment, many managers experience difficulties in framing the realistic inventory problems as they have not addressed how products are managed, supplied, and used to improve customer satisfaction. This paper provides optimal decision support tools for managers in the form of mathematical model that improve operational, tactical, and strategic decision making in the fuzzy supply chain system. This paper aims to raise the awareness of managers with regard to realistic inventory problems. A simple and classical differential calculus optimization technique is used to find the optimal solutions so as to easily pick up the proposed methodology by the managers.

6. Conclusion

An economic vendor-buyer supply chain model is presented in this study for a product with the consideration of vendor offers permissible delay period to buyer, the replenishment lead time is dependent on both lot size of the buyer and production rate of the vendor, buyer review his inventory using continuous review policy under fuzzy backorder environment. Here, we assumed that the lost sales rate is uncertain but possible to describe with a triangular fuzzy number. This assumption is natural, since triangular fuzzy numbers may be used to model many kinds of uncertainties in this field of study. Triangular fuzzy numbers are also easy to handle. This is important when analytical solutions are desired. A mathematical model is derived to investigate the effects of fuzzy backorder rate and trade credit policy on the continuous review integrated inventory system. A minimax approach similar to Song *et al.* [35] is applied to tackle the model and an iterative algorithm is established to obtain the optimal lot-size, production rate, and lead time to minimize the integrated expected total cost in the

TABLE 11. Effects of t_c on optimal solution.

t_c	(Δ_1, Δ_2)	Q	k	P	L	Π
0.1	(0.2, 0.2)	1278.5	2.4560	109 500	4.2617	969 530
	(0.1, 0.4)	1277.2	2.7194	109 500	4.2573	998 550
	(0.4, 0.1)	1280.2	2.1612	109 500	4.2673	937 050
			$R_Z = 2.99\%$		$\tilde{R}_Z = 3.35\%$	
0.2	(0.2, 0.2)	1746.5	2.0607	109 500	5.8217	1 134 600
	(0.1, 0.4)	1744.7	2.2913	109 500	5.8157	1 165 000
	(0.4, 0.1)	1748.8	1.8014	109 500	5.8283	1 100 300
			$R_Z = 2.68\%$		$\tilde{R}_Z = 3.02\%$	
0.3	(0.2, 0.2)	2325.8	1.7421	109 500	7.7527	1 362 500
	(0.1, 0.4)	2323.3	1.9477	109 500	7.7443	1 394 300
	(0.4, 0.1)	2329.0	1.5096	109 500	7.7633	1 326 100
			$R_Z = 2.33\%$		$\tilde{R}_Z = 2.74\%$	
0.4	(0.2, 0.2)	2951.7	1.5044	109 500	9.8390	1 616 500
	(0.1, 0.4)	2948.6	1.6925	109 500	9.8287	1 649 800
	(0.4, 0.1)	2955.9	1.2902	109 500	9.8530	1 578 000
			$R_Z = 2.06\%$		$\tilde{R}_Z = 2.38\%$	
0.5	(0.2, 0.2)	3600.0	1.3224	109 500	12.0000	1 882 900
	(0.1, 0.4)	3596.1	1.4982	109 500	11.9870	1 917 900
	(0.4, 0.1)	3605.1	1.1209	109 500	12.0170	1 842 100
			$R_Z = 1.86\%$		$\tilde{R}_Z = 2.17\%$	
0.6	(0.2, 0.2)	4260.5	1.1779	109 500	14.2017	2 156 100
	(0.1, 0.4)	4255.8	1.3447	109 500	14.1850	2 192 800
	(0.4, 0.1)	4266.7	0.9852	109 500	14.2223	2 112 900
			$R_Z = 1.70\%$		$\tilde{R}_Z = 2.00\%$	
0.7	(0.2, 0.2)	4928.2	1.0594	109 500	16.4273	2 433 200
	(0.1, 0.4)	4922.7	1.2197	109 500	16.4090	2 471 700
	(0.4, 0.1)	4935.5	0.8730	109 500	16.4517	2 387 500
			$R_Z = 1.58\%$		$\tilde{R}_Z = 1.88\%$	
0.8	(0.2, 0.2)	5600.7	0.9596	109 500	18.6690	2 712 800
	(0.1, 0.4)	5594.4	1.1149	109 500	18.6480	2 753 100
	(0.4, 0.1)	5609.2	0.7775	109 500	18.6973	2 664 400
			$R_Z = 1.49\%$		$\tilde{R}_Z = 1.78\%$	
0.9	(0.2, 0.2)	6276.4	0.8736	109 500	20.9213	2 994 000
	(0.1, 0.4)	6269.2	1.0253	109 500	20.8973	3 036 300
	(0.4, 0.1)	6286.1	0.6944	109 500	20.9537	2 942 700
			$R_Z = 1.41\%$		$\tilde{R}_Z = 1.71\%$	

fuzzy sense. A numerical example and sensitivity analysis have been carried out to illustrate the behaviors of the proposed model.

There are several extension of this work that could constitute future research related to this field. One immediate probable extension could be to discuss the effect of inflation. Also, we can consider multi-echelon supply chains such as; single-buyer multiple-vendor, multiple-buyer single-vendor and multiple-buyer multiple-vendor systems.

Appendix A.

Definition A.1. A fuzzy set $\tilde{A}$ is defined by a membership function $\mu_{\tilde{A}}(x)$ which maps each and every element of $\widetilde{X}$ to real interval $[0 \;\; 1]$, *i.e.*, $\mu_{\tilde{A}}(x) : \widetilde{X} \rightarrow [0 \;\; 1]$ *where* $\widetilde{X}$ *is the universal set.*

Definition A.2. A fuzzy set $\tilde{A}$ is called convex if its membership function is a convex function. In other words, if the relation $\mu_{\tilde{A}}\left(\lambda x_1 + (1-\lambda)x_2\right) \geq Min\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\}.$

Definition A.3. A fuzzy set $\tilde{A}$ is empty if its membership function is identically zero, *i.e.*, $\mu_{\tilde{A}}(x) = 0, \forall x \in \widetilde{X}$.

Definition A.4. A triangular fuzzy number is denoted by $\tilde{\delta} = (a, b, c)$ and defined by the following membership function

$$\mu_{\tilde{\delta}}(x) = \begin{cases} \frac{x-a}{b-a}, \text{ if } & a \leq x \leq b \\ \frac{c-x}{c-b}, \text{ if } & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

where $a, b, c \in R, \tilde{\delta} \in F_N, F_N$ is the set of triangular fuzzy numbers.

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