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On a conjecture of Kazamaki

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1 Introduction

The aim of this paper is to answer a question posed by N. Kazamaki in ([1],p.68) :

Does there exist a continuous decreasing function $\Phi : (1,\infty) \to (0,\infty)$, which satisfies the implication

$$d_2(M, L_\infty) < \Phi(p) \Rightarrow \mathcal{E}(M) \text{ satisfies } (R_p) \qquad \forall p > 1$$

obeying

$$lim_{p\to 1}\Phi(p) = +\infty \qquad lim_{p\to +\infty}\Phi(p) = 0 \quad ?$$

Here d_2 denotes the distance induced by the BMO_2-norm, which is defined as $\|M\|^2_{BMO_2} = sup_T\{\|\mathbb{E}[\langle M\rangle_\infty - \langle M\rangle_T|\mathcal{F}_T]\|_\infty\}$, M is a continuous BMO-martingale, L_∞ stands for the space of uniformly bounded martingales and (R_p) means the validity of the reverse Hölder inequality:

$$\mathcal{E}(M) \text{ satisfies } (R_p) \Leftrightarrow \mathbb{E}[\mathcal{E}(M)^p_\infty|\mathcal{F}_T] \le C_p\mathcal{E}(M)^p_T \qquad a.s.$$

for every stopping time T, with a constant C_p depending only on p.

There are two partial answers to this question. One has been given by W. Schachermayer in ([2], rem. 4.2). He explicitly constructs a function Φ, obeying all conditions except the left boundary condition $\Phi(1+) = \infty$. The other result, given by Kazamaki himself in ([1], Th. 3.9), is the following :

Let L^K_∞ denote the class of all martingales bounded by the positive constant K and let $1 < p < \infty$. If $d_2(M, L^K_\infty) < e^{-K}\Phi(p)$, then $\mathcal{E}(M)$ has (R_p), where Φ is a function fulfilling all conditions demanded above.

Despite these two positive results the conjecture of Kazamaki turns out to be wrong. This is shown by a counterexample in Section 2.

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2 The Counterexample

In order to answer the question of Kazamaki negatively, it is sufficient to construct a family of BMO-martingales $M^{(b)}$ ($b \in I\!R_+$), such that

$$d_1(L_\infty, M^{(b)}) \leq C,$$

with a constant C *independent* of b, and

$$\mathcal{E}(M^{(b)}) \text{ does not satisfy } (R_{p(b)}) \text{ with } lim_{b\to\infty}p(b) = 1$$

hold. Note that d_1 is induced by the BMO_1-norm, which is equivalent to the BMO_2-norm.

The main tools for constructing our counterexample are two classical results. The first one is formulated e.g. in ([1], p. 11).

Lemma 2.1 *Let $a, b > 0$ and $\tau = inf\{t|B_t \notin (-a,b)\}$, where B denotes standard Brownian motion. Then we have*

$$I\!E[exp(\frac{1}{2}\theta^2\tau)] = \frac{cos(\frac{a-b}{2}\theta)}{cos(\frac{a+b}{2}\theta)} \qquad (0 \leq \theta < \frac{\pi}{a+b}).$$

The second one is the celebrated Garnett-Jones theorem - in its martingale version due to N. Varopoulos and M. Emery (c.f. [1] Theorem 2.8) - which characterizes the BMO-distance of a continuous martingale M from L_∞ in terms of the critical exponent a(M), defined by

$$a(M) = sup\{a \in I\!R_+ | sup_T \| I\!E[exp(a|M_\infty - M_T|)|\mathcal{F}_T]\|_\infty < \infty\},$$

where T runs through all stopping times.

Theorem 2.1 *For a continuous $M \in BMO$ we have*

$$\frac{1}{4d_1(M, L_\infty)} \leq a(M) \leq \frac{4}{d_1(M, L_\infty)}.$$

Now we give the example mentioned above.

Example:

Let B be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$. For $b \in I\!R_+$ we define the stopping time $\tau^{(b)} = inf\{t : |B_t| = b\}$ and a stopped Brownian motion with drift as $M_t^{(b)} = -B_{t\wedge\tau^{(b)}} + t \wedge \tau^{(b)}$. Applying Girsanov's theorem yields that $M_t^{(b)}$ is a local P-martingale, if the density is given by $\frac{dP}{dQ} = exp(B_{\tau^{(b)}} - \frac{1}{2}\tau^{(b)})$. Further on, because $B^{\tau^{(b)}} \in L_\infty(Q)$ and therefore in $BMO(Q)$, we can infer from Theorem 3.6 in [1] that $M^{(b)} \in BMO(P)$.

The first step is to show : No matter how small (p-1) is, we can always find a constant b s.t. $M^{(b)}$ does not satisfy (R_p). It suffices to prove

Lemma 2.2 *If $M^{(b)}$ is the family of $BMO(P)$-martingales defined above, we have*

$$\|\mathcal{E}(M^{(b)})_\infty\|_{L^p(P)} = \infty \qquad for\ p \geq 1 + \frac{\pi^2}{4b^2}.$$

Proof :
For notational convenience we drop the superscript (b) in this proof.

A simple calculation gives

$$\begin{aligned} I\!E_P[\mathcal{E}(M)^p_\infty] &= I\!E_Q[exp(B_\tau - \frac{1}{2}\tau)exp(pM_\infty - \frac{p}{2}\langle M\rangle_\infty)] \\ &= I\!E_Q[exp(B_\tau - \frac{1}{2}\tau)exp(-pB_\tau + p\tau - \frac{p}{2}\tau)] \\ &= I\!E_Q[exp(B_\tau(1-p))exp(\tau(\frac{p-1}{2}))] \\ &\geq exp(-b|1-p|)I\!E_Q[exp(\tau(\frac{p-1}{2}))]. \end{aligned}$$

The last expectation is $+\infty$ by Lemma 2.1 for $\frac{p-1}{2} \geq \frac{\pi^2}{8b^2}$, completing the proof. □

Remark : It is worth mentioning that, if we change the slope of the drift of the Brownian motion from 1 to k ($k > \frac{1}{2}$), analogous calculations yield the result

$$\|\mathcal{E}(M^{(b)})_\infty\|_{L^p(P)} = \infty \qquad \text{for } p \geq \frac{k^2}{2k-1} + O(\frac{1}{b^2}),$$

and we note that the first term attains its minimum for $k = 1$. For $0 < k < \frac{1}{2}$ we have $\|\mathcal{E}(M^{(b)})_\infty\|_{L^p(P)} < \infty$ for all $p > 1$. So the first part of our example works only for $k = 1$.

The second step is to show that the BMO_1-distance of $M^{(b)}$ to L_∞ is uniformly bounded, which is done by

Lemma 2.3 *Let $M^{(b)}$ be the family of BMO-martingales defined above. Then we have*

$$d_1(M^{(b)}, L_\infty) \leq 8 \qquad \forall b \in I\!R_+.$$

Proof: In order to apply the Garnett-Jones theorem, we have to calculate the critical exponent $a(M^{(b)})$. As above we drop the superscript (b) in the following computations. For an arbitrary stopping time T we get

$$\begin{aligned} &I\!E_P[exp(\lambda|M_\infty - M_T|)|\mathcal{F}_T] = \\ &= I\!E_Q[exp(B_\tau - B_{\tau\wedge T} - \frac{1}{2}(\tau - \tau\wedge T))exp(\lambda| - B_\tau + B_{\tau\wedge T} + \tau - \tau\wedge T|\mathcal{F}_T] \\ &\leq e^{2b+2\lambda b} I\!E_Q[exp((\tau - \tau\wedge T)(\lambda - \frac{1}{2}))|\mathcal{F}_T] < \infty \qquad \text{a.s.} \quad \text{for} \quad \lambda \leq \frac{1}{2}. \end{aligned}$$

Therefore $a(M^{(b)}) \geq \frac{1}{2}$ holds, and the Garnett-Jones theorem yields

$$d_1(M^{(b)}, L_\infty) \leq \frac{4}{a(M^{(b)})} \leq 8,$$

finishing the proof. □

Lemma 2.2 and 2.3 together prove our assertion, formulated at the beginning of section 2.

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References

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