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K-THEORY AND TOEPLITZ-C*-ALGEBRAS - A SURVEY

par *Matthias LESCH*

This article is an elaborate version of a lecture given at Institut Fourier, April 91. Its aim is to give an overview over some developments in the theory of Toeplitz C*-algebras and to advertise that operator K-theory is a strong tool for their investigation. Since it is devoted to non-specialists in operator algebras, I have decided to put in a rough introduction to operator K-theory. As a standard reference for this I refer to [B].

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1. C*-algebras and K-theory

As for many objects in mathematics there are two definitions for a C*-algebra, an intrinsic and an extrinsic one:

DEFINITION (concrete C*-algebra). — *A concrete C*-algebra is a norm-closed *-subalgebra $\mathcal{A}$ of the algebra of bounded operators $\mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. *-subalgebra means, that with $T \in \mathcal{A}$ we also have $T^* \in \mathcal{A}$.*

DEFINITION (abstract C*-algebra). — *An abstract C*-algebra $\mathcal{A}$ is a complex algebra together with a Banach-space structure and an antilinear involution $*$ such that for $a, b \in \mathcal{A}$*

(i) $\|ab\| \leqslant \|a\|\,\|b\|$

(ii) $\|a^*a\| = \|a\|^2$.

Of course, every concrete C*-algebra is an abstract one with the usual *-operation, since (i),(ii) are easy to check for arbitrary bounded operators in a Hilbert space. The converse is non-trivial.

THEOREM (Gelfand-Naimark [GN]). — *Every abstract C*-algebra $\mathcal{A}$ is a concrete one, i. e. has an isometric embedding $\mathcal{A} \hookrightarrow \mathcal{L}(\mathcal{H})$.*

For a long time one was seeking for the right axioms guaranteeing this theorem. Gelfand and Naimark assumed in addition the invertibility of $I + a^*a$ for every $a \in \mathcal{A}$. Lateron one found out, that this extra condition is redundant. It is a remarkable fact, that this redundancy does not hold for real C*-algebras. As a counterexample consider the algebra $\mathbf{C}$ as a real algebra with trivial involution $*$=id. Then we have $1+i^*i = 1+i^2 = 0$ and hence this algebra cannot have a faithful *-representation on a real Hilbert space, i. e. is no real C*-algebra.

The main advantage of the abstract characterization of C*-algebras is that it is more easy to check if a given object is a C*-algebra, since in real life a C*-algebra does not come with a canonical representation.

Example. — Let X be a locally compact space and consider the algebra $C_0(X)$ of continuous functions vanishing at infinity with the usual pointwise involution and norm

$$||f|| := \sup\{|f(x)| \mid x \in X\}.$$

To obtain a representation we consider a Radon measure μ on X and put

$$M : C_0(X) \longrightarrow \mathcal{L}(L^2(X, \mu)), M_f g := fg.$$

This representation is faithful if and only if the support of μ is X.

This example is a very important one, since the C*-algebras obtained in this way are exactly the commutative ones. Namely, the functor $\mathcal{C}_0$ which assigns to every locally compact space its C*-algebra of continuous functions vanishing at infinity is a (contravariant) category equivalence between the category of locally compact spaces with proper continuous maps and the category of commutative C*-algebras with *-homomorphisms. This is an abstract formulation of the Gelfand representation theorem. The inverse of the functor $\mathcal{C}_0$ is the Gelfand functor $\mathcal{G}$ which assigns to every commutative C*-algebra $\mathcal{A}$ its spectrum $spec(\mathcal{A})$ of continuous nontrivial algebra homomorphisms into the complex numbers. In a canonical way $spec(\mathcal{A})$ is a locally compact subspace of the unit ball in the dual $\mathcal{A}'$ equipped with the weak-*-topology. Hence every topological notion has its reflection in the category of commutative C*-algebras. However, the translation is sometimes nontrivial. As an important example we intruduce K-theory, which is the appropriate cohomology theory for locally compact spaces.

Let X be a compact space and consider the set of isomorphism classes of complex vector bundles over X. The Whitney sum of vector bundles makes this set into a commutative monoid, denoted by V(X).

DEFINITION. — *$K^0(X)$ is the Grothendieck group of the monoid V(X). If X is locally compact, then $K^0(X) := \ker(K^0(X^+) \rightarrow K^0(+))$, where X^+ denotes the one-point compactification of X.*

K gives a cohomology theory for locally compact spaces, which satisfies all Eilenberg-Steenrood axioms except the dimension axiom. Now, how to translate K into terms of commutative C*-algebras? Of course, one has $K_0(C_0(X)) = K^0(X)$; but this is trivial and rather useless, since we would like to have an intrinsic characterization, that carries over to the non-commutative case. First of all we need a more algebraic characterization of vector bundles. This is obtained from the observation that a vector bundle can equally well be described by the set of its sections. For the moment let X be compact. The set of sections $C(E)$ of a vector bundle E over X is a $C(X)$-right-module in the following way

$$C(E) \times C(X) \to C(E), (sf)(x) := f(x)s(x) \tag{1}$$

for $s \in C(E), f \in C(X)$. If $E \simeq \mathbf{C}^n_X$ is trivial, then this module is free, i.e. $C(E) \simeq C(X)^n$. Those $C(X)$-right-modules that occur as sections of a vector bundle can be characterized algebraically:

THEOREM (Swan [At, Corollary 1.4.14], [K, Theorem I.6.5]). — *Let E be a vector bundle over the compact space X. Then there is a vector bundle F over X such that $E \oplus F$ is trivial.*

COROLLARY. — *To a $C(X)$-right-module $\mathcal{E}$ there exists a vector bundle E over X such that $\mathcal{E} \simeq_{C(X)} C(E)$ if and only if $\mathcal{E}$ is a direct summand of a finite-dimensional free $C(X)$-right-module, i. e. $\mathcal{E}$ is a finitely generated projective $C(X)$-right-module.*

Proof . — The only if part is Swan's theorem. To prove the if part, consider a finitely generated projective $C(X)$-right-module $\mathcal{E}$. Projective means, that there is another such module $\mathcal{F}$ with $\mathcal{E} \oplus \mathcal{F} \simeq C(X)^n$. Now let $p : C(X)^n \to \mathcal{E}$ be the projection with respect to this decomposition. Since this is a module homomorphism we have $p \in M_n(C(X)) = C(X, M_n(\mathbf{C}))$, i. e. p is a continuous family of projections. Then $E_x := \text{im } p(x), F_x := \ker p(x)$ define vector bundles over X with $C(E) \simeq \mathcal{E}$, $C(F) \simeq \mathcal{F}$ [K, Theorem I.6.3].

With this at hand, K-theory can be carried over to non-commutative C*-algebras. So let $\mathcal{A}$ be an arbitrary C*-algebra with unit and consider a finitely generated projective $\mathcal{A}$-right-module $\mathcal{E}$. Choose another such module $\mathcal{F}$ with

$$\mathcal{E} \oplus \mathcal{F} \simeq \mathcal{A}^n\,. \tag{2}$$

We identify $\mathcal{E}$ with its image in $\mathcal{A}^n$ and denote by $p : \mathcal{A}^n \to \mathcal{E}$ the projection with respect to the decomposition (2). Since this is a module homomorphism we have $p \in M_n(\mathcal{A})$ and $\mathcal{E} = p\mathcal{A}^n$. Since $\mathcal{A}$ is a C*-algebra we may assume that $p = p^*$ and it turns out that two modules $\mathcal{E}, \mathcal{F}$ are isomorphic if and only if the corresponding projections p, q are Murray-von Neumann equivalent, i. e. if there exists a partial isometry u such that $u^*u = p, uu^* = q$. Thus, as before, we consider the set $V(\mathcal{A})$ of equivalence classes of finitely generated projective $\mathcal{A}$-right-modules, which is a commutative monoid with direct sum as addition. The discussion above shows that $V(\mathcal{A})$ can alternatively be described as the set of Murray-von Neumann equivalence classes of projections in the infinite matrix algebra $M_\infty(\mathcal{A}) = \lim_{n\to\infty} M_n(\mathcal{A})$ over $\mathcal{A}$.

DEFINITION. — *$K_0(\mathcal{A})$ is the Grothendieck group of the monoid $V(\mathcal{A})$. If $\mathcal{A}$ has no unit, then $K_0(\mathcal{A}) := \ker(K_0(\mathcal{A}^+) \to K_0(\mathbf{C}))$, where $\mathcal{A}^+$ denotes $\mathcal{A}$ with an identity adjoined.*

Higher K-groups are defined as in algebraic topology via the suspension, namely one defines for $n \in \mathbf{N}$

$$S^n\mathcal{A} := C_0(\mathbf{R}^n) \otimes \mathcal{A} \simeq C_0(\mathbf{R}^n, \mathcal{A}) \tag{3a}$$

and puts

$$K_n(\mathcal{A}) := K_0(S^n\mathcal{A}). \tag{3b}$$

K_1 has another characterization, namely there is a canonical isomorphism $K_1(\mathcal{A}) \simeq \pi_0(U_\infty(\mathcal{A}))$, where $U_\infty(\mathcal{A})$ is the unitary group in $M_\infty(\mathcal{A})$. With (3b) one finds by induction

$$K_{n+1}(\mathcal{A}) \simeq \pi_n(U_\infty(\mathcal{A})). \tag{3c}$$

This relation is the reason for the significance of K-theory for the stable homotopy of the classical groups.

K_* is a homology theory on the category of C*-algebras [B,Chap. 21]. The property of K_* we mention first is one, that cannot be seen in the commutative category.

PROPOSITION. — *K_* is a stable functor, i. e. there is a canonical isomorphism $K_*(\mathcal{A}) \simeq K_*(\mathcal{A} \otimes \mathcal{K})$, where $\mathcal{K}$ denotes the ideal of compact operators on a (separable) Hilbert space.*

Example. — $\mathcal{A} = \mathbf{C}$: two projections in $M_n(\mathbf{C})$ are Murray-von Neumann equivalent if and only if they have the same rank, thus we have

$$V(\mathbf{C}) = \mathbf{N}_0, \quad K_0(\mathbf{C}) = \mathbb{Z}, \tag{4}$$

and since the unitary group $U_n(\mathbf{C})$ is connected we have $K_1(\mathbf{C}) = 0$. With the stabilization property we obtain

$$K_0(\mathcal{K}) \simeq \mathbb{Z}, \quad K_1(\mathcal{K}) \simeq 0. \tag{5}$$

The isomorphism $\Phi : K_0(\mathcal{K}) \to \mathbb{Z}$ is canonical and can be described as follows. Every element in $K_0(\mathcal{K})$ is of the form $[p] - [q]$ with finite dimensional projections $p, q \in \mathcal{K}$. Then one has

$$\Phi([p] - [q]) := tr(p) - tr(q) = rank(p) - rank(q). \tag{6}$$

The most important theorem at this stage is the Bott periodicity theorem, which in this context is due to Wood, Karoubi.

THEOREM (Bott periodicity, [W],[K]). — *The map which sends a projection $p \in M_n(\mathcal{A})$ to the map $t \mapsto e^{2\pi i t}p + (1-p)$ in $U_n((C_0(0,1), \mathcal{A})^+)$ induces an isomorphism $K_0(\mathcal{A}) \simeq K_1(S\mathcal{A})$, i. e. there are canonical isomorphisms*

$$K_n(\mathcal{A}) \simeq K_{n+2}(\mathcal{A}). \tag{7}$$

For a proof of this theorem see also [B,Chap. 9].

An important consequence of this theorem is that the long exact sequence in homology is in fact a six term cyclic exact sequence.

THEOREM. — *If* $0 \to \mathcal{J} \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \to 0$ *is an exact sequence of C*-algebras, then there is an exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{J}) & \xrightarrow{i_*} & K_0(\mathcal{A}) & \xrightarrow{\sigma_*} & K_0(\mathcal{B}) \\ \partial_1 \uparrow & & & & \downarrow \partial_2 \\ K_1(\mathcal{B}) & \xleftarrow{\sigma_*} & K_1(\mathcal{A}) & \xleftarrow{i_*} & K_1(\mathcal{J}) . \end{array} \tag{8}$$

Example. — With these theorems it is an easy exercise to compute the K-groups of spheres inductively from $K_*(\mathbf{C})$. It turns out that $K^*(S^n) \simeq K_*(\mathbf{C}) \oplus K_*(S^n\mathbf{C})$ and

$$K_k(S^n\mathbf{C}) \simeq K_{n+k}(\mathbf{C}) \simeq \begin{cases} 0 & \text{if } k+n \equiv 1(\text{mod}2) \\ \mathbb{Z} & \text{if } k+n \equiv 0(\text{mod}2) . \end{cases} \tag{9}$$

The non-trivial generator of $K^*(S^n)$ which corresponds to the generator of $K_*(S^n\mathbf{C})$ is called the Bott element.

As a last remark in this paragraph we want to point out a relationship between K-theory and index theory. The connecting homomorphism ∂_1 has an interpretation as an index.

PROPOSITION. — *Assume that the C*-algebra in (8) is represented in some Hilbert space $\mathcal{H}$, $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$, such that $\mathcal{J} = \mathcal{K}$. Then one has the following abstract "index theorem":*

An operator $T \in M_n(\mathcal{A})$ is a Fredholm operator if and only if its symbol $\sigma(T)$ is invertible in $M_n(\mathcal{B})$ and in this case we have the formula

$$\text{ind}\, T = \partial_1[\sigma(T)] \in K_0(\mathcal{K}) \simeq \mathbb{Z} , \tag{10}$$

where $[\sigma(T)]$ denotes the class of the invertible element $\sigma(T)$ in $K_1(\mathcal{B})$ and $\partial_1 : K_1(\mathcal{B}) \to K_0(\mathcal{K})$ is the connecting homomorphism in (8) (cf. [B,8.3.2]).

Example. — Consider a compact manifold M and let $CZ(M)$ be the Calderon-Zygmund algebra, which is the C*-algebra generated by the (scalar) pseudo-differential operators of order zero on M. It is well-known, that there is an exact sequence

$$0 \to \mathcal{K}(L^2(M)) \longrightarrow CZ(M) \xrightarrow{\sigma} C(S^*M) \to 0 , \tag{11}$$

where $C(S^*M)$ denotes the cosphere bundle over M and σ is the usual symbol map for pseudo-differential operators. We obtain from the preceding proposition, that a system of pseudo-differential operators is Fredholm iff it is elliptic and that the index is given by the connecting map $\partial_1 : K^1(S^*M) \to \mathbb{Z}$. Thus we have an expression for the analytical index of Atiyah-Singer and we obtain immediately that the index of an elliptic operator only depends on the stable homotopy class of the symbol in $M_\infty(C(S^*M))$. Of course the hard work is contained in proving the exact sequence (11) and it is another story to express the analytical index in topological terms, which was done by Atiyah-Singer.

2. Toeplitz-C*-algebras on bounded symmetric domains

A bounded domain $D \subset \mathbf{C}^n$ is called symmetric, if for every $z \in D$ there exists a holomorphic automorphism Φ_z of D with z as isolated fixed point and $\Phi_z \circ \Phi_z = \mathrm{id}$. We may assume that $0 \in D$ and D is circular, i. e.

$$\lambda r z \in D, \text{ for } \lambda \in S^1, r \in (0,1), z \in D. \tag{12}$$

The group Aut(D) of holomorphic automorphisms acts transitively on D. It is a Lie group that generalizes the Möbius transformations in case of the unit disc.

We consider the Banach-algebra

$$\mathcal{H}(\bar{D}) = \{f \in C(\bar{D}) \mid f_{|D} \text{holomorphic}\} \tag{13}$$

of bounded holomorphic functions on D that have a continuous extension to $\bar{D}$. A set $F \subset \bar{D}$ is called *determining* if for $f \in \mathcal{H}(\bar{D})$ $\|f\|_{\bar{D}} = \|f\|_F$. The set

$$\check{S}(D) = \bigcap_{F \subset \bar{D} \text{ determining}} F$$

is called the *Shilov boundary* of D. By the maximum principle we have $\check{S}(D) \subset \partial D$ and it is a standard theorem from the theory of commutative Banach-algebras that $\check{S}(D)$ is determining. Now denote by $K \subset$ Aut(D) the isotropy group of $0 \in D$. This is a compact group acting linearly on $\mathbf{C}^n$. The following proposition is important for determining the Shilov boundary.

PROPOSITION (Bott-Korányi [KW, Chap. 3]). — *K acts transitively on $\check{S}(D)$.*

COROLLARY. — *There exists a unique K-invariant propability measure μ on $\check{S}(D)$.*

Examples. —

1. Since a circular domain is star-shaped and hence simply-connected, by the Riemann mapping theorem, there is only one example in dimension one: the unit disc $D = \{z \in C \mid |z| < 1\}$. The automorphisms of D with fixed point 0 are the rotations, i. e. $K = S^1$. Since this group acts transitively on the whole topological boundary $\partial D = S^1$ we infer from the theorem of Bott-Korányi that $\check{S}(D) = S^1$. The invariant probability measure on S^1, of course, is the usual Lebesgue measure.

2. The direct generalization of the preceding example to higher dimensions are the unit balls in $\mathbf{C}^n$ $A_{n,1} := \{z \in \mathbf{C}^n \mid \|z\| < 1\}$. The denotion $A_{n,1}$ will become transparent in the next example. Here the results are essentially the same as before, one has $\check{S}(A_{n,1}) = \partial A_{n,1} = S^{2n-1}$, $K = U(n)$ and the invariant measure is the usual Lebesgue measure.
3. (Matrix domains) One class of matrix domains is defined for $n \geqslant m$ as follows

$$A_{n,m} := \{Z \in M_{n,m}(\mathbf{C}) \mid I_m - Z^*Z \text{ is positive definite}\}. \tag{14}$$

It is clear, that this is a circular domain. To describe its automorphisms we introduce

$$\begin{aligned} I_{n,m} &:= \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}, \\ U(n,m) &:= \{G \in GL(n+m,\mathbf{C})\, ;\ G^* I_{n,m} G = I_{n,m}\}, \\ SU(n,m) &:= U(n,m) \cap SL(n+m,\mathbf{C}). \end{aligned} \tag{15}$$

An element $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,m)$ induces an automorphism Φ_G of $A_{n,m}$, which is a generalized Möbius transformation, by $\Phi_G(Z) := (AZ+B)(CZ+D)^{-1}$. One can show that all automorphisms of $A_{n,m}$ are obtained in this way [H]. From this description one can construct the symmetries and $A_{n,m}$ is in fact a symmetric domain. $\Phi_G(Z)$ lies in the isotropy group of 0 if and only if $G = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ with $A \in U(n), B \in U(m)$, det(A)det(B)=1. Thus K is not transitive on the topological boundary if $m > 1$ and it turns out that

$$\check{S}(A_{n,m}) = \{Z \in M_{n,m}(\mathbf{C}) \mid Z^*Z = I_m\} = V_{n,m} \tag{16}$$

is the complex Stiefel manifold of orthonormal m-frames in $\mathbf{C}^n$, in particular $\check{S}(A_{n,n}) = U(n)$.

All these examples are irreducible bounded symmetric domains, i. e. they cannot be written as the product of some bounded symmetric domains. The irreducible bounded symmetric domains have been classified by E. Cartan [C]. Beside the examples above there are three other series of domains and two exceptional ones. With one of the other series, the Lie balls, we will become acquainted lateron.

Now we are going to introduce Toeplitz operators. We consider again an arbitrary bounded symmetric domain D.

DEFINITION. — *The Hardy space $H^2(D)$ is the closed subspace of $L^2(\check{S}(D),\mu)$ generated by $\{f_{|\check{S}(D)} \mid f \in \mathcal{H}(\bar{D})\}$. The orthogonal projection $P : L^2(\check{S}(D),\mu) \to H^2(D)$ is called the Szegö projection.*

DEFINITION. — *For $f \in C(\check{S}(D))$ the Toeplitz operator T_f is defined as*

$$H^2(D) \ni g \mapsto P(fg) \tag{17}$$

and the Toeplitz C-algebra $\mathcal{T}(D)$ of D is the C*-algebra generated by all T_f.*

These are interesting C*-algebras since they reflect much of the structure of the domain D. There are relationships to pseudo-differential operators [BM]. The structure of these algebras is completely determined by the following two theorems.

THEOREM (Upmeier [Up1,Lemma 3.1, Proposition 3.11], Schröder [S1, Satz 2.4.5, Satz 2.4.10]). — *Let $\mathcal{C}(D)$ be the closed *-ideal in $\mathcal{T}(D)$ generated by all commutators $[T_f, T_g]$. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{C}(D) \longrightarrow \mathcal{T}(D) \xrightarrow{\sigma} C(\check{S}(D)) \longrightarrow 0\,. \tag{18}$$

The map $f \mapsto T_f$ is a completely positive cross-section of σ. Moreover if D is irreducible, then $\mathcal{T}(D)$ acts irreducible on $H^2(D)$.

From now on, we restrict ourselves to irreducible D.

THEOREM (Upmeier [Up2, Theorem 3.12]). — *There are ideals*

$$\{0\} \subset \mathcal{J}_0 \subset \mathcal{J}_1 \subset \cdots \subset \mathcal{J}_r = \mathcal{T}(D)\,, \tag{19}$$

with $\mathcal{J}_0 = \mathcal{K}(H^2(\check{S}(D)))$, $\mathcal{J}_{r-1} = \mathcal{C}(D)$. More precisely,

$$\mathcal{J}_r/\mathcal{J}_{r-1} \simeq C(\check{S}(D))\,,\ \mathcal{J}_k/\mathcal{J}_{k-1} \simeq C(S_k, \mathcal{K})\,,\ k = 1, \cdots, r-1\,, \tag{20}$$

where r is the rank of D as a symmetric space and the S_k are the "strata" of the boundary ∂D.

I will not describe what the S_k are in general, since this would require the machinery of Jordan-algebras. I will explain it only in case of the three examples above. One can show, that the rank of $A_{n,m}$ is m, thus the rank of the Hilbert balls is one and these are the only irreducible domains of rank one. In this case we have $S_1 = S^{2n-1}$. In general, the boundary of $A_{n,m}$ contains all partial isometries in $M_{n,m}(\mathbf{C})$, which have a natural stratification by their rank. In fact, it turns out that

$$S_k(A_{n,m}) = \{u \in M_{n,m}(\mathbf{C}) \mid uu^*u = u, \mathrm{rank}(u) = k\}\,, k = 0, \cdots, m\,. \tag{21}$$

3. Index theory for Toeplitz operators

Since the Toeplitz algebra on an irreducible bounded symmetric domain contains the compact operators, one can ask for Fredholm criteria and index theorems. The answer depends on the rank of the domain. I will discuss here the rank one and two case and make some remarks on the general case. The rank one case is the most similar one to elliptic operators, since it is a special case of Boutet de Monvel's index theorem. Since in this case, the commutator ideal is the ideal of compact operators, the abstract index theorem of Chap. 1 applies. It remains to determine the index homomorphism.

THEOREM. — *For the Hilbert balls $A_{n,1}$ one has the exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}(A_{n,1}) \xrightarrow{\sigma_n} C(S^{2n-1}) \longrightarrow 0 . \tag{22}$$

An operator $T \in M_k(\mathcal{T}(A_{n,1}))$ is Fredholm iff $\sigma_n(T)$ is invertible and in this case one has

$$\operatorname{ind} T = (-1)^n b_{2n-1}[\sigma_n(T)] , \tag{23}$$

where $[\sigma_n(T)]$ denotes the class of the invertible element $\sigma_n(T)$ in $K^1(S^{2n-1})$ and $b_{2n-1} : K^1(S^{2n-1}) \to \mathbb{Z}$ is the Bott isomorphism in (9).

Thus we have found an extension of C*-algebras that represents up to sign the Bott isomorphism. In particular, it follows from this theorem, that in $M_k(\mathcal{T}(A_{n,1}))$ there exist Fredholm operators of arbitrary index. For $n = 1$ this theorem is the celebrated Gohberg-Kreĭn index theorem [GK], which is equivalent to the Atiyah-Singer index theorem on S^1. For $n > 1$, where Toeplitz operators are no longer pseudo-differential, this theorem is due to Venugopalkrishna [V]. Schröder [S1,S2] has extended it to operators whose symbols take values in a II_1-factor.

It should be noted, that with sequence (22) and $n = 1$ an alternative proof of Bott periodicity can be given [Cu, § 4].

Domains of higher rank are more complicated since the ideal of compact operators stands at the beginning of an ideal chain. It seems to me that spectral sequence techniques would be appropriate here [Sch],[SP]. Beside two other special domains there are two classes of domains of rank 2: the $A_{n,2}$'s and the Lie balls. The Lie balls B_n are defined as

$$B_n := \left\{ z \in \mathbf{C}^n \mid 1 - 2\sum_{j=1}^{n} |z_j|^2 + \left|\sum_{j=1}^{n} z_j^2\right|^2 > 0 , \left|\sum_{j=1}^{n} z_j^2\right| < 1 \right\} , \; n \geqslant 3 , \tag{24a}$$

and their Shilov boundaries are the so called Lie spheres

$$L_n := \check{S}(B_n) = \{z = \lambda x \in \mathbf{C}^n \mid \lambda \in S^1\ ,\ x \in S^{n-1}\} = S^1 \cdot S^{n-1} \subset \mathbf{C}^n\,. \tag{24b}$$

The Lie balls are exactly the tube domains of rank 2. Tube domains have a norm function on their Shilov boundary which generalizes the determinant. For the Lie spheres this norm function is defined as

$$N(z) = \sum_{j=1}^{n} z_j^2 = \lambda^2\,. \tag{24c}$$

By Upmeier's theorem one has the ideal chain

$$0 \subset \mathcal{K} \subset \mathcal{C}(L_n) \subset \mathcal{T}(L_n)\,.$$

This ideal chain had been discovered before by Berger, Coburn and Koranyi [BCK], who also had determined $\mathcal{C}(L_n)$. They had shown that $H^2(L_n) \simeq H^2(S^1) \otimes L^2(S^{n-1})$ and this isomorphism induces an embedding $\mathcal{T}(L_n) \subset \mathcal{T}(S^1) \otimes CZ(S^{n-1})$ under which $\mathcal{C}(L_n) \simeq \mathcal{K}_1 \otimes CZ(S^{n-1})$, where $\mathcal{K}_1 := \mathcal{K}(H^2(S^1))$. Thus (11) with $M = S^{n-1}$ yields $\mathcal{C}(L_n)/\mathcal{K}(H^2(L_n)) \simeq C(S^*S^{n-1}, \mathcal{K}_1)$ and hence $S_1(L_n) = S^*S^{n-1}$. Analyzing the various identifications involved one obtains the following huge commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & \mathcal{K}(H^2(L_n)) & & & & & & \\ & & \downarrow & & & & & & \\ 0 \to & & \mathcal{C}(L_n) & \to & \mathcal{T}(L_n) & \xrightarrow{\sigma} & C(L_n) & \to & 0 \\ & & \downarrow \sigma_0 & & \downarrow \sigma_0 & & \downarrow f^* & & \\ 0 \to & & C(S^*S^{n-1}, \mathcal{K}_1) & \to & C(S^*S^{n-1}, \mathcal{T}(S^1)) & \xrightarrow{\sigma_1} & C(S^1 \times S^*S^{n-1}) & \to & 0 \end{array} \tag{25}$$

where σ_1 is the symbol map in (22) with $n = 1$, tensored with $C(S^*S^{n-1})$; σ_0 is the symbol map for pseudo-differential operators on S^{n-1}, tensored with $\mathcal{K}_1$ (resp. $\mathcal{T}(S^1)$), and

$$f : S^1 \times S^*S^{n-1} \to L_n, (\lambda, x, \xi) \mapsto \frac{1}{2}(x - i\xi + \lambda(x + i\xi))\,. \tag{26}$$

It is possible to compute the induced map in K-theory $f^* : K^*L_n \to K^*S^1 \times S^*S^{n-1}$ [L, Prop. 2.8]. I cannot go into this here, since the computations are rather long and tedious. Basicly one has to construct explicit generators for the K-groups and to compute their pullbacks under f. But since it is more convenient to deal with differential forms, one has to plug in the Chern character. Details may be found in [L1, § 2]. Having done this, diagram chasing yields the following theorem, which is the key ingredient for the investigation of Fredholm operators in $\mathcal{T}(L_n)$.

THEOREM [L1, Theorem 3.2]. — *In the exact sequence of K-groups*

$$\begin{array}{ccccc} K_0(\mathcal{C}(L_n)) & \xrightarrow{i_*} & K_0(\mathcal{T}(L_n)) & \xrightarrow{\sigma_*} & K_0(\mathcal{C}(L_n)) \\ \partial_1 \uparrow & & & & \downarrow \partial_2 \\ K_1(\mathcal{C}(L_n)) & \xleftarrow{\sigma_*} & K_1(\mathcal{T}(L_n)) & \xleftarrow{i_*} & K_1(\mathcal{C}(L_n)) \end{array} \tag{27}$$

∂_1 is an isomorphism and ∂_2 is surjective with $\ker(\partial_2) = \mathbb{Z}[1_{L_n}]$, *in particular we have*

$$K_0(\mathcal{T}(L_n)) = \mathbb{Z}[1_{\mathcal{T}(L_n)}]\,, K_1(\mathcal{T}(L_n)) \simeq 0\,. \tag{28}$$

This theorem has some interesting consequences for the Fredholm operators in $M_k(\mathcal{T}(L_n))$, which may be derived completely abstract within the framework of K-theory. In order to formulate them, I have to introduce the notion of stable homotopy of Fredholm operators. Two Fredholm operators $T_1 \in M_{k_1}(\mathcal{T}(L_n))$, $T_2 \in M_{k_2}(\mathcal{T}(L_n))$ are called *stably homotopic* iff there is a path

$$\omega : [0,1] \to \{T \in M_{k'}(\mathcal{T}(L_n)) \mid T \text{ Fredholm}\} \tag{29}$$

for some $k' \geqslant \max(k_1, k_2)$ such that

$$\omega(0) = \begin{pmatrix} T_1 & 0 \\ 0 & I_{k'-k_1} \end{pmatrix}, \omega(1) = \begin{pmatrix} T_2 & 0 \\ 0 & I_{k'-k_2} \end{pmatrix}. \tag{30}$$

It is clear that stably homotopic Fredholm operators have the same index.

THEOREM [L1, Proposition 3.4, Theorem 3.5]. —

1. *Two Fredholm operators $T_1, T_2 \in M_k(\,T(L_n))$ are stably homotopic if and only if* ind $T_1 =$ ind T_2.
2. *A Fredholm operator $T \in M_k(\mathcal{T}(L_n))$ is stably homotopic to a Fredholm operator $T_1 \in 1_{\mathcal{T}(L_n)} + \mathcal{C}(L_n)$, especially one has*

$$[\sigma(T)]_{K^1(L_n)} = 0\,. \tag{31}$$

Since $\mathcal{C}(L_n) \simeq \mathcal{K}_1 \otimes CZ(S^{n-1})$ this theorem says that up to stable homotopy there are as many Fredholm operators in $\mathcal{T}(L_n)$ as there are elliptic operators over S^{n-1} and the index for Fredholm operators in $\mathcal{T}(L_n)$ is completely determined by the Atiyah-Singer index theorem on S^{n-1}. In some sense this result is a negative one, i. e. no new phenomena occur.

Recently, the author has proved that essentially the same results hold for the $A_{n,2}$'s, i. e. one has

$$K_0(\mathcal{T}(A_{n,2})) = \mathbb{Z}[1_{\mathcal{T}(A_{n,2})}]\,, K_1(\mathcal{T}(A_{n,2})) \simeq 0\,, \tag{32}$$

with similar consequences for the Fredholm operators in $\mathcal{T}(A_{n,2})$. So beside the two other existing domains of rank two, the situation in the rank two case is completely understood. Further computations show that there is some evidence for the following conjecture.

CONJECTURE. — $K_0(\mathcal{T}(D)) = \mathbb{Z}[1_{\mathcal{T}(D)}]$, $K_1(\mathcal{T}(D)) \simeq 0$ *for all irreducible domains, at least for domains of tube type.*

Beside the Toeplitz algebras on bounded symmetric domains, there is another Toeplitz construction, that comes from dynamical systems and it is a remarkable fact that always the above K-groups occur, if the algebras are irreducible in some natural sense ([JK],[MPX],[L2]). So, it is natural to state the following problem, which of course cannot be made precise in a mathematical sense.

PROBLEM. — *Find an interesting Toeplitz algebra, not having the K-groups above.*

One can ask conversely: what is the deeper reason for the occurence of these K-groups?

I'd like to finish with some further remarks concerning the index theorems of Upmeier [Up3], which also lead to some ideas for attacking the conjecture above. I start completely abstract and consider a filtered C*-algebra. This is a C*-algebra $\mathcal{T}$ with a chain of closed *-ideals

$$\mathcal{J}_0 \subset \mathcal{J}_1 \subset \cdots \subset \mathcal{J}_r = \mathcal{T}\,. \tag{33}$$

One immediately checks that

$$0 \longrightarrow \mathcal{J}_{k-1}/\mathcal{J}_{k-2} \longrightarrow \mathcal{J}_k/\mathcal{J}_{k-2} \longrightarrow \mathcal{J}_k/\mathcal{J}_{k-1} \longrightarrow 0 \tag{34}$$

is an exact sequence of C*-algebras. The k-index

$$\operatorname{ind}_k : K_*(\mathcal{J}_k/\mathcal{J}_{k-1}) \longrightarrow K_{*-1}(\mathcal{J}_{k-1}/\mathcal{J}_{k-2}) \tag{35}$$

is defined to be the connecting map in the six-term exact sequence of (34). The significance of these index maps stems from the fact that

$$\begin{aligned} E^1_{p,q} &:= K_{p+q}(\mathcal{J}_p/\mathcal{J}_{p-1}), \\ d^1_{p,q} &:= \operatorname{ind}_p : E^1_{p,q} \to E^1_{p-1,q} \end{aligned} \quad p = 0, \cdots, r; q \in \mathbb{Z}_2 \tag{36}$$

is the E^1-term of a spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ converging to $K_*(\mathcal{T})$. Hence complete knowledge of the index maps leads to complete knowledge of $K_*(\mathcal{T})$ up to group extension [Sch].

Now we turn back to the Toeplitz C*-algebra on an irreducible bounded symmetric domain. By Upmeiers theorem we have $\mathcal{J}_k/\mathcal{J}_{k-1} \simeq C(S_k, \mathcal{K})$, where $\mathcal{K}$ is the ideal of compact operators on some Hilbert space (possibly one dimensional). Hence

$$\operatorname{ind}_k : K^*(S_k) \longrightarrow K^{*-1}(S_{k-1}) \tag{37}$$

is a map between the K-groups of some nice homogeneous spaces. Upmeier has expressed ind_k in topological terms in complete generality. The general result again uses Jordan algebras. For simplicity we restrict to the $A_{n,m}{}'s$. Let

$$\Sigma^k_{n,m} := \{(u,v) \in S_{k-1}(A_{n,m}) \times S_1(A_{n,m}) \mid u \perp v\}, \tag{38}$$

where $u \perp v$ means that v is a partial isometry from $\ker u$ to $\operatorname{im} u^\perp$. The isotropy group K acts transitively on this space. For fixed $u_0 \in S_{k-1}(A_{n,m})$ the set

$$\{\lambda v \mid |\lambda| < 1, v \perp u_0\} \tag{39}$$

is a strongly pseudoconvex (singular) domain with boundary

$$\Sigma^k_{n,m,u_0} \simeq \{v \in S_1(A_{n,m}) \mid v \perp u_0\}. \tag{40}$$

The family index theorem of Boutet de Monvel [BM, Final Remarks] yields an index homomorphism

$$\chi^1(\Sigma^k_{n,m}) : K^*(\Sigma^k_{n,m}) \longrightarrow K^{*-1}(S_{k-1}(A_{n,m})). \tag{41}$$

Moreover, there is a canonical map

$$\beta_k : \Sigma^k_{n,m} \longrightarrow S_k(A_{n,m}),\ (u,v) \mapsto u+v. \tag{42}$$

Now one has the fascinating result

THEOREM (Upmeier, [Up3, Theorem 4.2]). —

$$\operatorname{ind}_k = \chi^1(\Sigma^k_{n,m}) \circ \beta_k{}^*. \tag{43}$$

With this at hand and the spectral sequence (36) it should be possible to attack the conjecture above, although the computations become combinatorially very complicated.

The details for the $A_{n,2}{}'s$ and some domains of higher rank will be worked out in a forthcoming publication.

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