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Summation of quasi-vectors on boolean tribes and its application to quantum theories. I. Mathematically precise theory of P.A.M. Dirac's delta function

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SUMMATION OF QUASI - VECTORS ON BOOLEAN TRIBES AND ITS APPLICATION TO QUANTUM THEORIES. I. MATHEMATICALLY PRECISE THEORY OF P. A. M. DIRAC'S DELTA FUNCTION

Memoria (*) di OTTON MARTIN NIKODÝM (*a Gambier, Ohio*) [1])

This paper may be considered as generalization and, at the same time, as simplification of the paper (14) by the author in which a mathematical apparatus for Quantum Mechanics is exhibited. Its background are Boolean lattices whose elements are closed subspaces in the separable and complete Hilbert-Hermite space. It yields a kind of generalized orthogonal system of coordinates (in this space) which is so well adapted to the continuous spectrum of selfadjoint operators, as the ordinary saturated orthogonal system of vectors is to the discontinuous spectrum. The theory deals with the notion of «trace» (french «lieu») and of « quasi - vectors », and uses a special kind of integrals which resemble the Burkill - integral. The theory, in the general setting, has given a simple canonical representation of maximal normal operators, (22), (11), (26) which has

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make possible to visualize the notion of multiplicity of the continuous spectrum.

The latter was till then wrapped in complicated formulas and thus was not transparent at all. The canonical representation of normal operators has supplied a very natural and simple theory (22) of Stone's « operational calculus » (16) for normal operators, and also a simple theory of permutable normal operators, (22).

The present paper develops several variants of the integration mentioned above which correspond to various kinds of approximations, [§ 1], and also yields a precise setting of the genuine Dirac's Delta Function (33), giving not only, [§ 7], a correct definition but also proofs of its basic properties. At the end of this paper the reader can find a list which compares the theorems as stated by Dirac with those which are proved in our paper.

The Dirac's Delta Function was always a very fascinating problem for mathematicians, because it seemed to escape every precise approach, (the L. Schwartz « Dirac's δ–measure » is far from being a suitable equivalent (34)), though it yielded, as if by a mysterious witchcraft, in the hands of physicists, many correct and important results.

To have our apparatus more useful, we have admitted a very general approach to it, by taking as substratum general Boolean lattices, and developping several kinds of approximations of elements of that lattice by special elements called « complexes » [§ 1]. These approximations yield, in turn, various sorts of integration. The theory of traces, which is only sketched in (11), (and in rather special setting in (14)), is now exhibited with detailed proofs [§ 3], and the same can be said of the general system of coordinates (sketched in (14)), which is now exhibited with details in [§ 5]. The same is with various kinds of integration, mentioned above, [§ 2] and [§ 4].

The analysis of the properties of the δ–function led to our opinion that it should be considered not as a function of a single variable but as the function $\delta(x, y)$ of two variables, which is however, translation-invariant. Indeed the function is a kind of a kernel of integral–equation. Another modification was needed: the variables x, y could not be numbers. In our setting they are « traces ».

The paper contains more than needed for the delta function, because the author intends to have some other application of this apparatus to Quantum Theories.

Numbers in fat parentheses refer to the list at the end of this paper.

1. - **Preliminaries.** We like to use the term tribe, **(35)**, (Boolean tribe) to denote the complementary and distributive lattice **(4)**. Thus the tribe will be conceived as an *ordering* (commonly called « partial ordering », « partially ordered set », **(37)**). The element of a tribe will be termed *soma*, **(36)**. The tribe will be termed *finitely, denumerably, completely additive* whenever all finite, denumerable, all lattice joins are meaningful. Of course, finite joins always exist. The ordering will be denoted by « $\leq$ » and lattice operations (somatic operations) by $+$, $\cdot$, $-$, co (complement), Σ, Π. If the somata of the tribe are sets, we shall use Bourbaki symbols **(38)**, $\smile$, $\frown$, $\backsim$, $\cup$, $\cap$. If we introduce the « *algebraic addition* » $\dot{+}$ (symmetric difference) defined by $a \dot{+} b \overline{\overline{\mathrm{df}}} (a - b) + (b - a)$, **(18)**, the tribe will be organized into a commutative ring with unit, (Stone's ring). The *zero–soma* and the *unit–soma* will be denoted O, I respectively. If the somata are sets, and the ordering relation the inclusion of sets $\subseteq$, the zero will be the empty set and denoted by $\emptyset$. Since the tribe can be conceived as a ring, the notion of *ideal* can be applied, **(5)**. The somata a, b are said to be disjoint whenever $a \cdot b = O$.

2. - **Remark.** Notice that somatic operations depend not only on somata operated but on the totality of the tribe.

3. - Every theory, axiomatised or constructed, has a specific ***notion of equality***, with respect to which the genuine operations and relations should be ***invariant***.

E. g. for additions of somata we have: « if $a = a'$, $b = b'$, $c = c'$, $a + b = c$, then $a' + b' = c'$ ». According to the situation, that notion of equality may be axiomatised, constructed or taken over from another theory.

We shall call it «*governing equality*». (3), (2), (10).

4. - Let F', F'' be two tribes. We shall denote with prime, (double prime), the notions in F', (F''). We say that F' is a *finitely genuine subtribe of* F'' whenever the following takes place (5): 1) The elements of F' are also elements of F'', 2) the following are equivalent for somata a, b, c of F':

a) $a +' b =' c$ and $a +'' b ='' c$

b) $a \cdot' b = c$ and $a \cdot'' b ='' c$

c) $O' ='' O''$

d) $I' ='' I''$.

These four conditions are independent from one another. (If similar equivalences also hold true for denumerable (all) operations, we say that F' is *denumerably (completely) genuine subtribe of* F''. The isomorphism $\mathfrak{A}$ which attaches to a and its $='$-equals in F' the element a and its $=''$-equals in F'', plunges in some way F' into F''. If the elements of F' do not belong to F'', but there is a correspondence $\mathfrak{B}$ preserving operations such that $\mathfrak{B}F'$ is a finitely genuine subtribe of F'', we say that F' is a *finitely genuine subtribe of* F'' *through isomorphism* $\mathfrak{B}$. If the equality $='$ is just the equality $=''$ restricted to F' we say that F' is a *finitely genuine strict subtribe of* F''.

5. - E. g. Let F' be the tribe whose somata are finite unions of half-open « intervals » $(\alpha, \beta\rangle$ where $0 \leq \alpha, \beta \leq 1$, with ordering relation $\leq'$ defined as inclusion $\subseteq$ of sets. Let F'' be the tribe of all Lebesgue-measurable subsets of $(0, 1\rangle$ with ordering relation $\leq''$ defined by:

$$E \leq'' F \cdot \overline{\overline{\scriptstyle df}} \cdot \operatorname{meas}(E - F) = 0.$$

The governing equality $='$ on F' is the identity of sets, that $=''$ on F'' is equality modulo the ideal of nullsets in $(0, 1\rangle$. Here F' is a finitely genuine subtribe of F''. The correspondence $\mathcal{A}$ attaches to every set a of F' the sets $a+M-N$ where meas $M =$ meas $N = 0$.

If we take the tribe F_1' whose ordering relation is $\leq''$ and somata are $a+M-N$, with $a \in F'$, then F_1' will be a finitely genuine strict subtribe of F''.

Notice that F' is not a denumerably genuine subtribe of F''.

6. - Concerning the notion of homomorphism and isomorphism we refer to our paper (3), where some subtle possible confusions are clarified.

7. - Let F be a tribe. Let us attach to every soma $a \in F$ a non negative number [2] $\mu(a)$ with the conditions: if $a \cdot b = O$, then $\mu(a+b) = \mu(a) + \mu(b)$, $a = b$ implies $\mu(a) = \mu(b)$. We shall call the function $\mu(a)$ *finitely additive measure on F.*

If $a_1, a_2, \ldots, a_n, \ldots$ denumerable in number are disjoint, and $\sum_{n=1}^{\infty} a_n$ has a meaning and $\mu\left(\sum_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$, then we say that μ is a *denumerably additive measure on F.* Usually we consider denumerably additive measures on tribes, which are themselves denumerably additive, (see (5)). The measure on Boolean tribes was introduced independently in (36) and (6).

The measure is said to be *effective* whenever $\mu(a) = 0$ implies $a = 0$. The usual term « measure algebra » will be not applied.

8. - In this paper we shall pay special attention to tribes whose somata are closed subspaces of a Hilbert-Hermite space, (1), (14).

9. - Russell-Whitehead «relations» (40) will be termed *correspondences.* Functions will be considered as correspon-

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[2]) $a \in F$ means that a is an element of the *domain of the tribe.* Though an ordering and its domain are logically different notions, we shall be allowed to use the same letter F for the tribe and for its domain.

dences. If we like to emphasize that a letter denotes a variable we provide it with *dot*, ex. $f(\dot{x})$. If R is a correspondence, its ***domain*** will be denoted by $\mathrm{Ɑ}R$ and its ***range*** by $\mathrm{D}R$. The correspondence R, ***restricted in the domain to the set*** E, will be denoted (40) by $E \,1\, R$. The symbol $\{\varphi(x) \mid w(\cdot x \cdot)\}$ will denote the set of all $\varphi(x)$ such that x satisfies the condition $w(\cdot x \cdot)$. The sign $\overline{\overline{df}}$ means « equal by definition ».

10. - We shall consider ***Hilbert-Hermite spaces*** (16), (41), which may be of finite or denumerably infinite dimensions. We shall consider only the case where the space H is separable and complete. The ***norm*** of the vector $\vec{X}$ will be denoted by $\|\vec{X}\|$, the (Hermitean) scalar product will be written $(\vec{X}, \vec{Y})$.

11. - By ***linear variety in H*** we shall understand a non empty set E of vectors such that if $x, y \in E$, then $\alpha x + \beta y \in E$ whatever the complex numbers α, β may be.

A closed linear variety will be termed ***subspace***, or simply, ***space***. Thus $(\vec{0})$ and H are subspaces. A vector x is said to be ***orthogonal to*** the ***space*** a if $x \perp y$ for all $y \in a$; we write $x \perp a$, $a \perp x$. Two spaces a, b are said to be ***orthogonal***, $a \perp b$, if $x \perp y$ for every $x \in a$ and every $y \in b$.

If a is a space, then the set b of all vectors orthogonal to a is also a space and is termed (orthogonal) ***complement*** of a. We write $b = \mathrm{co}\, a$.

12. - If x is a vector and a a space, there exists a unique decomposition $x = x' + x''$, where $x' \in a$, $x'' \in \mathrm{co}\, a$. The vector x' is termed ***projection of x on a*** and denoted by $\mathrm{Proj}_a\, x$ or $\mathrm{Proj}\,(a)x$. The operation of projection is a selfadjoint hermitean operation which carries H onto a. The following properties are known:

$\mathrm{Proj}_a\, x \in a$; $\|\mathrm{Proj}_a\, x\| \leq \|x\|$, $(\mathrm{Proj}_a\, x, y) = (x, \mathrm{Proj}_a\, y)$,

$\mathrm{Proj}_a\,(\alpha x + \beta y) = \alpha\, \mathrm{Proj}_a\, x + \beta\, \mathrm{Proj}_a\, y$; if $x_n \to x$, then $\mathrm{Proj}_a\, x_n \to \mathrm{Proj}_a\, x$;

If $a \subseteq b$, then $\mathrm{Proj}_a\, \mathrm{Proj}_b\, x = \mathrm{Proj}_a\, x$.

13. - Stone **(16)** has studied the following operations on spaces. If $a_1, a_2, \ldots$ are spaces, then by *their sum* $a_1 + a_2 + \ldots$, $\Sigma_i a_i$ we understand the smallest space containing them all, and by *their product* $a_1 \cdot a_2, \ldots$, $\Pi_i a_i$ we understand the greatest space included in each of them, i. e. their set-intersection. These operations are also considered for any not empty class of spaces.

If we denote the whole space H by I, write O instead of $(\vec{O})$, and put $a - b \underset{df}{=} a \cdot \text{co}\, b$, we have the following rules:

$$a + b = b + a \quad , \quad a + (b + c) = (a + b) + c \quad , \quad a + a = a,$$

$$a \cdot b = b \cdot a \quad , \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad , \quad a \cdot a = a.$$

$$a \cdot O = O \quad , \quad a \cdot I = a \quad , \quad a + O = a \quad , \quad a + I = I.$$

De Morgan's laws are valid:

$$\text{co}\,(a + b) = \text{co}\, a \cdot \text{co}\, b \quad , \quad \text{co}\,(a \cdot b) = \text{co}\, a + \text{co}\, b.$$

The distributive law $(a + b) \cdot c = a \cdot c + b \cdot c$ is not in general true, neither $a - b = a - a \cdot b$.

14. - In **(14)** we have introduced the following notion: two spaces a, b are said to *be compatible* (with one another) if $(a - ab) \perp (b - ab)$. The following properties are equivalent: 1) a, b are compatible; 2) $\text{Proj}_a\, b \subseteq b$; 3) $\text{Proj}_a\, b \subseteq a \cdot b$; 4) $\text{Proj}_a\, b = a \cdot b$; 5) $\text{Proj}_a\, b = \text{Proj}_b\, a$; 6) $a - b = a - ab$; 7) $a - ab \subseteq a - b$; 8) $a = ab + a \cdot \text{co}\, b$; 9) $\text{Proj}_a \text{Proj}_b\, x = \text{Proj}_b \text{Proj}_a\, x$ for every vector x, **(42)**; 10) there exists a space c such that $\text{Proj}_a \text{Proj}_b\, \vec{x} = \text{Proj}_c\, \vec{x}$ for every vector $\vec{x}$.

15. - Concerning compatibility, the following theorems hold true: If a, b are compatible with one another and with c, then $a + b$, $a \cdot b$, $\text{co}\, a$, $a - b$, $a \dot{+} b$ are also compatible with c. If all spaces of a non empty collection $\{ N \}$ are compatible with one another and with c, then $\Sigma\, a$, $(a \in \{ N \})$ and $\Pi\, a$, $(a \in \{ N \})$ are also compatible with c.

16. - Let T be a non empty set of spaces such that 1) if $a, b \in T$, then $a + b \in T$, 2) if $a \in T$, then $\text{co}\, a \in T$, 3) if

$a, b \in T$ and $a \cdot b = 0$, then $a \perp b$. This set T if ordered by the inclusion-relation $\subseteq$ of sets, is organized into a Boolean finitely additive tribe T, where the lattice operation coincide with spacial operations introduced above **(39)**. The spaces of T are compatible with one another. The distributive law holds true within T.

If the condition 1) is replaced by the following one:

if $a_1, a_2, \ldots, a_n, \ldots \in T$, then $\sum_{n=1}^{\infty} a_n \in T$, T will be denumerably additive. If any sum of spaces of T belongs to T, the tribe will be completely additive.

If T is a non empty set of spaces such that 1) if $a, b \in T$, then $a + b \in T$; 2) if $a \in T$, then $\operatorname{co} a \in T$, then a necessary and sufficient condition that T be a tribe set, is that all spaces of T be compatible with one another.

17. - The following theorems are proved in **(39)**:

If T is a tribe of spaces, then there exists a denumerably additive tribe T' of spaces, such that T is a subset of T' and is closed in T with respect to finite addition and complementation:

If T is a denumerably additive tribe of spaces, then it is also completely additive. (For a more general theorem for abstract tribes, see **(12)**).

18. - If T is a tribe of spaces, p a space, then a necessary and sufficient condition that there axist a tribe T' of spaces such that $T \subseteq T'$, $p \in T'$, is that p be compatible with T (i. e. with all spaces $\in$ T), **(14)**. The same condition is required, if T is denumerably additive. The smallest T' containing both T and p is the set of all spaces $a \cdot p + b \cdot \operatorname{co} p$ theorems where $a, b \in T$.

19. - A tribe T of spaces is said to be *saturated* if for every tribe T' with $T \subseteq T'$ we have $T = T'$. The following hold true:

For every T there exists a saturated T' such that $T \subseteq T'$.

Every saturated tribe is completely additive.

20. - Let T be a tribe of spaces and E a not empty set of vectors. By the *radiation space of E with respect to T*, (champs de rayonnement **(14)**) we understand the set $M_T(E)$ of all vectors x such that for every $\varepsilon > 0$ there exist complex numbers λ_i, vectors $\xi_i \in E$ and spaces $a_i \in T$, $i = 1, 2, \ldots, n)$, $(n = 1, 2, \ldots)$, where $\| x - \Sigma_i \lambda_i \operatorname{Proj}_{a_i} \xi_i \| \leq \varepsilon$.

$M_T(E)$ is the smallest space containing all vectors

$$\operatorname{Proj}_a \xi \quad \text{where} \quad a \in T \quad , \quad \xi \in E.$$

For sets E composed of a single vector see Stone's book **(16)**, the general case is mentioned in **(43)**.

A necessary and sufficient condition that a space p be compatible with a tribe T is that there exists a set E such that $p = M_T(E)$.

21. - A vector $\vec{\omega}$ is called *generating vector of the space H with respect to T*, if $M_T((\vec{\omega})) = I = H$. The following two properties are equivalent: 1) there exists a generating vector of H with respect to T. 2) for every set $E \neq \emptyset$ of vectors there exists a vector ξ such that

$$M_T(E) = M_T((\xi)).$$

If $\vec{\omega}$ is a generating vector, then for every $a \in T$ we have $a = M_T((\operatorname{Proj}_a \vec{\omega}))$.

A necessary and sufficient condition that there exists a generating vector of I with respect to T, is that T be saturated, **(16)**, **(44)**.

22. - For every denumerably additive tribe T of spaces there exists an effective, non negative measure on it.

23. - If $\vec{\omega}$ is a generating vector of the space with respect to T, and if we put $\mu(a) = \| \operatorname{Proj}_a \|^2$ for $a \in T$, we obtain a denumerably additive, effective non negative, (finite) measure on T. (Every denumerably additive measure can be given by an analogous formula).

§ 1. - Approximation of somata by complexes.

This [§ 1] is of auxiliary capacity and contains definitions of some notions and proofs of their properties which are needed for foundation of a kind of integration in [§ 2]. First we introduce the notion of the base B of a finitely additive tribe F and give some of its properties. The tribe F will be extended to a denumerably additive tribe G which admits a denumerably additive, non negative measure μ. The notion of μ-distance between two somata of G will be introduced [5] and studied. Later, G will be admitted to be the Lebesguean extension of F. The notion of a complex will be introduced [9] and special approximations of somata of G by complexes will be discussed. The reader interested mainly in application of our theory, may omit reading quite complicated proofs of theorems [18.1] and [19.1].

1. - In this [§ 1] we admit the following hypotheses referred to as **Hyp FBG**, (compare (11)).

F is a non trivial, finitely additive tribe; its somata $f, g, h, \ldots$ will be termed *figures*.

B is a subset of F, satisfying the conditions:

1) $O \in B,\ I \in B$,

2) if $a, b \in B$. then $a \cdot b \in B$,

3) if $f \in F$, then there exists a finite number of somata of B whose sum is f.

The somata $a, b, c, \ldots, p, q, \ldots$ of B will be termed *bricks*. The set B will be termed *base of F* [3]).

G is a denumerably additive tribe. We suppose that F is its finitely genuine subtribe and denote by $\mathfrak{A}$ the corresponding isomorphism from F into G.

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[3]) Though the tribe is an ordering, and hence it differs logically from the set of all its elements, nevertheless, to avoid complications in notation, we shall use the same letter to denote the tribe and the set of all its somata.

2. - We shall need some lemmas valid for any tribe S. The letters will denote its somata.

2.1. - Lemma. $p + q = p + q \operatorname{co} p$.

2.2. - Lemma. Let $p_1, \dots, p_{n+1} \in S$, $(n \geq 1)$. Put

$$q_1 \underset{df}{=} p_1 \quad , \quad q_2 \underset{df}{=} p_2 - p_1, \dots, \; q_{n+1} \underset{df}{=} p_{n+1} - (p_1 + \dots + p_n);$$

then $p_1 + \dots + p_{n+1} = q_1 + \dots + q_{n+1}$. We prove this by induction.

2.3. - Lemma. The somata $q_1, \dots, q_{n+1}$ in [2.2] are disjoint.

Proof. Let $2 \leq i < k$. We have

$$q_k = p_k - (p_1 + \dots + p_{k-1}) \quad , \quad q_i = p_i - (p_1 + \dots + p_{i-1}).$$

Hence $q_k = p_k \operatorname{co} (p_1 + \dots + p_{k-1}) = p_k \operatorname{co} p_1 \dots \operatorname{co} p_{k-1}$, which gives

$$(1) \qquad q_k \leq \operatorname{co} p_i .$$

We have $q_i \leq p_i$. hence, by (1),

$$q_i q_k \leq p_i \operatorname{co} p_i = O, \qquad \text{so} \qquad q_i, \; q_k \text{ are disjoint.}$$

Now let $i = 1$. We have $q_k = p_k - (p_1 + \dots + p_{k-1})$; hence $q_k \leq \operatorname{co} p_1$. Since $q_1 = p_1$, il follows $q_k q_1 \leq \operatorname{co} p_1 \cdot p_1 = O$. The lemma is proved.

3. - We shall consider the following new hypotheses:

(Hyp. Ad.). If $a \in B$, then $\operatorname{co} a$ can be represented as a denumerable sum of mutually disjoint bricks, where the infinite summation is taken over from G. This statement shall be understood as follows: If $a \in B$, then there exist disjoint bricks $a_1, a_2, \dots, a_n, \dots$, denumerable in number, such that $\operatorname{co}^G (\mathfrak{A} a) =^G \mathfrak{A} a_1 +^G \mathfrak{A} a_2 +^G \dots$, where $\mathfrak{A}$ is the isomorphism mentioned in [1].

Another hypothesis is:

(Hyp. Af.). If $a \in B$, then there exists a finite number of disjoint bricks $a_1, \dots, a_n$ such that $\operatorname{co} a = a_1 + \dots + a_n$.

Of course (Hyp. Af.) implies (Hyp. Ad.). For our main purpose (Hyp. Af) is sufficient.

However. since we have further generalizations in mind, we prefer to work under the less restrictive hypothesis (Hyp. Ad.). The complication which our approach will imply, is not too great.

3.1. - Example. Let F be the tribe whose somata are O and finite unions of half-open segments (α, β) where $0 \leq \alpha < \beta \leq 1$. Bricks are just those segments. (Hyp. Af.) is satisfied.

3.2. - Theor. If

1) F is a finitely additive tribe,

2) $B' \subseteq F$,

3) $O \in B'$, $I \in B'$,

4) if a, $b \in B'$, then $a \cdot b \in B'$.

5) if $a \in B'$, then co a is a finite sum of mutually disjoint elements of B',

6) F is the smallest strict subtribe of F containing B',

then

1) every $p \in F$ is a finite sum of mutually disjoint elements of B',

2) every finite sum of mutually disjoint elements of B' belongs to F,

3) B' is a base of F.

Proof. Denote by F_1 the set of all somata of F which are finite disjoint sums of elements of B'. We shall first prove that F_1 is a strict subtribe of F.

Let $p \in F_1$. We have $p = a_1 + \ldots + a_n$ where $a_i \in B'$, $n \geq 1$. Hence $\text{co}\, p = \text{co}\,(a_1 + \ldots + a_n) = \text{co}\, a_1 \ldots \text{co}\, a_n$. By hyp. 5 we have for $i = 1, 2, \ldots$, $\text{co}\, a_i = a_{i1} + a_{i2} + \ldots$ where the sum is finite and the elements are disjoint. Hence

$$\text{co}\, p = \Sigma_{\alpha_1, \ldots, \alpha_n} a_{1\alpha_1} a_{2\alpha_2} \ldots a_{n\alpha_n} .$$

The terms of this sum are, by hyp. 4, elements of B'

and they are disjoint. Indeed $a_{1\alpha_1} \dots a_{n\alpha_n} \cdot a_{1\beta_1} \dots a_{n\beta_n} = O$ all the time when one of the inequalities $\alpha_1 \neq \beta_1, \dots, \alpha_n \neq \beta_n$ is true; hence different terms are disjoint.

Consequently $\operatorname{co} p \in F_1$.

Thus we have proved that

(1) $$\text{if } p \in F_1, \quad \text{then} \quad \operatorname{co} p \in F_1 .$$

Now let $p, q \in F_1$. We have

$$p = c_1 + c_2 + \dots \quad , \quad q = d_1 + d_2 + \dots$$

where the terms of each sum are disjoint elements of B', and the sums are finite.

We get $p \cdot q = \Sigma_{i,j} c_i d_j$. The terms of this sum are disjoint elements of B'.

Thus we have proved that if $a, b \in F_1$, then $a \cdot b \in F_1$. This result and (1) imply that F_1 is a tribe. Its unit is I and its zero is O. It is a strict subtribe of F and contains B'. Hence, by hyp. 6, $F = F_1$. It follows that B' is a base of F.

3.3. - There exists a tribe F and a base B of F such that (Hyp. Ad.) is not satisfied-hence (Hyp. Af.) neither.

Example. We shall consider various half open subintervals $\langle\alpha, \beta)$ of $\langle 0, 1)$. Denote by $a(0)$, b, $a(1)$ the intervals $\left\langle 0, \frac{1}{3}\right)$, $\left\langle \frac{1}{3}, \frac{2}{3}\right)$, $\left\langle \frac{2}{3}, 1\right)$ respectively. an put $b' \underset{df}{=} a(0) \cup b$. $b'' \underset{df}{=} b \cup a(1)$. We divide a(0) into three equal parts $a(0, 0)$, $b(0)$, $a(0, 1)$, and we do the same with $a(1)$, getting $a(1, 0)$. $b(1)$, $a(1, 1)$. We put

$$b'(0) \underset{df}{=} a(0, 0) \cup b(0) \quad , \quad b''(0) \underset{df}{=} b(0) \cup a(0, 1)$$

$$b'(1) \underset{df}{=} a(1, 0) \cup b(1) \quad ; \quad b''(1) \underset{df}{=} b(1) \cup a(1, 1).$$

Suppose we have already defined all $a(\alpha_1, \alpha_2, \dots, \alpha_r)$ where $\alpha_1, \alpha_2, \dots, \alpha_r$ are 0 or 1 and r is given. We divide this interval into three equal parts:

(1) $$a(\alpha_1, \dots, \alpha_r, 0), \; b(\alpha_1, \dots, \alpha_r), \; a(\alpha_1, \dots, \alpha_r, 1),$$

and define:

$$(2) \qquad \begin{cases} b'(\alpha_1, \dots, \alpha_r) \underset{df}{=} a(\alpha_1, \dots, \alpha_r, 0) \cup b(\alpha_1, \dots, \alpha_r), \\ b''(\alpha_1, \dots, \alpha_r) \underset{df}{=} b(\alpha_1, \dots, \alpha_r) \cup a(\alpha_1, \dots, \alpha_r, 1). \end{cases}$$

Thus we have defined inductively all $a(\alpha_1, \dots, \alpha_n)$ for $n = 1, 2, \dots$. Then (1) defines all $b(\alpha_1, \dots, \alpha_n)$ and (2) defines all $b'(\alpha_1, \dots, \alpha_n)$ and $b''(a_1, \dots, \alpha_n)$.

Denote by B the class of sets composed of O, $(0, 1)$, b, b', b'', and all $b(\alpha_1, \dots, \alpha_n)$, $b'(\alpha_1, \dots, \alpha_n)$, $b''(\alpha_1, \dots, \alpha_n)$. Denote by F the smallest tribe of sets which contains B. Under the above circumstances B is a base of F, but (Hyp. Ad.) is not satisfied.

3.4. - There exist F and B where (Hyp. Ad.) is satisfied but (Hyp. Af.) is not satisfied.

Example. We consider the segments denoted by b, without or with primes and indices as before, but of all kinds: open, closed, half open on the right and half open on the left. They, and all their endpoints [4]) will constitute the base.

If we consider the smallest tribe containing that base, we have satisfied (Hyp. Ad.), but not (Hyp. Af.).

3.5. - After this preliminary discussion, we shall prove some lemmas under (Hyp. Ad.) or (Hyp. Af.). To simplify writing of formulas which involve infinite operations, we shall make the following agreement. We shall write $\sum_{n=1}^{\infty} f_n$ and $\prod_{n=1}^{\infty} f_n$ instead of $\sum_{n=1}^{\infty}{}^{G} \mathfrak{a} f_n$ and $\prod_{n=1}^{\infty}{}^{G} \mathfrak{a} f_n$ respectively. If special clarity is needed we shall say that the infinite denumerable operations, performed on somata of F, « *are taken from* G ».

3.5. - **Lemma.** Under (Hyp. Ad.) if $a_1, \dots, a_n$ are bricks, $(n \geq 2)$, then $\mathrm{co}(a_1 + a_2 + \dots + a_n)$ is the sum of a denume-

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[4]) We mean single point-sets.

rable number of mutually disjoint bricks (infinite summation are taken from G).

Proof. We have $\text{co}\,(a_1 + \ldots + a_n) = \text{co}\,a_1 \ldots \text{co}\,a_n$, and by (Hyp. Ad.) $\text{co}\,a_i = \sum_{s=1}^{\infty} a_{is}$, $(i = 1, \ldots, n)$, where $a_{i1}, a_{i2}, \ldots$ are disjoint bricks. Hence

$$(1) \qquad \text{co}\,a_1 \ldots \text{co}\,a_n = \sum_{k_1=1}^{\infty} a_{1k_1} \ldots \sum_{k_n=1}^{\infty} a_{nk_n} = \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} a_{1k_1} \ldots a_{nk_n}.$$

Now $(a_{1k_1} \ldots a_{nk_n}) \cdot (a_{1j_1} \ldots a_{nj_n}) = 0$ all the times when at least one of the inequalities $k_1 \neq j_1, \ldots, k_n \neq j_n$ holds true. Hence all the termes in (1) are disjoint. They are bricks on account of [1]. The lemma is proved.

3.5.1. - Lemma. Under (Hyp. Af.) if $a_1, \ldots, a_n$ are bricks $(n \geq 2)$, then $\text{co}\,(a_1 + \ldots + a_n)$ is the sum of a finite number of mutually disjoint bricks.

Proof. Similar to that of [Lemma 3.5.].

3.6. - Theor. Under (Hyp. Ad.), if $f \in F$, then f is the sum of a denumerable number of mutually disjoint bricks. (Infinite summation is taken from G).

Proof. By [1] we have $f = a_1 + \ldots + a_n$ where a_i are bricks. We may suppose $n \geq 2$, since we can add as many zero somata as we like. Indeed, O is a brick.

Put $g_1 \underset{df}{=} a_1$, $g_2 \underset{df}{=} a_2 - a_1, \ldots, g_n \underset{df}{=} a_n - (a_1 + \ldots + a_{n-1})$. We have by [2.2], and [2.3],

$$(1) \qquad f = g_1 + \ldots + g_n \quad \text{with} \quad g_1, \ldots, g_n \quad \text{disjoint.}$$

Since $g_i = a_i\,\text{co}\,(a_1 + \ldots + a_{i-1})$, we have, by [3.5], $g_i = a_i \cdot \sum_{s=1}^{\infty} a_{i-1,s}$, where $a_{i-1,1}, a_{i-1,2}, \ldots$ are disjoint bricks. We get $g_i = \sum_{s=1} (a_i a_{i-1,s})$, so, by (1), the lemma is proved.

3.6.1. - Theor. Under (Hyp. Af.), if $f \in F$, then f is the sum of a finite number of disjoint bricks.

Proof. Similar to that of [3.6. Lemma].

4. - Def. By a *covering (brick-covering)* we shall understand an at most denumerable sum of bricks, [i. e. $\mathfrak{A}$-images of bricks] [5]). Hence O is a covering.

We shall prove that, under (Hyp. Ad.), a covering can always be considered as $\sum_{n=1} a_n$, where a_n is an infinite sequence of disjoint bricks. To do that we need some lemmas.

4.1. - Lemma. If S is a denumerably additive tribe, then for its somata $E_1, E_2, \ldots, E_n, \ldots$ we have $E_1 + E_2 + \ldots E_n + \ldots = E_1 + (E_1 \dotplus E_2) + (E_1 + E_2 + E_3) +$

Proof. Put $A \underset{df}{=} E_1 + E_2 + \ldots$, $B \underset{df}{=} E_1 + (E_1 + E_2) + \ldots$

We have $E_1 \leq B$, $E_2 \leq B, \ldots$, hence, by definition of the lattice sum, $A \leq B$. On the other hand we have $E_1 \leq A$, $E_1 + E_2 \leq A, \ldots$, hence $B \leq A$, so the lemma is proved.

4.2. - Lemma. If S is a denumerably additive tribe, then for an infinite sequence of its somata $E_1, E_2, \ldots, E_n, \ldots$ we have:

1) $F_1 + F_2 + \ldots + F_n + \ldots = E_1 + E_2 + \ldots + E_n + \ldots$,
where $F_1 \underset{df}{=} E_1$, $F_2 \underset{df}{=} E_2 - E_1, \ldots$, $F_{n+1} \underset{df}{=} E_{n+1} - (E_1 + \ldots + E_n)$;

2) The somata $F_1, F_2, \ldots, F_n, \ldots$ are all disjoint.

Proof. The thesis 2) follows from [2.3]. We have, by [4.1], $E_1 + E_2 + \ldots = E_1 + (E_1 + E_2) + (E_1 + E_2 + E_3) + \ldots$; hence, by [2.2], $E_1 \dotplus E_2 + \ldots = F_1 + (F_1 + F_2) + (F_1 + F_2 + F_3) + \ldots$, and then, by [4.1], $E_1 + E_2 + \ldots = F_1 + F_2 + \ldots$.

4.2.1. - If $L \in G$, $L = \mathfrak{A}f_1 + \mathfrak{A}f_2 + \ldots + \mathfrak{A}f_n + \ldots$ where $f_n \in F$, then L can be considered as the sum

$$L = \mathfrak{A}g_1 + \mathfrak{A}g_2 + \ldots + \mathfrak{A}g_n + \ldots$$

where $g_n \in F$ and all g_n are disjoint.

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5) There must be at least one brick in the sum, since we do not consider « empty » sums-for reasons given in (3). We can define a covering as $\sum_{n=1}^{\infty} a_n$ where $\{a_n\}$ is an infinite sequence of bricks. Since O is a brick, we always can replace finite sums by infinite ones.

Proof. This follows from [4.2. Lemma].

4.3. - **Theor**. Under (Hyp. Ad), if L is a covering, then it is a denumerable sum of mutually disjoint bricks.

Proof. We have, by [Def. 4], $L = a_1 + a_2 + ... + a_n + ...$, where a_i are bricks.

Put $f_1 \underset{df}{=} a_1$, $f_2 \underset{df}{=} a_2 - a_1$, $f_3 \underset{df}{=} a_3 - (a_1 + a_2)$, ...

We have, by [4.2], $L = f_1 + f_2 + ...$, where all f_i are disjoint. On the other hand, since $f_i \in F$, the figure f_i is, by [3.6], a sum of a denumerable number of disjoint bricks. This completes the proof.

4.4. - **Cor**. Under (Hyp. Ad.), if $L = f_1 + f_2 + f_3 + ... + f_n + ...$, where f_n are figures, then there exists an infinite sequence of mutually disjoint bricks $a_1, a_2, ..., a_n, ...$ such that 1) $L = a_1 + a_2 + a_3 + ...$ 2) for every i there exists j such that $a_i \leq f_j$.

Proof. We take over the proof of [Theor. 4.3], getting $L = g_1 + g_2 + ... + g_n + ...$ where $g_1, g_2, ..., g_n, ...$ are figures and where $g_1 \leq f_1$, $g_2 \leq f_2, ...$.

Now, by [3.6], we can decompose each g_n into disjoint bricks $g_n = a_{n1} + a_{n2} + ... (n = 1, 2, ...)$.

Since the set $\{a_{ik}\}$, $i, k = 1, 2, ...$ is denumerable it can be represented by an infinite sequence. We have $a_{nk} \leq g_n \leq f_n$, so the theorem is proved.

5. - In this subsection till [12], but [12] excluded, we shall consider a general denumerably additive, non trivial tribe G. We shall not admit neither (**Hyp**. *FBG*), nor (**Hyp**. **Ad**.). The topic we shall deal with will be later applied to the circumstances conditioned by (**Hyp**. *FBG*) and (**Hyp**. **Ad**.).

We admit the ***hypothesis*** (**Hyp**. $G\mu$): G admits a non negative measure μ which is not trivial i.e. we have $\mu(I) > 0$. The measure should be $\overset{G}{=}$ invariant, i.e. if $E \overset{G}{=} F$, then $\mu(E) = \mu(F)$.

We take such a measure and will keep it fixed.

The measure may be effective on G or not. It induces

in a subtribe F of G a measure, also denoted by μ, which is finitely additive on F. It may be effective on F without being effective on G. We do not suppose anything about the effectiveness of μ.

5.1. - **Def.** Under (**Hyp.** $G\mu$) we define on G the notion *of μ distance between two somata E, F* of S; (see **(7)**): We define it by:

$$|E, F|_\mu \underset{df}{=} \mu(E \dot{+} F) = \mu(E - F) + \mu(F - E),$$

where $\dot{+}$ denotes algebraic addition in the Stone's ring G (see Preliminaries).

The notion $|E, F|_\mu$ is G - equality - invariant, i. e., if $= E_1$, $F = F_1$, then $|E, F|_\mu = |E_1, F_1|_\mu$.

5.2. - We shall have some theorems concerning that notion of distance. They are based on some properties of the algebraic addition.

We mention the following ones: $E \dot{+} F \leq E + F$, $E \dot{+} E = O$, the associative and commutative law for the algebraic addition [6]).

We have $\mu(E) = |O, E|_\mu$.

5.3. - $|E, F| = |F, E|$.

5.4. - $|E, E| = 0$.

5.5. - $|E, F| \leq |E, G| + |G, F|$.

5.6. - $\mu(E \dot{+} F) \leq \mu(E) + \mu(F)$, $|E, F| \leq \mu(E) + \mu(F)$.

5.7. - $(E_1 + F_1) \dot{+} (E_2 + F_2) \leq (E_1 \dot{+} E_2) + (F_1 \dot{+} F_2)$, which gives by induction:

5.7.I. - $(E_1 + \ldots + E_n) \dot{+} (F_1 + \ldots + F_n) \leq (E_1 \dot{+} F_1) + (E_2 \dot{+} F_2) + \ldots + (E_n \dot{+} F_n)$ for $n \geq 2$.

5.8. - $E_1F_1 \dot{+} E_2F_2 \leq (E_1 \dot{+} E_2) + (F_1 \dot{+} F_2)$.

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[6]) We shall usually write $|E, F|$ instead of $|E, F|_\mu$, when no ambiguity can result.

5.9. - co $E_1 \dotplus$ co $E_2 = E_1 \dotplus E_2$.

5.10. - $(E_1 - F_1) \dotplus (E_2 - F_2) \leq (E_1 \dotplus E_2) + (F_1 \dotplus F_2)$.

5.10.I. - $(E_1 . E_2 \dots E_n) \dotplus (F_1 F_2 \dots F_n) \leq (E_1 \dotplus F_1) + \dots + (E_n \dotplus F_n)$.

Proof. By [5.9], $(E_1 \dots E_n) \dotplus (F_1 .. F_n) =$ co $(E_1 \dots E_n) \dotplus$ co $(F_1 \dots F_n) = ($co $E_1 + \dots +$ co $E_n) + ($co $F_1 + \dots +$ co $F_n)$.

Applying [5.7.I] and afterwords [5.9], we prove the statement.

The above lemmas yield proofs for the following properties of the distance of somata of G.

5.11. - $| E_1 + F_1, E_2 + F_2 | \leq | E_1, E_2 | + | F_1, F_2 |$. Proof by [5.7].

5.11.I. - $E_1 + \dots + E_n, F_1 + \dots + F_n | \leq | E_1, F_1 | + \dots + | E_n, F_n$. Proof by [5.7.I].

5.12. - $| E_1 F_1, E_2 F_2 | \leq | E_1, E_2 | + | F_1, F_2 |$. Proof by [5.8].

5.12.I. - $| E_1 E_2 \dots E_n, F_1 F_2 \dots F_n | \leq | E_1 F_1 | + \dots + | E_n F_n |$. Proof by [5.10.I].

5.13. - $| E, F | = |$ co E, co $F |$. Proof by [5.9].

5.14. - $| E_1 - F_1, E_2 - F_2 | \leq | E_1, E_2 | + | F_1, F_2 |$. Proof by [5.10].

5.15. - $| EH, FH | \leq | E, F |$.

Proof. By [5.12] we have $| EH, FH | \leq | E, F | + | H, H |$, from which, by [5.4], the statement follows.

5.16. - $| E + F, H | \leq | E, H | + | F, H |$.

Proof. $| E + F, H | = | E + F, H + H | \leq$, by [5.II], $| E, H | + | F, H |$.

5.17. - If $\eta > 0$, $| E, F | < \eta$, then $| \mu E - \mu F | < 2\eta$.

Proof. We have

$$\text{(I)} \qquad \begin{aligned} \mu(E) &= \mu(E - F) + \mu(E . F) \\ \mu(F) &= \mu(F - E) + \mu(E . F). \end{aligned}$$

Since $\mu(E-F)+\mu(F-E)<\eta$, we have $\mu(E-F)<\eta$ and $\mu(F-E)<\eta$. Hence for some θ with $|\theta|<I$ and some θ' with $|\theta'|<I$ we have $\mu(E-F)=\eta\theta$, $\mu(F-E)=\eta\theta'$. It follows $\mu(E)-\mu(F)=\eta(\theta-\theta')$: hence $|\mu(E)-\mu(F)|=$ $=\eta.|\theta-\theta'|<2\eta$.

5.18. - If $|E, F|<\eta$, then $\mu(E)-\eta\leq\mu(E\cdot F)\leq\mu(E)$, $\mu(F)-\eta\leq\mu(E\cdot F)\leq\mu(F)$.

Proof. We have $E=EF+(E-F)$, where both terms on the right are disjoint somata. Hence $EF=E-(E-F)$, which gives, as $E-F\leq E$, $\mu(E.\ F)=\mu(E)-\mu(E-F)$.

From the hypothesis it follows that $\mu(E-F)<\eta$, hence we get $\mu(EF)\geq\mu(E)-\eta$. The proof of the second thesis is similar.

5.19. - If $\eta>0$, $|E_1, F_1|\leq\eta$, $|E_2, F_2|\leq\eta$. $E_1\cdot E_2=O$, then $\mu(F_1\cdot F_2)\leq 2\eta$.

Proof. By [5.12] we have $|E_1.\ E_2, F_1\cdot F_2|\leq|E_1, F_1|+$ $+|E_2, F_2|\leq 2\eta$. Hence $|O, F_1F_2|\leq 2\eta$. The thesis follows by [5.2].

5.20. - If $E\cdot F=O$, $\eta>0$, $|E, E'|\leq\eta$, $|F, F'|\leq\eta$, then $|E, E'-F'|\leq 3\eta$, $|F, F'-E'|\leq 3\eta$, and $E'-F'$, $F'-E'$ are disjoint.

Proof. We have $|E, E'-F'|=|E-O, E'-E'F'|\leq$, by [5.14], $|E, E'|+|O, E'\cdot F'|$.

By hypothesis, [5.12] and [5.19] it follows that $|E, E'-$ $-F'|\leq\eta+2\eta=3\eta$. Similarly we get the inequality $|F,$ $F'-E'|\leq 3\eta$.

The second thesis follows from $(E'-F')(F'-E')=$ $=E'$ co $F'\cdot F'$ co $E'=O$.

6. Theor. For somata of G the following are equivalent:

I. μ is an effective measure on G.

II. $|E, F|_\mu=0$ implies $E\overset{G}{=}F$.

Proof. Suppose II is not true. Then there exist E, F such that

$$|E, F|=0 \text{ and } E\neq^{G}F. \tag{1}$$

Hence

(2) $$\mu(E - F) = \mu(F = E) = 0.$$

I say that either $E - F \neq O$ or $F - E \neq O$. Indeed, if not, we would have $E - F = O$, $F - E = O$, and then $E \cdot \operatorname{co} F = O$. $F \operatorname{co} E = O$. Since $E = EF + E \operatorname{co} F$, $F = FE + F \operatorname{co} E$, we would have $E = EF$. $F = FE$, and then $E = F$, which has been excluded in (1).

If $E - F \neq O$ we get, by (2), $\mu(E - F) = 0$, and if $F - - E \neq 0$, we get, by (2), $\mu(F - E) = 0$, so μ is not effective. The above arguments show that I implies II. To prove that II implies I, suppose I is not true.

There exists $E \neq^G O$ such that $\mu(E) = 0$. Hence, [5.2], $O, E| = 0$. Hence, by II, $O \overset{G}{=} E$ which is a contradiction. The theorem is proved.

6.1. - Suppose that μ is effective. We have the following:

$$|E, F|_\mu = |F, E|_\mu, \ |E, F|_\mu \leq |E, G|_\mu + |G, F|_\mu,$$

and $|E, F|_\mu = 0$ is equivalent to $E = F$.

This all by [5.3], [5.5], [6] and because the notion of distance is G–equality invariant.

Hence the notion of distance of somata organizes the tribe G into a HAUSDORFF - metric space, hence into a topology (7).

6.2. Def. Now let us drop the hypothesis that μ is an effective measure on G. Introduce for somata of G the following new notion of *equality* $E =^\mu F$, defined by

$$|E, F|_\mu = 0.$$

6.2.1. - This notion is invariant with respect to the G–equality. It satisfies the usual axioms of identity.

Indeed, we have, by [5.4], $E =^\mu E$. If $E =^\mu F$, then $F =^\mu E$, [5.3].

If $E =^\mu F$ and $F =^\mu G$, then $E =^\mu G$.

Proof. Suppose that $E =^\mu F$ and $F =^\mu G$. We have $\mu(E \dotplus F) = 0$, $\mu(F \dotplus G) = 0$. Now $E \dotplus G = E \dotplus F \dotplus F \dotplus G =$

$= (E \dot{+} F) \dot{+} (F \dot{+} G) \leq (E \dot{+} F) + (F \dot{+} G)$. Hence $\mu(E \dot{+} G) \leq \leq \mu(E \dot{+} F) + \mu(F \dot{+} G) = 0$, so $E =^{\mu} G$.

6.2.2. - The notion of distance $|E, F|_{\mu}$ is invariant with respect to the μ-equality. Indeed, suppose that $E =^{\mu} E'$, $F =^{\mu} F'$. We have

$$\mu(E' \dot{+} F') = \mu[(E \dot{+} E') \dot{+} (F \dot{+} F') \dot{+} (E \dot{+} F)] \leq$$
$$\leq \mu(E \dot{+} E') + \mu(F \dot{+} F') + \mu(E \dot{+} F);$$

hence $\mu(E' \dot{+} F') \leq \mu(E \dot{+} F)$.

Similarly we get $\mu(E \dot{+} F) \leq \mu(E' \dot{+} F')$, which completes the proof.

6.2.3. - The above shows that the notion of distance $|E, F|_{\mu}$ organizes G into a HAUSDORFF metric topology, in which however not $=^{G}$ is the governing equality, but $=^{\mu}$.

Indeed the relation $|E, F|_{\mu} = 0$ is equivalent to $E =^{\mu} F$.

6.2.4. - The notion of measure μ is invariant with respect to $=^{\mu}$, i. e. if $E =^{\mu} F$, then $\mu(E) = \mu(F)$.

Indeed we have $\mu(E) = |E, O|_{\mu}$, $\mu(F) = |F, O|_{\mu}$. Since the notion of distance is $=^{\mu}$-invariant, we have $|E, O|_{\mu} = |F, O|_{\mu}$, hence $\mu(E) = \mu(F)$.

6.3. - The set J of all somata E with $\mu(E) = 0$ is a denumerably additive ideal in the tribe G.

Proof. If $E_1, E_2, \dots, E_n, \dots \in J$, then $\mu(E_1 + E_2 + \dots) = 0$, and then $E_1 + E_2 + \dots \in J$.

On the other hand we have: if $E \in J$ and $E' \leq E$, then $E' \in J$. Indeed $\mu(E') \leq \mu(E) = 0$.

6.3.1. - The equality $E =^{\mu} F$, defined as G in [6.2] coincides with the equality modulo J in the tribe G, i. e. with $E =^{J} F$, (see Preliminaries).

Proof. We define $E =^{J} F$ as $E \dot{-} F \in J$, where $\dot{-}$ is the algebraic subtraction in the Stone's ring G. Since the subtraction coincides with the addition, $E \dot{-} F \in J$, can be written as $E \dot{+} F \in J$ i. e. $\mu(E \dot{+} F) = 0$.

6.3.2. - **Def.** If we introduce on G the *μ-inclusion* defined: by « $E \subseteq^{\mu} F$ means $\mu(E - F) = 0$ », then $E \subseteq^{\mu} F$ will be an ordering, and G will be organized into a BOOLEAN lattice with governing equality $=^{\mu}$. (See Preliminaries).

The operations $E +^{\mu} F$, $E \cdot^{\mu} F$, $\mathrm{co}^{\mu} F$, $E -^{\mu} F$, induced by the ordering $\subseteq^{\mu}$ have the properties: $E +^{\mu} F =^{\mu} E + F$, $E \cdot^{\mu} F =^{\mu} E \cdot F$, $\mathrm{co}^{\mu} F =^{\mu} \mathrm{co}\, F$, $E -^{\mu} F =^{\mu} E - F$ and we also have $I^{\mu} =^{\mu} I$, $O^{\mu} =^{\mu} O$.

In addition to that: $E_1 +^{\mu} E_2 +^{\mu} \ldots =^{\mu} E_1 + E_2 + \ldots$, and similarly for infinite products.

The measure μ is denumerably additive and effective on the tribe G_J, i. e. on G taken module J, (see [6.3]).

The above auxiliaries and some remarks, given in Preliminaries yield the following:

6.3.3. - **Theorem.** The relation $E \subseteq^{\mu} F$, defined by $\mu(E - F) = 0$, in an ordering on G. It organizes G into a denumerably additive BOOLEAN lattice G_J with $E =^{\mu} F$, defined in [6.2], as the governing equality. The operations in G_J satisfy the conditions [6.3.2.]. G_J is just the tribe G, taken modulo the ideal J of all somata of G whose μ-measure equals 0. The measure μ is $=^{\mu}$-invariant, denumerable additive and effective on G_J. The notion of distance $|E, F|_{\mu}$ is $=^{\mu}$—invariant. It organizes G_J into a HAUSDORFF-metric space. The author has proved in (7) that this space is complete.

6.4. - **Def.** The notion of distance induces in G the notion *of limit of an infinite sequence of somata of G*.

If $E, E_1, E_2, \ldots, E_n, \ldots \in G$, we say that the sequence $\{E_n\}$ *μ-tends to E*, whenever $\lim |E_n, E|_{\mu} = \lim \mu(E_n \dot{+} E) = 0$. The limit, if it exists, is $=^{\mu}$—unique. We write

$$E_n \rightarrow^{\mu} E \text{ or } \lim_{\mu} E_n =^{\mu} E.$$

6.5. - The notion of limit is $=^{\mu}$-invariant.

6.6. - Using methods of the theory of metric-spaces and properties [5.11]–[5.14] we prove the following: If $E_n \to^\mu E$, $F_n \to^\mu F$, then co $E_n \to^\mu$ co E, $E_n + F_n \to^\mu E + F$, $E_n \cdot F_n \to^\mu$ $\to^\mu E \cdot F$, $E_n - F_n \to^\mu E - F$ and $E_n \dot{+} F_n \to^\mu E \dot{+} F$.

6.7. - If $E_n \to^\mu E$, $F_n \to^\mu F$, then $|E_n, F_n|_\mu \to |E, F|_\mu$.

Proof. By [5.5] we have $0 \leq |E_n . F_n| \leq |E_n, E| + |E, F| + |F, F_n|$. Hence, since $|E_n, E| \to 0$, $|F_n, F| \to 0$, [Def. 6.4], we get $\overline{\lim} |E_n, F_n| \leq |E, F|$. We also have $|E, F| \leq |E, E_n| + |E_n, F_n| + |F_n, F|$; hence $|E, F| \leq \underline{\lim} |E_n, F_n|$.

Consequently $|E, F| \leq \underline{\lim} |E_n, F_n| \leq \overline{\lim} |E_n, F_n| \leq |E, F|$, which completes the proof.

6.8. - If $E_n \to^\mu E$, then $\mu(E_n) \to \mu(E)$.

Proof. - We have $E_n \to^\mu E$, $O \to^\mu O$, hence, by [6.7], $\lim |E_n, O| = |E, O|$ i. e. $\lim \mu E_n = \mu E$.

7. - This subsection and some following ones (7–11.2) are devoted to various kinds of extension of BOOLEAN tribes. The character of this discussion is general. We need it because the topic which is rather subtle, will be applied later in our main discussion. The hypotheses we admit are the following: There are two tribes F and G, where F is finitely additive and a finitely genuine subtribe of G which is supposed to be denumerably additive. Let $\mathfrak{A}$ be the corresponding isomorphism from F into G. We do not consider any base B of F.

7.1. Def. - By *the Borelian extension of F within G* we shall understand the smallest tribe F^b whose somata $\in G$, and such that:

1) if $f \in F$, then $\mathfrak{A} f \in G$,

2) F^b is a denumerably genuine strict subtribe of G. (See Preliminaries), so the notions of ordering, equality, zero, unit and finite and denumerable operations in F^b are taken from G,

3) if $A \in F^b$, then co $A \in F^b$,

4) if $A_1, A_2, \ldots, A_n, \ldots \in F^b$ then $\sum_{n=1}^{\infty} A_n \in F^b$.

7.1. - The existence of F^b can be proved by taking the intersections of all strict subtribes G' of G having the above properties. This can be done, because G satisfies the above conditions.

7.2. - The tribe F^b can be constructed by transfinite recurrence, by defining the sets A^α, B^α of somata of G, where $\alpha \geq 1$ are ordinals, [7]) as follows: $A^1 \underset{df}{=} \mathcal{A}F$, $B^1 \underset{df}{=} \mathcal{A}F$, A^2 is defined as the set of all denumerable sums $P_1 + P_2 + \ldots + P_n + \ldots$ where $P_n \in A^1 \cup B^1$, and B^2 is defined as the set of all co P, where $P \in A^2$.

In general, let $\alpha \geq 2$ be an ordinal. We define A^α as the set of all sums $P_1 + P_2 + \ldots + P_n + \ldots$ where $P_n \in \in \bigcup_{\beta<\alpha} [A^\beta \cup B^\beta]$, and we define B^α as the set of all somata co P, where $P \in A^\alpha$.

We have for $\alpha_1 < \alpha_2$:

$$A^{\alpha_1} \subseteq A^{\alpha_2},\ B^{\alpha_1} \subseteq A^{\alpha_2}, \tag{1}$$

Now, we define:

$$G' \underset{df}{=} \bigcup_{\alpha} A^\alpha \tag{2}$$

where the union is taken over denumerable ordinals.

7.3. - We shall prove that $G' = F^b$.

Proof. First we shall show that G' is a denumerably additive tribe, and, at the same time organized into a denumerably genuine strict subtribe of G. Indeed, let $P_1, P_2, \ldots, P_n, \ldots \in G'$. There exists, by (2), ordinals $\alpha(1), \alpha(2), \ldots, \alpha(n), \ldots < \Omega$ such that $P_n \in A^{\alpha(n)}$. There exists $\alpha \geq \alpha(n)$, $(n = 1, 2, \ldots)$, with $\alpha < \Omega$. We have, by (1), $A^{\alpha(n)} \subseteq A^\alpha$. Hence $\sum_{n=1}^{\infty} P_n \in A^{\alpha+1} \subseteq \subseteq G'$. Let $P \in G'$, we have for some $\alpha < \Omega$, $P \in A^\alpha$. Hence co $P \in B^\alpha \subseteq A^{\alpha+1} \subseteq G$.

Thus G' is denumerably additive. If we take on G' the ordering $\leq^G$, restricted to G', we get the statement to be proved, because $\mathcal{A}F \subseteq G'$.

([7]) Concerning a precise setting of the theory of ordinals. see (8).

Now $F^b \subseteq G'$, because F^b has the minimum-property [7]. On the other hand we can prove, by induction, that $G' \subseteq F^b$. Indeed, $A^1 \subseteq F^b$, because $A^1 = \mathfrak{A}F \subseteq F^b$, [7.2]. Suppose that for all ordinals $\beta < \alpha$, where $\alpha \geq 2$, we have $A^\beta \subseteq F^b$. It follows that $B^\beta \subseteq F^b$, and then $A^\beta \cup B^\beta \subseteq F^b$.

Take $P \in A^\alpha$. We have $P = P_1 + P_2 + \dots + P_n + \dots$ where $P_1 \in A^{\alpha(1)} \cup B^{\alpha(1)}$, $P_2 \in A^{\alpha(2)} \cup B^{\alpha(2)}$, ... where $\alpha(1)$, $\alpha(2)$, ... are some ordinals $< \alpha$. We get, by (1), P_1, P_2, ... $\in F^b$, and then $P \in F^b$. Thus we get $A^\alpha \subseteq F^b$, and then, by induction, we have proved that for $\alpha < \Omega$ we have $A^\alpha \subseteq F^b$. Taking account of (2), we obtain $G' \subseteq F^b$. Thus we have proved that $G' = F^b$.

7.4. - We can prove that the set F^b is the smallest class G'' of somata of G, which satisfies the conditions 1), 2), 3) of [7], but instead of 4), the following one:

4') if P_1, P_2, ..., P_n, ... $\in G''$, and are all disjoint, then $\sum_{n=1}^{\infty} P_n \in G''$.

7.5. - We can get F^b by inductive construction, fitting [7.4], as follows. We define $C^1 \underset{df}{=} \mathfrak{A}F$, $D^1 \underset{df}{=} \mathfrak{A}F$, and we define C^α as the collection of all somata of G, having the form $P_1 + P_2 + \dots + P_n + \dots$ where $P_n \in \bigcup_{\beta<\alpha} [C^\beta \cup D^\beta]$ and where all P_n are disjoint. We define D^α as the collection of all somata co P, where $P \in C^\alpha$. We prove that

$$F^b = \bigcup_{\alpha<\Omega} [C^\alpha \cup D^\alpha] = \bigcup_{\alpha<\Omega} C^\alpha.$$

7.6. - **Remark**. The above is general and can be applied whenever we have a finitely additive tribe which is a finitely genuine subtribe of a denumerably additive tribe. If we take for G a wider tribe, but with not changed notion of equality, F^b will not change. We emphasize that in F^b the governing equality and operations are taken from G. The tribe F^b is a denumerably genuine strict subtribe of G.

8. - *The μ-null-somata-extension of* F^b *within* G.

Let us admit that G possesses a denumerably additive measure $\mu'E$). The set J of all E, for which $\mu(E)=0$, is a denumerably additive ideal in G, (see (5), [6.3] and [6.3.3]). Consider the set F^{μ} of all somata of G having the form $P+P'-P''$ where $P \in F^b$, $\mu(P')=\mu(P'')=0$, and $P', P'' \in G$. In other words F^{μ} is the set of all somata of G which are equivalent modulo J to somata of F^b. The set F^{μ} is, within G, organized into a tribe with ordering, equality and operations taken from G. It is a denumerably additive tribe. The set F^{μ} is the set of all $E \in G$ such that there exists $P \in F^b$ with $|E, P|_{\mu}=0$. The null-sets in F^{μ} are the same as in G.

Remark - If we amplify G with preservation of measure. but with not changed equality on G, then F^{μ} may also be amplified, since we may have more null-sets. Thus the μ-null-set-extension of F^b within G depends on G.

9. - *The Lebesgue-covering μ-extension of F within G.* In this subsection we shall refer to [1] but we shall not admit any measure on the denumerably additive tribe G. We shall suppose a finitely additive non negative measure μ on F, which is a finitely genuine subtribe of G through the isomorphism $\mathcal{A}$ which preserves the equality of somata and finite operations. We shall imitate the LEBESGUE'S extension-device from F into G by means of some kind of coverings. We refer to our paper (2). We define $\mu(\mathcal{A}f) \underset{df}{=} \mu(f)$ for all $f \in F$. Let $E \in G$. We define the «*exterior*» *measure of E* by

$$\mu_e(E) \underset{df}{=} \inf \sum_{n=1}^{\infty} \mu(\mathcal{A}f_n) = \inf \sum_{n=1}^{\infty} \mu(f_n),$$

where the infimum is taken over all sequences, $f_1, f_2, \ldots, f_n, \ldots \in F$ with $E \subseteq \sum_{n=1}^{\infty} \mathcal{A}f_n$.

We define the «*interior*» *measure of E* by:

$$\mu_i(E) \underset{df}{=} \mu(I) - \mu_e(E).$$

If $\mu_e(E) = \mu_i(E)$, the soma E will be termed *μ-Lebesgues-coverings-measurable or* L-*measurable* and we put

$$\mu^L(E) \underset{df}{=} \mu_i(E) = \mu_e(E).$$

It will be termed *L-measure.*

9.1. - Let E be measurable in the above sense. There exists an infinite sequence of infinite sequences

$$f_{k1}, f_{k2}, \ldots, f_{kN}, \ldots \in F, \ (k = 1, 2, 3, \ldots),$$

such that

$$E \subseteq \mathfrak{a} f_{k1} + \mathfrak{a} f_{k2} + \ldots + \mathfrak{a} f_{kn} + \ldots$$

for all k, and

$$\lim_{k \to \infty} (\mu(f_{k1}) + \mu(f_{k2}) + \ldots + \mu(f_{k,n}) + \ldots) = \mu^L E.$$

Put

$$g_{k1} \underset{df}{=} f_{k1}, \ g_{k2} = f_{k2} - f_{k1}, \ \ldots,$$

$$g_{kn} = f_{kn} - (f_{k1} + f_{k2} + \ldots + f_{k,n-1}) \text{ for } n = 2, 3, \ldots .$$

By [Lemma 4.2] we have

$$\mathfrak{a} f_{k1} + \mathfrak{a} f_{k2} + \ldots + \mathfrak{a} f_{kn} + \ldots = \mathfrak{a} g_{k1} + \mathfrak{a} g_{k2} + \ldots \mathfrak{a} g_{kn} + \ldots .$$

Hence

$$E \subseteq \mathfrak{a} g_{k1} + \mathfrak{a} g_{k2} + \ldots + \mathfrak{a} g_{kn} + \ldots .$$

The somata $g_{k1}, g_{k2}, \ldots, g_{kn}, \ldots$ are disjoint. We have:

$$\mu^L(E) \leq \mu g_{k1} + \mu g_{k2} + \ldots + \mu g_{kn} + \ldots .$$

Since $g_{kn} \subseteq f_{kn}$ we have $\mu g_{kn} \leq \mu f_{kn}$. It follows that

$$\mu^L(E) = \lim_{k \to \infty} [\mu g_{k1} + \mu g_{k2} + \ldots + \mu g_{kn} + \ldots].$$

We have $g_{kn} \in F$.

Thus we have proved that if E is measurable in the sense considered, then there exists an infinite sequence of infinite sequences

(1) $$g_{k1}, g_{k2}, \ldots, g_{kn}, \ldots (k = 1, 2, \ldots)$$

of somata of F such that $g_{kn} \in A^1$, [7.2], the somata (1) are disjoint. $E \subseteq P_k$ where

$$P_k = g_{k1} + g_{k2} + \dots + g_{kn} + \dots \in A^2$$

and where

$$\mu^L(E) = \lim_{n \to \infty} [\mu g_{k1} + \mu g_{k2} + \dots + \mu g_{kn} + \dots].$$

The sequence of sequences $\{g_{kn}\}$, $(k = 1, 2, \dots)$ yields the infimum spoken of in [9].

9.2. - In (2) we have proved that for every $f \in F$ the soma $\mathfrak{C}f$ is L-measurable. We have also proved that the set F^L of all measurable somata of G constitutes a denumerably additive tribe. The equality governing on F^L, is that of G, so is the zero, the unit of F^L, so is the ordering on F^L, and so are all finite and denumerably operations in F^L. The tribe F is a finitely genuine subtribe of F^L, and F^L is denumerably genuine strict subtribe of G. The μ^L-null sets in F^L are those sets E, that for every $\eta > 0$ there exists an infinite sequence $f_1, f_2, \dots, f_n, \dots$ with $E \subseteq \sum_{n=}^{\infty} \mathfrak{C}f_n$, $\sum_{n=1}^{\infty} \mu(f_n) \leq \eta$. Using the device, shown in [9.1], we prove that the above condition is equivalent to the following: For every $\eta > 0$ there exists an infinite sequence $f_1, f_2, \dots, f_n, \dots$ of disjoint somata of F, such that

$$E \subseteq \sum_{n=1}^{\infty} \mathfrak{C}f_n, \quad \sum_{n=1}^{\infty} \mu(f_n) \leq \eta.$$

The measure μ^L is denumerably additive on F^L.

9.3. - Under the general circumstances considered it is not true, that if $f \in F$, then $\mu^L(\mathfrak{C}f) = \mu f$, (see (2)): We only have $\mu^L(\mathfrak{C}f) \leq \mu f$.

Now, we have proved, in (2), that a necessary and sufficient condition that $\mu^L(\mathfrak{C}f) = \mu f$ is the following ***Fréchet's condition***, stated in (9):

If $f_1 \geq f_2 \geq \dots \geq f_n \geq \dots \in F$ with $\prod_{n=1}^{\infty} (\mathfrak{C}f_n) = 0$, then $\mu f_n \to 0$, (of course if $\mu(f) = \mu(\mathfrak{C}f)$, for $f \in \boldsymbol{F}$.

9.3.1. - Now let us go back to our main case, conditioned by hypotheses in [1] and (Hyp. G_μ) in [5]. There a measure μ was given in $\boldsymbol{G}$, and it induced the measure in $\boldsymbol{F}$. Take that measure on $\boldsymbol{F}$ and extend it, by special coverings as in [9], in $\boldsymbol{G}$. Since μ is denumerably additive in $\boldsymbol{G}$, μ satisfies FRÉCHET'S condition: if $E_1 \geq E_2 \geq \dots \geq E_n \geq \dots$ are somata of $\boldsymbol{G}$ with $\prod_{n=1}^{\infty} E_n = 0$, then $\mu E_n \to 0$. Hence, a fortiori, this condition is satisfied by $f_1 \geq f_2 \geq \dots \geq f_n \geq \dots \in \boldsymbol{F}$ with $\prod_{n=1}^{\infty} (\mathfrak{C}f_n) = 0$. Consequently, by [9.3], $\mu^L(\mathfrak{C}f) = \mu(f)$ for $f \in \boldsymbol{F}$. In the sequal we shall be interested only in that case, though it does not mean that $\boldsymbol{G}$ coincides with the set of all L-measurable somata.

9.4. - Concerning the collection of all μ^L-null sets and that of all μ-null sets, we must remark, that they may differ: there may be more μ-null sets, than μ^L-null sets.

9.5. - We shall prove, by induction, that the borelian extension $\boldsymbol{F}^b$ is a denumerably genuine strict subtribe of $\boldsymbol{F}^L$, (see [7]).

Proof. From [9.2] we know that if $f \in \boldsymbol{F}$, then $\mathfrak{C}f \in \boldsymbol{F}^L$. We take over the topic of [7.2]. Suppose that if $P \in A^\beta \cup B^\beta$, then $P \in \boldsymbol{F}^L$, this for all β less than a given ordinal α where $1 < \alpha < \Omega$. Let $Q \in A^\alpha$. There exists a sequence $\{Q_n\}$ where $Q_n \in A^{\beta(n)} \cup B^{\beta(n)}$ for some $\beta(n) < \alpha$, and such that $Q = Q_1 + \dots + Q_n + \dots$. By hypothesis $Q_n \in \boldsymbol{F}^L$. Since $\boldsymbol{F}^L$ is denumerably additive, [9.2], we get $Q \in \boldsymbol{F}^L$. Thus we have proved that if $P \in A^\alpha \cup B^\alpha$, then $P \in \boldsymbol{F}^L$. Since $\boldsymbol{F}^b = \bigcup_{\alpha<\Omega} [A^\alpha \cup B^\alpha]$, it follows that $\boldsymbol{F}^b \subseteq \boldsymbol{F}^L$. Q.E.D.

9.6. - Under (Hyp. $G\mu$) we shall prove that if $P \in \boldsymbol{F}^b$, then $\mu^L P = \mu P$, (see [9.3.1]).

Proof. By [9.3.1] we have: if $f \in \boldsymbol{F}$, then $\mu^L(\mathfrak{C}f) = \mu f =$

$= \mu(\mathcal{C}f)$. We take over the topic in [7.4] and [7.5]. Suppose that $1 \leq \alpha < \Omega$ and that if $P \in [C^\beta \cup D^\beta]$ for $1 \leq \beta < \alpha$, then $\mu^L(P) = \mu(P)$. We shall prove that if $Q \in [C^\alpha \cup D^\alpha]$, then $\mu^L(Q) = \mu(Q)$. Let $Q \in C^\alpha$. We have $Q = Q_1 + Q_2 + \dots + Q_n + \dots$ with disjoint terms, where $Q_n \in C^{\beta(n)} \cup D^{\beta(n)}$ for some ordinals $\beta(n) < \alpha$. By hypothesis we have $\mu^L(Q_n) = \mu(Q_n)$. Since both measures are denumerably additive [9.2], we have $\mu^L(Q) =$ $= \mu^L(Q_1) + \mu^L(Q_2) + \dots = \mu(Q_1) + \mu(Q_2) + \dots = \mu(Q)$. Now let $Q \in D^\alpha$. We have $Q = \operatorname{co} Q'$, where $Q' \in (C)^\alpha$. Now, as we have proved, $\mu^L(Q') = \mu(Q')$, we get $\mu^L(Q) = \mu(Q)$. This completes the proof.

10. - If $\mu^L(E) = 0$, $E \in G$, then $\mu(E) = 0$.

Proof. Let $\eta > 0$. There exist $f_1, f_2, \dots, f_n, \dots \in F$ such that $E \subseteq \sum_{n=1}^{\infty} \mathcal{C}f_n$, $\sum_{n=1}^{\infty} \mu(f_n) < \eta$. Hence

$$\mu E \leq \mu\left(\sum_{n=1}^{\infty} \mathcal{C}f_n\right) \leq \sum_{n=1}^{\infty} \mu(\mathcal{C}f_n) = \sum_{n=1}^{\infty} \mu(f_n) < \eta.$$

Since this happens for every $\eta > 0$, we get $\mu E = 0$.

10.1. - If $|E, F|_{\mu^L} = 0$, then $|E, F|_\mu = 0$.

Proof. Let $|E, F|_{\mu^L} = 0$. Hence $\mu^L(E \dot{+} F) = 0$: hence by [10], $\mu(E \dot{+} F) = 0$, and then $|E, F|_\mu = 0$.

11. - We shall prove that if $E \in F^L$, then $\mu^L E = \mu E$.

Proof. Let $E \in F^L$. By [9.1] there exists an infinite sequence of infinite sequences of somata of F:

$$(1) \qquad g_{k1}, g_{k2}, \dots, g_{kn}, \dots, \quad (k = 1, 2, \dots)$$

such that for a fixed k, they are disjoint, $E \leq P_k$, where

$$P_k \underset{df}{=} \mathcal{C}g_{k1} + \mathcal{C}g_{k2} + \dots,$$

where $\mu^L E = \lim_{n \to \infty} (\mu g_{k1} + \mu g_{k2} + \dots \mu g_{kn} + \dots)$, and where the sequence $\{P_k\}$ yields the infimum, spoken of in [9.1]. We have

$$\mu^L = \lim_{n \to \infty} [\mu(\mathfrak{a}g_{k1}) + \mu(\mathfrak{a}g_{k2}) + \dots + \mu(\mathfrak{a}g_{kn}) + \dots].$$

Since μ is denumerably additive on G and $g_{k1}, g_{k2} \dots$ disjoint, we have

$$\mu^L(E) = \lim_{n \to \infty} \mu(P_n), \; E \leq P_n.$$

Put

$$Q_1 \underset{df}{=} P_1, \; Q_2 \underset{df}{=} P_1 P_2, \; \dots, \; Q_n \underset{df}{=} P_1 P_2 \dots P_n.$$

We have $\mu(Q_n) \leq \mu(Q_{n-1})$ and $E \leq \prod\limits_{n=1}^{\infty} Q_n$. We have

$$\mu^L(E) \leq \lim_{n \to \infty} \mu(Q_n) \leq \lim_{n \to \infty} \mu(P_n).$$

Hence

$$\mu^L(E) = \lim_{n \to \infty} \mu(Q_n) = \mu\left(\prod_{n=1}^{\infty} Q_n\right).$$

On account of [9.6] we have $\mu Q_n = \mu^L(Q_n)$, hence

$$(2) \qquad \mu^L(E) = \mu^L\left(\prod_{n=1}^{\infty} Q_n\right).$$

Consequently

$$\left| E, \prod_{n=1}^{\infty} Q_n \right|_{\mu^L} = \mu^L\left(\prod_{n=1}^{\infty} Q_n - E\right),$$

because

$$E \subseteq \prod_{n=1}^{\infty} Q_n.$$

Hence, by (2),

$$\left| E, \prod_{n=1}^{\infty} Q_n \right|_{\mu^L} = \mu^L\left(\prod_{n=1}^{\infty} Q_n\right) - \mu^L E = 0.$$

By [10.1] it follows that

$$\left| E, \prod_{n=1}^{\infty} Q_n \right|_{\mu} = 0,$$

and hence

$$\mu\left(\prod_{n=1}^{\infty} Q_n - E\right) = 0,$$

which gives

$$\mu \prod_{n=1}^{\infty} Q_n = \mu E.$$

From (2) it follows $\mu^L(E) = \mu(E)$.

11.1. - We shall prove that if we extend F^b by adjunction of all μ^L-null-somata of G, we get F^L i.e. F^L is the μ^L-nullset-extension of F^b within G, (see [8]).

Proof. Denote this extension by F'. If $E \in F'$, then $E = P + P' - P''$, where $P \in F^b$, $P', P'' \in G$ with $\mu^L P' = \mu^L P'' = 0$. Since $P \in F^L$, by [9.5], and since $P', P'' \in F^L$, it follows that $E \in F^L$. Thus

$$(1) \qquad F' \subseteq F^L.$$

Conversely, let $E \in F^L$. Referring to arguments in [11], we have

$$\left| E, \prod_{n=1}^{\infty} Q_n \right|_{\mu^L} = 0,$$

for some somata Q_n which belong to F^b. Hence E equals $\prod_{n=1}^{\infty} Q_n$ modulo the ideal of somata whose L-measure is zero. Hence, (see Preliminaries),

$E = \prod_{n=1}^{\infty} Q_n + Q' - Q''$, where $\mu^L(Q') = \mu^L(Q'') = 0$. Hence $E \in F'$.

We have proved that

$$(2) \qquad F^L \subseteq F'$$

From (1) and (2) the statement follows.

11.2. - **Remark.** Suppose we have, in a denumerably additive tribe G, a denumerably additive ideal J, and introduce on G the new equality $\overset{J}{=}$, i.e. that modulo J. Let $E'_1, E'_2, \dots, E'_n, \dots$ be an infinite sequence of somata of G which are mutually J-disjoint, i.e. $E'_i \cdot E'_j \overset{J}{=} O$, for $i \neq j$.

Then we can find an infinite sequence

$$(1) \qquad E_1, E_2, \dots, E_n, \dots \in G,$$

such that

$E_i \stackrel{J}{=} E'_i$, and all somata are mutually disjoint, i.e. $E_i \cdot E_j = O$. To prove it, we define in G:

$$E_1 \underset{df}{=} E'_1, \quad E_2 \underset{df}{=} E'_2 - E'_1, \quad E_3 \underset{df}{=} E'_3 - (E'_1 + E'_2), \ldots .$$

12. - We are going back to the circumstances conditioned by hypotheses (Hyp. *BFG*) in [1]. We admit (Hyp. *Ad*) [3] and (Hyp. $G\mu$) [5], and also the folloving hypothesis (Hyp. $L\mu$):

(**Hyp.** $L\mu$) The tribe G, its measure μ and the subtribe F satisfy the condition

$$G = F^L \text{ i. e.}$$

G coincides whith the LEBESGUE covering extension of F. as defined in [9].

12.1. - The μ-null somata in G coincide with the μ^L-null somata. G is the μ-null-somata-extension of F^b within G, [8]. We have $\mu^L E = \mu E$ for all $E \in G$.

12.2. - The following are equivalent for E_n, $E \in G$. ($n = 1, 2, \ldots$), I. $E_n \xrightarrow{\mu} E$, [6.4], II. From each subsequence $\{E_{k(n)}\}$ of $\{E_n\}$ it can be extracted another one $\{E_{kl(n)}\}$ such that

$$\overline{\lim_{n \to \infty}}\; E_{kl(n)} =^{\mu} \underline{\lim_{n \to \infty}}\; E_{kl(n)} =^{\mu} E,$$

where $=^{\mu}$ is the equality modulo the ideal of all μ-null-somata in G, (see (**1**)) (8).

(8) The theorem is proved in (1) for denumerably additive tribes whose somata are sets of abstract elements, and where on the tribe a non negative measure is admited, which is denumerably additive. Now, since the proof in (1) does not use the relation $\in$ of belonging of an element to a class, the proof is valid for all abstract denumerably additive tribes.

$\overline{\lim\limits_{n \to \infty}}\, A_n$ means $(A_1 + A_2 + A_3 + \ldots) \cdot (A_2 + A_3 + \ldots) \cdot (A_3 + \ldots) \ldots$

$\underline{\lim\limits_{n \to \infty}}\, A_n$ means $(A_1 \cdot A_2 \cdot A_3 \ldots) + (A_2 \cdot A_3 \ldots) + (A_3 \ldots) + \ldots$.

Compare (**4**).

12.3. - Under hypotheses [12], if $E \in G$ and $\eta > 0$, then there exists an infinite sequence $g_1, g_2, \ldots, g_n, \ldots$ of disjoint figures (i. e. $\in F$), such that

$$|E, \sum_{n=1}^{\infty} \mathcal{A} g_n|_\mu < \eta. \quad E \subseteq \sum_{n=1}^{\infty} \mathcal{A} g_n.$$

Proof. By [9.1] and [12.1] there exists a sequence $g_1, g_2, \ldots, g_n, \ldots$ of disjoint figures, such that

(1) $$E \subseteq \sum_{n=1}^{\infty} \mathcal{A} g_n$$

and

$$0 \leq \sum_{n=1}^{\infty} \mu(g_n) - \mu(E) < \eta.$$

Since g_n are disjoint, we have

$$0 \leq \mu\left(\sum_{n=1}^{\infty} \mathcal{A} g_n\right) - \mu(E) < \eta,$$

and, then, by (1),

$$|E, \sum_{n=1}^{\infty} \mathcal{A} g_n|_\mu < \eta.$$

12.4. - Under hypotheses [12], if $E \in G$ and $\eta > 0$, then there exists a figure f, such that $|E, \mathcal{A} f|_\mu < \eta$.

Proof. By [12.3] we can find disjoint figures $g_1, g_2, \ldots, g_n, \ldots$ such that

$$|E, \sum_{n=1}^{\infty} \mathcal{A} g_n|_\mu < \frac{\eta}{2}, \quad E \subseteq \sum_{n=1}^{\infty} A g_n.$$

If we put

(1) $$P = \mathcal{A} f_1 + \mathcal{A} f_2 + \ldots,$$

we have

(2) $$|E, P|_\mu < \frac{\eta}{2}.$$

Now, since the series (1) converges, there is an index

n such that

$$\mu[P-(Ag_1+\mathfrak{A}g_2+\ldots+\mathfrak{A}g_n)]<\frac{\eta}{2}.$$

Hence

$$|P,\ \mathfrak{A}\sum_{k=1}^{n} g_k|_\mu<\frac{\eta}{2}. \tag{3}$$

The soma $f \underset{df}{=} \sum_{k=1}^{n} g_n$ is a figure. Hence from (3) and (2) we get

$$|E,\ f|_\mu<\eta. \qquad \text{Q. E. D.}$$

12.5. - Under hypotheses [12], if $E\in G$ and $\eta>0$, there exists an infinite sequence of disjoint bricks $a_1, a_2, \ldots, a_n, \ldots$, such that

$$|E,\ \sum_{k=1}^{\infty}\mathfrak{A}a_k|_\mu<\eta. \quad E\subseteq\sum_{k=1}^{\infty}\mathfrak{A}a_k.$$

Proof. By [12.3], there is an infinite sequence $\{g_n\}$ of figures with

$$E-\sum_{n=1}^{\infty} g_n, \quad |E,\ \sum_{n=1}^{\infty}\mathfrak{A}g_n|_\mu<\eta.$$

Now, by (Hyp *Ad*)

$$\mathfrak{A}g_n=\mathfrak{A}a_{n1}+\mathfrak{A}a_{n2}+\ldots,$$

where $a_{n1}, a_{n2}, \ldots$ are disjoint bricks. Since the set $\{a_{nk}\}$, $(n=1, 2, \ldots)$, $(k=1, 2, \ldots)$ is denumerable, the theorem follows.

12.6. - Under hypotheses [12], if $E\in G$ and $\eta>0$, there exists a finite number of disjoint bricks $a_1, \ldots, a_n$, $(n\geq 1)$ with $|E,\ \mathfrak{A}a_1+\ldots+\mathfrak{A}a_n|_\mu<\eta$.

Proof. By [12.4], we find a figure f, such that

$$|E,\ \mathfrak{A}f|_\mu<\frac{\eta}{2}. \tag{1}$$

By (Hyp *Ad*) we have

$$\mathfrak{A}f=\mathfrak{A}a_1+\mathfrak{A}a_2+\ldots+\mathfrak{A}a_n\ldots$$

where $a_1, a_2, \ldots, a_n, \ldots$ are disjoint bricks. We can find n such that

$$0 \leq \mu\mathfrak{A}f - \mu(\mathfrak{A}a_1 + \mathfrak{A}a_2 + \ldots + \mathfrak{A}a_n < \frac{\eta}{2},$$

hence

$$(2) \qquad |\mathfrak{A}f, \mathfrak{A}a_1 + \ldots + \mathfrak{A}a_n|_\mu < \frac{\eta}{2}$$

From (1) and (2) the theorem follows.

13. - In the sequel we shall use brick-coverings of somata of G, defined in [4], and often apply the theorems [12.3 — 12.6]. By *a covering of* $E \in G$ we shall understand any covering L such that $E \leq L$. To simplify notations, we shall write, for figures, f instead of $\mathfrak{A}f$, and the same will be for bricks. In the case of infinite sums we shall take summations from G, as explained in [3.5].

13.1. - **Remark.** It does not seem true that in the case where the measure μ in effective, the (Hyp *Ad*) follows.

Indeed, if $f \in F$ we can find a brick $a_1 \leq f$, again in $f - a_1$ another brick a_2, etc. But $f - (a_1 + a_2 + \ldots)$ may have a positive measure, though it may not contain any brick.

13.2. - The following two theorems can be proved, under hypotheses [12]: If $E \in G$, then there exists an infinite sequence of coverings of E, $L_1 \geq L_2 \geq \ldots \geq L_n \geq \ldots$ such that $\mu E = \lim \mu(L_n)$ and $\lim^\mu L_n =^\mu E$.

13.3. - Under hypotheses [12], if

1. $E, F \in G$,
2. $L_1 \geq L_2 \geq \ldots \geq L_n \geq \ldots$ are coverings of E,
3. $M_1 \geq M_2 \geq \ldots \geq M_n \geq \ldots$ are coverings of F,
4. $E \cdot F =^\mu O$,
5. $|E, L_n|_\mu \to 0$, $|F, M_n|_\mu \to 0$, then $\mu(L_n \cdot M_n) \to 0$.

13.4. - As we mentioned in [6.23], the notion of distance $|E, F|_\mu$ organizes the tribe G, with governing equality $=^\mu$, into a metric space, hence into a topology. This topology is is necessarily complete [9], but it may be not separable [10].

Later on, in [21], we shall discuss the condition of separability.

14. - Hypothesis. To facilitate the discussion we shall often admit that the measure μ is effective on G. This hypothesis does not affect much the generality. Indeed, if the measure μ is not effective, we replace G by the same tribe, taken modulo the ideal J, of null-sets in G.

To get general theorems from those which were derived under Hyp [14], we only need to change $=$, $\leq$ into $=^J$, $\leq^J$ i.e. $=^\mu$, $\leq^\mu$ respectively. The relation $E \leq^\mu F$ means $\mu(E - F) = 0$, [6.3.2.].

14.0. - Theorem. Under hypotheses [12] and [14], the tribes F^b and F^L coincide.

14.1 - Def. By a *complex* we shall understand a finite (even empty) set of mutually disjoint bricks. A not empty complex P will be denoted by $\{p_1, p_2, \dots, p_n\}$, $(n \geq 1)$ or $\{p_i\}$, where p_i are bricks.

By the *soma of the complex* we shall understand $p_1 + p_2 + \dots + p_n$ where $n \geq 1$, and the soma O, if the complex is empty. We shall write som P, if P denotes the complex.

By the *measure of the complex* P, we shall understand $\mu(\text{som } P)$. We shall write $\mu\{p_i\}$, $\mu(P)$ or $\mu(\text{som } P)$.

If P, Q are complexes and $E \in G$, then by $|P, Q|$, $|E, P|$, $|P, E|$ we shall understand $|\mathfrak{Cl} \text{ som } P, \mathfrak{Cl} \text{ som } Q|_\mu$, $|E, \mathfrak{Cl} \text{ som } P|_\mu$, $|\mathfrak{Cl} \text{ som } P, E|_\mu$ respectively.

14.2. - Under hypotheses [12] and [14], if $E \in G$, $\eta > 0$, then there exists a complex P such that $|E, P|_\mu < \eta$.

Proof. This follows from [12.6].

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(9) This means, that the existence of the limit, [6.4], $\lim^\mu E_n$ for $E_n \in G$ is equivalent to the *Cauchy condition*: for every $\eta > 0$ there exists n_0 such that if $n \geq n_0$, $m \geq n_0$ we have $|E_n, E_m|_\mu \leq \eta$.

(10) i. e. it may be not true that there exists a denumerable set of somata $\in G$ which is everywhere dense in G.

14.3 - Under hypotheses [12] and [14], if $E \in G$ and A is a brick-covering of E, $\eta > 0$, then there exists a complex P such that $| E, P |_\mu < \eta$, som $P \leq A$.

Proof. By [14.2] we find a complex Q with $| E, Q |_\mu < \frac{\eta}{2}$. We have $E \leq A$. By [4.3] there exists a sequence of mutually disjoint bricks $a_1, a_2, \ldots, a_n, \ldots$ with $A = \Sigma a_n$. Let $Q = \{ q_1, q_2, \ldots, q_m \}$ where q_i are disjoint bricks, $(m \geq 1)$. Consider the bricks $a_n q_i$, for all n and i. They are disjoint. We have som $Q \cdot A = \sum_{n,i} a_n q_i$, because som $Q = \sum_i q_i$. We have

$$| EA, \text{ som } Q \cdot A | \leq \frac{\eta}{2}, \text{ [5.15]},$$

$$\text{(1)} \qquad \text{i.e. } | E, \text{ som } Q \cdot A | \leq \frac{\eta}{2}.$$

Let us arrange the bricks $a_n g_i$, into a sequence; denote it by $p_1, p_2, \ldots$. If it is finite, the complex $\{ p_1, p_2, \ldots \}$ yields the thesis. If it is infinite, take m such that $| \sum_{k=1}^{\infty} p_k, \sum_{k=1}^{m} p_k | \leq \frac{\eta}{2}$ i.e.

$$\text{(2)} \qquad | \text{som } Q \cdot A, \sum_{k=1}^{m} p_k | \leq \frac{\eta}{2}.$$

From (1) and (2) we get $| E, \sum_{k=1}^{m} p_k | \leq \eta$, $\sum_{k=1}^{m} p_k \leq A$: the complex $P \underset{df}{=} \{ p_1, \ldots, p_m \}$ yields the thesis.

15. - **Lemma.** If

1. $E_1, \ldots, E_n \in G$, $n = 1, 2, \ldots$,
2. $E = E_1 + E_2 + \ldots + E_n + \ldots$,
3. all E_n are disjoint,
4. $\eta \underset{df}{=} \sup_{n=1,2,\ldots} \mu(E_n) \cdot \eta > 0$,

then

1) there exists i with $\mu(E_i) = \eta$.

2) the number of all j, for which $\mu(E_i) = \eta$ is finite.

Proof. Suppose that there does not exist any index i with $\mu(E_i) = \eta$. There exists an infinite sequence of indices $\alpha(1) < \alpha(2) < \dots$ with $\lim_{i \to \infty} \mu(E_{\alpha(i)}) = \eta$, $\mu(E_{\alpha(i)}) < \eta$. Hence, starting from some k we have $\mu(E_{\alpha(k)}) > \frac{\eta}{2}$, $\mu(E_{\alpha(k+1)}) > \frac{\eta}{2}, \dots$. Since $\mu(E) = \mu(E_1) + \mu(E_2) + \dots$ it follows that $\mu(E) \geq \mu(E_{\alpha(k)}) + \mu(E_{\alpha(k+1)}) + \dots$ and then $\mu(E) \geq m \cdot \frac{\eta}{2}$ for all $m = 1, 2, \dots$, which is impossible. Hence there exists i with $\mu(E_i) = \eta$. Now the number of those indices i must be finite, because if not, we would have $\mu(E) \geq m \cdot \eta$ for all $m = 1, 2, \dots$ which is impossible.

15.1. - **Def.** By a *partition of a soma* $E \in G$ we shall understand an at most denumerable sequence of mutually disjoint subsomata $E_1, E_2, \dots, E_n, \dots$ of E, with $E = E_1 + E_2 + \dots$, but where we do not take care of the order in which $E_1, E_2, \dots$ are written. (For a more precise setting see [(10), § 2]).

15.2. - **Def.** Given two partitions

A: $E = E_1 + E_2 + \dots$ and

B: $E = F_1 + F_2 + \dots$ of a given soma $E \in F$;

by *the product* $A \cdot B$ *of them* we shall understand the partition $E = \sum_{ik} E_i F_k$ where the terms $E_i F_k$ may be arranged, in any way, into a sequence.

15.3. - Given two partitions $\sum_i E_i$ and $\sum_k F_k$ of E, we say that the second partition is a *subpartition of the first*, whenever for every k with $F_k \neq O$ there exists i with $F_k \leq E_i$. This index i is unique for a given k.

15.4. - If $\sum_i E_i$, $\sum_i F_i, \dots$ is an infinite sequence of partitions of E, we say that this is a *nested sequence of partitions of* E, whenever every partition, starting from the second, is a subpartition of the preceding one.

15.5. - If A, B are partitions of E, then their product is a subpartition of A and of B.

15.6. - We shall be interested in partitions whose elements are either bricks or figures.

15.7. - **Def.** Given a partition E_i or a complex $P\{p_i\}$, by the ***net-number*** $\mathfrak{N}\{E_i\}$, $\mathfrak{N}P$ *of the given partition or complex* we shall understand the maximum of the numbers $\mu(E_i)$, $\mu(p_1)$ respectively. If the complex P is empty we define $\mathfrak{N}P$ as the number 0.

15.8. - If P is a not empty complex $\{p_1, p_2, \dots, p_n\}$, $n \geq 1$, its bricks constitute a partition of som P. If P' is a complex which constitutes a subpartition of P, then $\mathfrak{N}P' \leq \mathfrak{N}P$.

16. - In § 2 we shall introduce a kind of integration which will be based on approximation of the given soma of G by complexes. The integration requires « small particles » i.e. complexes whose elements should have « small » measures. This is, however, impossible in the case where G possesses atoms. To master this difficulty, several lemmas will be introduced concerning special notions of « smallness » when atoms are taken into account. We start with the case where there are no atoms in G.

16. - **Lemma.** 1) Under hypothesis [12], hence especially (Hyp $L\mu$), and (Hyp Ad), we have the following: if 2) the tribe G has no *measure-atoms* [11] 3) a is a brick, $\mu a > 0$. 4) $\eta > 0$, then there exists a partition of a: $a = a_1 + a_2 + \dots + a_n + \dots$ into an at most denumerable number of disjoint bricks a_i, such that

$$\max_{n=1,2,\dots} \mu(a_n) < \eta,$$

i. e. the net number $\mathfrak{N}\{a_i\}$ of the partition is $< \eta$.

16.a - **Proof.** We may admit that μ is effective, so the measure atoms coincide with genuine atoms. First of all

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(11) This means that if $E \in G$ and $\mu E > 0$, then there exist E_1, $E_2 \in G$ with $E_1 \cdot E_2 = 0$, $\mu(E_1) > 0$, $\mu(E_2) > 0$, $E_1 + E_2 = E$. If μ is effective, measure atoms coincide with ordinary atoms.

we recall, that for any partition $A: a = a_1 + a_2 + ... + a_n + ...$ of a into an at most denumerable number of bricks, the set of all numbers $\mu(a_n)$ admits a maximum [15 Lemma], which we shall denote by $\mathfrak{M}(A)$ [15.7]. The theorem says that there exists a decomposition A of a with $\mathfrak{M}(A) < \eta$. Suppose this be not true. Hence, if we take all possible above decompositions of a we shall have

$$\alpha \underset{df}{=} \inf \mathfrak{M}(A) \geq \eta \tag{0}$$

Hence there exists an infinite sequence $A_1, A_2, ..., A_n, ...$ of partitions of a such that

$$\mathfrak{M}(A_1) \geq \mathfrak{M}(A_2) \geq ... \to \alpha \geq \eta > 0.$$

Take the sequence of partitions $B_1 \underset{df}{=} A_1$, $B_2 \underset{df}{=} A_1 \cdot A_2$, $B_3 \underset{df}{=} A_1 \cdot A_2 \cdot A_3$, ... (the products of partitions [15.2]). They make up a nested sequence of partitions ([15.4], [15.5], B_1, B_2, B_3, ... where B_{n+1} is a subpartition of $B_1, ..., B_n$, $(n = 1, 2, ...)$. We have

$$\mathfrak{M}(B_1) \geq \mathfrak{M}(B_2) \geq ... \to \alpha.$$

16. b. - Denote by λ_n the number of disjoint bricks b in B_n with $\mu(b) \geq \alpha$. Their number is finite and ≥ 1. Indeed, we have $\lambda_n \cdot \alpha \leq \mu(a)$, hence

$$\lambda_n \leq \frac{\mu(a)}{\alpha} \tag{1}$$

If λ_n were $= 0$, all bricks in B_n would have the measure $< \alpha$, which is excluded. In the infinite sequence

$$\lambda_1, \lambda_2, ..., \lambda_n, ...$$

we have

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$$

because $\{B_n\}$ is nested.

Hence, by (1), starting from an index n_0, all λ_n are equal. Put $\lambda \underset{df}{=} \lambda_{n_0} = \lambda_{n_0+1} = ...$ We have $\lambda \geq 1$. We shall call λ, temporarily in this proof, *characteristic number of the sequence* $\{B_n\}$.

Thus we have proved that if the theorem is not true, then there exists, among all sequences yielding the infimum α, a nested sequence of partitions $\{B_n\}$ of a with $\mathfrak{M}(B_1) \geq \mathfrak{M}(B_2) \geq \dots \to \alpha$ with characteristic number $\lambda \geq 1$.

Consider such a nested sequence with the smallest characteristic number. Denote this number by λ. We have $\lambda \geq 1$. Denote the sequence of partitions by $\{D_n\}$. We have

$$\mathfrak{M}(D_1) \geq \mathfrak{M}(D_2) \geq \dots \to \alpha \tag{2}$$

Denote the bricks in D_n whose measure $\geq \alpha$ by d_{n1}, d_{n2}, ..., $d_{n\lambda}$. Since the partitions D_n are nested, we can admit that

$$\begin{array}{l} d_{22} \geq d_{21} \geq \dots \geq d_{n1} \geq \dots \\ d_{12} \geq d_{22} \geq \dots \geq d_{n2} \geq \dots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ d_{1\lambda} \geq d_{2\lambda} \geq \dots \geq d_{n\lambda} \geq \dots \end{array}$$

Put

$$c_1 \underset{df}{=} \prod_{n=1}^{\infty} d_{n1}, \ \dots, \ c_\lambda \underset{df}{=} \prod_{n=1}^{\infty} d_{n\lambda}.$$

These somata may not be bricks, but they $\in G$. We have $\mu c_1 \geq \alpha$, ..., $\mu c_\lambda \geq \alpha$.

16.c. - We shall prove that for at least one index k we have $\mu c_k = \alpha$. Suppose this be not true. Hence for all indices $i = 1, \dots, \lambda$ we have $\mu c_i > \alpha$. Hence there exists $\delta < 0$ with

$$\mu c_i > \alpha + \delta \quad \text{for all} \quad i = 1, \dots, \lambda. \tag{3}$$

Consider the partition D_n. There are only λ bricks for which $\mu(b) \geq \alpha$, namely d_{n1}, ..., $d_{n\lambda}$; for all other bricks b of D_n we have $\mu b < \alpha$. Now the numbers μd_{n1}, ..., $\mu d_{n\lambda}$ are all $> \alpha + \delta$. Hence $\mathfrak{M}(D_n) > \alpha + \delta$ which contradicts (2).

16.d. - Take k such that

$$\mu c_k = \alpha. \tag{4}$$

Since $\mu c_k > 0$ and G has no measure-atoms, there exist two somata E, F of G with $\mu E > 0$, $\mu F > 0$, $E \cdot F = O$, $E + F = c_k$. We have, by (4),

$$\text{(5)} \qquad \mu E + \mu F = \alpha.$$

Take such sets E, F. Take $\eta > 0$ such that

$$\text{(6)} \qquad \eta < \frac{1}{24} \min [\mu(E), \ \mu(F)].$$

Since $\mu(E) < \alpha$, $\mu(F) < \alpha$, we have $\eta < \frac{\alpha}{24}$, i. e.

$$\text{(7)} \qquad \frac{\alpha}{2} - 12\eta > 0.$$

Take $\delta > 0$ such that

$$\text{(8)} \qquad \delta < \frac{\alpha}{2} - 12\eta,$$

and consider n such that

$$\text{(9)} \qquad \mu(d_{kn} - c_k) \leq \delta.$$

16.e. - Having that, consider figures e, f, such that, [12.4], $|E, e|_\mu < \eta$, $|F, f|_\mu < \eta$. Put $e' \underset{df}{=} e \cdot d_{nk}$, $f' \underset{df}{=} f \cdot d_{nk}$. Since $E \leq d_{nk}$, $F \leq d_{nk}$, we have $e' \leq d_{nk}$, $f' \leq d_{nk}$.

We have, [5.15],

$$\text{(10)} \qquad |E, e'|_\mu = |Ed_{nk}, ed_{nk}|_\mu \leq |E, e| < \eta.$$

Similarly $|F, f'|_\mu < \eta$. Put $e'' \underset{df}{=} e' - f'$, $f'' \underset{df}{=} f' - e'$. We have $e'' \leq d_{nk}$, $f'' \leq d_{nk}$, $e'' \cdot f'' = O$ and by [5.20], $|E, e''|_\mu \leq 3\eta$. $|F, e''|_\mu \leq 3\eta$. Hence, by [5.17], $|\mu(e'') - \mu(E)| \leq 6\eta$, $|\mu(f'') - \mu(F)| \leq 6\eta$. This gives

$$\text{(11)} \qquad \begin{aligned} \mu(E) - 6\eta &\leq \mu(e'') \leq \mu(E) + 6\eta \\ \mu(F) - 6\eta &\leq \mu(f'') \leq \mu(F) + 6\eta \,. \end{aligned}$$

From (6) we get $6\eta < \frac{1}{2} \min [\mu(E) \cdot \mu(F)]$. Hence

$$\text{(11.1)} \qquad 6\eta < \frac{1}{2} \mu(E), \ 6\eta < \frac{1}{2} \mu(F),$$

and then

$$(12)\qquad \begin{aligned} \mu(E) - 6\eta &> \mu(E) - \frac{1}{2}\mu(E) = \frac{1}{2}\mu(E), \\ \mu(F) - 6\eta &> \mu(F) - \frac{1}{2}\mu(F) = \frac{1}{2}\mu(F). \end{aligned}$$

Hence, by (11),

$$(13)\qquad \begin{aligned} 0 &< \frac{1}{2}\mu(E) < \mu(e''), \\ 0 &< \frac{1}{2}\mu(F) < \mu(f''). \end{aligned}$$

Taking the right-hand-side inequalities in (11), we get, by (11.1) and (5), $\mu(e'') \leq \mu(E) + 6\eta \leq \alpha - \mu(F) + 6\eta \leq \alpha - \mu(F) + \frac{1}{2}\mu(F) \leq \alpha - \frac{1}{2}\mu(F)$ and similarly $\mu(f'') \leq \alpha - \frac{1}{2}\mu(E)$. This together with (13) gives

$$(14)\qquad \begin{aligned} 0 &< \frac{1}{2}\mu(E) < \mu(e'') < \alpha - \frac{1}{2}\mu(F), \\ 0 &< \frac{1}{2}\mu(F) < \mu(f'') < \alpha - \frac{1}{2}\mu(E). \end{aligned}$$

We have from (11) $\alpha - 12\eta \leq \mu(e'') + \mu(f'')$.

Hence $\mu(d_{nk} - (e_k'' + f_k'')) = \mu d_{nk} - \mu(e_k'' + f_k'') \leq \mu d_{nk} - (\alpha - 12\eta)$, by (9), $\leq \mu(c_k) + \delta - (\alpha - 12\eta)$, by (4), $\leq \alpha + \delta - (\alpha - 12\eta) = \delta + 12\eta$, by (8) $< \frac{\alpha}{2}$. Thus we have $\mu(d_{nk} - (e_k'' + f_k'')) < \frac{\alpha}{2}$ and, by (14),

$$(15)\qquad \begin{aligned} \mu(e_k'') &< \alpha - \frac{1}{2}\mu(F), \\ \mu(f_k'') &< \alpha - \frac{1}{2}\mu(E). \end{aligned}$$

16.f. - **The brick d_{nk} is thus decomposed into three disjoint figures: $g \underset{df}{=} d_{nk} - (e_k'' + f_k'')$, e_k'' and f_k'', all with a measure smaller than α. If we decompose, (see (Hyp Ad)), each of the figures g, e_k'', f_k'' into bricks, the measure of every brick will be surely $< \alpha$. If we do this, the partition D_n will be changed into another partition D' of a, where the part**

outside d_{nk} is not changed, and only d_{nk} is replaced by a partition with max. measure of bricks less than α.

Consider the sequence D_1D', D_2D', ..., D_nD', ...

This is a sequence of nested partitions of a with $\mathfrak{N}(D_nD') \rightarrow \alpha$, but with characteristic number $< \lambda$ or without any one, which is a contradiction, since λ is the minimal characteristic number.

16.1. - Under hyp. [12] and [14] suppose that 1) G has no measure-atoms, 2) we have a partition A of a soma of G into an at most denumerable number of disjoint bricks, 3) $\eta < 0$.

Then there exists a subpartition B of A into bricks such that the net-number $\mathfrak{N}B < \eta$.

Proof. This follows from [Lemma 16].

17.1. - **Lemma.** Under [12] and [14]. If A is an atom of G, $E \in G$, than either $A \leq E$ or $A \leq$ co E, disjointedly.

Proof We have $A = AE + A$ co E. If $AE = O$, then $A = A$ co E and then $A \leq$ co E. If A co $E = O$, we get $A \leq E$. The remaining case is $AE \neq O$ and A co $E \neq O$ is impossible. Indeed we would have $\mu(AE) > 0$, $\mu(A$ co $E) > 0$. so A were not an atom.

17.2. - **Lemma.** Under Hyp. [12] and [14], if

1. A is an atom in G.
2. $p_1, p_2, \dots, p_n, \dots$ are disjoint different bricks.
3. $A \leq \Sigma p_n$, then there exists one and only one index n such that $A \leq p_n$.

Proof. First of all we cannot have two different indices i, j with $A \cdot p_i \neq O$, $A \cdot p_j \neq O$. Indeed, by [Lemma 17.1], we would have $A \leq p_i$, hence $A \leq$ co p_j because $p_i \leq$ co p_j, and the bricks $\{p_i\}$ are different. Hence, since $A \leq p_j$, [17.1], we would have $A \leq p_j \cdot$ co $p_j = O$, i. e. A is not an atom. Now, by hyp. 3., there exists an index n with $A \cdot p_n \neq O$. Hence, [17.1], we have $A \subseteq p_n$. This index n is unique by virtue of what just has been proved. The lemma is established.

17.3. - Lemma. Under Hyp. [12] and [14], if 1. A is an atom in G, 2. $E \in G$, 3. $|A, E|_\mu \leq \eta$, $\eta > 0$, 4. $\eta < \mu(A)$, then $A \leq E$.

Proof. We have

$$(1) \qquad \mu(E - A) + \mu(A - E) \leq \eta.$$

Suppose that the inclusion $A \leq E$ is not true. Then, [17.1], $A \leq \text{co}\, E$; hence $A \cdot E = O$. Since $A = (A - E) + AE$, we get $A = A - E$, and then, by (1), $\mu(E - A) + \mu(A - E) = \mu(E - A) + \mu A \leq \eta$ hence $\mu A \leq \eta$, which contradicts hyp. 4. The lemma is proved.

17.4. - Lemma. Under Hyp. [12] and [14], if A_1, A_2 are two different atoms of G, then there exists a partition of I into different bricks $\{a_n\}$ such that A_1, A_2 are lying in distinct bricks: $A_1 \leq a_i$, $A_2 \leq a_j$, where $a_i \cdot a_j = O$.

Proof. We have $\mu A_1 > 0$, $\mu A_2 > 0$. Take $\eta > 0$ such that $\eta < \frac{1}{3} \min [\mu A_1, \mu A_2]$. Find, [14.2], complexes P_1, P_2 such that $|A_1, P_1| < \eta$, $|A_2, P_2| < \eta$. Since $A_1 A_2 = O$ we get, [5.20], $|A_1, \text{som} P_1 - \text{som}\, P_2| \leq 3\eta$, $|A_2, \text{som}\, P_2 - \text{som}\, P_1| \leq 3\eta$. We also have $3\eta < \mu(A_1)$, $3\eta < \mu(A_2)$. Appliyng [17.3], we get $A_1 \leq \text{som}\, P_1 - \text{som}\, P_2$, $A_2 \leq \text{som}\, P_2 - \text{som}\, P_i$.

Since $\text{som}\, P_1 - \text{som}\, P_2$ and $\text{som}\, P_2 - \text{som}\, P_1$ are figures, they can be, partitioned, (Hyp. *Ad*), into an at most denumerable number of disjoint different bricks. By [17.2] there exists in $\text{som}\, P_1 - \text{som}\, P_2$ one brick of the partition, which contains A_1, and in $\text{som}\, P_2 - \text{som}\, P_1$ there exists one brick, containing A_2.

Since $\text{som}\, P_1 - \text{som}\, P_2$ and $\text{som}\, P_2 - \text{som}\, P_1$ are disjoint, the mentioned bricks are also disjoint and different. Having this, decompose the figure $I - [(\text{som}\, P_1 - \text{som}\, P_2) + (\text{som}\, P_2 - \text{som}\, P_1)]$ into a denumerable number of disjoint bricks, (Hyp. *Ad*). Thus we shall have a partition of I into bricks, such that A_1, A_2 are lying in two different bricks of that partition. The lemma is proved.

17.5. - Lemma. Under hypotheses [12] and [14], if 1) P is a partition of I into bricks. 2) $A_1, A_2, \dots, A_n, \dots$ are some or all atoms of G, finite or infinite in number, 3) these atoms are lying in distinct bricks of P. (Never two atoms in one brick). 4) Q is a subpartition of P, then the above atoms are also lying in distinct bricks of Q.

Proof. By [17.2].

17.6. - Lemma. Under hypotheses [12], [14] let $A_1, A_2, \dots, A_n$, $(n \geq 2)$ be some different atoms of G. (They may be all atoms or not.). Then there exists a partition P of I such that all $A_1, \dots, A_n$ are lying in distinct bricks of the partition. (Never two atoms in one bricks).

Proof. Consider a not ordered couple (A_i, A_j) where $A_i \neq A_j$. Take, by [17.4], a partition P_{ij} of I such that A_i and A_j are lying in different bricks of P_{ij}.

The product $P' \underset{df}{=} \prod_{(i,j)} P_{ij}$, taken for all different above couples of indices, is a partition of I into a denumerable number of bricks. [15.2]. and is, [15.5], a subpartition of all P_{ij}. By [17.5], the atoms $A_i, A_j, (i \neq j)$ are lying in different bricks of P'. This being true for any couple i, j of indices, the lemma is established.

17.7. - Lemma. Admit the hypotheses [12] and [14]. If

1. $A_1, A_2, \dots, A_n$, $(n \geq 1)$, are some, (or all), different atoms of G.

2. $\eta > 0$,

then there exists a partition of I into different bricks:

$$a_1, a_2, \dots, a_m, \dots \tag{1}$$

such that:

1) The atoms $A_1, A_2, \dots, A_n$ are lying in different bricks (1).

2) if $A_i \leq a_m$, then $0 \leq \mu(a_m) - \mu(A_i) < \eta$.

Proof. Relying on [17.6], find a partition P of I into different bricks such that each atom $A_1, \dots, A_n$ is lying in a separate brick of P, i. e. two different above atoms are lying in different bricks. Let

(1) $$A_1 \leq a_1, \ldots, A_n \leq a_n,$$

where $a_1, \ldots, A_n$ are bricks of P. Take $\eta' > 0$ such that $\eta' < \min[\eta, \mu A_1, \ldots, \mu A_n]$. By [14 2] find complexes $P_1, \ldots, P_n$ such that

(2) $$|A_1, P_1| \leq \eta', \ldots, |A_n, P_n| < \eta'.$$

Since $\eta' < \mu A_i$ and $|A_i, P_i| < \eta'$, we get, by [17.3],

(3) $$A_i \leq \text{som } P_i.$$

From (1) and (3) we get

(4) $$A_i \leq a_i \text{som } P_i.$$

From (2) we have, by [5.15], $|A_i a_i, a_i \text{ som } P_i| \leq \eta'$.

Hence, by (1),

(4.1) $$|A_i, a_i \text{ som } P_i| \leq \eta'.$$

Hence, by (4),

(5) $$0 < \mu(a_i \text{ som } P_i) - \mu(A_i) \leq \eta',$$

Having this, replace the brick a_i, $(i = 1, 2, \ldots, n)$, by the two different figures a_i som P_i, $a_i - a_i$ som P_i, whose sum is a_i, and partition these figures into bricks. Since a_i belongs to P, we get, in this way, a subpartition P' of P. Since the atoms $A_1, \ldots, A_n$ are lying in separate bricks of P, therefore they are also lying, by [17.5], in separate bricks of P', say in $a'_1, \ldots, a'_n$ respectively. I say that $a'_i \leq a_i$ som P_i.

Indeed, by (4), $A_i \leq a_i$ som P_i and, by (1), $A_i \leq a_i$. The brick a'_i is contained either in a_i som P_i or in a_i–som P_i. In the second case we would have $A_i \leq$ co som P_i which is a contradiction. Thus

(6) $$a'_i \leq a_i \text{ som } P_i.$$

We have, by (6), $\mu(a'_i) - \mu(A_i) \leq \mu(a_i \text{ som } P_i - A_i) =$, by (5), $= \mu(a_i \text{ som } P_i) - \mu(A_i) \leq \eta' < \eta$,

Thus we have got a partition P' wich satisfies the requirements of the thesis.

18. - Def. There exists in G an at most denumerable number of mutually disjoint measure-atoms, say β_1, β_2, ..., β_n, ... Put $\beta \underset{df}{=} \beta_1 + \beta_2 + \dots$. Let $P = \{p_1, p_2, \dots, p_n\}$ be a non empty complex. Then by *the reduced net-number of* P, we understand the number

$$\mathfrak{N}_R(P) \underset{df}{=} \max\,[\mu(p_1 - \beta), \dots, \mu(p_n - \beta)].$$

A similar definition we admit for any at most denumerable partition into disjoint bricks, (see [15]).

18.1. - **Theorem.** If we admit hyp. [12] and [14], then for every $\eta > 0$ there exists a partition of I into bricks such that $\mathfrak{N}_R(P) > \eta$. [Def. 18].

18.1.a - **Proof.** First consider the case, where G is purely atomic i. e., every soma of G, which $\neq O$, is the sum of an at most denumerable number of somata $A_1, A_2, \dots, A_n$, ... with $\mu(A_n) > 0$ for $n = 1, 2, \dots$ where all A_n are different measure-atoms. In that case $\beta = I$. We have $p_i - \beta = O$, hence $\mathfrak{N}_R(P) = 0$ for any partition P. In that case the theorem is true.

18.1b. - Suppose that G has no atoms at all. Then we are in the conditions of [16.1], applied to I. Hence if $\eta > 0$, there exists a partition P of I such that $\mathfrak{N}(P) < \eta$. If $P = \{p_1, p_2, \dots\}$, we have $\max\limits_i \mu(p_i) = \max\limits_i \mu(p_i - \beta) < \eta$. Hence $\mathfrak{N}_R(P) < \eta$, so the theorem is true in our case.

18.1c. - The case where G is not purely atomic, but has only a finite number of atoms, will constitute a simplified version of arguments which will follow in the discussion of the case where we have an infinite number of atoms.

18.1d. - Thus we direct our attention to the case where G is not purely atomic, but has a denumerable infinite number of different atoms, say: $A_1, A_2, \dots, A_m, \dots$. We may suppose that

(1) $$\mu(A_1) \geq \mu(A_2) \geq \dots \geq \mu(A_m) \geq \dots.$$

Let $\eta > 0$. We can find n such that

(2) $$\mu(A_{n+1} + A_{n+2} + \dots) < \eta\,.$$

18.1e. - Applying [17.1], take a partition P of I into different bricks $a_1, a_2, \dots$, such that

(4) $$A_1 \leq a_1, \dots, A_n \leq a_n$$

and $\mu(a_i - A_i) < \eta$.

Putting

(5) $$\beta \underset{df}{=} A_1 + A_2 + \dots + A_n + A_{n+1} + \dots,$$

we get

(6) $$\mu(a_i - \beta) < \eta \text{ for } i = 1, 2, \dots, n, \dots.$$

We have $A_1 + \dots + A_n \leq a_1 + \dots + a_n$, but there may also exist atoms among

(7) $$A_{n+1}, A_{n+2}, \dots,$$

which are included in $a_1 + \dots + a_n$. If all atoms (7) are included in $a_1 + \dots + a_n$, the arguments which follows will be simplified. Let

(8) $$A_{k(1)}, A_{k(2)}, \dots,$$

finite, or infinite in number, be all atoms taken from (7) which are not included in $a_1 + \dots + a_n$. Then they must lie in co $[a_1 + \dots + a_n]$; this by [17.1].

Let

(9) $$A_{l(1)}, A_{l(2)}, \dots$$

be all atoms among (7) which are lying in $[a_1 + \dots + a_n]$. The sets (8) and (9) make up the whole set (7). These sets are disjoint.

18.1f. - We have

(10) $$A_{k(1)} + A_{k(2)} + \dots \leq \text{co}\,(a_1 + \dots + a_n).$$

Find, by [13.2], a covering B of $A_{k(1)} + A_{k(2)} + \dots$ such that

(11) $$|A_{k(1)} + A_{k(2)} + \dots, B|_\mu < \eta.$$

The soma $\text{co}\,(a_1 + \dots + a_n)$ is a figure, hence it is also a covering [Def. 4]. Hence

(12) $$B' \underset{df}{=} \text{co}\,(a_1 + \dots + a_n) \cdot B$$

is a covering of $A_{k(1)} + A_{k(2)} + \dots$, such that

(13) $$|A_{k(1)} + A_{k(2)} + \dots, B'|_\mu < \eta,$$

and

(14) $$A_{k(1)} + A_{k(2)} + \dots \leq B'.$$

Let us partition B' into disjoint different bricks

(15) $$b_1, b_2, \dots.$$

Since, by (2), $\mu(A_{k(1)} + A_{k(2)} + \dots) < \eta$, we get, by [5.17], from

(13), $\mu B' < 2\eta$, and then $\mu b_k < 2\eta$. It follows that

$$\mu(b_k - \beta) < 2\eta \tag{16}$$

for all bricks (15).

18.1g. - Define

$$C \underset{df}{=} \mathrm{co}\,(B' + a_1 + \dots + a_n). \tag{17}$$

Since $A_1 + A_2 + \dots + A_n \leq a_1 + \dots + a_n$, $A_{l(1)} + A_{l(2)} + \dots$ $\dots \leq a_1 + \dots + a_n$, and $A_{k(1)} + A_{k(2)} + \dots \leq B'$, we have

$$C - \beta = C, \tag{18}$$

i. e. all atoms of G are disjoint with C. We have

$$C \leq \mathrm{co}\,(a_1 + \dots + a_n), \tag{19}$$

If $\mathrm{co}\,(a_1 + \dots + a_n) = O$, the thesis is proved, and so is if $C = O$. Suppose that $\mathrm{co}\,(a_1 + \dots + a_n) \neq O$ and that $C \neq O$. Consider the tribe

$$G' \underset{df}{=} \mathrm{co}\,(a_1 + \dots + a_n) 1\, G. \tag{20}$$

its zero is O and its unit $I' = \mathrm{co}\,(a_1 + \dots + a_n)$. The tribe G' is the Lebesgue's-μ covering extension of the tribe

$$F' \underset{df}{=} I' 1\, F. \tag{21}$$

The set B' of all somata $a \cdot I'$, where $a \in B$, constitute a base of F'. Its bricks are the bricks of B contained in I'. We have supposed that $C \neq O$. Consider the tribes

$$\begin{aligned} G'' &\underset{df}{=} C 1\, G = C 1\, G', \\ F'' &\underset{df}{=} C 1\, F = C 1\, F'. \end{aligned} \tag{22}$$

Take account of (19), and notice that C may not be a figure. Denote by B'' the set of all somata $a \cdot C$ where $a \in B$. We see that G'' is the Lebesgue's-μ-covering extension of F'', and B'' is a base of F''.

18.1h. - The tribe G'' has no atoms, so we can apply [Lemma 16]. By its virtue the B''-brick C (which is the unit I'' of G''), can be partitioned into a denumerable number of disjoint B''-bricks $p_1, p_2, \dots, p_n, \dots$, such that

$$\mu(p_n) < \eta \text{ for } n = 1, 2, \dots. \tag{23}$$

Now, $p_n = q_n \cdot C$ where q_n is a B'-brick. We have $q_n \in G'$. Since C is the unit of G'', we have $C \in G''$, hence $C \in G'$. Consequently

$$p_n \in G'. \tag{24}$$

18.1i. - There exists in G' a covering of p_n : $Q_n = \{q_{n1}, q_{n2}, \dots\}$ such that

$$|Q_n, p_n| < \eta, \tag{25}$$

and then q_{nk} are disjoint F'-bricks, hence F-figures. By (23), (25), by virtue of [5.17], we get $\mu Q_n < 2\eta$, and then $\mu(q_{nk}) < 2\eta$ for all indices n, k. Let us decompose every q_{nk}, which is an F-figure, into disjoint B-bricks $q_{nk} = \{r_{nk1}, r_{nk2}, \dots\}$.

In this way every p_n is covered by a denumerable number of B-bricks r_{nkj}, such that

$$\mu(r_{nkj}) < 2\eta. \tag{26}$$

Thus C is covered by a denumerable number of B-bricks. The soma B' is also covered, by [18.1f], by a denumerable number of B-bricks. The soma $a_1 + \dots + a_n$ is also covered by the bricks $a_1, \dots, a_n$. For all those bricks c we have either $\mu(c) < 2\eta$ or $\mu(c - \beta) < 2\eta$, (see (16), (6), (26)). Hence for all of them we have $\mu(c - \beta) < 2\eta$.

Thus we have a denumerable number of bricks c whose sum equals $I \in G$. Applying [cor. 4.4], we get I decomposed into a denumerable number of disjoint bricks $d_1, d_2, \dots$ such that each d_n is lying in one of the bricks c. Hence we get for every n: $\mu(d_n - \beta) < 2\eta$, so the theorem is established.

18.2. - **Remark.** The presence of atoms hinders making partitions with small «meshes», so the notion of reduced net-number helps. Now, if the tribe G is composed of atoms only, the reduced net number will always be $= 0$, so another kind of net-number shall be introduced - just to cover all possibilities. Therefore we introduce the following definition:

19. - **Def.** Under hypotheses [12] and [14], if $a \neq O$ is a brick, put

$$\mathfrak{N}'(a) \underset{df}{=} \mu(a) - \max_{A \leq a} \mu(A),$$

where the maximum is taken over all atoms A included in a; of course, this happens, if there exists an atom A included in a. If a does not contain any atom, we put $\mathfrak{N}'(a) \underset{df}{=} \mu(a)$. If P is a complex $\{p_1, \ldots, p_n\}$, $(n \geq 1)$, then by the *atom-net-number of P* we shall understand the non negative number

$$\mathfrak{N}_A(P) \underset{df}{=} \max(\mathfrak{N}'(p_1), \ldots, \mathfrak{N}'(p_n)).$$

If P is an empty complex, we define $\mathfrak{N}_A(P) \underset{df}{=} 0$.

A similar definition is admitted for partitions into bricks. We can do this because the maximum spoken of always exists (compare [Lemma 15]).

19.1. - **Theorem.** We admit the hypotheses [12] and [14]. For every $\eta > 0$ there exists a partition P of I into disjoint bricks, such that $\mathfrak{N}_A(P) < \eta$.

19.1a - **Proof.** If G has no atoms, then the theorem follows from [Lemma 16]. If G has a finite numbers of atoms, the theorem follows from [Lemma 17.7].

19.1b. - Suppose that G has an infinite number of atoms. It must be denumerable. Let $A_1, A_2, \ldots, A_n, \ldots$ be all different atoms, arranged so as to have

$$\mu(A_1) \geq \mu(A_2) \geq \ldots \geq \mu(A_n) \geq \ldots. \tag{1}$$

This can be done, because the number of atoms whose measure is $\geq \varepsilon$, where $\varepsilon > 0$, is at most finite. Let $\eta > 0$. Find n such that, if we put $A \underset{df}{=} A_{n+1} + A_{n+2} + \ldots$, we have $\mu A < \eta$.

Relying on [17.7] we can find a partition P of I into disjoint bricks, such that the atoms $A_1, A_2, \ldots, A_n$ are lying in separate bricks, say $a_1, \ldots, a_n$, so that we have $A_1 \leq a_1, \ldots, A_n \leq a_n$, and that $\mu(a_i) - \mu(A_i) < \eta$ for $i = 1, 2, \ldots, n$. On account of (1) we have

$$\mu(a_i) - \max_{A_s \leq a_i} \mu(A_s) < \eta, \tag{2}$$

because there does not exists any atom in a_i whose measure were $> \mu A_i$. Suppose that $A \neq I$, because if $A = I$ the thesis follows.

Consider the tribe $G' \underset{df}{=} \text{co}\, A \rceil G$. The unit I' of G' is co A; it is a figure. The atoms of G' are $A_{n+1}, A_{n+2}, \ldots$ if any. By [18.1] we can find a partition $Q = \{q_1, q_2, \ldots\}$ of $\text{co}\, A = I'$, so that $\max[\mu(q_1 - A'), \mu(q_2 - A'), \ldots] < \eta$, where A' is the sum of all atoms of G'. Since $A' \leq A$, we get $\mu A' < \eta$ and $\max[\mu(q_1 - A), \mu(q_2 - A), \ldots] < \eta$.

The only atoms contained in q_j are among those of A'. Since $\mu A' < \eta$, it follows $\mu(q_j) = \mu(q_j - A') + \mu(q_j A') < \eta + \eta = 2\eta$.

Hence

$$(3) \qquad 0 \leq \mu(q_j) - \max_{A_s \leq q_j} \mu(A_s) < \mu(q_j) < 2\eta.$$

Now $\{a_1, a_2, \ldots, a_n, q_1, q_2, \ldots\}$ is a partition of I into bricks. From (2) and (3) it follows that if b is any brick of this partition, we have $\mu(b) - \max_{A_s \leq b} \mu(A_s) < 2\eta$, so the theorem is proved.

19.2. - If 1. A is a partition of I into bricks, 2. $\mathfrak{N}_B A < \eta$, 3. B is a subpartition of A into bricks, then $\mathfrak{N}_B(B) < \eta$.

Proof. Let $b \in B$. There exists $a \in A$ with $b \leq a$. We have $\mu(a - \beta) < \eta$, consequently $\mu(b - \beta) < \eta$. This being true for any $b \in B$, the lemma follows.

19.3. - If

1. A is a partition of I into bricks $a_1, a_2, \ldots$,
2. $\mathfrak{N}_A(a_n) < \eta$ for all $n = 1, 2, \ldots$,
3. $B = \{b_1, b_2, \ldots\}$ is a subpartition of A into bricks,

then $\mathfrak{N}_A(b_n) < \eta$ for $n = 1, 2, \ldots$.

Proof. Let $b \in B$. There exists $a \in A$ with $b \leq a$. Now $\mu(a) - \max_{A \leq a} \mu(A) < \eta$, where the maximum is taken for all atoms A which are included in a, if they exist. We have $\mu(a) < \eta$ if no atoms, lying in a, are available. Let

$$(1) \qquad A_1, A_2, \ldots, A_s, \quad s \geq 0$$

be all different atoms included in a with $\mu A_1 = \ldots = \mu A_s = \max_{A \leq a} \mu(A) \underset{df}{=} \delta$.

Now $b \leq a$. Concerning the atoms $A_1, \ldots, A_s$, each one of them is either included in b or in $a - b$. Suppose that one at least of (1), say A_k, is in $a - b$. Since $\mu(a) - \delta < \eta$ and since $b \leq a - A_k$, we have $\mu(b) < \eta$, and then a fortiori $\mu(b) - \max\limits_{A \leq b} \mu(A) < \eta$.

Now suppose that no one of (1) is in $a - b$. Hence they all are in b. Hence $\max\limits_{A \leq b} \mu(A) = \max\limits_{A \leq a} \mu(A)$. Hence $\mu(a) - - \max\limits_{A \leq b} \mu(A) < \eta$ and then $\mu(b) - \max\limits_{A \leq b} \mu(A) < \eta$.

The remaining case is, where no atom is included in a, then no atom is included in b; hence $\mu(b) \leq \mu(a) < \eta$. The theorem is proved.

20. - Theorem. Under hyp. [12] and [14], if $E \in G$ and $\eta > 0$, then there exists a complex $P = \{p_1, \ldots, p_n\}$ such that 1) $|E, P|_\mu < \eta$, 2) if β is the sum of all atoms of G in the case they exists, and $\beta = O$, if there are no atoms in G, then $\mu(p_1 - \beta), \ldots, \mu(p_n - \beta) < \eta$ i. e. $\mathfrak{N}_R(P) < \eta$.

Proof. First we find a complex P such that $|E, P|_\mu < \frac{\eta}{2}$. co som P is a figure; hence it can be partitioned into an at most denumerable number of disjoint bricks. This partition, together with $P = \{p_1, p_2, \ldots\}$, make up a partition Q of I into an at most denumerable number of disjoint bricks.

By [Theor. 18.1] we can find a partition S of I into disjoint bricks such that $\mathfrak{N}_R(S) < \eta$. Take the product $Q \cdot S$. This is a subpartition of S and of P. Hence, by [19.2], $\mathfrak{N}_R(QS) < \eta$. If we confine that partition to som P, we have partitioned som P into a denumerable number of disjoint bricks with reduced net-number $< \eta$. Let this partition of som P be $p'_1, p'_2, \ldots$. For sufficiently great n we have $|P, p'_1 + \ldots + p'_n|_\mu < \frac{\eta}{2}$, and if we put $P' \underset{df}{=} \{p'_1, \ldots, p'_n\}$, we have $|P, P'|_\mu < \frac{\eta}{2}$; hence $|E, P'| < \eta$ and $\mathfrak{N}_R(P') < \eta$. Thus the theorem is proved.

20.1. - Theorem. Under hypoth. [12] and [14], if $E \in G$

and $\eta > 0$, then there exists a complex $P = [p_1, \ldots, p_n]$ such that: 1) $|E, P|_\mu < \eta$; 2) if $A_1, A_2, \ldots$, are all atoms of G, then $\mu(p_i) - \max\limits_{A_j \leq p_i} A_j < \eta$, $(i = 1, \ldots, n)$, and $\mu(p_i) < \eta$, if there does not exist any above $A_j \leq p_i$, i. e., $\mathfrak{N}_A(P) < \eta$.

Proof. The proof is similar to that of [Theor. 20]. The difference is that instead of relying on [18.1], we rely on [19.1] and [19.3].

20.2. - Under hyp. [12] and [14], if $E \in G$ and $\eta > 0$, then there exists a complex P such that 1) $|E, P|_\mu < \eta$, 2) $\mathfrak{N}_A(P) < \eta$. 3) $\mathfrak{N}_R(P) < \eta$.

Proof. The proof follows the pattern of the two preceding proofs. Take a complex P such that $|E, P|_\mu < \frac{\eta}{2}$. The soma co som P is a figure; hence it can be partitioned into an at most denumerable number of disjoint bricks, getting a partition Q of I. By theor. [18.1] we can find a partition S_1 of I into bricks such that $\mathfrak{N}_R(S_1) < \eta$. By theor. [19.1] we can find a partition S_2 of I into bricks such that $\mathfrak{N}_A(S_2) < \eta$. The product $R \underset{df}{=} Q \cdot S_1 \cdot S_2$ is a partition of I into a denumerable number of bricks. By theor. [19.2] we have $\mathfrak{N}_A(R) < \eta$, and by theor. [19.3] we have $\mathfrak{N}_A(R) < \eta$. Let us confine the partition R to som P; we get a partition $P' \underset{df}{=} \{p'_1, p'_2, \ldots\}$ of som P. For sufficiently great n we have putting $P'' \underset{df}{=} \{p'_1, \ldots, p'_n\}$, $|P', P''|_\mu < \frac{\eta}{2}$ which gives $|P''E|_\mu < \frac{\eta}{2}$. We also have $\mathfrak{N}_A(P'') < \eta$, $\mathfrak{N}_R(P'') < \eta$. So the theorem is established.

20.3. - **Theorem**. Under hyp. [12], [14], if $E \in G$, A is a brick-covering of E, (hence $E \leq A$), $\eta > 0$, then there exists a complex P such that

1) som $P \leq A$, 2) $|P, E|_\mu \leq \eta$, 3) $\mathfrak{N}_A(P) \leq \eta$, 4) $\mathfrak{N}_R(P) \leq \eta$.

Proof. We rely on [14.3] and apply [20.2].

21. - The μ-topology on G is always complete but it may be not separable, (see [13.4]). In farther sections we shall need an important property (see [21.2]) of complexes approximating somata of G, this property being strongly related to separability. Therefore, in what follows in this § 1, we shall admit, in addition to hypotheses [12] and [14], the following hypothesis (Hyp. S) of separability:

21.1. - **(Hyp. S).** There exists a denumerable sequence

$$(1) \qquad M_1, M_2, \ldots, M_n, \ldots,$$

of somata of G such that for every soma $E \in G$ we can find a subsequence $\{M_{k(n)}\}$ of I such that $|M_{k(n)}, E|_\mu \to 0$, for $n \to \infty$.

We have

The following important theorem:

21.2. - **Theorem.** Under hypotheses [12], [14] and (Hyp. S), if 1) $E \in G$, 2) A is a brick-covering of E, then there exists an infinite sequence $\{T_n\}$ of complexes such that

1) $\mathfrak{N}_R(T_n) \to 0$, $\mathfrak{N}_A(T_n) \to 0$,

2) som $T_n \leq A$,

3) $|E, T_n|_\mu \to 0$,

4) for every soma $F \leq E$ there exists for every n a partial complex R_n of T_n, (i. e. $R_n \subseteq T_n$), such that $|F, R_n|_\mu \to 0$.

21.2. - **Proof.** Since the μ-topology is separable, there exists a denumerable set of G,

$$(1) \qquad M_1, M_2, \ldots, M_n, \ldots \qquad \text{as in [21.1].}$$

Let $F \in G$ and $F \leq E$. We can find a subsequence $\{M_{k(n)}\}$, $(n = 1, 2, \ldots)$, of (1) such that $|M_{k(n)}, F|_\mu \to 0$. We have, by [5.15], $|M_{k(\alpha)} \cdot E, F \cdot E| \leq |M_{k(\alpha)}, F| \to 0$; hence $|M_{k(n)} \cdot \cdot E, F| \to 0$, for $n \to \infty$. Thus the set of all $M_n \cdot E$ is everywhere dense in the topology restricted to E.

21.2b. - Having this, let $\eta > 0$ and find a complex P_1 such that $P_1 \leq A$, and where

$$(1.1) \qquad |E, P_1| \leq \eta,$$

[14.2]. Find another complex Q_1 with som $Q_1 \leq A$, [14.3], such that

$$(1.2) \qquad |M_1 \cdot E, Q_1| \leq \eta.$$

Put

(1.3) $\quad B_1 \underset{df}{=} \text{som } P_1 \cdot \text{som } Q_1, \; C_1 \underset{df}{=} \text{som } P_1 - \text{som } Q_1.$

B_1, C_1 are figures. We have

(2) $$B_1 C_1 = 0. \; B_1 + C_1 = \text{som } P_1 \cdot$$

Let R_1, S_1 be complexes such that, [20.3],

(3) $$\text{som } R_1 \leq B_1, \quad \text{som } S_1 \leq C_1,$$

(4) $$| R_1, B_1 | \leq \eta, \qquad | S_1, C_1 | \leq \eta.$$

$\mathfrak{N}_R(R_1) \leq \eta$. $\mathfrak{N}_R(S_1) \leq \eta$, $\mathfrak{N}_A(R_1) \leq \eta$, $\mathfrak{N}_A(S_1) \leq \eta$. The set

(4.1) $$T_1 \underset{df}{=} R_1 \cup S_1$$

is a complex because of (2) and (3). From (4) it follows $| \text{som } R_1 + \text{som } S_1, B_1 + C_1 | \leq 2\eta$, [5.11], i.e. $| T_1, B_1 + C_1 | \leq \leq 2\eta$. Hence, since $B_1 + C_1 = \text{som } P_1$, we have $| T_1, P_1 | \leq 2\eta$, and hence, by (1.1),

(5) $$| E, T_1 | \leq 3\eta.$$

21.2.c - We have (1.2): $| M_1 \cdot E, Q_1 | < \eta$, $| E, P_1 | \leq \eta$. Hence $| M_1 \cdot E \cdot E, \text{som } P_1 \cdot \text{som } Q_1 | \leq | M_1 \cdot E, Q_1 | + | E, P_1 | \leq 2\eta$, [5.12], i.e., $| M_1 \cdot E, \text{som } P_1 \cdot \text{som } Q_1 | \leq 2\eta$, i.e., (1.3),

(5.1) $$| M_1 \cdot E, B_1 | \leq 2\eta.$$

Since, $| M_1 \cdot E, R_1 | \leq | M_1 \cdot E, B_1 | + | R_1, B_1 |$, we get, by (4) and (5.1),

(6) $$| M_1 \cdot E, R_1 | \leq 3\eta.$$

Now R_1 is, by (4.1) a partial complex of T_1.

Thus we have found a complex T_1 such that $\text{som } T_1 \leq A$, $| E, T_1 | \leq 3\eta$, $\mathfrak{N}_A(T) \leq \eta$, $\mathfrak{N}_R(T) \leq \eta$, and found a partial complex R_1 of T_1 such that $| M_1 \cdot E, R_1 | \leq 3\eta$. This can be done for every $\eta > 0$.

21.2d. - Let us consider the somata E, $M_1 E$ and $M_2 E$. We like to find a complex T_2 within A, approximating E

and such that it would contain one partial complex R_2, approximating M_2E and another one approximating M_2E.

First find complexes P_2, Q_{21}, Q_{22}, all within A, and such that

$$(6.1)\quad |P_2,\ E| < \eta,\ |M_1\cdot E,\ Q_{21}| < \eta,\ |M_2\cdot E,\ Q_{22}| < \eta.$$

Let us agree, for the sake of simplicity, to use the same symbol for the complex and its soma. Consider the figures

$$(7)\qquad P_2Q_{21}Q_{22},\ P_2\bar{Q}_{21}Q_{22},\ P_2Q_{21}\bar{Q}_{22},\ P_2\bar{Q}_{21}\bar{Q}_{22}$$

where, in general, X means co X. They are all disjoint figures, which may be also null-figures. We have

$$(7.1)\qquad P_2 = P_2Q_{21}Q_{22} + P_2Q_{21}Q_{22} + P_2Q_{21}Q_{22} + P_2Q_{21}\bar{Q}_{22}$$

and

$$(7.2)\quad P_2Q_{21} = P_2Q_{21}Q_{22} + P_2Q_{21}\bar{Q}_{22},\ P_2Q_{22} = P_2\bar{Q}_{21}Q_{22} + P_2Q_{21}Q_{22}.$$

Find complexes

$$(8)\qquad R_{12},\ R_{\bar{1}2},\ R_{1\bar{2}},\ R_{\bar{1}\bar{2}},$$

contained in the figures $P_2Q_{21}Q_{22}$, $P_2\bar{Q}_{21}Q_{22}$, $P_2Q_{21}\bar{Q}_{22}$, $P_2\bar{Q}_{21}\bar{Q}_{22}$ respectively, such that their atom-net-numbers are $< \eta$, their reduced net-numbers $< \eta$ and that

$$(8.1)\qquad \begin{aligned} &|R_{12},\ P_2Q_{21}Q_{22}| < \eta,\ |R_{\bar{1}2},\ P_2\bar{Q}_{21}Q_{22}| < \eta,\\ &|R_{1\bar{2}},\ P_2Q_{21}\bar{Q}_{22}| < \eta,\ |R_{\bar{1}\bar{2}},\ P_2\bar{Q}_{21}\bar{Q}_{22}| < \eta. \end{aligned}$$

Since the figures (7) are all disjoint, so are the complexes. (8). Hence if we put $T_2 \underset{df}{=} R_{12} \cup R_{\bar{1}2} \cup R_{1\bar{2}} \cup R_{\bar{1}\bar{2}}$, we get a complex, for which all (8) are partial complexes.

21.2e. - Now we have, by (7.1) and [5.11.1],

$$|P_2,\ T_2| \le |P_2Q_{21}Q_{22},\ R_{12}| + |P_2\bar{Q}_{21}Q_{22},\ R_{\bar{1}2}| + |P_2Q_{21}\bar{Q}_{22},\ R_{1\bar{2}}| + \\ + |P_2\bar{Q}_{21}\bar{Q}_{22},\ R_{\bar{1}\bar{2}}| \le 4\eta.$$

Hence, since by (6.1), $|P_2,\ E| < \eta$, we get, [5.7],

$$(9)\qquad |E,\ T_2| \le 5\eta.$$

Now we have, (6.1), $|M_1 \cdot E,\ Q_{21}| < \eta$, $|M_2 \cdot E,\ Q_{22}| < \eta$, $|E,\ P_2| < \eta$.

Hence $|E \cdot M_1E,\ P_2Q_{21}| < 2\eta$ and $|E \cdot M_2E,\ P_2Q_{22}| < 2\eta$. Hence, by (7.2),

$$(10) \quad |M_1E,\ P_2Q_{21}Q_{22} + P_2Q_{21}\bar{Q}_{22}| < 2\eta,\ |M_2E,\ P_2\bar{Q}_{21}Q_{22} + \\ + P_2Q_{21}Q_{22}| < 2\eta.$$

Since, by (8.1),

$$|P_2Q_{21}Q_{22} + P_2Q_{21}\bar{Q}_{22},\ R_{12} \cup R_{1\bar{2}}| < 2\eta,$$

$$|P_2\bar{Q}_{21}Q_{22} + P_2\bar{Q}_{21}\bar{Q}_{22},\ R_{\bar{1}2} \cup R_{\bar{1}\bar{2}}| < 2\eta,$$

we get from (10):

$$(11) \qquad |M_1E,\ R_{12} \cup R_{1}^{-}| < 4\eta,\ |M_2E,\ R_{\bar{1}2} \cup R_{\bar{1}\bar{2}}| < 4\eta.$$

Since $R_{12} \cup R_{1\bar{2}}$ and $R_{\bar{1}2} \cup R_{\bar{1}\bar{2}}$ are partial complexes of T_2, we have found a complex T_2 such that $|E,\ T_2| \leq 4\eta$, the atom-and-reduced-net number of T_2 are $< \eta$, and which contains a partial complex $R_1^{(1)}$ with $|M_1E,\ R_1^{(1)}| < 4\eta$, and which also contains another partial complex $R_2^{(1)}$ with $|M_2E,\ R_2^{(1)}| < 4\eta$. This can be done for every $\eta > 0$.

21.2f. - Using a similar method we can prove that for $\eta > 0$ there exists a complex T_n contained in A, with $|E,\ T_n| \leq (2^n + 1)\eta$, with the atom-and-reduced-net-number $< \eta$. T_n is such that it contains partial complexes: $R_1^{(n)}$, $R_2^{(n)}, \dots, R_n^{(n)}$ such that $|R_i^{(n)},\ M_iE| \leq 2^n\eta$. This is true for every $\eta > 0$. We leave to the reader to develop this argument with details.

21.2g. - Taking $\eta \underset{df}{=} \frac{1}{2^n + 1} \cdot \frac{1}{n}$, we can find for every n a complex T_n such that som $T_n \leq A$, $|T_n,\ E| \leq \frac{1}{n}$ with reduced-and atom-net number $\leq \frac{1}{n}$, and such that it contains partial complexes $R_i^{(n)}$, $(i = 1, 2, \dots, n)$, such that

$$|R_i^{(n)},\ M_iE| \leq \frac{1}{n},\ (i = 1, 2, \dots, n).$$

21.2h. - Having the sequence $\{T_n\}$, let $F \leq E$, and take $\varepsilon > 0$. Find m_0 such that

$$(12) \qquad |F, EM_{m_0}| \leq \frac{\varepsilon}{2}.$$

Take $n > m_0$ with $\frac{1}{n} \leq \frac{\varepsilon}{2}$, $|T_n, E| \leq \frac{1}{n}$. There exists a partial complex R_n of T_n such that $|EM_{m_0}, R_n| \leq \frac{1}{n} \leq \frac{\varepsilon}{2}$. By (12) we get $|F, R_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for sufficiently great n. The theorem is established.

21.3. - **Def.** The sequence $\{T_n\}$ having the properties 1., 3., 4., expressed in [21.2], will be termed *completely distinguished for E.*

The property 4. will be termed *property* (S) *for E.*

21.3.1. - **Corollary.** A slightly modified argument yields the following theorem: Under hypotheses [12], [14] and (Hyp. S), if 1. $E \in G$, 2. $A_1, A_2, \dots, A_n, \dots$ is an infinite sequence of coverings of E, then there exists a completely distinguished sequence $\{T_n\}$ of complexes for E such that som $T_n \leq A_n$.

21.4. - **Remark.** The validity of theor. [21.2] implies the hypothesis (Hyp. S).

21.5. - Considering the item [21.2d], in the proof of [21.2], we had (6.1): $|P_2, E| < \eta$, $|M_1E, Q_{21}| < \eta$ and $|M_2E, Q_{22}| < \eta$ and we have found the complex $T_2 = R_{12} \cup R_{1\bar{2}} \cup R_{\bar{1}2} R_{\bar{1}\bar{2}} \leq$ $\leq$ som P_2 with $|T_2, E| \leq 4\eta$, (in [21.2e]), getting partial complexes $R_{12} \cup R_{1\bar{2}}$ and $R_{\bar{1}2} \cup R_{\bar{1}\bar{2}}$, approximating M_1E, M_2E respectively up to 4η, [21.2e].

A similar remark can be said of the general case where $M_1E, M_2E, \dots, M_nE$ are approximated by subcomplexes $R_1^{(n)}, \dots, R_n^{(n)}$ of a complex T_n, where som $T_n \leq P_n$. We get $|R_i^{(n)}, M_iE| \leq 2^n\eta$. $(i = 1, 2, \dots, n)$ and $|E, T_n| \leq (2^n + 1)\eta$.

Now let $\{Q_n\}$ be a sequence of complexes such that $|Q_n, E| \to 0$. We can find a subsequence $\{Q_{k(n)}\}$ of $\{Q_n\}$ such that $|Q_{k(n)}, E| \leq \frac{1}{n} \cdot \frac{1}{2^n+1}$. Taking the corresponding $\{T_{k(n)}\}$, we have $|T_{k(n)}, E| \to 0$, and in each $T_{k(n)}$ we can find partial complexes $\{R_i^{n(k)}\}$ with $|R_i^{n(k)}, M_i E| \to 0$, $(i = 1, 2, \dots, n(k))$. This allows to state the following:

21.5. - **Corollary.** If $|P_n, E| \to 0$, then there exists a subsequence $k(n)$ of indices, and a sequence of complexes $\{T'_n\}$ such that som $T'_n \leq$ som $P_{k(n)}$, and that $\{T'_n\}$ is completely distinguished with respect to E.

21.6. - If $\{T_n\}$ has the property (S) for $E \in G$, then any subsequence $\{T_{k(n)}\}$ of it also has that property.

21.7. - If 1. $E \cdot F = 0$, 2. $\{P_n\}$ has the property (S) for E, 3. Q_n has the property (S) for F, 4. som $P_n \cdot$ som $Q_n = O$, then $\{P_n \cup Q_n\}$ has the property (S) for $E + F$.

Proof. Let $H \leq E + F$, $H \in G$. We have $H \cdot E \leq E$, $H \cdot F \leq F$. By theorem [21.2] there exist partial complexes P'_n, Q'_n of P_n, Q_n respectively, such that

$$(1) \qquad |H \cdot E, P'_n| \to 0, \qquad |H \cdot E, Q'_n| \to 0.$$

Since som $P'_n \leq$ som P_n, som $Q'_n \leq$ som Q_n, we have som $P'_n \cdot$ $\cdot$ som $Q'_n = O$. Hence $P'_n \cup Q'_n$ is a partial complex of $P_n \cup Q_n$. We have $|P_n, E| \to 0$, $|Q_n, F| \to 0$. Hence $|P_n \cup Q_n, E + F| \to 0$, and from (1) we have $|HE + HF, P'_n \cup Q'_n| \to 0$. i. e., $|H, P'_n \cup Q'_n| \to 0$, which completes the proof.

21.8. - If 1. $E \in G$, 2. The sequence $\{P_n\}$ of complexes has the property (S) with respect to E. 3. $F \leq E$, $F \in G$, 4. $Q_n \subseteq P_n$, $n = 1, 2, \dots$, 5. $|Q_n, F|_\mu \to 0$ for $n \to \infty$, then $\{Q_n\}$ has the property (S) with respect to F.

Proof. Let $G \in G$, $G \leq F$. Since $\{P_n\}$ has the property (S) with respect to E and since $G \leq E$, there exists, for every n, a partial complex S_n of P_n with

$$(1) \qquad |S_n, G|_\mu \to 0 \text{ for } n \to \infty.$$

We have

$$S_n \subseteq P_n, \quad Q_n \subseteq P_n, \quad S_n \frown Q_n \subseteq Q_n, \tag{2}$$

and $\operatorname{som}(S_n \frown Q_n) = \operatorname{som} S_n \cdot \operatorname{som} Q_n$. By (hyp S) and (1) we get $|\operatorname{som} S_n \operatorname{som} Q_n, FG| = |\operatorname{som} S_n \operatorname{som} Q_n, G| \to 0$. Hence

$$| S_n \frown Q_n, G | \to 0. \tag{3}$$

From (2) and (3) the theorem follows.

21.9. - If 1. $\{P_n\}$ is a completely distinguished sequence, for E, 2. $F \leq E$. 3. $Q_n \subseteq P_n$ for all $n=1, 2, \ldots$. 4. $|Q_n, F| \to 0$ then $\{Q_n\}$ is a completely distinguished sequence for F.

Proof. Follows from [21.8].

21.10. - If 1. $E \cdot F = O$, 2. $\{P_n\}$ is a completely distinguished sequence for E. 3. $\{Q_n\}$ is a completely distinguished sequence for F. 4. $\operatorname{som} P_n \cdot \operatorname{som} Q_n = O$, $(n = 1, 2, \ldots)$, then $\{P_n \smile Q_n\}$ is a completely distinguished sequence for $E + F$.

Proof. Follows from [21.7].

21.11. - If 1. $E \in G$. 2. $\{P_n\}$ has the property (S) for E and $|P_n, E| \to 0$ for $n = 1, 2, \ldots$. 3. $\{P'_n\}$ is a complex whose every brick is contained in a brick of $\{P_n\}$. 4. $|P'_n, P_n| \to 0$ for $n = 1, 2, \ldots$, then $\{P'_n\}$ also has the property (S) for E.

Proof. Let $F \leq E$. By hyp. 2, we can find a partial complex Q_n of P_n such that $|F, Q_n| \to 0$ for $n = 1, 2, \ldots$. Denote by Q'_n the maximal partial complex of P'_n whose bricks are contained in the bricks of Q_n. We shall prove that

$$| Q'_n, F | \to 0. \tag{1}$$

Let $P_n = \{p_{n1}, p_{n2}, \ldots\}$. Suppose that k is such that p_{nk} contains at least one brick of P'_n. Denote these bricks by $p'_{nk1}, p'_{nk2}, \ldots$ and their sum by p'_{nk}. If p_{nk} does not contain any brick of P'_n, put $p'_{nk} = O$. We have som P_n — som $P'_n =$ $= \Sigma_k(p_{nk} - p'_{nk})$. Since the somata $p_{nk} - p'_{nk}$, $(k = 1, 2, ...)$ are disjoint, we have $\mu(\text{som } P_n - \text{som } P'_n) = \Sigma_k \mu(p_{nk} - p'_{nk})$. If we confine ourselves to those indices k' only, for which $p_{nk} \in Q_n$ we shall get $\mu(\text{som } Q_n - \text{som } Q'_n) = \Sigma_{k'} \mu(p_{nk'} -$ $- p'_{nk'}) \leq \Sigma_k \mu(p_{nk} - p'_{nk})$. Since $|P_n, P'_n| \to 0$, we get

(2) $$| Q_n, Q'_n | \to 0.$$

From (1) and (2) we deduce that $|F, Q'_n| \to 0$, which completes the proof.

21.12. - If 1. $E \in G$. 2. The sequence $\{P_n\}$ of complexes has the property (S) with respect to E,

then we can find complexes $\{Q_n\}$ such that

1) each brick of Q_n is contained in some brick of P_n,

2) $\{Q_n\}$ is completely distinguished for E.

Proof. We partition every brick p of P_n into a denumerable number of bricks so as to have the reduced net number and the atom net number tending to 0 for $n \to \infty$. Having this we take for each brick p a finite number of meshes of partition, sufficiently great so as to have p approximated with error $< \dfrac{1}{n \cdot k(n)}$, where $k(n)$ denotes the number of bricks in P_n. The theor. [21.8] will complete the proof.

21.13. - If. 1. $E \in G$. 2. $\{P_n\}$ is a completely distinguished sequence for E, then there exists a subsequence $k(n)$ of indices and a sequence $\{Q_{k(n)}\}$ such that

1) som $P_{k(n)} \cdot$ som $Q_{k(n)} = O$,

2) $\{P_{k(n)} \cup Q_{k(n)}\}$ is a completely distinguished sequence for I.

3) $Q_{k(n)}$ is completely distinguished for co E.

Proof. We have $|P_n, E| \to O$. Hence, [5.13],

(1) $$|\text{co som } P_n, \text{co } E| \to 0.$$

The soma co som P_n is a figure, hence a covering. Hence, [4.3], it can be represented as a denumerable sum of mutually disjoint bricks, say $a_{n1}, a_{n2}, \dots$. Hence there exists a finite number of them, say $b_{n1}, b_{n2}, \dots$, such that $|a_{n1} + a_{n2} + \dots, b_{n1} + b_{n2} + \dots| \leq \frac{1}{n}$.

It follows from (1), that $|b_{n1} + b_{n2} + \dots,$ co $E| \to 0$. $R_n \underset{df}{=} \{b_{n1}, b_{n2} \dots\}$ is a complex, for which $|R_n,$ co $E| \to 0$. Hence we can apply [21.5], by virtue of which there exists a subsequence $k(n)$ of indices, and a complex $Q_{k(n)}$ such that

1) som $Q_{k(n)} \leq$ som $R_{k(n)} \leq$ co som $P_{k(n)}$,

2) $\{Q_{k(n)}\}$ is completely distinguished for co E.

Now $\{P_{k(n)}\}$ is, [21.6], completely distinguished for E, $\{Q_{k(n)}\}$ is completely distinguished for co E, som $P_{k(n)} \cdot$ som $Q_{k(n)} = O$, hence $P_{k(n)} \frown Q_{k(n)} = \emptyset$. Hence, [21.10], the sequence $\{P_{k(n)} \smile Q_{k(n)}\}$ is completely distinguished for I. The theorem is established.

21.14. - If 1. $p \neq O$ is a figure. 2. P_n is a completely distinguished sequence for *1*. 3. Q_n is a partial complex of P_n with $|Q_n, p| \to 0$, $Q_n = \{q_{n1}, q_{n2}, \dots\}$, then, by [21.9], $\{Q_n\}$ is completely distinguished for p. 4. Let a_{ni} and e_{ni} be those among q_{nk} for which $a_{ni} \leq p$, and $e_{ni} \cdot p \neq O$, e_{ni} cop $\neq O$ respectively, then $|\Sigma a_{ni} + \Sigma e_{ni} p, p| \to 0$, $\mu(\Sigma e_{ni}$ co $p) \to 0$.

Proof. Since $|Q_n, p| \to 0$, we have $|p \cdot$ som $Q_n, p| \to 0$. Now the bricks of Q_n whose soma of their sum contributes to $p \cdot$ som Q_n are $a_{n1}, a_{n2}, \dots$ and $e_{n1}, e_{n2}, \dots$. Hence $p \cdot$ som $Q_n = \Sigma a_{nk} + p \Sigma e_{nk}$, so the first part of the thesis is proved. Now we have

$$|\Sigma_k a_{nk} + \Sigma_k b_{nk} + \Sigma_k e_{nk}, p| \to 0, \tag{1}$$

where b_{nk} are all bricks in Q_n for which $p \cdot b_{nk} = O$. Hence

$$|(\Sigma_k a_{nk} + \Sigma_k b_{nk} + \Sigma_k e_{nk})\Sigma_k b_{nk},$$

$$p \cdot \Sigma_k b_{nk}| \to 0, \text{ i. e. } |\Sigma_k b_{nk}, O| \to 0. \tag{2}$$

By subtraction of (1) and (2) and by [5.14] we get $|\Sigma_k a_{nk} + + \Sigma_k e_{nk}, p| \to 0$. Hence $|\Sigma_k a_{nk} + \Sigma_k e_{nk} p + \Sigma_k e_{nk} \operatorname{co} p, p| \to 0$. By subtraction we get $|\Sigma_k e_{nk} \operatorname{co} p, O| = 0$, hence $\mu(\Sigma_k e_{nk} \cdot \operatorname{co} p) \to 0$, wich completes the proof.

§ 2. - Vector fields on a tribe and their summation.

1. - We shall consider the tribes F, G and the base B as before under (Hyp. *FBG*), [§ 1; 1]. We also admit the (Hyp. *Ad*), [§ 1; 3]. To simplify arguments we admit that F is a finitely genuine strict subtribe of G, and that μ is an effective (11) denumerably additive measure on G. The tribe G is supposed to be the μ-figure-covering-Lebesgue's extension of F, [§ 1; 9, 9.3.1, 12, (Hyp $L\mu$)]. The hypothesis of separability of the μ-topology on G, [§ 1; 6.1, 21, (Hyp S)] will be especially important.

1.1. - **Hypothesis.** Let V be a F. Riesz-S. Banach-vector space, complete. Its elements $\vec{x}, \vec{y}, \ldots$ will be termed *vectors*. The norm of $\vec{x}$ will be denoted by $\|\vec{x}\|$, (20), (21).

2. - **Def.** By a *V-field on B* we shall understand any function $\vec{\varphi}(a)$ where a varies over B and $\vec{\varphi}(a) \in V$.

2.1. - **Def.** In [§ 1] we have studied infinite sequences $\{P_n\}$ of complexes which have approximated a given soma E of G. They have the following property: $|E, P_n|_\mu \to 0$. We shall call this property *D-property*. We may subject the sequence $\{P_n\}$ to additional conditions, as $\mathfrak{M}_R(P_n) \to 0$, $\mathfrak{M}_A(P_n) \to 0$, [§ 1: 18, 19], called (R), (A)-*properties*, and if (Hyp S) is admitted. the sequence $\{P_n\}$ may have the property 4), expressed in [§ 1; Theor. 21.2, Def. 21.3], called (S)-*property*.

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(11) The hypothesis that F is a strict subtribe of G is a non-essential restriction. It can always be obtained by taking $\mathfrak{A}(F)$ and $\mathfrak{A}(B)$ instead of F and B respectively, (see [§ 1; 1]). A similar remark can be made concerning the effectiveness of measure, (see [§ 1; 14]).

We shall consider various kinds of sequences $\{P_n\}$, but all with (D)-property; we shall call then *distinguished sequences for* E and denote then by (D), (DR), (DA) (DAR), (DS), (DAS), (DRS), $(DARS)$ according to the specific properties admitted. The $(DARS)$-sequences will be termed *completely distinguished*, as in [§ 1; 21.3].

2.1.1. - To each of these kind of approximating sequences there will correspond a notion of summation of vector-fields, to be soon introduced. The existence of (DRA)-distinguished sequences has been proved in [§ 1; 20.3], and under hypothesis (S) of separability, the existence of $(DRAS)$-distinguished sequences has been proved in [§ 1; 21.1]. If we shall speak in general of a distinguished sequence without specifying its character, we shall say simply « *distinguished* » (D'-*sequence*).

2.1.2. - **Remark.** We do not know whether (DS) does imply $(DARS)$, or not. At present we do not need to be interested in this question.

2.2. - **Def.** Let $\vec{\varphi}$ be a V-vector field on B and $E \in G$. We say that $\vec{\varphi}$ *is summable on* E *with respect to the given kind* (D') *of distinguished sequences,* whenever for every D'-distinguished sequence $\{P_n\} = \{p_{n1}, p_{n2}, \dots\}$ for E the sum $\vec{\varphi}(P_n) \underset{df}{=} \sum\limits_i \vec{\varphi}(p_{ni})$ converges for $n \to \infty$ in the topology of V. We call this limit « *sum of the field* $\vec{\varphi}$ *on* E (*with respect to* D') » and denote it by $\mathbf{S}_E \vec{\varphi}$ or $\mathbf{S}_{E,(D')} \vec{\varphi}$.

2.3. - **Remark.** If instead of G we consider the tribe $f \upharpoonright G$ restricted to a given figure, and suppose that $E \leq f$, the notion of summability in E may change. Thus the notion depends on the totality of the vector-field.

3. - We shall be mainly interested in distinguished sequences $(DARS)$, so the theorems which follow will concern that case. Changes in statements will be given in remarks.

The sums introduced above constitute, some way, a generalization of Weierstrass-Burkill integrals (17). They are more general than these in (14).

3. - **Theorem.** Under hypotheses [§ 1; 12, 14] and [1.1], if 1. $E \in G$, 2. $\vec{\varphi}$ is a V-vector field in B, 3. $\vec{\varphi}$ is (DRA)-summable in E, 4. $\eta > 0$, then there exists $\delta > 0$ such that if $\mathfrak{M}_A(P) \leq \delta$, $\mathfrak{M}_R(P) \leq \delta$, $|E, P|_\mu \leq \delta$, where P is a complex, then

$$\|\vec{\varphi}(P) - \mathrm{S}_{E,(DRA)}\vec{\varphi}\| < \eta.$$

Proof. Suppose the theorem not true. For every $\delta > 0$ there exists a complex P such that $\mathfrak{M}_R(P) \leq \delta$, $\mathfrak{M}_A(P) \leq \delta$ and $|E, P|_\mu \leq \delta$, but nevertheless

$$\|\vec{\varphi}(P) - \mathrm{S}_E\vec{\varphi}\| \geq \eta.$$

Take $\delta = \frac{1}{n}$, $(n = 1, 2, \ldots)$, and find $P_1, P_2, \ldots, P_n, \ldots$ according to the above. We have

$$\text{(1)} \qquad \|\vec{\varphi}(P_n) - \mathrm{S}_E\vec{\varphi}\| \geq \frac{1}{n}, \ |E, P_n|_\mu \leq \frac{1}{n},$$

$$\mathfrak{M}_R(P) \leq \frac{1}{n}, \ \mathfrak{M}_A(P) \leq \frac{1}{n}.$$

The sequence $\{P_n\}$ is (DRA)-distinguished for E. Hence, by Hyp. 3,

$$\lim_{n \to \infty} \vec{\varphi}(P_n) = \mathrm{S}_E\vec{\varphi}.$$

This, however, contradicts (1). The theorem is proved.

3.1. - **Remark.** A similar theorem holds for the following kinds of distinguished sequences (D), (DA), (DR), but the above proof cannot be used for the sequences (DS), (DAS), (DRS), $(DARS)$.

4. - **Theor.** Let us admit hypotheses [§ 1, 12, 14], (Hyp. S) and [1.1]. We shall consider $(DARS)$-summations.

If $E \in G$, 2. $F \leq E$, $F \in G$, 3. $\mathrm{S}_{E,(DARS)} \overrightarrow{\varphi}$ exists, then $\mathrm{S}_{F,(DARS)} \overrightarrow{\varphi}$ also exists.

Proof. Suppose that $\mathrm{S}_F \overrightarrow{\varphi}$ does not exist. There exists a completely distinguished sequence for F, [2.1],

$$Q_1, Q_2, \ldots, Q_n, \ldots \tag{1}$$

such that $\varphi(Q_n)$ does not tend to any limit. Hence, since V is complete, there exists $\eta > 0$ and a subsequence

$$Q'_1, Q''_1, Q'_2, Q''_2, \ldots, Q'_n, Q''_n, \ldots \tag{2}$$

of (1) such that

$$\|\overrightarrow{\varphi}(Q'_n) - \overrightarrow{\varphi}(Q''_n)\| \geq \eta. \tag{3}$$

(2) is completely distinguished for F, [§ 1; 21.6]. We have

$$|Q'_n, F| \to 0, \ |Q''_n, F| \to 0. \tag{3.1}$$

Hence by [§ 1, 5.16]

$$|\operatorname{som} Q'_n + \operatorname{som} Q''_n, F| \to 0. \tag{4}$$

The soma

$$\operatorname{som} Q'_n + \operatorname{som} Q''_n \tag{4.0}$$

is a figure.

Consider any sequence $P_1, P_2, \ldots, P_n, \ldots$ completely distinguished for E with

$$\|\overrightarrow{\varphi}(P_n) - \mathrm{S}_E \overrightarrow{\varphi}\| \to 0. \tag{5}$$

We have

$$|E, P_n| \to 0. \tag{6}$$

From (6) and (4), by virtue of [§ 1, 5.14],

$$|\operatorname{som} P_n - (\operatorname{som} Q'_n + \operatorname{som} Q''_n), E - F| \to 0.$$

Put

$$A_n = \text{som}\, P_n - (\text{som}\, Q'_n + \text{som}\, Q''_n).$$

The soma A_n is a figure. We have

$$(7) \qquad | A_n ,\ E - F | \to 0.$$

By [§ 1, Theor. 21.5] there exists a subsequence $k(n)$ of indices and there exists a completely distinguished sequence $\{ s_n \}$ for $E - F$ such that

$$\text{som}\, s_n \cdot \text{som}\, Q'_{k(n)} = 0, \quad \text{som}\, s_n \cdot \text{som}\, Q''_{k(n)} = 0.$$

Hence $s_n \cup Q'_{k(n)}$, $s_n \cup Q''_{k(n)}$ are complexes. Since $\{ s_n \}$ is completely distinguished for $E - F$, and since $Q'_{k(n)}$, $Q''_{k(n)}$ are both completely distinguished for F, it follows, [§ 1, 21.7], that $\{ s_n \cup Q'_{k(n)} \}$ and $\{ s_n \cup Q''_{k(n)} \}$ are both completely distinguished for E. Hence

$$\overrightarrow{\varphi}(s_n) + \overrightarrow{\varphi}(Q'_{k(n)}) \to \mathbf{S}_E \overrightarrow{\varphi}, \quad \overrightarrow{\varphi}(s_n) + \overrightarrow{\varphi}(Q''_{k(n)}) \to \mathbf{S}_E \overrightarrow{\varphi};$$

hence $\| \overrightarrow{\varphi}(Q'_{k(n)}) - \overrightarrow{\varphi}(Q''_{k(n)}) \| \to 0$, which contradicts (3).

4.1. - **Corollaries.** The theorem [4] holds true for any of the summations with character (DS), (DAS), (DNS), and the proof is similar. Denoting these categories by I, II, III respectively, we make the following changes in the proof of [4], respectively. Instead of the completely distinguished sequence $\{ Q_n \}$ in (1) we suppose its character to be (D) in I, (DAS) in II, (DNS) in III. The sequences $\{ Q'_n \}$, $\{ Q''_n \}$ have the same character respectively. The sequence $\{ P_n \}$ will be supposed to be (DS), (DAS), (DNS) respectively. The remaining arguments will be not chanched.

The theorem [4] is also true for the summations of character (D), (DA), (DN), (DAN). Denote then by I′, II′, III′, IV′ respectively. We take $\{ Q_n \}$ of the given character. The character of $\{ Q'_n \}$, $\{ Q''_n \}$ will be the same. Having obtained the relation (7), we shall not need to select a subsequence $k(n)$, but we shall stay with A_n. We shall choose s_n with som $s_n \leq A_n$ and with properties I′, II′, III′, IV′ respectively. The sequences $\{ s_n \cup Q'_n \}$, $\{ s_n \cup Q''_n \}$ will be the required distinguished sequences, yielding the final contradiction. Thus we can state the following theorem:

4.1a. - **Theorem.** Admitting the hypotheses [§ 1, 12, 14] and (Hyp S), if needed, take any character D' of summation. If $F \leq E$ and $\mathbf{S}_{E,(D')}\vec{\varphi}$ exists, then $\mathbf{S}_{F,(D')}\vec{\varphi}$ exists too.

5. - **Lemma.** If 1. $\vec{\varphi}, \vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_n \ldots$ are vectors of V, 2. from every sequence $\{\vec{\varphi}_{k(n)}\}$ another subsequence $\{\vec{\varphi}_{kl(n)}\}$ can be extracted, such that $\lim \vec{\varphi}_{kl(n)} = \vec{\varphi}$, then $\lim \vec{\varphi}_n = \vec{\varphi}$.

Proof. Suppose that $\{\vec{\varphi}_n\}$ does not converge. Then, since V is complete, there exists $\eta > 0$ and subsequences $\{\vec{\varphi}_{s(n)}\}$, $\{\vec{\varphi}_{t(n)}\}$ such that

$$1)\ t(n-1) < s(n) < t(n) < s(n+1),\ (n \geq 2),$$

$$2)\ \|\vec{\varphi}_{s(n)} - \vec{\varphi}_{t(n)}\| \geq \eta,\ (n = 1, 2, \ldots). \tag{1}$$

Extract from $\{\vec{\varphi}_{s(n)}\}$ a subsequence $\{\vec{\varphi}_{s(s'n)}\}$ with

$$\lim \vec{\varphi}_{s(s'n)} = \vec{\varphi}. \tag{2}$$

We get from (1)

$$\|\vec{\varphi}_{ss'(n)} - \vec{\varphi}_{ts'(n)}\| \geq \eta \text{ for } n = 1, 2, \ldots . \tag{3}$$

Now from $\{\vec{\varphi}_{ts'(n)}\}$ another subsequence $\{\vec{\varphi}_{ts't'(n)}\}$ can be extracted with

$$\lim \vec{\varphi}_{ts't'(n)} = \vec{\varphi}. \tag{4}$$

From (3) we get

$$\|\vec{\varphi}_{ss't'(n)} - \vec{\varphi}_{ts't'(n)}\| \geq \eta. \tag{5}$$

Since, by (2), $\vec{\varphi}_{ss't'(n)} \rightarrow \vec{\varphi}$, we get from (4): $0 \geq \eta$ which is a contradiction.

Hence $\vec{\varphi}_n$ converges. Put $\vec{\psi} \underset{df}{=} \lim \vec{\varphi}_n$. There exists a partial sequence tending to $\vec{\varphi}$. Hence $\vec{\psi} = \vec{\varphi}$.

5.1. - **Theorem.** Admit the hypotheses [§ 1; 12, 14], [1.1] and (Hyp S). We shall consider $(DARS)$-sums. Let 1. $E, E_1, E_2, \ldots, E_n, \ldots \in G$, 2. $E_n \leq E$ for $n = 1, 2, \ldots$, 3. $\mu(E_n) \rightarrow 0$,

4. $\vec{\varphi}$ is $(DRAS)$-summable on E,

$$\text{then } \mathrm{S}_{E_n,(DRAS)}\vec{\varphi} \to \vec{O}.$$

Proof. Let $\{A_n\}$ be a sequence of complexes with

$$|A_n, E| \to 0. \tag{0}$$

By [Theor. 4], φ is summable on E_n. Find, for every $n = 1, 2, \ldots$ a complex P_n such that

$$\|\vec{\varphi}(P_n) - \mathrm{S}_{E_n}\vec{\varphi}\| \leq \frac{1}{n}, \tag{0.1}$$

$$\mathfrak{M}_A(P_n) \to 0,\ \mathfrak{M}_R(P_n) \to 0,\ |E_n| \leq \frac{1}{n}.$$

Let $\{P_{k(n)}\}$ be a partial sequence of $\{P_n\}$. We have

$$\|\vec{\varphi}(P_{k(n)} - \mathrm{S}_{E_{k(n)}}\vec{\varphi}\| \leq \frac{1}{k(n)} \leq \frac{1}{n}, \tag{1}$$

$$|E_{k(n)}, P_{k(n)}| \leq \frac{1}{n},\ \mathfrak{M}_A(P_{k(n)}) \to 0,\ \mathfrak{M}_R(P_{k(n)}) \to 0. \tag{2}$$

Consider the figure

$$Q_{k(n)} \underset{df}{=} A_{k(n)} - \operatorname{som} P_{k(n)}. \tag{3}$$

Since $\mu(E_{k(n)}) \to 0$, we have $|E_{k(n)}, O| \to 0$, hence, by (2),

$$|P_{k(n)}, O| \to 0. \tag{4}$$

It follows, [§ 1; 5.14], from (0) and (4): $|A_{k(n)} - \operatorname{som} P_{k(n)}, E - O| \to 0$, i. e.

$$|Q_{k(n)}, E| \to 0. \tag{5}$$

By [§ 1; 21.5] there exists a sequence $l(n)$ of indices 1, 2, ... and a complex $T_n \leq Q_{kl(n)}$ such that $\{T_n\}$ has the property (S) with respect to E, and in addition to that;

$$\mathfrak{M}_R(T_n) \to 0,\ \mathfrak{M}_A(T_n) \to 0. \tag{6}$$

It follows that

$$\vec{\varphi}(T_n) \to \mathrm{S}_E\vec{\varphi}. \tag{6.1}$$

Since $Q_{kl(n)}$ and som $P_{kl(n)}$ are disjoint, it follows that som T_n and som $P_{kl(n)}$ are disjoint.

$$(7) \qquad R_n \underset{df}{=} T_n \cup P_{kl(n)}$$

is a complex. By (2) and (6), we have

$$(8) \qquad \mathfrak{N}_A(R_n) \to 0, \ \mathfrak{N}_B(R_n) \to 0.$$

Since $|E, T_n| \to 0$ and, by (4), $|O, P_{kl(n)}| \to 0$, it follows [§ 1; 21.7], $|E, T_n \cup P_{kl(n)}| \to 0$ i. e.

$$(9) \qquad |E, R_n| \to 0.$$

T_n has the property that for every $F \in G$, $F \leq E$ there exists a partial complex T'_n, $(n = 1, 2, ...)$ with $|T'_n, F| \to 0$. Now T'_n is also a partial complex of $R_n = T_n \cup P_{kl(n)}$. Hence, by (8), (9), $\{R_n\}$ has the property (S) with respect to E. It follows $\vec{\varphi}(R_n) \to \mathrm{S}_E \vec{\varphi}$, i. e.

$$(10) \qquad \vec{\varphi}(T_n) + \vec{\varphi}(P_{kl(n)}) \to \mathrm{S}_E \vec{\varphi}.$$

Hence, by (6.1) $\vec{\varphi}(P_{kl(n)}) \to \vec{O}$. If we take account of (0.1) we get

$$(11) \qquad \mathrm{S}_{E_{kl(n)}} \vec{\varphi} \to \vec{O}.$$

Thus we have proved that from every increasing sequence $k(n)$ of natural number another sequence $kl(n)$ can be extracted so as to have (11). It follows, [Lemma 5], that $\mathrm{S}_{E_n} \vec{\varphi} \to \vec{O}$.

5.2. - Corollaries. In [Theor. 5.1] we have considered $(DARS)$-sums. Now the theorem holds true for any one of the following summations: (DS), (DAS), (DRS). The proof is almost the same. In these three cases, denoted by I, II, III, respectively, we shall drop $\mathfrak{N}_A(P_n) \to 0$, $\mathfrak{N}_B(P_n) \to 0$ in I, we shall drop $\mathfrak{N}_B(P_n) \to 0$ in II and $\mathfrak{N}_A(P_n) \to 0$ in III. The theorem holds also true for summations I′, II′, III′, IV′ of the character (D), (DA), (DR) and (DAR) respectively. The proof is even simpler. We omit the conditions $\mathfrak{N}_A(P_n) \to 0$, $\mathfrak{N}_B(P_n) \to 0$ in I′, omit $\mathfrak{N}_B(P_n) \to 0$ in II′, omit $\mathfrak{N}_A(P_n) \to 0$ in III′. We do not choose any partial sequence $\{P_{k(n)}\}$, but stay with $\{P_n\}$. So we get, instead of (5), $|Q_n, E| \to 0$. Now, instead of (6) we find $T_n \leq Q_n$, with $|T_n, E| \to 0$, som $T_n \leq Q_n$, $\mathfrak{M}_B(T_n) \to 0$, $\mathfrak{M}_A(T_n) \to 0$. Thus we get $\vec{\varphi}(T_n) \to \mathrm{S}_E \vec{\varphi}$, and putting $R_n \underset{df}{=} T_n \cup P_n$, we get $\vec{\varphi}(T_n) + \vec{\varphi}(P_n) \to \mathrm{S}_E \vec{\varphi}$, and then $\vec{\varphi}(P_n) \to O$, so the proof can be completed. Thus we can state the general theorem:

5.3. - **Corr.** Admitting hypotheses [1.1], [§ 1; 12, 14] and (Hyp S), if needed, consider sums of any kind (D'). If 1., 2. as before and 3'., $\vec{\varphi}$ is (D')-summable, then

$$\mathrm{S}_{E_n,(D')} \vec{\varphi} \to \vec{O}.$$

6. - **Theorem.** Take the hypotheses [1.1], [§ 1: 12, 14], and (Hyp S) if needed. Suppose that $E \in G$. Let (D') be any kind of summation. Suppose that $\mathrm{S}_{E,(D')} \vec{\varphi}$ exists, then

$$\mathrm{S}_{O,(D')} \vec{\varphi} = \vec{O}.$$

Proof. We apply the [Theor. 5.1], taking $E_n = O$. We get $\mathrm{S}_{O,(D')} \vec{\varphi} \to \vec{O}$. Hence, since this is a constant sequence, we have

$$\mathrm{S}_{O,(D')} \vec{\varphi} = \vec{O}.$$

6.1. - **Theorem.** If $\mathrm{S}_{O,(D')} \vec{\varphi}$ exists, then $\mathrm{S}_{O,(D')} \vec{\varphi} = \vec{O}$.

Proof. This follows from [6], because $O \leq O$.

6.2. - **Theorem.** If $\mathrm{S}_{O,(D')} \vec{\varphi}$ exists, then $\vec{\varphi}(O) = \vec{O}$.

Proof. The sequence $\{O\}, \{O\}, \{O\}, \ldots$ is a sequence of complexes each of which being composed of the single brick O. This sequence is distinguished of any character considered. Let $\vec{\varphi}_0 \underset{df}{=} \vec{\varphi}(0)$. We have $\lim \vec{\varphi}(\{O\}) = \vec{\varphi}_0$. Since $\mathrm{S}_{O,(D')} \vec{\varphi}$ exists, we have $\mathrm{S}_{O,(D')} \vec{\varphi} = \vec{\varphi}_0$. Hence, by [6.1], $\vec{\varphi}_0 = \vec{O}$, which proves the theorem.

6.3. - **Theorem.** If $\mathrm{S}_{E,(D')} \vec{\varphi}$ exists for some E, then $\vec{\varphi}(O) = \vec{O}$.

Proof. By [4], the sum $\mathbf{S}_{O,(D')}\vec{\varphi}$ exists; hence, by [6.2] $\vec{\varphi}(O) = \vec{O}$.

7. - **Theorem.** Consider the hypotheses [1.1], [§ 1; 12, 14 and (Hyp S). We shall consider $(DARS)$-summation.

If 1. $E_1, E_2 \in G$, 2. $E_1 \cdot E_2 = O$, 3. $\mathbf{S}_{E_1+E_2,(DARS)}\vec{\varphi}$ exists, then

$$\mathbf{S}_{E_1+E_2,(DARS)}\vec{\varphi} = \mathbf{S}_{E_1,(DARS)}\vec{\varphi} + \mathbf{S}_{E_2,(DARS)}\vec{\varphi}.$$

Proof. By [4], the sums $\mathbf{S}_{E_1}\vec{\varphi}$, $\mathbf{S}_{E_2}\vec{\varphi}$ exist. By [§ 1; 13.2] we can find coverings $L_{11} \geq L_{12} \geq L_{13} \geq \dots \geq L_{1n} \geq \dots$ of E such that $E_1 \leq L_{1n}$, $(n = 1, 2, \dots)$ and

$$\lim \mu(L_{1n}) = \mu(E_1). \tag{1}$$

Similarly we can find coverings $L_{21} \geq$, L_{22}, $\geq \dots \geq L_{2n} \geq \dots$ of E_2 with $E_2 \leq L_{2n}$, $(n = 1, 2, \dots)$ and

$$\lim \mu(L_{2n}) = \mu(E_2). \tag{2}$$

Since $E_1 \cdot E_2 = O$, we have, [§ 1; 13.3],

$$\mu(L_{1n} \cdot L_{2n}) \to 0. \tag{3}$$

Find complexes P'_1. $P'_2, \dots, P'_n, \dots$ such that

$$\text{som}\ P'_n \leq L_{1n},\ |P'_n, L_{1n}| \to 0, \tag{4}$$

and complexes $P''_1, P''_2, \dots, P''_n, \dots$ such that

$$\text{som}\ P''_n \leq L_{2n},\ |P''_n, L_{2n}| \to 0. \tag{5}$$

Since $\text{som}\ P'_n \leq L_{1n}$, $\text{som}\ P''_n \leq L_{2n}$, we have, by (3),

$$\lim \mu(\text{som}\ P'_n \cdot \text{som}\ P''_n) \to 0. \tag{6}$$

Consider the figures

$$Q'_n \underset{df}{=} \text{som}\ P'_n - \text{som}\ P''_n,\quad Q''_n \underset{df}{=} \text{som}\ P''_n - \text{som}\ P'_n. \tag{7}$$

They are disjoint.

By (4) we have $|P'_n, L_{1n}| \to 0$, and by (1), $|L_{1n}, E_1| \to 0$; hence $|P'_n, E_1| \to 0$. Since, by (6), $|\text{som}\, P'_n \cdot \text{som}\, P''_n, O| \to 0$, we get, by [§ 1; 5.14], $|\text{som}\, P'_n - \text{som}\, P'_n \cdot \text{som}\, P''_n, E_1 - O| \to 0$, i.e.

$$(8) \qquad |Q'_n, E_1| \to 0.$$

In the same way we obtain

$$(9) \qquad |Q''_n, E_2| \to 0.$$

Having this, apply theor. [§ 1; 21.5], getting a subsequence $k(n)$ of indices and a completely distinguished sequence $\{R'_{k(n)}\}$ for E_1, such that

$$(9.1) \qquad \text{som}\ R'_{k(n)} \leq Q'_{k(n)}.$$

Since, by (9),

$$(10) \qquad |Q''_{k(n)}, E_2| \to 0,$$

we get, by the same [§ 1; 21.5], a subsequence $\{kl(n)\}$ of the indices $\{(kn)\}$, and a completely distinguished sequence $\{R''_{kl(n)}\}$ for E_2, such that

$$(11) \qquad R''_{kl(n)} \leq Q''_{kl(n)}.$$

Since som Q'_n, som Q''_n are disjoint, so are som $Q'_{kl(n)}$, som $Q''_{kl(n)}$, and then, by (9.1) and (11), so are $R'_{kl(n)}$, $R''_{kl(n)}$. By [§ 1; 21.6], $R'_{kl(n)}$ is a completely distinguished sequence for E_1, and $R''_{kl(n)}$ is a completely distinguished sequence for E_2. Hence, by [§ 1; 21.10], $\{R_n\} \underset{df}{=} \{R'_{kl(n)} \cup R''_{kl(n)}\}$ is a completely distinguished sequence for $E_1 + E_2$. By hyp. 3, $\lim R_n(\overrightarrow{\varphi})$ exists and equals $\mathbf{S}_{E_1+E_2} \overrightarrow{\varphi}$. Since $\mathbf{S}_{E_1} \overrightarrow{\varphi}$, $\mathbf{S}_{E_2} \overrightarrow{\varphi}$ exist, and since $\{R'_{kl(n)}\}$ and $\{R''_{kl(n)}\}$ are completely distinguished sequences for E_1, E_2 respectively, we have

$$\lim R'_{kl(n)} = \mathbf{S}_{E_1} \overrightarrow{\varphi}, \quad \lim R''_{kl(n)} = \mathbf{S}_{E_2} \overrightarrow{\varphi}.$$

Since

$$\lim R_n(\overrightarrow{\varphi}) = \lim R'_{kl(n)}(\overrightarrow{\varphi}) + \lim R''_{kl(n)}(\overrightarrow{\varphi}),$$

it follows that

$$\mathbf{S}_{E_1+E_2}\overrightarrow{\varphi} = \mathbf{S}_{E_1}\overrightarrow{\varphi} + \mathbf{S}_{E_2}\overrightarrow{\varphi}. \qquad \text{Q. E. D.}$$

7.1. - The above theorem is valid for the following kinds of summation (DS), (DAS), (DRS). The proof is almost the same. Indeed, instead of taking completely distinguished sequences $R'_{kl(n)}$, $R''_{kl(n)}$ we take only those of the characters, (DS), (DAS), (DRS) respectively

The theorem also holds true for any of the following kinds of summations:

$$(1) \qquad (D), (DA), (DN), (DAN)$$

The proof will be even simpler. Having obtained $\{Q'_n\}$ and $\{Q''_n\}$ and the relations (7) and (8): $|E_1, Q'_n| \to 0$, $|E_2, Q''_n| \to 0$, we shall not need to consider subsequences, but we shall find R'_n, R''_n so as to satisfy the corresponding condition (1) with som $R'_n \leq Q'_n$, som $R''_n \leq Q''_n$. Thus we can state the

7.1. - Coroll. Under hypotheses [1.1], [§ 1; 12, 14] and (Hyp S) if needed, consider any kind (D') of summation.

If $E_1 \cdot E_2 = O$, $\mathbf{S}_{E_1\dot{+}E_2,(D)}\overrightarrow{\varphi}$ exists, then

$$\mathbf{S}_{E_1\dot{+}E_2,(D')}\overrightarrow{\varphi} = \mathbf{S}_{E_1,(D')}\overrightarrow{\varphi} + \mathbf{S}_{E_2,(D')}\overrightarrow{\varphi}.$$

8. - Theorem. Let us admit the hypotheses [1.1], [1; 12, 14]. (Hyp S). We shall consider $(DARS)$-summations.

If 1. $E, F \in G$, 2. $E \cdot F = O$, 3. $\overrightarrow{\varphi}$ is $(DARS)$-summable on E and on F, then $\overrightarrow{\varphi}$ is $(DARS)$-summable on $E+F$. Hence, by [7]: $\mathbf{S}_{E+F}\overrightarrow{\varphi} = \mathbf{S}_E\overrightarrow{\varphi} + \mathbf{S}_F\overrightarrow{\varphi}$.

Proof. Consider a completely distinguished sequence $\{P_n\}$ of complexes for $E+F$. There exists a partial complex R_n of P_n, $(n = 1, 2, \ldots)$, such that $|R_n, E|_\mu \to 0$. Let $P_n = R_n \cup S_n$ where $R_n \cap S_n = \Theta$. We have $|P_n, E+F| \to 0$ and $|R_n, E| \to 0$. Hence, [§ 1; 5.14],

$$|P_n - R_n, (E+F) - E| \to 0 \text{ i.e. } |S_n, F| \to 0.$$

By virtue of [§ 1: 21.9], $\{R_n\}$ is a completely distinguished sequence for E and $\{S_n\}$ a completely distinguished sequence for F. By hyp. 3 we have

$$(1) \qquad \lim \vec{\varphi}(R_n) = \mathbf{S}_E \vec{\varphi}, \quad \lim \vec{\varphi}(S_n) = \mathbf{S}_F \vec{\varphi}.$$

Since $P_n = R_n \cup S_n$, we have $\vec{\varphi}(P_n) = \vec{\varphi}(R_n) + \vec{\varphi}(S_n)$.

It follows, from (1), that $\lim \vec{\varphi}(P_n)$ exists. Since the completely distinguished sequence $\{P_n\}$ was any one, the field $\vec{\varphi}$ is summable on $E + F$. Applying [7] we get the thesis.

8.1. - Corollaries. The theorem is valid for summation of the character (DAS), (DNS) and (DS). For proof it is sufficient to drop, in the forgoing proof, the condition $\mathfrak{N}_R(P_n) \to 0$, $\mathfrak{N}_A(P_n) \to 0$, or both respectively.

8.2. - Remark. The theorem 7 is not true for summations (D), (DA), (DR), (DAR), even if we admit (Hyp$S$). The following example shows it:

Let $\boldsymbol{B}$ be composed of all half-open rectangles

$$(x, y) \left\{ \begin{matrix} a \leq x < b \\ c \leq x < d \end{matrix} \right\} \quad \text{where} \quad 0 \leq a.\ b,\ c,\ d \leq 1.$$

$\boldsymbol{F}$ is defined as the finite union of those rectangles and $\boldsymbol{G}$ as the tribe of all Lebesgue's measurable subsets of the square $I \overline{\overline{\scriptstyle df}} \{(x, y) \mid 0 \leq x < 1,\ 0 \leq y < 1\}$, μ will be the Lebesgue's measure. Define

$$E \overline{\overline{\scriptstyle df}} \left\{ (x, y) \mid 0 \leq x < 1,\ 0 \leq y < \frac{1}{2} \right\},$$

$$F \overline{\overline{\scriptstyle df}} \left\{ (x, y) \mid 0 \leq x < 1,\ \frac{1}{2} \leq y < 1 \right\}.$$

We have $E \cdot F = \emptyset$, $E + F = I$. Let $\vec{\varphi}_0 \in V$, $\vec{\varphi}_0 \neq \vec{O}$. If $a = \{(x, y) \mid \alpha \leq x < \beta,\ 0 \leq y < 1\}$, when $0 \leq \alpha < \beta \leq 1$, put $\vec{\varphi}(a) = \vec{\varphi}_0$, and for all other bricks b put $\vec{\varphi}(b) = \vec{O}$. The sums $\mathbf{S}_E \vec{\varphi} = \mathbf{S}_F \vec{\varphi} = \vec{O}$ exist, but $\mathbf{S}_I \vec{\varphi}$ does not exist if we do not use the (S)-property for complexes, yielding the sum.

9. - **Theorem.** If 1. $E \in G$, 2. $\mathrm{S}_{E,(D')}\vec{\varphi}$ exists, 3. $\{P_n\}$ is a sequence of complexes with $\mu(P_n) \to 0$ for $n \to \infty$, then $\vec{\varphi}(P_n) \to \vec{O}$.

Proof. $\{P_n\}$ is distinguished of every kind for O. Since $\mathrm{S}_{E,(D')}\vec{\varphi}$ exists, therefore, [6], $\mathrm{S}_{O,(D')}\vec{\varphi} = \vec{O}$. We have $\mathrm{S}_{O,(D')}\vec{\varphi} = \lim \vec{\varphi}(P_n)$; hence, $\lim \vec{\varphi}(P_n) = \vec{O}$.

9.1. - **Theorem.** If 1. $E \in G$, 2. $\mathrm{S}_{E,(D')}\vec{\varphi}$ exists, 3. $\alpha > 0$, then there exists $\beta > 0$ such that if P is a complex with $\mu P \leq \beta$, then $\|\vec{\varphi}(P)\| < \alpha$.

Proof. Suppose that the thesis is not true. Then for every $\beta > 0$ we can find a complex P such that $\mu P \leq \beta$ and

$$\|\vec{\varphi}(P)\| \geq \alpha. \tag{1}$$

Putting $\beta = \frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ and finding the corresponding P_n, we have $\mu(P_n) \to 0$: hence, by [9], $\vec{\varphi}(P_n) \to \vec{O}$ which contradicts (1).

9.2. - If 1. $E \in G$, 2. $\mathrm{S}_{E,(D')}\vec{\varphi}$ exists, 3. $\alpha > 0$, then there exists $\beta > 0$ such that, if $\mu(F) \leq \beta$, $F \in G$, then $\|\mathrm{S}_{F,(D')}\vec{\varphi}\| \leq \beta$.

Proof. Similar to that of [9.1], through contradiction with [5.1].

10. - **Theorem.** Under hypotheses [1.1], [§ 1; 12, 14] and (Hyp S), if needed, we have: If 1. $E_1, E_2, \ldots, E_n, \ldots, F \in G$, 2. $\mathrm{S}_{F,(D')}\vec{\varphi}$ exists, where (D') is any fixed of summation, 3. $E \underset{df}{=} E_1 + E_2 + \ldots + E_n + \ldots$, 4. $E \leq F$, 5. $E_1, E_2, \ldots$ are disjoint, then

$$\mathrm{S}_{E,(D')}\vec{\varphi} = \sum_{n=1}^{\infty} \mathrm{S}_{E_n,(D')}\vec{\varphi},$$

where Σ is understood as convergent in the V-topology.

Proof. By hyp. 2 and by virtue of [4.1a], the sums $\mathrm{S}_E\vec{\varphi}$ $\mathrm{S}_{E_n}\vec{\varphi}$ exist, $(n = 1, 2, \ldots)$. Put $A_n \underset{df}{=} E - (E_1 + \ldots + E_n)$. We have $E = E_1 + \ldots + E_n + A_n$. By [4.1a] the sum $\mathrm{S}_A\vec{\varphi}$

exists. By [7.1],

$$(1) \qquad \mathrm{S}_E \overrightarrow{\varphi} = \mathrm{S}_{E_1} \overrightarrow{\varphi} + \dots + \mathrm{S}_{E_n} \overrightarrow{\varphi} + \mathrm{S}_{A_n} \overrightarrow{\varphi}.$$

Since G is denumerably additive, we have $\mu A_n \to 0$; hence, by [9.2], $\| \mathrm{S}_{A_n} \overrightarrow{\varphi} \| \to 0$. Consequently $\| \mathrm{S}_E \overrightarrow{\varphi} - (\mathrm{S}_{E_1} \overrightarrow{\varphi} + + \dots + \mathrm{S}_{E_n} \overrightarrow{\varphi}) \| \to 0$, i. e.

$$\mathrm{S}_E \overrightarrow{\varphi} = \sum_{n=1} \mathrm{S}_{E_n} \overrightarrow{\varphi}. \qquad \text{Q. E. D.}$$

10.1. - Theorem. Under the same hypotheses as in [10], if 1. $E \in G$ and $\mathrm{S}_{E,(D')} \overrightarrow{\varphi}$ exists, 2. we put for all $F \leq E$, $F \in G$, $\overrightarrow{K}(F) \underset{df}{=} \mathrm{S}_F \overrightarrow{\varphi}$, then $\overrightarrow{K}(F)$ with variable $F \leq E$ is denumarably additive. Hence $\overrightarrow{K}$ is a kind of vector valued measure, (see (6)).

Proof. Follows from [10].

11. - Remark. We can prove the following: Under hypotheses [1.1], [§ 1; 12, 14] and (Hyp S) consider any of the summations (DS), (DAS), (DRS), ($DARS$). Suppose that 1. E_1, $E_2, \dots, E_n, \dots$ are all disjoint. 2. Put $E \underset{df}{=} E_1 + E_2 + \dots + E_n + \dots$. 3. Suppose that $\mathrm{S}_{E_n} \overrightarrow{\varphi}$ exists for all $n = 1, 2, \dots$, then $\mathrm{S}_E \overrightarrow{\varphi}$ exists too, and we have

$$\mathrm{S}_E \overrightarrow{\varphi} = \sum_{n=1}^{\infty} \mathrm{S}_{E_n} \overrightarrow{\varphi}.$$

The theorem is not true for the summations (D), (DA), (DR), (DAR), even if (Hyp S) is admitted.

12. - Theorem. Under hypotheses [1.1], [§ 1; 12, 14] and (Hyp S), if needed, consider any kind (D') of summations. Let $E_1, E_2, \dots, E_n, \dots, E \in G$, $F \in G$. Suppose that $E_n \leq F$, $E \leq F$. If 1. $\mathrm{S}_F \overrightarrow{\varphi}$ exists, 2. $|E_n, E|_\mu \to 0$, then $\mathrm{S}_{E_n} \overrightarrow{\varphi} \to \mathrm{S}_E \overrightarrow{\varphi}$ for $n \to \infty$ in the V-topology.

Proof. We rely on [4.1a] concerning the existence of sums to be now considered. We have

$$\mathrm{S}_E \overrightarrow{\varphi} = \mathrm{S}_{E - E_n} \overrightarrow{\varphi} + \mathrm{S}_{E \cdot E_n} \overrightarrow{\varphi},$$

by [7.1 cor]. Since $\mu(E - E_n) \to 0$, we have, by [5.3],

$$\lim \mathrm{S}_{E-E_n} \vec{\varphi} = \vec{O}.$$

Hence

$$\lim \mathrm{S}_{E \cdot E_n} \vec{\varphi} = \mathrm{S}_E \vec{\varphi}. \tag{1}$$

We have

$$\mathrm{S}_{E_n} \vec{\varphi} = \mathrm{S}_{E_n - E} \vec{\varphi} + \mathrm{S}_{E_n \cdot E} \vec{\varphi},$$

by [7.1 cor]. Since $\mu(E_n - E) \to 0$, we have, by [5.5],

$$\lim \mathrm{S}_{E_n - E} \vec{\varphi} = \vec{O}.$$

Hence, by (1), we get

$$\lim \mathrm{S}_{E_n} \vec{\varphi} = \mathrm{S}_E \vec{\varphi}. \qquad \text{Q. E. D.}$$

12.1. - **Remark.** The theorem [12] says that $\mathrm{S}_E \vec{\varphi}$, considered as the function of E, is (μ-limit)-(V-limit)-continuous for any kind of summation considered.

13. - **Theorem.** Let (D') be any kind of summation of $\vec{\varphi}$. Suppose that $\vec{\varphi}$ is (D')-summable on I. Put $\vec{K}(a) \underset{df}{=} \mathrm{S}_a \vec{\varphi}$ for every brick a, then $\vec{K}$ is (D')-summable on I and we have for every $E \in G$, $\mathrm{S}_E \vec{K} = \mathrm{S}_E \vec{\varphi}$.

Proof. Let $E \in G$. Consider a (D')-distinguished sequence of complexes $\{P_n\}$, $\{P_n\} = \{p_{n1}, p_{n2}, \ldots\}$ for E. Put $\vec{K}(P_n) \underset{df}{=} \sum_i \vec{K}(p_{ni})$. We have

$$\vec{K}(P_n) = \sum_i \mathrm{S}_{p_{ni}} (\vec{\varphi}) = \mathrm{S}_{\mathrm{som}\, P_n} \vec{\varphi},$$

[4.1a], [7.1 cor]. Since $|\mathrm{som}\, P_n, E|_\mu \to 0$, we have, by [12], $\mathrm{S}_{\mathrm{som}\, P_n} \vec{\varphi} \to \mathrm{S}_E \varphi$, i. e.

$$\vec{K}(P_n) \to \mathrm{S}_E \vec{\varphi}. \tag{1}$$

This being true for any (D')-distinguished sequence $\{P_n\}$

for E, it follows that $\overrightarrow{K}$ is summable on E, and we have

(2) $$\overrightarrow{K}(P_n) \rightarrow \mathrm{S}_E \overrightarrow{K}.$$

From (1) and (2), the theorem follows.

14. - **Theorem.** Admit the hypotheses [1.1], [§ 1; 12, 14] and (Hyp S), if needed. We shall consider any kind (D') of summation. If 1. $\overrightarrow{\varphi}$ is (D')summable on E, where $E \in G$, 2. λ is a number (real or complex, depending on whether V is a real vector space or a complex one). 3. We define for all bricks a the vector field $\overrightarrow{\psi}(a) \underset{df}{=} \lambda \overrightarrow{\varphi}(a)$; then $\overrightarrow{\psi}$ is also (D')-summable on E, and

$$\mathrm{S}_E \overrightarrow{\psi} = \lambda\, \mathrm{S}_E \overrightarrow{\varphi}.$$

15. - Under previous hypotheses, if $\overrightarrow{\varphi_1}$ and $\overrightarrow{\varphi_2}$ are two vector fields, both (D')-summable on E, then the vector-field $\overrightarrow{\varphi_3}(a) \underset{df}{=} \overrightarrow{\varphi_1}\, a) + \overrightarrow{\varphi_2}(a)$ is also (D')-summable on E, and we have

$$\mathrm{S}_E \overrightarrow{\varphi_3} = \mathrm{S}_E \overrightarrow{\varphi_1} + \mathrm{S}_E \overrightarrow{\varphi_2}.$$

Proof. Both [14] and [15] follow from the linearity of the vector-field V.

16. - The existence of the (D)-sum $\mathrm{S}_E \overrightarrow{\varphi}$ requires more than the existence of any other (D')-sum on E. Generally, the « addition » of a letter « increases » the size of summable fields.

17. - Let $\eta > 0$ and let A be an atom of G, then, by [§ 1; 14.2], there exists a complex P such that $|P, A|_\mu < \eta$. If we take $\eta < \frac{1}{2}\mu(A)$, then, [§ 1; 17.3], we get $A \leq P$. Hence, by [§ 1; 17.2], there exists one and only one brick p of P with

(1) $$A \leq p.$$

Since $|P, A|_\mu < \eta$, $A \leq P$, we have $\mu P - \mu A < \eta$; hence, by (1),

$$\mu p - \mu A < \eta.$$

Thus, if A is an atom of G, then for every $\eta < 0$ sufficiently small there exists a brick p such that $A \leq p$, $\mu(p - A) < \eta$.

17.1. - Let $\eta_n \to 0$. We can find, by [17], a brick p_n with $A \leq p_n$, $\mu(p_n - A) < \eta_n$. If we put $q_n = p_1 \dots p_n$, $(n = 1, 2, \dots)$, we get bricks

$$q_1 \geq q_2 \geq \dots \geq q_n \geq \dots \tag{2}$$

with

$$A \leq q_n\,,\ \mu(q_n - A) \to 0. \tag{3}$$

Thus if A is an atom of G, there exists an infinite sequence of bricks (2), satisfying the condition (3).

17.2. - Now, suppose that A is an atom of G. Suppose that $\overrightarrow{\varphi}$ is (D)-summable on I. Then $\overrightarrow{\varphi}$ is $(DARS)$-summable on the set $\{A\}$ composed of the single soma A, [4.1a]. Consider the sequence (2) in [17.1] with properties (3). The sequence of complexes

$$\{q_1\},\ \{q_2\}, \dots,\ \{q_n\} \tag{4}$$

satisfies the conditions (R) and (A), and also (S) for $\{A\}$ because the only subsoma of A, differing from A is O. Thus (4) is a completely distinguished sequence for $\{A\}$. It follows that

$$\mathrm{S}_A \overrightarrow{\varphi} = \lim_{n \to \infty} \overrightarrow{\varphi}(q_n).$$

17.3. - Let $A_1, A_2, \dots, A_m \dots$ be a finite or infinite sequence of different atoms, (some ones or all), of G. Suppose that $\overrightarrow{\varphi}$ is $(DARS)$-summable on I. Then, by [10], [7.1],

$$\mathrm{S}_{A_1} \overrightarrow{\varphi} + \mathrm{S}_{A_2} \overrightarrow{\varphi} + \dots + = \mathrm{S}_{A_1 + A_2 + \dots} \overrightarrow{\varphi}\,.$$

17.4. - Since the hypothesis of existence of the $(DARS)$-sum on I is less restrictive that each one concerning (DS), (DAS), (DRS), the above holds true for those summations too. But, even if (Hyp S) is not supposed, our arguments are valid for (DAN)-sums; hence for any one of the sums of character (D), (DA), (DN) too.

18. - Suppose that $\vec{\varphi}$ is (D)-summable on I. Let a be a brick. Since $\{a\}, \{a\}, \ldots$ is a (D)-distinguished sequence for $\{a\}$ we have

$$S_a \vec{\varphi} = \vec{\varphi}(a).$$

If $a = a_1 + a_2 \ldots + a_n + \ldots$ is a finite or denumerable sum of disjoint bricks, we have [10]:

$$\vec{\varphi}(a) = \vec{\varphi}(a_1) + \vec{\varphi}(a_2) + \ldots + \vec{\varphi}(a_n) + \ldots,$$

a quite strong condition imposed on $\vec{\varphi}(a)$. Dealing with bricks and atoms we can get examples showing that the different kinds of summations do not coincide.

§ 3. - Measurable sets of traces and integration of functions of a variable trace.

We shall need an auxiliary theory which aims at the foundation of a general orthogonal system of co-ordinates in Hilbert space, [§ 6], that system being adapted to discontinuous spectrum of hermitian and normal operators as well as to their continuous spectrum. The auxiliary theory to be now developped deals with the notion of trace (13) and constitutes a generalization of a similar notion defined in our Comptes Rendus notes: (23), (24), (25), and in (14). The notion of a trace we shall deal with is just the notion spoken of in (11), (26) and used in (22). Since in all these papers the theory has been sketched only, and many theorems have not been accompanied with proof, therefore it seems to be in order to supply now the proof in the present paper.

1. - We admit the hypotheses [§ 1, 1] and take over the topic and notations of [§ 1; 1 - 4.4], but we do not admit neither (Hyp A_d) nor (Hyp A_f). Thus we shall consider the tribes F, G and the base B of F. To avoid non essential complications, we shall admit that F is a finitely genuine strict

(13) The french term is « lieu ».

subtribe of G, and that G admits an effective, denumerably additive non negative measure μ.

2. - We shall consider infinite descending sequences $a_1 \geq a_2 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$ of bricks. If $\{a_n\}$, $\{b_n\}$ are two such sequences, we say that $\{a_n\}$ *is included in* $\{b_n\}$: $\{a_n\} \leq \{b_n\}$, if for every n there exists m such that $a_m \leq b_n$. We see that the sequence $\{O, O, \dots\}$ is included in every sequence, and every sequence is included in $\{I, I, \dots\}$. If $\{a_n\} \leq \{b_n\}$, $\{b_n\} \leq \{c_n\}$, then $\{a_n\} \leq \{c_n\}$, We have $\{a_n\} \leq \{a_n\}$, We say that $\{a_n\}$ *is equivalent to* $\{b_n\}$:

$$\{a_n\} \backsim \{b_n\}, \quad \text{if } \{a_n\} \leq \{b_n\} \quad \text{and } \{b_n\} \leq \{a_n\}.$$

The notion of equivalence obeys the formal rules of identity, and the notion of «being included» is invariant with respect to the equivalence.

If the sequences $\{a_n\}$, $\{b_n\}$ differ only by a finite number of elements, they are equivalent.

2.1. - **Def.** A descending sequence $\{a_n\}$ of bricks is said *to be minimal* if the following conditions are satisfied;

1) $\{a_n\}$ is not equivalent to $\{O, O, \dots\}$,

2) if $\{b_n\} \leq \{a_n\}$, then either $\{b_n\} \backsim \{O, O, \dots\}$ or else $b_n\} \backsim \{a_n\}$.

2.2. - **Remark.** In the general case we cannot prove the existence of minimal sequences without supplementary hypotheses concerning B. Thus, in what follows we shall admit the existence of at least one minimal sequence.

2.3. - **Def.** A saturated class of mutually equivalent minimal sequences of bricks will be termed *trace* (14), and each of those sequences *representative of that trace.* All elements of a representative of a trace are $\neq O$.

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(14) The notion of trace is related to ultrafilters (27), (28) and also to maximal ideals in Stone's-rings (Boolean rings), (18), (29). However, the notion of trace seems to be more adapted to application to Quantum Physics than ultrafilters.

2.3.1. - **Remark.** Notice that there does not exist anything like a «null-trace».

2.4. - **Def.** Two traces x, y are said to *be equal* $x = y$, if their representatives are equivalent.

2.4.1. - This notion of equality obeys the formal laws of identity. It coincides with the identity of classes of sequences, suitably restricted.

2.5. - **Def.** We say that the *brick* a *covers the trace* x, whenever there exists a representative $\{a_n\}$ of x such that $a_1 \leq a$. (Of course it follows that all $a_n \leq a$).

3. - In [§ 1; 4] we have defined a *covering* as an at most denumerable sum of bricks. In this chapter we take over that definition.

3.1. - If $L_1, L_2, \ldots, L_n, \ldots$ are coverings, finite or infinite denumerable in number, then $\sum_n L_n$ is also covering.

3.2. - If L_1, L_2 are coverings, so is $L_1 \cdot L_2$. Indeed, if $L_1 = \sum_n a_n$, $L_2 = \sum_m b_n$, we have $L_1 \cdot L_2 = \sum_n \sum_m a_n b_m$, and $a_n b_m$ is a brick.

4. - **Def.** Let X be a set of traces (it may be even empty), and L a covering. We say that *X is covered by L* if the following implication is true: «if $x \in X$, then there exists a brick a such that 1) $a \leq L$, and 2) x is covered by a, [Def. 2.5]».

4.1. - The empty set of traces is covered by any covering. Every set X posseses a covering, namely the soma I.

4.2. - The following are equivalent:

I. The trace x is covered by the brick a,

II. The set (x), composed of the single trace x, is covered by the covering a.

5. - The following lemmas hold true: Let X, Y be sets

of traces and L a covering. If $X \subseteq Y$, and L is a covering of Y, then L is also a covering of X.

5.1. - Let L, M be coverings and X a set of traces. If $L \leq M$, and L is a covering of X, then M is also a covering of X.

5.2. - Let $S \neq \emptyset$ be a collection of indices. Suppose that for every $i \in S$ the set X_i of traces is covered by the covering L, then $\cup_i X_i$, $(i \in S)$ is also covered by L.

5.3. - If X_1, X_2, ..., at most denumerable in number, are covered by L_1, L_2, ... respectively, then $\cup_n X_n$ is covered by $\Sigma_n L_n$.

5.4. - **Theor.** If L, M are coverings of the sets of traces X, Y respectively, then $L \cdot M$ is a covering of $X \cap Y$.

Proof. Let $x \in X \cap Y$. There exist bricks a, b, both covering x, and such that $a \leq L$, $b \leq M$. Since x is covered by a, there exists a representative $\{a_n\}$ of x such that $a_n \leq a$, $(n = 1, 2, ...)$. Since x is covered by b, there exists a representative $\{b_n\}$ of x with $b_n \leq b$, $(n = 1, 2, ...)$. The sequences $\{a_n\}$, $\{b_n\}$ are equivalent, [2]. Hence there exists p with $b_p \leq a_1$. Hence $b_p \leq a \cdot b \leq N \cdot M$. Since $\{b_p, b_{p+1}, ...\}$ is equivalent to $\{b_1, b_2, ...\}$, it is equivalent to $\{a_n\}$. Since $a \cdot b$ is a brick, it follows that x is covered by $L \cdot M$. The lemma is proved.

6. - Denote by W the set of all traces. Il X is a set of traces, denote by co X the complement of X with respect to W, i.e. $W - X$.

We admit following two hypotheses:

Hyp. I. - If X is a set of traces, L, M, are coverings of X and co X respectively, then $L + M = I$.

Hyp. II. - If a is a brick and X the set of all traces covered by a, then co a is the covering of co X.

6.1. - Suppose that B has at least one element b differing from O and I. From (Hyp. I) it follows that there exists at least one trace. Indeed, by [4.1] the empty set of traces is covered by any covering. If $W = \Theta$, we would have $W = \text{co}\, W$. The brick b would cover both W and co W, but $b + b \neq I$.

From the hypotheses (Hyp. I) and (Hyp. II) it follows that, if $a \neq O$ is a brick, but not an atom, there exists a trace covered by a. Indeed, suppose that there does not exist any trace covered by a. Then, by (Hyp. II) W is covered by co a. There exists a brick b with $O < b < a$. Now co $W = \Theta$ is covered by b. Hence, by (Hyp. I), $b + \text{co}\, a = I$, which is not true.

If a is an atom, then $\{a, a, \ldots\}$ is a minimal sequence, hence representing a trace covered by a.

If F and B are composed of O and unit only, I is an atom.

Thus we have proved, that if $f \in F$ is any figure $\neq O$, then there exists a trace covered by f.

7. - The above hypotheses (Hyp. I) and (Hyp. II) and the existence of an effective measure on G will make possible to develop a theory of measurability of sets of traces. The theorems which will not involve the measure μ explicitely, will be indipendent of the choice of the effective measure.

We emphasize that, given the tribe F, the notion of traces depends on the choice of the base B of F, so we may call them ***B*-traces in *F***.

7.1. - We shall rely on the following theorem by Wecken (12): If G is a denumerably additive tribe admitting a denumerably additive, non negative and effective measure, then G is completely additive.

7.2. - **Def.** If X is a set of traces, then by its *outer coat* we shall understand the soma of G: $[X]^* = \Pi L$, where the product is extended over all coverings L of X.

7.3. - **Def.** If X is a set of traces, then by its *inner* *coat* we shall understand the soma of G:

$$[X]_* = I - [Y]^* = \text{co}\,[Y]^*,$$

where $Y = \text{co}\,X$.

7.4. - **Theorem.** For every set X of traces we have

$$O \leq [X]_* \leq [X]^*.$$

Proof. Put $Y = \text{co}\,X = W - X$. If L is a covering of X, and M a covering of Y, we have, by (Hypothesis I):

$$L + M = I.$$

Multiplyng both sides by co M, we get $L \cdot \text{co}\,M = \text{co}\,M$; hence $\text{co}\,M \leq L$. This being true for a given L and any M, we get

$$\Sigma\,\text{co}\,M \leq L \tag{1}$$

where the summation is extended over all coverings of Y. The inclusion (1) being valid for any covering L of X, we get $\Sigma\,\text{co}\,M \leq \Pi L$ i. e. $\text{co}\,\Pi M \leq \Pi L$, hence $\text{co}\,[Y]^* \leq [X]^*$, and then $[X]_* \leq [X]^*$.

7.4.1. - **Theorem.** If $X \leq Y$ are sets of traces, we have for their outer and inner coats the inclusions:

$$[X]^* \leq [Y]^*,\ [X]_* \leq [Y]_*.$$

7.5. - **Theorem.** If X is a set of traces, then there exists a denumerable sequence of coverings of X:

$$L_1 \geq L_2 \geq \dots \geq L_n \geq \dots$$

such that $[X]^* = \Pi_n L_n$.

For such a sequence we have $\mu([X]^*) = \lim \mu(L_n)$.

In addition to that we have $\mu([X]^*) = \inf \mu(M)$ where the infimum is taken for all coverings M of X.

7.5a. - **Proof.** First suppose that $X = \mathsf{O}$. Then all coverings L will be coverings of X, (see [4.1]). Hence the soma

O is also a covering of X, (see [§ 1, 4] and [§ 1, 1]). Hence $\inf \mu(M) = \mu(O) = 0$, where the infimun is taken over all coverings M of X. Put $L_n = 0$, $n = 1, 2, \ldots$. We have $L_1 \geq L_2 \geq \geq \ldots$ and $\lim \mu(L_n) = \inf \mu(M)$. Now $\Pi M \leq O$, hence $[X]^* = O$, and then $\mu[X]^* = 0$.

Thus in the case where $X = \Theta$ the theorem is proved.

7.5a′. - Let $X \neq \Theta$, so if L is a covering of X, we have $L \neq O$. Denote by $\{L\}$ the class of all coverings of X. We have $[X]^* = \Pi L$ where $(L \in \{L\})$. We order well the different elements of $\{L\}$:

$$(1) \qquad L_1 L_2, \ldots, L_\delta, \ldots$$

where the indices are consecutive ordinals. Denote the ordering of these indices by $\mathfrak{S}$.

7.5b. - Suppose $\mathfrak{S}$ is finite:

$$(1) \qquad L_1, L_2, \ldots, L_n.$$

Then $L \underset{df}{=} L_1 \cdot L_2 \ldots L_n$ is a covering of X, (see [3.2] and [5.4]). We have $[X]^* = \Pi L = L_1 \ldots L_n = L$. This soma is one of the somata (1). If we put

$$L_1' = L_2' = \ldots \underset{df}{=} L,$$

we get $\lim \mu(L_n') = \mu[X]^* = \inf \mu(M)$ where the infimum is taken for all coverings M of X. The theorem is established in the case considered.

7.5b′. - Let us suppose that $\mathfrak{S}$ is infinite. Put, for every δ, where $\delta \in \mathfrak{S}$,

$$(2) \qquad p_\delta = \Pi_{\eta \leq \delta} L_\eta .$$

This product is meaningful because G is completely additive: $p_\delta \in G$.

If $\delta' \leq \delta''$, we have $p_{\delta'} \geq p_{\delta''}$, and then $\mu(p_{\delta'}) \geq \mu(p_{\delta''})$.

We say that the following are equivalent:

$$\mathrm{I} \cdot \mu(p_\alpha) = \mu(p_\beta), \qquad \mathrm{II} \cdot p_\alpha = p_\beta .$$

Indeed, the implication II $\rightarrow$ I is evident. Suppose I, and $\alpha \geq \beta$. It follows $p_\alpha \leq p_\beta$, and then $\mu(p_\beta - p_\alpha) = \mu(p_\beta) - \mu(p_\alpha) = 0$, which gives, on account of the effectiveness of the measure μ, $p_\beta - p_\alpha = 0$. Hence $p_\beta \leq p_\alpha$. It follows $p_\alpha = p_\beta$, i. e. II.

7.5c. - Denote by $\{p\}$ the class of all somata p_δ for $\delta \in \mathfrak{S}$. For each $q \in \{p\}$ consider the smallest ordinal $\lambda(q)$ such that $p_{\lambda(q)} = q$.

If $q, q' \in \{p\}$, and $q < q'$, we have $\lambda(q) > \lambda(q')$, and conversely, if $\lambda(q) > \lambda(q')$, we have $q < q'$.

7.5d. - The ordering $\mathfrak{S}$ of indices, if restricted to the set of all $\lambda(q)$, $(q \in \{p\})$, is a partial well ordering, say $\mathfrak{S}'$:

$$1 = \tau(1) < \tau(2) < \ldots < \tau(\beta) < \ldots, \tag{3}$$

where the variable β varies over a range of consecutive ordinals. Denote the ordering of these indices by $\mathfrak{M}$. Put $r_\beta = p_{\tau(\beta)}$ for all $\beta \in \mathfrak{M}$.

There is one — to — one correspondence between the somata $q \in \{p\}$ and the indices $\beta \in \mathfrak{M}$, where

$$r_\beta = p_{\tau(\beta)} = p_{\lambda(q)} = q, \tag{4}$$

and where the following are equivalent:

(5) I. $q' > q''$; II. $\tau(\beta') = \lambda(q') < \lambda(q'') = \tau(\beta'')$; III. $\beta' < \beta''$.

Thus we have $L_1 = p_1 = r_1 > r_2 > \ldots > r_\beta > \ldots$ i. e.

$$p_{\tau(1)} > p_{\tau(2)} > \ldots > p_{\tau(\beta)} > \ldots.$$

7.5e. - We notice at once that the well ordering $\mathfrak{M}$ is at most denumerable. Indeed, we have $\mu(r_1) > \mu(r_2) > \ldots > \mu(r_\beta) > \ldots$ and $r_1 > r_2 > \ldots > r_\beta > \ldots$, so all $r_s - r_{s-1}$ are disjoint and have positive measures, for μ is effective. If their number were non denumerable, we would have a contradiction, for μ is denumerably additive.

Let us emphasize the following remark: If $q \in \{p\}$, $q \gtrsim r_\beta$, the unique index η, for which $q = r_\eta$, has the property

$$\eta \gtrsim \beta. \tag{6}$$

7.5f. - Now we shall prove that every r_β is a denumerable product of coverings of X. Suppose that the statement is not true. Clearly r_1 is a denumerable product of coverings of X, hence the smallest ordinal η, for which this does not take place, is > 1.

Suppose that $\eta - 1$ exists. We have $\eta - 1 \geq 1$ and $r_{\eta-1}$ is a denumerable product of coverings of X. We have, by (4),

$$r_{\eta-1} = p_{\tau(\eta-1)}, \; r_\eta = p_{\tau(\eta)}.$$

We say that if

$$\tau(\eta - 1) < \xi < \tau(\eta), \tag{7}$$

we must have $p_\xi = r_{\eta-1}$.

To prove this, notice that $r_{\eta-1} \geq p_\xi \geq r_\eta$. By (6) we get $\eta - 1 < \eta' \leq \eta$. Hence $\eta' = \eta$ which gives $p_\xi = r_{\eta'} = r_\eta = p_{\tau(\eta)}$ where $\xi < \tau(\eta)$, by [7.5c] and (7). Now this is impossible, because $\tau(\eta)$ is the smallest index v for which $p_v = p_\xi$. Thus we have proved that (7) implies $p_\xi = r_{\eta-1}$.

7.5g. - Take ξ' such that

$$\tau(\eta - 1) < \xi' < \tau(\eta). \tag{7.1}$$

We have, by [7.5f.],

$$p_{\tau(\eta-1)} = p_{\xi'} = \Pi_{\alpha \leq \xi'} L_\alpha, \tag{7.2}$$

Consequently, by varging ξ' in (7.1), and by taking the product of all equalities (7.2), we get

$$p_{\tau(\eta-1)} \leq \Pi_{\tau(\eta-1)<\xi<\tau(\eta)} L_\xi,$$

which gives

$$p_{\tau(\eta-1)} \cdot \Pi_{\tau(\eta-1)<\xi<\tau(\eta)} L_\xi = p_{\tau(\eta-1)}. \tag{8}$$

Now

$$p_{\tau(\eta)} = \Pi_{\alpha\leq\tau(\eta)} L_\alpha = \Pi_{\alpha\leq\tau(\eta-1)} L_\alpha \cdot \Pi_{\tau(\eta-1)<\alpha<\tau(\eta)} L_\alpha \cdot L_{\tau(\eta)} = p_{\tau(\eta-1)} \cdot$$
$$\cdot \Pi_{\tau(\eta-1)<\alpha<\tau(\eta)} L_\alpha \cdot L_{\tau(\eta)},$$

and hence, by (8) and (4),

$$r_\eta = p_{\tau(\eta)} = r_{\eta-1} \cdot L_{\tau(\eta)}.$$

Since $L_{\tau(\eta)}$ is a covering, and $r_{\eta-1}$ has been supposed to be a denumerable product of coverings, it follows that also r_η is a denumerable product of coverings. The contradiction thus obtained shows that η must be a limit ordinal.

7.5h. - Since $\mathfrak{N}$ is a denumerable well ordering [7.5e], it follows that there exists an infinite sequence $\eta_1 < \eta_2 < \ldots < \eta_n < \ldots < \eta$ such that

$$(9) \qquad \eta = \lim_n \eta_n .$$

For every n the soma r_{η_n} is a denumerable product of coverings of X. We have, by (3), $\tau(\eta_1) < \tau(\eta_2) < \ldots < \tau(\eta_n) < \ldots < \tau(\eta)$. Let τ' be the smallest ordinal $\geq$ than all $\tau(\eta_n)$. We have

$$(9.0) \qquad \tau(\eta_n) < \tau' \leq \tau(\eta)$$

for all n. We shall prove that $\tau' = \tau(\eta)$. To do this, notice that

$$(9.1) \qquad p_{\tau(\eta_n)} \geq p_{\tau'} \geq p_{\tau(\eta)} .$$

Determine the unique β for which $p_{\tau'} = r_\beta$. We have $r_{\eta_n} \geq r_\beta \geq r_\eta$, which gives, by (6),

$$(9.2) \qquad \eta_n \leq \beta \leq \eta$$

for $n = 1, 2, \ldots$. Suppose that $p_{\tau'} > p_{\tau(\eta)}$. We get, by (6), $\beta < \eta$, and then, by (9), there exists m with $\beta < \eta_m$. Thus we obtain, from (9.2), $\eta_m \leq \beta < \eta_m$ which is impossible. It follows that $p_{\tau'} = p_{\tau(\eta)}$. Since $\tau(\eta)$ has the minimum property, it follows, from (9.0), that $\tau' = \tau(\eta)$.

7.5i. - The equality $\tau' = \tau(\eta)$ being established, we get $\tau(\eta) = \lim_n \tau(\eta_n)$. It follows that if $1 \leq \delta < \tau(\eta)$, there exists m with

$$(10) \qquad \tau(\eta_m) < \delta.$$

Having this, we can write

$$r_\eta = p_{\tau(\eta)} = \Pi_{1 \leq \alpha \leq \tau(\eta_1)} L_\alpha \cdot \Pi_{1 \leq \alpha \leq \tau(\eta_2)} L_\alpha \ldots L_{\tau(\eta)} ,$$

i. e.

$$r_\eta = r_{\eta_1} \cdot r_{\eta_2} \ldots L_{\tau(\eta)} .$$

Since $L_{\tau(\eta)}$ is a covering, and each of the remaining factors in the above product is a denumerable product of coverings of X, it follows that r_{τ_1} is so too. The contradiction thus obtained proves that every $q \in \{p\}$ is a denumerable product of coverings.

7.5j. - Now we have $[X]^* = \Pi_\alpha L_\alpha$, the index α ranging over $\mathfrak{S}$, hence $[X]^* = \Pi_\beta p_\beta$, the index β ranging over $\mathfrak{M}$. Since the number of factors in this product is denumerable, it follows that $[X]^*$ is a denumerable product of coverings of X. Let us write

$$[X]^* = \Pi_{n=1}^{\infty} M'_n$$

where M'_n are coverings of X. If we put $M_n = \Pi_{i=1}^{n} M'_i$, we obtain

$$[X]^* = \Pi_{n=1}^{\infty} M_n \quad \text{with} \quad M_1 \geq M_2 \geq \dots, \tag{10.1}$$

and M_n is a covering of X, (by virtue of Lemma 5.4). Thus the first part of the thesis is proved.

7.5k. - If $[X]^* = \Pi_{n=1}^{\infty} M_n$, where M_n are coverings of X with $M_1 \geq M_2 \geq \dots$, it follows, on account of the denumerable additivity of the measure μ, that

$$\mu([X]^*) = \lim \mu(M_n).$$

7.5l. - To prove the last statement of the thesis, put

$$\lambda \underset{df}{=} \inf \mu(L), \tag{11}$$

where the infimum is taken for all coverings L of X. Take the sequence $\{M_n\}$ from (10.1). We have, [7.5k], $\lambda \leq \lim \mu(M_n) = \mu([X]^*)$.

Suppose that $\lambda < \mu([X]^*)$. There exists $\varepsilon > 0$ such that $\lambda < \mu([X]^*) - \varepsilon$. By (11) there exists a covering L' such that $\lambda \leq \mu(L') < \mu([X]^*) - \varepsilon$.

Since $M \cdot L'$ is also a covering of X, we have

$$\lambda \leq \mu(M_n L') \leq \mu(L') < \mu([X]^*) - \varepsilon. \tag{12}$$

Now, by (10.1), $[X]^* \leq \Pi_{n=1}^{\infty} M_n L' \leq \Pi_{n=1}^{\infty} M_n [X^*]$; hence $[X]^* = \Pi_{n=1}^{\infty}(M_n L')$.

In addition to that we have $M_1 L' \leq M_2 L' \geq \dots$, and these somata are coverings of X. It follows, on account of [7.5k], that $\mu([X]^*) = \lim \mu(M_n L')$. On the other hand we get from (12),

$$\mu([X]^*) < \mu([X]^*) - \varepsilon,$$

which is false. Thus we have proved that $\lambda = \mu([X]^*)$, and this completes the proof of the whole theorem [7.5].

8. - **Measurable sets of traces. Defin.** A set X of traces is said *to be measurable* whenever its inner coat and its outer coat coincide: $[X]_* = [X]^*$. In this case we speak of the *coat of* X and denote it by $[X]$. We have $[X] = [X]^* = [X]_*$.

8.1. - **Theorem.** If X is a sets of traces, $Y = \text{co}\, X$, then the following are equivalent: I. X is measurable, II. $[X]^* \cdot [Y]^* = O$, III. Y is measurable.

In this case we have $[X] = \text{co}\,[Y]$.

8.2. - **Theorem.** The empty set Θ of traces is measurable. The total set W of traces is measurable. They are different.

Proof. By [3.3] we have for $X = \Theta$:

$$O \leq [X]_* \leq [X]^*.$$

We have $[X]^* = O$ (compare part [7.5a] of the proof of the theor. [7.5]). Hence $[X]_* = [X]^*$, so X is measurable. By [8.1], since $W = \text{co}\,\Theta$, and since Θ is measurable, so is W. We have $[\Theta] = O$ and $[W] = \text{co}\,[\Theta] = \text{co}\, O = I$, and $O \neq I$. Since $[\Theta] \neq [W]$, the sets Θ and W are different.

Remark. Soon we shall prove that if a is a brick, then the set of all traces covered by a is measurable.

8.3. - **Defin.** If X is *a* measurable set of traces, then the number

$$\mu([X]) = \mu([X]^*) = \mu([X]_*)$$

is called *measure of* X and denoted by $\mu(X)$.

8.4. - Theorem. If X is a measurable set of traces, $[X] = \Pi_{n=1}^{\infty} L_n$ where $L_1 \geq L_2 \geq \dots$, are coverings of X, then $\mu(X) = \lim \mu(L_n)$.

8.5. - Theorem. Let X be a set of traces, $Y = \mathrm{co}\, X$, and let L_n, M_n be coverings of X, Y respectively with

$$L_{n+1} \leq L_n, \ M_{n+1} \leq M_n, \ (n = 1, 2, \dots),$$

where $[X]^* = \Pi_n L_n$, $[Y]^* = \Pi_n M_n$. If X is measurable, then $\lim \mu(L_n \cdot M_n) = 0$.

Proof. By theorem [8.1], $[X]^* \cdot [Y]^* = O$, hence $O = \Pi_n L_n \cdot \Pi_n M_n = \Pi_n (L_n \cdot M_n)$. We have $L_{n+1} \cdot M_{n+1} \leq L_n \cdot M_n$ which completes the proof.

8.6. - Theorem. Let X be a set of traces, $Y = \mathrm{co}\, X$. Let L_n, M_n, $(n = 1, 2, \dots)$ be coverings of X, Y respectively. If $\lim \mu(L_n \cdot M_n) = 0$, then X is measurable, and we have $[X] = \Pi_n L_n$.

Proof. Put $L'_n = L_1 \dots L_n$, $M'_n = M_1 \dots M_n$, $(n = 1, 2, \dots)$. We have $L'_n \cdot M'_n \leq L_n \cdot M_n$, and then

$$\text{(1)} \qquad \lim \mu(L'_n \cdot M'_n) = 0.$$

Now $[X]^* \leq L'_n$, $[Y]^* \leq M'_n$; hence

$$\text{(2)} \qquad [X]^* \cdot [Y]^* \leq \Pi_n (L'_n \cdot M'_n).$$

Since $L'_{n+1} \cdot M'_{n+1} \leq L'_n \cdot M'_n$, we get, by (1),

$$\mu(\Pi_n (L'_n \cdot M'_n)) = \lim \mu(L'_n \cdot M'_n) = 0,$$

and hence, by the effectiveness of μ, $\Pi_n (L'_n \cdot M'_n) = O$, which gives, by (2), $[X]^* \cdot [Y]^* = O$, and then, by virtue of theor. [8.1], the measurability of X, and of Y.

To prove the second part of the thesis, notice that $\mathrm{co}\,[X] = [Y] \leq M_n$; hence $L_n \cdot \mathrm{co}\,[X] \leq L_n \cdot M_n$, $\mu(L_n - [X]) \leq \mu(L_n \cdot M_n)$. Since $\lim \mu(L_n \cdot M_n) = 0$ it follows

$$\text{(3)} \qquad \lim \mu(L_n - [X]) = 0.$$

Put

$$p = \Pi_n L_n. \tag{4}$$

We have $[X] \leq p \leq L_n$; hence $\mu(p - [X]) \leq \mu(L_n - [X])$. This gives, on account of (3), $p - [X] = O$, i. e. $p \leq [X]$, and then $p = [X]$. Consequently, by (4), $[X] = \Pi_n L_n$, which is the second thesis. The theorem is established.

8.6.1. - The measure of the empty set Θ of traces is 0, the measure of the total set W of traces is $\mu(I)$.

8.7. - **Theorem.** If a is a brick and X the set of all traces covered by a, then X is measurable. We have $[X] = a$, $\mu([X]) = \mu(a)$.

Proof. By (Hypothesis II), co X is covered by co a. Put $L_n = a$, $M_n = \text{co } a$, $(n = 1, 2, ...)$. We get, by theor. [8.6], that X is measurable, $[X] = a$, and $\mu(X) = \mu(a)$.

8.8. - **Theorem.** For every brick $a \neq O$ there exists a trace covered by a.

Proof. Denoting by X the set of all traces covered by a, suppose that $X = \Theta$. The complement co X, is covered, by (Hypothesis II), by co a. Now O is a covering for Θ. It follows, by (Hypothesis I), that co $a + O = I$, which gives $a = O$, thus a contradiction. The theorem is proved; (see [6.1]).

8.9. - **Remark.** The theory of measure of sets of traces is similar to the Lebesgue's classical theory of measure of point-sets, and it can be developped similarly. We shall apply the original Lebesgue's device (13) in proving that the denumerable union of measurable sets is also measurable. This device can be, however, greatly simplified owing to the fact that μ is a denumerably additive measure on G. The same device is used in the authors paper (14).

8.10. - **Theorem.** If X, Y are measurable sets of traces, then $X \cup Y$ is measurable and

$$[X \cup Y] = [X] + [Y].$$

Proof. Put $X' = \text{co}\, X$, $Y' = \text{co}\, Y$. Let L_n, L'_n, M_n, M'_n be coverings of X, X', Y, Y' respectively with $L_{n+1} \leq L_n$, $L'_{n+1} \leq L'_n$, $M_{n+1} \leq M_n$, $M'_{n+1} \leq M'_n$, and

(1) $[X] = \Pi_n L_n$, $[X'] = \Pi_n L'_n$, $[Y] = \Pi_n M_n$, $[Y'] = \Pi_n M'_n$.

By Lemma [5.3] and [5.4] $L_n + M_n$, $L'_n \cdot M'_n$ are coverings of

(2) $$X \cup Y,\ X' \cap Y'$$

respectively.

We have

(3) $$L_{n+1} + M_{n+1} \leq L_n + M_n, \qquad L'_{n+1} \cdot M'_{n+1} \leq L'_n \cdot M'_n$$

and

(4) $$X' \cap Y' = \text{co}\,(X \cup Y).$$

In addition to that,

(5) $$\mu[(L_n + M_n) \cdot L'_n \cdot M'_n] \leq \mu(L_n \cdot L'_n) + \mu(M_n \cdot M'_n).$$

Since X, Y are measurable, we have, by (1) and theor. [8.5],

$$\lim \mu(L_n \cdot L'_n) = 0, \qquad \lim \mu(M_n \cdot M'_n) = 0,$$

hence, by (5):

$$\lim \mu[(L_n + M_n) \cdot L'_n \cdot M'_n] = 0.$$

If we take account of (2), (3), and apply theor. [8.6], we obtain the measurablity of $X \cup Y$, and in addition to that,

(5.1) $$[X \cup Y] = \Pi_n (L_n + M_n).$$

To prove the second part of the thesis, notice that (1) gives

$$[X] \leq \Pi_n (L_n + M_n), \qquad [Y] \leq \Pi_n (L_n + M_n),$$

which implies

(6) $$[X] + [Y] \leq [X \cup Y].$$

On the other hand the relations $L_{n+1} \leq L_n$, $M_{n+1} \leq M_n$ yield

$$\Pi_{n=1}^{\infty}(L_n + M_n) = \Pi_{n=m}^{\infty}(L_n + M_n) \leq \Pi_{n=m}^{\infty}(L_n + M_m) = \Pi_{n=m}^{\infty} L_n + M_m.$$

Hence, by (5.1),

$$[X \cup Y] \leq [X] + M_m, \qquad (m = 1, 2, \ldots).$$

It follows

$$[X \cup Y] \leq \Pi_{m=1}^{\infty}([X] + M_m) = [X] + \Pi_{m=1}^{\infty} M_m = [X] + [Y],$$

which together with (6), completes the proof.

8.10.1 - **Theorem.** If X is measurable, then $\mathrm{co}\, X = W - X$ is also measurable, and $[\mathrm{co}\, X] = \mathrm{co}\, [X] = I - X$, $\mu(\mathrm{co}\, X) = \mu(I) - \mu(X)$.

Proof. By [8.1] and [8.3].

8.11. - **Theorem.** If X, Y are measurable sets of traces, then $X \cap Y$, $X - Y$, $X \dot{+} Y$ are also measurable, and their coats are $[X] \cdot [Y]$, $[X] - [Y]$, $[X] \dot{+} [Y]$ respectively.

Proof. We use de Morgan laws and theorems [8.1], [8.10].

8.12. - **Theorem.** If X, Y are measurable sets of traces, $X \subseteq Y$, then $[X] \leq [Y]$ and $\mu(X) \leq \mu(Y)$.

Proof. $X \subseteq Y$ is equivalent to $X = X \cap Y$. We use theor. [8.11].

8.13. - **Theorem.** If X, Y are measurable sets of traces, and $X \cap Y = \mathbf{O}$, we have $\mu(X \cup Y) = \mu(X) + \mu(Y)$.

Proof. This follows from theor. [8.11], [8.10].

8.14. - **Theorem.** If

1) $X_1, X_2, \ldots, X_n, \ldots$ are measurable sets of traces,

2) $X_i \cap X_j = \mathbf{O}$ for $i \neq j$,

3) $X \doteq \cup_{n=1}^{\infty} X_n$,

then

1) X is also measurable,

2) $[X] = \Sigma_{n=1}^{\infty}[X_n]$,

3) $\mu(X) = \Sigma_{n=1}^{\infty}\mu(X_n)$.

8.14a. - **Proof.** Let $\delta > 0$. Choose positive numbers $\delta_1 > \delta_2 > \dots$ such that $\Sigma_{p=1}^{\infty}\delta_p \leq \delta$. Fix p and consider X_p and $X'_p = \operatorname{co} X_p$. By theor. [7.5] and [8.5], there exist coverings L_n, L'_n of X_p and X'_p respectively, such that

(1) $$L_{n+1} \leq L_n, \quad L'_{n+1} \leq L'_n,$$

(2) $$[X_p] = \Pi_n L_n, \quad [X'_p] = \Pi_n L'_n,$$

and

(3) $$\lim_{n \to \infty} \mu(L_n \cdot L'_n) = 0.$$

Hence there exists $n = N(p)$ such that

(4) $$\mu(L_n \cdot L'_n) \leq \delta_p \text{ for all } n \leq N(p).$$

8.14b. - On the other hand we have for $n = 1, 2, \dots$: $[X_p] \leq L_n$, $L_n - [X_p] = L_n \cdot [X'_n] = L_n \cdot \Pi_m L'_m$; hence $L_n - [X_p] \leq L_n \cdot L'_n$, which gives $\mu(L_n - [X_p]) \leq \mu(L_n \cdot L'_n)$, and then, by (3), $\lim_{n \to \infty} (L_n - [X_p]) = 0$.

Hence there exists $n = M(p)$ such that

(5) $$\mu(L_n - [X_p]) \leq \delta_p \text{ for all } n \geq M(p).$$

8.14c. - Combining (4) and (5) we can say, that for every p there exist coverings M_p, M'_p of $[X_p]$, $[X'_p]$ respectively, such that

(6) $$\mu(M_p \cdot M'_p) \leq \delta_p, \quad \mu(M_p - [X_p] \leq \delta_p.$$

Let us fix these coverings.

8.14d. - We have $\operatorname{co} X = \bigcap_{n=1}^{\infty} \operatorname{co} X_n \subseteq \operatorname{co} X_m = X'_m$ for $m = 1, 2, \dots$. As M'_m is a covering of $\operatorname{co} X'_m$, it is also (by Lemma [5]), a covering of $\operatorname{co} X$. Hence $M'_1 \cdot M'_2 \cdot \ldots \cdot M'_m$ is, by Lemma [5.4], a covering of $\operatorname{co} X$ for any m. On the other hand $M_1 + M_2 + \dots$ is (by Lemma [5.3]) a covering of

X. We have [15] for a fixed m:

$$(M_1 + M_2 + \dots + M_n + \dots) \cdot M'_1 \cdot M'_2 \dots M'_m = M_1 \cdot (M'_1 \dots M'_m) + \\ + \dots + M_m \cdot (M'_1 \dots M'_m) + M_{m+1} \cdot (M'_1 \dots M'_m) + \dots \leq M_1 \cdot M'_1 + \\ + M_2 \cdot M'_2 + \dots + M_m \cdot M'_m + M_{m+1} + M_{m+2} \dots .$$

Since μ is denumerably additive and non negative, it follows

$$(7) \quad \mu[(M_1 + M_2 + \dots) \cdot M'_1 \dots M'_m] \leq \mu(M_1 \cdot M'_1) + \dots + \mu(M_m M'_m) + \\ + \mu(M_{m+1}) + \mu(M_{m+2}) + \dots ;$$

this inequality holding even if, by chance, the right hand series were divergent. Now, by (6), we have $\mu(M_p \cdot M'_p) \leq \delta_p$, $(p = 1, \dots, m)$ and $\mu(M_{m+k} - [X_{m+k}]) \leq \delta_{m+k}$, $(k = 1, 2, \dots)$; hence, since $[X_{m+k}] \leq M_{m+k}$, we get $\mu(M_{m+k}) - \mu(X_{m+k}) \leq \leq \delta_{m+k}$ i. e. $\mu(M_{m+k}) \leq \mu(X_{m+k}) + \delta_{m+k}$, $(k = 1, 2, \dots)$.

Consequently (7) gives:

$$(8) \quad \mu[(M_1 + M_2 + \dots) \cdot M'_1 \dots M'_m] \leq \delta_1 + \dots + \delta_m + [\mu(X_{m+1}) + \\ + \delta_{m+1}] + [\mu(X_{m+2}) + \delta_{m+2}] + \dots, \quad (m = 1, 2, \dots).$$

8.14e. - We shall get help from the denumerable additivity of μ. We have supposed that $X_i \frown X_j = 0$ for $i \neq j$. Hence, by theorem [8.11] and [8.2], we also have $[X_i] \cdot [X_k] = 0$. It follows that $\mu(\Sigma_{n=1}^{\infty}[X_n]) = \Sigma_{n=1}^{\infty}\mu([X_n]$, and hence, there exists m such that

$$\Sigma_{i=m+1}^{\infty}\mu(X_i) \leq \delta.$$

For such an m we get, from (8),

$$(8.1) \quad \mu[(M_1 + M_2 + \dots) \cdot (M'_1 \dots M'_m)] \leq \delta_1 + \dots + \delta_m + \delta_{m+1} + \dots + \delta \leq 2\delta.$$

Thus for every $\delta > 0$ there exists m with (8.1). Applying Theor. [8.6], and what has been said at the beginning of [8.14d], we deduce the measurability of X.

8.14f. - We have, by (6), $\mu(M_n - [X_n]) \leq \delta_n$, $(n = 1, 2, \dots)$;

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[15] This is the Lebesgue's device (13).

hence $\mu(M_n) \leq \mu(X_n) + \delta_n$. It follows

$$(9) \qquad \Sigma_{n=1}^{\infty}\mu(M_n) \leq \Sigma_{n=1}^{\infty}\mu(X_n) + \delta.$$

Since $\Sigma_{n=1}^{\infty}M_n$ is a covering of X, (by Lemma [5.3]), we have $\mu(X) \subseteq \mu(\Sigma_{n=1}^{\infty}M_n)$; hence, by (9), $\mu(X) \leq \Sigma_{n=1}^{\infty}\mu(X_n) + \delta$.

This being true for any $\delta > 0$, we obtain

$$(10) \qquad \mu(X) \leq \Sigma_{n=1}^{\infty}\mu(X_n).$$

8.14g. - On the other hand we have $[X_n] \leq [X]$, since $X_n \subseteq X$, (theor. [8.12]); hence

$$(11) \qquad \Sigma_{n=1}^{\infty}[X_n] \leq [X], \quad \mu(\Sigma_{n=1}^{\infty}[X_n]) \leq \mu(X).$$

Now, by hypothesis, all $[X_n]$ are disjoint; hence

$$(12) \qquad \Sigma_{n=1}^{\infty}\mu(X_n) \leq \mu(X).$$

From (10) and (12) we obtain

$$(13) \qquad \Sigma_{n=1}^{\infty}\mu(X_n) = \mu(X).$$

8.14h. - From (13) we get $\mu(X) - \mu(\Sigma_{n=1}^{\infty}[X_n]) = 0$; hence $\mu([X] - \Sigma_{n=1}^{\infty}[X_n]) = 0$. It follows

$$(14) \qquad [X] - \Sigma_{n=1}^{\infty}[X_n] = O \text{ i. e. } [X] \leq \Sigma_{n=1}^{\infty}[X_n].$$

This combined with (11) gives $[X] = \Sigma_{n=1}^{\infty}[X_n]$, so all items of the thesis are proved.

8.15. - **Theorem.** If $X_1, X_2, \dots, X_n, \dots$ are measurable sets of traces, then $X \overline{\overline{df}} \bigcup_{n=1}^{\infty} X_n$ is also measurable, and we have $[X] = \Sigma_{n=1}^{\infty}[X_n]$.

Proof. Put $Y_n = \bigcup_{i=1}^{n} X_i$, $(n = 1, 2, \dots)$. We have

$$X_1 \cup X_2 \cup \dots \cup X_n = Y_1 \cup (Y_2 - Y_1) \cup (Y_3 - Y_2) \cup \dots$$
$$\dots \cup (Y_n - Y_{n-1}), \quad (n = 2, 3, \dots).$$

Hence

$$(2) \qquad X = Y_1 \cup (Y_2 - Y_1) \cup (Y_3 - Y_2) \cup \dots .$$

The set X_n being measurable, so are Y_n, (Theor. [8.10]; hence $Y_{n+1} - Y_n$ are so, (Theor. [8.11]). Since all terms in (2) are disjoint, we can apply Theor. [8.14], which gives the measurability of X.

In addition to that we have $[Y_{n+1} - Y_n] = [Y_{n+1}] - [Y_n]$, (by Theor. [8.11]), and $[Y_n] = \Sigma_{i=1}^{n}[X_i]$, (by Theor. [8.10]). It follows

$$[X] = \Sigma_{n=1}^{\infty} [X_n],$$

so the theorem is proved.

8.16. - Theorem. If $X_1, X_2, \ldots, X_n, \ldots$ are measurable sets of traces, then

$$X \underset{df}{=} \bigcap_{n=1}^{\infty} X_n$$

is also measurable, and we have

$$[X] = \Pi_{n=1}^{\infty} [X_n].$$

Proof. By de Morgan laws.

9. - Defin. A set X of traces is called *null-set of traces* if its outer coat is O.

A null set is measurable, since $O \leq [X]_* \leq [X]^* = O$; its measure is 0; $\emptyset$ is a null set, W is not a null set. We shall state some theorems whose proofs we omit.

9.1. - If X is a null set, $Y \subseteq X$, then Y is also a null set.

9.2. - If $X_1, X_2, \ldots, X_n, \ldots$ are null sets of traces, so is $\bigcup_{n=1}^{\infty} X_n$.

9.3. - For a set X of traces the following are equivalent: I. X is a null set. II. $\mu(X) = 0$.

9.4. - Theor. If X is measurable and N a null set, then $X - N$ is measurable, and $[X - N] = [X]$, and $\mu(X - N) = \mu(X)$.

Proof. $X - N$ is measurable on account of [9] and [8.11]. We have $(X - N) \cup (N \cap X) = X$ and the sets $X - N$, $N \cap X$ are disjoint.

Hence by [8.13], $\mu(X - N) + \mu(N \frown X) = \mu(X)$. Now $N \frown \frown X \subseteq N$, hence $\mu(N \frown X) = 0$. It follows $\mu(X - N) = \mu(X)$.

9.5. - Theorem. If X is measurable and N is a null-set, then $X \smile N$ is measurable, $[X \smile N] = [X]$ and $\mu(X \smile N) = \mu(X)$.

Proof. $X \smile N$ is measurable because of [9] and [8.11]. We have $X \smile N = X \smile (N - X)$, where X and $N - X$ are disjoint sets. Hence, by [8.13], $\mu(X \smile N) = \mu(X) + \mu(N - X)$. Since $N - X \subseteq N$, we have $\mu(N - X) = 0$. It follows:

$$\mu(X \smile N) = \mu(X). \qquad \text{Q. E. D.}$$

9.6. - Consider the class $\boldsymbol{T}$ of all measurable sets of traces. We have $\mathbf{O} \in \boldsymbol{T}$ and $W \in \boldsymbol{T}$. We have $\mathbf{O} \neq W$.

If $X \in \boldsymbol{T}$, then $\operatorname{co} X = W - X \in \boldsymbol{T}$.

If $X_1, X_2, \ldots, X_n, \ldots \in T$, then $\bigcup_n X_n \in \boldsymbol{T}$.

It follows that $\boldsymbol{T}$ is organized into a Boolean tribe with identity of sets as governing equality and inclusion $\subseteq$ of sets as ordering correspondence. The tribe is denumerably additive. The class $\boldsymbol{J}$ of all null-sets is a denumerably additive ideal in $\boldsymbol{T}$. Hence $\boldsymbol{T}$ can be reorganized into another denumerably additive tribe $\boldsymbol{T}_J$ with $\overset{J}{=}$ as governing equality and $\overset{J}{\subseteq}$, defined by $X \overset{J}{\subseteq} Y \cdot \underset{df}{=} \cdot X - Y \in \boldsymbol{J}$ as the governing lattis ordering.

9.7. - The correspondence $\mathfrak{B}$ which attaches to a variable measurable set X of traces the coat $[X]$ is pluri-one; because if $X \overset{J}{=} Y$, then $[X] = [Y]$, (this follows from [9.4] and [9.5]; see also Preliminaries). The correspondence $\mathfrak{B}$ is invariant with respect to the identity of sets in the domain and equality of somata in the range. It preserves finite and denumerable operations, carries the null set into O and the set W into I. It also preserves the measure. $\mathfrak{B}$ is a homomorphism from $\boldsymbol{T}$ into $\boldsymbol{G}$. The tribe $\mathfrak{B}\boldsymbol{T}$ is a denumerably genuine, denumerably additive strict subtribe of $\boldsymbol{G}$.

9.8. - The correspondence $\mathfrak{B}$ is also invariant with respect to $\overset{J}{=}$ in the domain, and as such one constitutes an isomorphism from $\boldsymbol{T}_J$ into $\boldsymbol{G}$, with preservation of measure.

9.9. - The tribe $\mathfrak{B}(T)$ contains all bricks, hence it contains the tribe F, and then the borelian extension of F within G. (Remember that we have supposed that F is a strict subtribe of G).

Theor. If we suppose that G coincides with the Lebesgue's covering extension F^L of F within G, then $\mathfrak{B}(T)$ coincides with G.

Proof. This follows from [§ 1; 12.1].

9.10. - **Theorem.** If $A \in G$, then there exists a measurable set X of traces such that $A = [X]$. All sets X, for which $[X] = A$, can be obtained from one of one of then, say X_1, by taking $X_1 - N_1 \cup N_2$, where N_1, N_2 are null sets. If $A \neq O$, then X is not empty.

Proof. The existence follows from [9.9]. If $A \neq 0$, then $\mu A \neq 0$, because the measure μ is effective on G. We cannot have $X = \Theta$ because $\mu X = \mu[A] \neq 0$. The remaining thesis follows from that $\mathfrak{B}$ is $1 \to 1$ from T_J into G.

10. - Admit Hyp. [12, 14] of [§ 1]. We shall have some theorems on single traces.

Def. By *neighborhood of the trace* t we shall understand any brick which covers t, (see [Def. 4]). Denote by $v(t)$ the set of all neighborhoods of t.

10.1. - If $a \in v(t)$ and $b \in v(t)$, then $a \cdot b \neq O$.

Proof. Suppose that $a \cdot b = O$. Since a covers t, there exists a representative

$$a = a_1 \geq a_2 \geq \dots \text{ of } t. \tag{1}$$

Since b covers t, there exists a representative

$$b = b_1 \geq b_2 \geq \dots \text{ of } t. \tag{2}$$

As $a \cdot b = O$, we have $a_n \cdot a_m = O$ for all n, m. Now the sequences $\{a_n\}$, $\{b_n\}$, are equivalent, so we have $\{a_n\} \leq \{b_n\}$.

Hence, [2], there exists m with $a_m \leq b_1$. Since $a_m \cdot b_1 = O$, it follows that $a_m = O$ and then $a_m = a_{m+1} = ... = O$, so $\{a_n\}$ is equivalent to $(O, O, ...)$, which is impossible.

10.2. - If a sequence $\{a_n\} \backsim \{O, O, O, ...\}$, [2], then there exists m such that $a_m = a_{m+1} = ... = O$.

Indeed for O there exists n with $a_n \leq O$ hence $a_n = O$ and then $a_n = a_{n+1} = ... = O$.

10.3. - If $a'_1 \geq a'_2 \geq ...$ is a representative of t', $a''_1 \geq a''_2 \geq$ $\geq ...$ is a representative of t'', $b_n = a'_n a''_n \neq O$ for all n, then $t' = t''$.

Proof. We get $b_1 \geq b_2 \geq ...$. Since for every n we have $b_n \leq a'_n$, there exists m with $b_m \leq a'_n$; hence $\{b_n\} \leq$ $\leq \{a'_n\}$. Similarly we get $\{b_n\} \leq \{a''_n\}$. Since $\{a'_n\}$ is a minimal sequence [2.1], we have either $\{b_n\} = \{O, O, ...\}$ or $\{a'_n\} \backsim \{b_n\}$. The first alternative is impossible, hence $\{a'_n\} \backsim \{b_n\}$. Similarly we get $\{a''_n\} \backsim \{b_n\}$; hence $\{a'_n\} \backsim \{a''_n\}$, so $t' = t''$.

10.4. - If $t' \neq t''$, then there exist neighborhoods a' of t', and a'' of t'', such that $a' \cdot a'' = O$.

Proof. Let $a_1 \geq a_2 \geq ...$, $b_1 \geq b_2 \geq ...$ be representatives of t', t'' respectively. There exists, [10.3], at least one n with $a_n \cdot b_n = O$; a_n is a neighborhood of t'; b_n is a neighborhood of t''. The theorem is proved.

10.5. - If $v(t') = v(t'')$, then $t' = t''$.

Proof. Suppose $t' \neq t''$. By [10.4] there exist neighborhoods a', a'' of t', t'' respectively, such that $a' \cdot a'' = O$.

Now $a' \in v(t)$, $a'' \in v(t'')$. Since $v(t') = v(t'')$, we get $a', a'' \in v(t')$. Hence, by [10.1], $a' \cdot a'' \neq O$ which is a contradiction. The theorem is proved.

10.6. - From [10.5] it follows - since $t' = t''$ implies $v(t') = v(t'')$ - that the set of all neighborhoods $v(t)$ completely characterizes the trace t. Different traces have different total sets of neighborhoods.

11. - Def. We say that t is *an elusive trace* whenever $\mu a_n \to 0$, where $\{a_n\}$ is a representative of t. We say that *t is a heavy trace* whenever $\mu a_n \to \alpha > 0$.

11.1. - Theor. If 1) $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots$ is a minimal sequence of bricks, [2.1],

2) $\lambda > 0$, 3) $\mu(a_n) \geq \lambda$,

then $b \underset{df}{=} \Pi_n a_n$ is an atom in G.

Proof. Suppose b is not an atom, we have $\mu b \geq \lambda$. Hence there exists a decomposition $b = b' + b''$ where $b' \cdot b'' = O$, $\mu b' > 0$, $\mu b'' > 0$. Applying an argument similar to that applied in [§ 1; 16.d, 16.e, 16.f], we obtain a brick c, such that $c < b$, $\mu c > 0$. Putting $c_n \underset{df}{=} c$ for $n = 1, 2, \dots$, we get a sequence

$$c_1 \geq c_2 \geq \dots \geq c_n \geq \dots,$$

with $\{c_n\} \leq \{a_n\}$, where $\{c_n\}$ is not equivalent to $\{O, O, \dots\}$.

Since $\{a_n\}$ is a minimal sequence, it follows that $\{c_n\} \backsim \{a_n\}$. Hence there exists n with $a_n \leq c$. This is, however impossible, because $\mu c < \mu b \leq \mu a_n$. Thus we have proved that b is an atom in G. In the above proof we have taken into account the circumstance that μ is effective on G.

11.2. - Theor. If b is an atom in F^L then there exists a minimal sequence $\{a_n\}$ such that

$$b = \Pi_n a_n, \ \mu b = \lim \mu a_n > 0.$$

Proof. Let A be an atom. By [§ 2, 17.1] there exists an infinite sequence of bricks $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots \geq A$, such that $\mu(a_n) \to \mu(A) > 0$. We have $A \leq \Pi_{n=1}^{\infty} a_n$. Since the measure is effective, we cannot have $A < \Pi_{n=1}^{\infty} a_n$. Thus we have

$$A = \Pi_{n=1}^{\infty} a_n.$$

Consider the set U of all traces each of which being covered by all bricks a_n. We shall prove that the set U

is not empty. Denote by U_n the set of all traces which are covered by a_n. We have $\mu U_n = \mu a_n > 0$; hence $U_n \neq 0$, [8.7]. Since $U = \bigcap_n U_n$, and $U_{n+1} \subseteq U_n$, we have $\mu U = \lim \mu U_n = \lim \mu a_n = \mu A > 0$. Hence $U \neq \mathbf{0}$.

The traces belonging to U may be elusive or not. Denote by U_e the subset of U composed of all elusive traces.

We shall prove that U_e is null-set.

Let $t \in U_e$. Let $c_1 \geq c_2 \geq \ldots \geq c_m \geq \ldots$ be a representative of t.

Since t in elusive, we have $\lim \mu c_n = 0$. Hence there exists m_0 such that for all $m \geq m_0$ we have

$$\mu(c_m) < \frac{\mu(A)}{2}. \tag{1}$$

We have $c_m A \leq A$. Since A is an atom, we have either $c_m \cdot A = O$ or $c_m \cdot A = A$. In the second case we get $A \leq c_m$, and hence $\mu(A) \leq \mu(c_m)$, which contradicts (1). Hence $c_m A = O$, i. e.

$$c_m \leq \operatorname{co} A. \tag{2}$$

Since t is covered by c_m and by a_n, and $c_m \cdot a_n$ is a brick, it follows that there exists a brick $c_n(t)$ which covers t and is contained in $\operatorname{co} A$ and a_n. Hence

$$c_n(t) \leq a_n - A. \tag{3}$$

Such a brick can be found for every $t \in U_e$ and for every n.

By the axiom of choice we can find for a given n such set of bricks, $c_n(t)$. We have

$$\sum_{t \in U_e} c_n(t) \leq a_n - A. \tag{4}$$

Let L_n be a covering of $a_n - A$ with $\mu(L_n - (a_n - A)) < \frac{1}{n}$.

Such a covering exists, [7.5]. L_n is also a covering of U_e. Indeed, if $t \in U_e$, $c_n(t)$ is its covering and $c_n(t) \leq a_n - A \leq L_n$. Hence $[U_e]^* \leq L_n$. Now, since $\lim \mu(a_n - A) = 0$, and $\lim \mu(L_n - (a_n - A)) = 0$, we get $\lim \mu L_n = 0$; hence

$$\mu[U_e]^* = 0,$$

which proves that U_e is a null-set.

Denote by U_h the subset of all heavy traces contained in U. Since $U = U_h \cup U_e$, $U_h \cap U_e = \Theta$ and $\mu U_e = 0$, it follows that $\mu U_h = \mu U = \mu A$. We shall prove that U_h contains only one trace. Suppose that t', $t'' \in U_h$, $t' \neq t''$. Let $b'_1 \geq \geq b'_2 \geq \dots$; $b''_1 \geq b''_2 \geq \dots$ be representatives of t', t'' respectively. Since the trace t' is covered by a_m, there exists n with $b'_n \leq a_m$. Thus we can find a subsequence $\{b'_{k(n)}\}$ of $\{b'_n\}$ such that $b'_{k(n)} \leq a_n$ for $n = 1, 2, \dots$, and with $k(1) \geq k(2) \geq \dots$. Putting $c'_n \underset{df}{=} b'_{k(n)}$, we have $c'_1 \geq c'_2 \geq \dots$, and $c'_n \leq a_n$ · $\{c'_n\}$ is a representative of t'. In a similar way we shall find a subsequence $\{c''_n\}$ of $\{b''_n\}$ such that $c''_1 \geq c''_2 \geq \dots$, $c''_n \leq a_n$, and where $\{c''_n\}$ is a representative of t''. Since $t' \neq t''$, there exists, by [10.3], an index n_0 such that $c'_{n_0} \cdot c''_{n_0} = = O$. The sequences $c'_{n_0} \geq c''_{n_0+1} \geq \dots$, $c''_{n_0} \geq c''_{n_0+1} \geq \dots$ are also representatives of t', t'' respectively. The bricks of the first sequence are disjoint with the bricks of the second one. Since t', t'' are heavy traces, we have

$$\lambda' \underset{df}{=} \lim_{n \to \infty} \mu(c'_n) > 0, \qquad \lambda'' \underset{df}{=} \lim_{n \to \infty} \mu(c''_n) > 0;$$

hence, by [11.1], the somata

$$A' \underset{df}{=} \Pi_{n=1}^{\infty} c'_n, \qquad A'' \underset{df}{=} \Pi_{n=1}^{\infty} c''_n$$

are atoms in G.

They are disjoint. Since $c'_n \leq a_n$, $c''_n \leq a_n$, it follows that

$$A' \leq \Pi_{n=1} a_n = A, \qquad A'' \leq \Pi_{n=1} a_n = A.$$

Thus we have $A' + A'' = A$, $\mu A' > 0$, $\mu A'' > 0$, which contradicts the hypothesis that A is an atom. Thus we have proved that U_e has the measure 0 and U_h is composed of a single heavy trace; denote it by t_0. Let $d_1 \geq d_2 \geq \dots$ be a representative of t_0. We can, as before, derive from it another sequence

$$d'_1 \geq d'_2 \geq \dots,$$

representative of t_0, with $d'_n \leq a_n$, because t_0 is covered by the bricks a_n. We get

$$B \underset{df}{=} \Pi_{n=1} d'_n \leq \Pi_{n=1} a_n = A;$$

hence $B \leq A$.

Since t_0 is heavy, we have $\lim_{n \to \infty} \mu(d'_n) > 0$; hence $\mu B > 0$. Now A is an atom; consequently $B = A$. Thus we have proved that if t_0 the unique heavy trace covered by bricks a_n, then for every its representative $\{a_n\}$ we have $\Pi_{n=1}^{\infty} d_n = A$. The theorem is proved.

11.3. - Coroll. If b is an atom in F^L and $a_1 \geq a_2 \geq \dots$ is a sequence of bricks with $\Pi_{n=1}^{\infty} a_n = b$, then the set U_e of all elusive traces which are covered by all a_n is a null-set, and the set U_h composed of all heavy traces which are covered by all a_n, is confined to a single trace.

11.4. - Remark. The theorems [11.1], [11.2], and [11.3], yield information concerning the relation between atoms and heavy traces. They are not the same, of course, — but there is a 1 — 1 correspondence between them. Concerning the proof of [11.2], we notice that J_e may be empty.

11.5. - Remark. Aplying a theorem by Stone concerning ultrafilters, (rather on maximal ideals), we can prove the following:

Every infinite decreasing sequence $a_1 \geq a_2 \geq \dots$ of bricks with $a_n \neq O$, $\lim \mu(a_n) = 0$, contains a minimal sequence b_n, [2.1], though $\{a_n\}$ may not be minimal.

11.6. - Theor. The set (t) composed of a single elusive trace t is measurable. Its measure is 0.

Proof. Let a_n be a representative of t. We have $\mu a_n \to 0$ for $n \to \infty$. Let X_n be the set of all traces covered by a_n.

By [8.7], X_n is measurable, and we have $\mu X_n = \mu a_n$. We have

$$(t) \subseteq \bigcap_{n=1}^{\infty} X_n .$$

Now

$$\mu(\bigcap_{n=1}^{\infty} X_n) = \lim \mu(X_n) = \lim \mu(a_n) = 0.$$

Hence $\bigcap_{n=1}^{\infty} X_n$ is a null set of traces. Consequently (t) is a null set of traces; hence (t) is measurable.

11.7. - **Theor.** The set (t) composed of a single heavy trace t is measurable. Its measure is positive.

Proof. Let $a_1 \geq a_2 \geq \dots$ be a representative of (t). We have

$$\lim_{n \to \infty} \mu(a_n) > 0.$$

Let X_n be the set of all traces which are covered by a_n. We have $\mu(X_n) = \mu(a_n)$. We have

(1) $$(t) \subseteq \bigcap_{n=1}^{\infty} X_n .$$

The set $\bigcap_{n=1} X_n$ is measurable. By [11.1] the soma $A \underset{df}{=} \Pi_{n=1}^{\infty} a_n$ is an atom. Hence, by [11.3 Coroll.], the set U_e of all elusive traces covered by the bricks a_n is a null set, and the set U_h of all heavy traces covered by all a_n is confined to a single trace. We have $U_e \cup U_h = \bigcap_{n=1}^{\infty} X_n$; hence $U_h = \bigcap_{n=1}^{\infty} X_n - U_e$. Hence U_h is measurable. From (1) it follows that $(t) = U_h$. Hence (t) is measurable. We have $(t) = \bigcap_{n=1}^{\infty} X_n - U_e$, and then

$$\mu(t) = \mu \bigcap_{n=1}^{\infty} X_n = \mu \Pi_{n=1}^{\infty} a_n = \mu A > 0.$$

The theorem is proved.

11.8. - (Hyp. I) and (Hyp. II) are necessary conditions for having a measure theory of sets of traces.

12. - **Measurable functions of traces and integration.** The class T' of all measurable sets [9.6] possesses the properties: 1°. if $X \in \boldsymbol{T}$, then co $X \in \boldsymbol{T}$, 2°. if $X_1, X_2, \dots, X_n, \dots \in \boldsymbol{T}$, then $\bigcup_{n=1}^{\infty} X_n \in \boldsymbol{T}$, and there is available a denumerably additive non negative measure $\mu(X)$ defined for all $X \in \boldsymbol{T}$.

These properties enable us to apply Fréchet's theory of measurable functions $f(x)$ of all traces, and, in addition to that consider Fréchet's integrals $\int f(x) d\mu$, (15), (7). [16] This theory follows the known features of the Lebesgue's integration theory. In our case of number valued trace-functions we shall confine ourselves to a sketch only, referring for detailed proofs to (7), (16), (17).

12.1. - Let $f(x)$ be a real valued function defined *almost μ-everywhere* (a. e.) in W. (This means that $\mu(\text{co}\, \mathbb{C}\, f) = 0$). We say that $f(x)$ ***fits*** $\boldsymbol{T}$ or that $f(x)$ ***is measurable***, if whatever the real number λ may be, the set

$$\{ x \mid f(x) \leq \lambda \}$$

belongs to $\boldsymbol{T}$ (i. e. is measurable).

12.2. - This condition is equivalent to each of the following ones:

1°. for every λ the set $\{ x \mid f(x) < \lambda \}$ is measurable,

2°. for every λ the set $\{ x \mid f(x) \geq \lambda \}$ is measurable,

3°. for every λ the set $\{ x \mid f(x) > \lambda \}$ is measurable.

12.3. - **Def.** A measurable function $\varphi(x)$ is called ***simple***, if it is defined almost μ-everywhere and admits an at most denumerable number of values. The following properties hold true:

12.4. - If $f(x)$ is measurable, then there exists an infi-

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(16) Integrals of functions defined on abstract sets are currently called Lebesgue's integrals, but Fréchet (15) was the first who has liberated the integration theory from topological and metrical notions, and this step in that time was a tremendous progress, and the corresponding idea far from the intuition of contemporaneous mathematicians.

nite sequence

$$\varphi_1(x) \geq \varphi_2(x) \geq \dots \geq \varphi_n(x) \geq \dots$$

of simple functions converging on W a. e. uniformly to $f(x)$, and conversely, a uniform a. e. limit of a sequence $\varphi_1(x) \geq \geq \varphi_2(x) \geq \dots$ of simple function is measurable.

12.5. - If $f(x)$ is measurable, so is $|f(x)|$.

12.6. - If $f(x)$, $g(x)$ are measurable, so are are also $f(x) + g(x)$, $f(x) \cdot g(x)$, $f(x) \cup g(x)$, $f(x) \cap g(x)$. The last two functions are defined as $\max[f(x), g(x)]$, $\min[f(x), g(x)]$ for a. e. point x separately.

12.7. - If $\{f_n(x)\}$ is an infinite sequence of measurable functions, and if $\overline{\lim} f_n(x)$, $[\underline{\lim} f(x)]$, is defined a. e., then it is a measurable function.

13. - Let $\varphi(x)$ be a simple function, admitting the values $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ on the measurable and disjoint sets $X_1, X_2, \dots, X_n, \dots$, respectively with $\mu(\text{co} \cup_n X_n) = 0$.

Def. We say that $\varphi(x)$ is μ-*summable*, if the series $\Sigma_{n=1}^{\infty} \varphi_n \mu(X_n)$ converges absolutely.

13.1. - **Def.** The following definition is a generalization of the above one. A measurable function $f(x)$ is said to be μ-*summable*, if there exists two simple μ-summable functions $\varphi(x)$, $\psi(x)$ such that a. e.

$$\psi(x) \leq f(x) \leq \varphi(x).$$

If $f(x)$ is summable,

$$\varphi_1(x) \geq \varphi_2(x) \geq \dots$$

is a sequence of μ-summable function tending a. e. uniformly to $f(x)$, and if $\varphi_n(x)$ admits the values $\varphi_{1n}, \varphi_{2n}, \dots$ on the measurable sets $X_{1n}, X_{2n}, \dots$ respectively, then

$$\lim_{n \to \infty} \Sigma_{k=1}^{\infty} \varphi_{kn} \mu(X_{kn})$$

exists and does not depend on the choice of $\{\varphi_n(x)\}$. This limit is denoted by $\int f(x)d\mu$, and termed ***Frechet's integral of*** $f(x)$ *on* W.

13.2. - For a simple function $\varphi(x)$, admitting the values φ_k in the sets X_k we have

$$\int \varphi(x)d\mu = \Sigma_{k=1}^{\infty} \varphi_k \mu(X_k).$$

13.3. - **Remark.** The integral $\int f(x)d\mu$ can also be defined similarly as did Lebesgue in his « Leçons sur l'integration », (13), and thus can be given an equivalent definition.

13.4. - **Theorem.** (Lebesgue). If

1. $f_n(x)$ are μ–summable.
2. $f_n(x) \leq f_{n+1}(x)$ a. e.,
3. $f(x) = \lim_{n \to \infty} f_n(x)$ a. e., then the following are equivalent:

I. $f(x)$ is μ–summable;
II. $\lim_{n \to \infty} \int f_n(x)d\mu = \int f(x)d\mu$.

14. - The notion of measurability of functions and of their integrals can be extended to complex valued functions. A complex valued function $F(x)$ defined a. e. on W is said to *be measurable* (or else, *fitting* T), if, in the representation

$$F(x) = f(x) + ig(x),$$

the real valued functions $f(x)$, $g(x)$ are both measurable.

14.1. - Every measurable complex-valued function can be uniformly approximated a. e. by complex-valued simple functions.

15. - **Def.** If $f(x)$, $g(x)$ are both μ–summable, the function $F(x)$, in [14], is also termed μ–*summable* and we define

$$\int F(x)d\mu = \int f(x)d\mu + i\int g(x)d\mu.$$

The corresponding notions for real valued functions are but a particular case of the above more general notions.

16. - The notion of integral can be further generalized by introducing integrals over measurable subsets of W. If X is a measurable set of traces, $F(x)$ is a measurable complex valued function defined a. e. on W, then the function $F_1(x)$ defined by setting $F_1(x) = F(x)$ whenever $x \in X$, and $F_1(x) = 0$ whenever $x \in \mathrm{co}\, X$, is also measurable.

16.1. - **Def.** We define, in the case of summability of $F_1(x)$, [16]:

$$\int_X F(x) d\mu = \int_W F_1(x) d\mu,$$

and we say that $F(x)$ is *μ-summable on X*.

16.2. - The following theorems are true:

If $F(x)$ is μ-summable and X is measurable, then $F(x)$ is μ-summable on X.

16.3. - If $F(x)$, $G(x)$ are μ-summable on a measurable set X of traces, and α, β are complex numbers, then $\alpha F(x) + \beta G(x)$ is also μ-summable on X, and we have

$$\int_X (\alpha F(x) + \beta X(x)) d\mu = \alpha \int_X F(x) d\mu + \beta \int_X G(x) d\mu.$$

16.4. - If $F(x)$ is μ-summable on W, $X_1, X_2, \dots X_n, \dots$ are mutually disjoint measurable sets, then if we put $X = \Sigma_n X_n$, we have

$$\int_X F(x) d\mu = \Sigma_n \int_n F(x) d\mu.$$

This theorem says that the complex-valued set-function $A(X) \underset{\mathrm{df}}{=} \int_x F(x) d\mu$, defined for all $X \in \boldsymbol{T}$, is denumerably additive on $\boldsymbol{T}$.

16.5. - If X is measurable and $F(x)$ is summable on X, so is $|F(x)|$, and we have

$$\left| \int_X F(x) d\mu \right| \leq \int_X |F(x)| d\mu.$$

16.6. - (Lebesgue), If

1) $F_n(x)$ are μ-summable functions on a measurable set X,

2) $g(x)$ is a real valued μ-summable function on X,

3) $|F_n(x)| \lessgtr g(x)$, $(n = 1, 2, \ldots)$, a. e. on X,

4) $\lim\limits_{n \to \infty} F_n(x) = F(x)$ a. e. on X,

$$\text{then } \lim_{n \to \infty} \int_X F_n(x) d\mu = \int_X F(x) d\mu.$$

16.7. - If $F_n(x)$ are μ-summable functions on X, and if $F_n(x)$ tends a. e. on X, uniformly to $F(x)$, we have

$$\int_X F(x) d\mu = \lim \int_X F_n(x) d\mu.$$

17. - **Def.** We direct our attention to μ-square summable complex valued functions $F(x)$. We shall state definitions and theorems for functions defined a. e. on W, but the analogous statements will be valid and useful also in the case where $F(x)$ are defined a. e. on a measurable set X of traces.

A function $F(x)$ is said to be *μ-square summable* if

1) $|F(x)|^2$ is μ-summable on W,

2) $F(x)$ is measurable on W.

17.1. - If $F(x)$, $G(x)$ are μ-square-summable on W, so is with $\alpha F(x) + \beta G(x)$, where α, β are complex constants.

17.2. - If $F(x)$, $G(x)$ are μ-square summable on V, then $F(x) \cdot G(x)$ is μ-summable on V, and we have the *Cauchy-Schwartz inequality*

$$\left| \int F(x) \cdot G(x) d\mu \right|^2 \leq \int |F(x)|^2 d\mu \cdot \int |G(x)|^2 d\mu,$$

and the *Minkowski-inequality*

$$\sqrt{\int |F(x) + G(x)|^2 d\mu} \leq \sqrt{\int |F(x)|^2 d\mu} + \sqrt{\int |G(x)|^2 d\mu}.$$

17.3. - If $F(x)$ is μ-square summable, then there exists a sequence $\{\Phi_n(x)\}$, $(n = 1, 2, \ldots)$ of complex simple μ-square summable functions such that $\lim\limits_{n \to \infty} \Phi_n(x) = F(x)$ .a.e. uniformly.

18. - **Def.** Let $\{F_n(x)\}$ be an infinite sequence of μ-square summable functions. We say that it *converges in μ-square mean,* if for every $\varepsilon > 0$ there exists N such that if $n \geq N$, $m \geq N$, we have

$$\int | F_n(x) - F_m(x) |^2 d\mu \leq \varepsilon .$$

18.1. - If $F_n(x)$ converges in μ-square mean, then there exists a square μ-summable function $F(x)$ determined uniquely up to a null-set of traces, such that $\lim\limits_{n \to \infty} \int | F_n(x) - F(x) |^2 d\mu = 0$. We say that $\{F_n(x)\}$ *converges in μ-square mean toward* $F(x)$.

18.2. - If $\{F_n(x)\}$ converges in μ-square mean, then there exists $M \geq 0$ such that $\int | F_n(x) |^2 \, d\mu \leq M$ for $n = 1, 2, \ldots$.

18.3. - If $\{F_n(x)\}$ tends to $F(x)$ in μ-square mean, then every subsequence $\{F_{k(n)}\}$ contains another subsequence $\{F_{kl(n)}\}$ converging toward $F(x)$ almost μ-everywhere.

18.4. - If $\{F_n(x)\}$, $\{G_n(x)\}$ tend in μ-square mean respectively to $F(x)$, $G(x)$, then $\alpha F_n(x) + \beta G_n(x)$ tends in μ-square mean toward $\alpha F(x) + \beta G(x)$, and

$$\lim_{n \to \infty} \int F_n(x) G_n(x) d\mu = \int F(x) G(x) d\mu .$$

18.5. - If $\{F_n(x)\}$ converges on V in μ-square mean toward $F(x)$, if X is a measurable set of traces, then $F_n(x)$, if restricted to X, converges in μ-square mean toward $F(x)$ restricted to X.

18.6. - **Def.** If $F(x)$, $G(x)$ are μ-square summable functions defined a. e., we say that $F(x)$ *is equivalent to* $G(X)$, $F(x) =$ a. e. $G(x)$, if the set

$$(\complement F \dotplus \complement G) \cap \{ x \mid F(x) \neq G(x) \}$$

is a null set.

18.7. - The equivalence possesses the formal properties of the identity. The addition of two functions, the multiplication of a function by a complex number, the integral and the μ–square mean convergence, are invariant with respect to equivalence.

18.8. - The square summable functions with this equivalence considered as a kind of equality, make up a separable and complete Hilbert–Hermite–space (16), with the equality-invariant scalar product $(F, G) = \int \overline{F(x)} G(x) d\mu$ where $\overline{F(x)}$ denotes the conjugate imaginary. Indeed all axioms for H.H.–space (16) are satisfied.

18.9. - If X is a measurable set of traces, then the collection of all square summable functions $F(x)$ such that $F(x) = 0$ a.e. in co X constitutes a (closed) subspace of this H.H.–space. If we vary X these spaces make up a Boole'an denumerably additive saturated tribe of spaces.

§ 4. - Quasi-vectors and their summation.

1. - We take the hypothesis (*FBG*) and terminology of [§ 1; 1] concerning the finitely additive tribe $\boldsymbol{F}$, its basis $\boldsymbol{B}$, and the denumerably additive extension $\boldsymbol{G}$ of $\boldsymbol{F}$. The hypothesis (Hyp. *Ad*), [§ 1; 3], will be admitted. To avoid non-essential complications we shall admit, as in [§ 3; 1], that $\boldsymbol{F}$ is a finitely genuine strict subtribe of $\boldsymbol{G}$, and that the denumerably additive, non negative measure μ on $\boldsymbol{G}$ is effective. In addition to that we shall admit that $\boldsymbol{G}$ is the Lebesguean–covering–extension of $\boldsymbol{F}$ within $\boldsymbol{G}$. It follows that the borelian extension $\boldsymbol{F}^b$ of $\boldsymbol{F}$ within $\boldsymbol{G}$ coincides with $\boldsymbol{G}$, (see [§ 1; 12.1]). We shall take over the theory of $\boldsymbol{B}$–traces in $\boldsymbol{F}$ and admit (Hyp. I) and (Hyp. II) [§ 3; 6], to have the whole measure theory of sets of traces at our disposal.

2. - Let $\boldsymbol{V}$ be a F. Riesz–Banach normed and complete linear space. Its elements $\vec{x}, \vec{y}, \ldots$ will be termed vectors, as in [§ 2].

3. - **Def.** Denote, as in [§ 3; 10], by $v(x)$ the set of all neighborhoods of the trace x. By a *quasi-vector* $\overrightarrow{f_x}(p)$, (or $\overrightarrow{f_x}$) *with support* x, we shall understand any vector-valued function defined for all bricks p belonging to $v(x)$.

If V is the space of real or complex numbers, we shall use also the term *quasi-number with support* x. We know [§ 3; 10.6] that $v(x)$ determines uniquely the trace, hence the support is well determined by a quasi-vector.

3.1. - Various operations can be performed on quasi-vectors with the same support:

Let $\overrightarrow{f_x}$, $\overrightarrow{g_x}$ be two quasi-vectors. By their *sum (difference)* $\overrightarrow{f_x} + \overrightarrow{g_x}$, $(\overrightarrow{f_x} - \overrightarrow{g_x})$ we shall understand the function $\overrightarrow{h_x}(p)$ defined by $\overrightarrow{h_x}(p) \underset{df}{=} \overrightarrow{f_x}(p) \pm \overrightarrow{g_x}(p)$ for all neighborhoods p of x.

By $\lambda \overrightarrow{f_x}$ we shall understand the function $\overrightarrow{h_x}(p)$ defined by $\overrightarrow{h_x}(p) = \lambda \overrightarrow{f_x}(p)$ for all $p \in v(x)$. The number λ is real or complex according to the character of the space V.

If $F(\overrightarrow{X})$ is a number-valued functional or a vector-valued operator, $\overrightarrow{F}(\overrightarrow{X})$, defined for all $\overrightarrow{X} \in V$, we define $\overrightarrow{F}(f_x)$, $[\overrightarrow{F}(\overrightarrow{f_x})]$, as the number-valued function $h(p)$ defined by $h(p) \underset{df}{=} F(\overrightarrow{f}(p))$, [vector-valued function $\overrightarrow{h}(p)$ defined by $\overrightarrow{h}(p) = \overrightarrow{F}(\overrightarrow{f}(p))$], for all $p \in v(x)$. As a particular case we have the ***norm*** of the quasi-vector, $\|\overrightarrow{f}\|$, defined as $\|\overrightarrow{f_x}(x)\|$ for all $p \in v(x)$.

4. - Let $E \neq 0$ be a set of traces. If we have defined, for every $x \in E$, a quasi-vector $\overrightarrow{f_x}$ with support x, we shall say that we have *a set of quasi-vectors with support* E, $\{ f_x \mid x \in E \}$. The construction can also be considered as a quasi-vector-valued function defined on E.

4.1. - We shall be mainly interested in sets of quasi-vectors with support W, i.e. with the set of all traces as support. Such a set of quasi-vectors will be termed ***total***.

4.2. - A set of quasi-vectors with support E can be conceived as a function $\overrightarrow{F}(x, p)$ of two variables: x varying in E, and p varying over the whole set $v(x)$. It is not true that if p is the common neighborhood of two different traces x', x'', we must have $\vec{F}(x', p) = \vec{F}(x'', p)$. Thus to every neighborhood p, which is taken into account, there corresponds a set $T(p)$ of traces $x \in E$, such that $p \in v(x)$. Hence to every p there corresponds a set $\Phi(p)$ of vectors $\vec{f}_x(p)$ where x varies in $T(p)$.

4.3. - Thus we have a function $\vec{F}_1(x, \dot{p})$ which attaches to every p considered whole set of vectors

$$\{\vec{F}(x, p) \mid x \in T(p)\}. \tag{1}$$

Def. If this set is composed of single vector for every $x \in E$, we shall call the set of quasi-vectors, *regular* on E.

4.4. - Especially, if the set of quasi-vectors is total and regular on W, the set of quasi-vectors yields a vector-field $\vec{\varphi}(p)$ defined for all bricks p, [§ 2, def. 2]. If is it not regular on E, we can select in many ways, for each brick p, a trace $x = \alpha(p)$ and consider the vector $\vec{f}_{\alpha(p)}(p)$ which is well determined by the quasi-vector $\vec{f}_{\alpha(p)}(\dot{p})$. If we do that for a total set of quasi-vectors for every brick p, we shall have defined a vector-valued function $\vec{f}_{\alpha(\dot{p})}(\dot{p})$, thus constituting a vector-field defined for all bricks. If, in the case of a total quasi-vectors set, we consider all possible selection of $\alpha(p)$, we shall get various vector-fields $\vec{\Phi}_\alpha(p)$. We shall call them *selected vector-fields* or *generated by the given total set of quasi-vectors*.

5. - **Def.** We shall go over to the summation of a given set of quasi-vectors with support E, where E is a measurable set of traces [§ 3; 8]. We refer to [§ 2]. Let $\vec{f}_x$ be a total set of quasi-vectors. Consider one of the vector-fields $\vec{\Phi}_\alpha(\dot{p})$, defined for all bricks p, and generated by the given total quasi-vectors-set $\vec{f}_x$. Suppose that $\vec{\Phi}_\alpha(\dot{p})$ is summable on $[E]$ (17) in the sense of [§ 2] with respect to a kind (D')

(17) $[E]$ means the coat of E, [§ 3: 8].

of summation. Now, if whatever the choice of $\overrightarrow{\Phi}_\alpha(\dot{p})$ may be, the vector-field $\overrightarrow{\Phi}_\alpha(\dot{p})$ is (D')-summable on E, and the sum $\mathbf{S}_{[E]}\overrightarrow{\Phi}_\alpha$ has for all choices of $\overrightarrow{\Phi}_\alpha$ the same value, we say that the total set $\overrightarrow{f_x}$ of quasi-vectors *is (D')-summable on E, (over E)*, we denote the sum by

$$\mathbf{S}_E \overrightarrow{f_x},$$

and call it « *sum of f_x on E, or (over E)* ».

5.1. - We shall be only interested in $(DARS)$ sums [§ 2; 2.1 2.2] and admit the (Hyp S), [§ 1; 21.1]. We leave the discussion of other kinds of sums to the reader.

5.2. - For $(DARS)$-summation we shall prove the theorem: If for all choices of Φ_α, the sum $\mathbf{S}_{[E]}\overrightarrow{\Phi}_\alpha$ exists, then all these sums must be equal.

Proof. Let $\overrightarrow{\Phi}'(\dot{p})$, $\overrightarrow{\Phi}''(\dot{p})$ be different vector-fields generated by the given total set $\overrightarrow{f_x}$ of quasi-vectors [4.4], and suppose that $\overrightarrow{A}' \neq \overrightarrow{A}''$, where

$$\overrightarrow{A}' \underset{df}{=} \mathbf{S}_{[E]}\overrightarrow{\Phi}', \quad \overrightarrow{A}'' \underset{df}{=} \mathbf{S}_{[E]}\overrightarrow{\Phi}''.$$

Let $\{P_n\}$ be a completely distinguished sequence of complexes for E, [§ 1; 21.3]. Put

$$P_n \underset{df}{=} \{p_{n1}, p_{n2}, \ldots\}, \quad (n = 1, 2, \ldots).$$

Consider the set of all bricks p_{nk}, $(n = 1, 2, \ldots)$, $(k = 1, 2, \ldots)$. For such a brick p_{nk} there is possibly, a double choice of the vector attached to it:

$$\overrightarrow{\Phi}'(p_{nk}), \quad \overrightarrow{\Phi}(p_{nk}).$$

If p_{nk} is an atom, there exists one and only one trace covered by p_{nk}, hence in this case $\overrightarrow{\Phi}'(p_{nk}) = \overrightarrow{\Phi}''(p_{nk})$, so the choice is well determined.

We call p_{nk} *single choice-brick* or *double choice-brick* according to the case, whether $\overrightarrow{\Phi}'(p_{nk})$, $\overrightarrow{\Phi}''(p_{nk})$ are equal or different.

Bricks which are atoms are always of single choice; other bricks may be single choice bricks or not.

We shall find a partial sequence $l(n)$ of indices, as follows. We put $l(1) = 1$. We take the choice $\overrightarrow{\Phi}'(p_{11})$, $\overrightarrow{\Phi}'(p_{12})$, ...

The bricks p_{11}, p_{12}, ... are finite in number.

I say that if n is sufficiently great, then not a single brick among p_{11}, p_{12}, ... will occur in the complex P_n, excepting, perhaps, when the brick is an atom. Suppose this be not true. Then there exists an infinite sequence $t(n)$ of indices, such that in every $P_{t(n)}$ there is available at least one of the non atomic bricks p_{1k}. The number of those bricks p_{1k} is finite. Hence there exists a non atomic brick, say p_{1m}, and a subsequence $\{p_{s(n)}\}$ of $\{p_{t(n)}\}$ such that p_{1m} is a mesh in every complex $P_{s(n)}$, $(n = 1, 2, ...)$: $p_{1m} \in P_{s(n)}$, for $n = 1 \; 2, ...$

Since p_{1m} is not an atom, and since μ is effective, there exist somata A, B such that $\mu A > 0$, $\mu B > 0$, $A \cdot B = O$, $A + B = p_{1m}$. Now $\{p_{s(n)}\}$ is completely distinguished for E [§ 1; 21.6]. We have $|E, P_{s(n)}| \to 0$ for $n \to \infty$. Hence $|Ep_{1m}, \text{som}\, P_{s(n)} \cdot p_{1m}| \to 0$ for $n \to \infty$, i. e. $|Ep_{1m}, p_{1m}| \to 0$.

It follows, as this is a constant sequence, $|Ep_{1m}, p_{1m}| = 0$ and hence $Ep_{1m} = p_{1m}$, which gives $p_{1m} \le E$.

Il follows that $A \le E$. Hence there exists a partial complex Q_n of $P_{s(n)}$ with $|Q_n, A| \to 0$, for $n \to \infty$. Hence $|\text{som}\, Q_n \cdot p_{1m}, Ap_{1m}| \to 0$;

$$(1) \qquad |\text{som}\, Q_n \cdot p_{1m}, A| \to 0,$$

because $A \le p_{1m}$.

Since Q_n is a partial complex of $P_{s(n)}$ to which p_{1m} belongs as a mesh, we have either $\text{som}\, Q_n \cdot p_{1m} = O$ or $\text{som}\, Q_n \cdot p_{1m} = p_{1m}$.

The first alternative cannot occur for an infinite number of indices n, because from (1) we would get $|O, A| \to 0$ i. e. $A = O$ which is not true. Hence the relation spoken of can occur only for an at most finite number of indices n. Hence, for sufficiently great n we have surely

$$\text{som}\, Q_n \cdot p_{1m} = p_{1m},$$

and hence (1) yields $|p_{1m}, A| \to 0$, hence $|p_{1m}, A| = 0$ and then $p_{1m} = A$ which is not true. The obtained contradiction proves that the supposition, stating that $p_{1m} \in P_{s(n)}$ for $n = 1, 2, \ldots$, is not true.

Hence, our statement that if n is sufficiently great, then not a single brick among

$$(2) \qquad p_{11}, p_{12}, \ldots$$

will occur in $p_{n1}, p_{n2}, \ldots$ excepting, perhaps, when a brick (2) is an atom, is proved. Thus we can find an index $l(2) > l(1)$ such that not a single non-atomic brick of $P_{l(1)}$ will occur in any P_n when $n \geq l(2)$.

Considering $P_{l(2)}$ we shall repeat our argument, finding an index $l(3)$, such that if $n \geq l(3)$ non a single non atomic brick of $P_{l(2)}$ will occur in P_n. By induction we shall find an infinite sequence of indices $l(1) < l(2) < \ldots < l(n) < \ldots$ such that if $k < n$, not a single non-atomic brick occuring in:

$$(3) \qquad P_{l(1)}, P_{l(2)}, \ldots, P_{l(k)}$$

will occur in $P_{l(n)}$. Thus if we consider any brick which occurs in all (3), we see that either this is an atom, and can occur in many complexes, or also it occurs only once in (3). The sequence $P_{l(1)}, P_{l(2)}, \ldots$ is completely distinguished for E. Consider the vectors

$$\begin{array}{lll} \vec{\Phi}'(p_{l(1),1}), & \vec{\Phi}'(p_{l(1),2}), & \ldots \\ \vec{\Phi}''(p_{l(2),1}), & \vec{\Phi}''(p_{l(2),2}), & \ldots \\ \vec{\Phi}'(p_{l(3),1}), & \vec{\Phi}'(p_{l(3),2}), & \ldots \\ \vec{\Phi}''(p_{l(4),1}), & \vec{\Phi}''(p_{l(4),2}), & \ldots \end{array}$$

which are defined for bricks $p_{l(n),k}$, ($n = 1, 2, \ldots$ and $k = 1, 2, \ldots$). We denote these vectors by

$$\vec{\Phi}'''(p_{l(n),k}),\ n = 1, 2, \ldots, k = 1, 2, \ldots .$$

They are defined only for bricks $p_{l(n),k}$. There may be some remaining bricks in B, b_1, b_2, ... We put

$$\overrightarrow{\Phi}'''(b_n) = \overrightarrow{\Phi}'(b_n), \quad n = 1, 2, \dots .$$

Thus $\overrightarrow{\Phi}'''(p)$ is defined for all bricks $\in B$ and they constitute a selection of a vector-field generated by the total set $\overrightarrow{f_t}$ of quasi-vectors. Now $\overrightarrow{\Phi}'''(P_{l(n)})$ has a limit, say $\overrightarrow{A}$, by hypothesis. because $\{p_{l(n)}\}$ is completely distinguished for E. We have:

$$\overrightarrow{\Phi}'''(P_{l(2n-1)}) \to \overrightarrow{A}' \quad \text{and} \quad \overrightarrow{\Phi}'''(P_{l(2n)}) \to \overrightarrow{A}''.$$

Hence $\overrightarrow{A}' = \overrightarrow{A}$, $\overrightarrow{A}'' = \overrightarrow{A}$, and then $\overrightarrow{A}' = \overrightarrow{A}''$, which contradicts the hypothesis that $\overrightarrow{A}' \neq \overrightarrow{A}''$. The theorem is established.

5.3 - Remark. The theorem [5.2] is true for (DS), (DAS), and (DNS)-summation, but it is not true for other kind of summation.

6. - The fundamental theorems [6.1 — 6.10] on sums of quasi-vectors will be given for $(DARS)$-summations only, hence we admit $(Hyp\ S)$, [§ 1; 21.1].

These theorems are direct consequence of the corresponding theorems in [§ 2].

6.1. - Considering $(DARS)$-summations, suppose that

1) $\overrightarrow{f_\tau}$ is a total set of quasi-vectors,

2) E, F are measurable sets of traces,

3) $F \subseteq E$, 4) $\overrightarrow{f_\tau}$ is summable on E,

then $\overrightarrow{f_\tau}$ is summable on F.

Proof. Let $\varphi_\alpha(\dot{p})$ be a selected vector-field, generated by $\overrightarrow{f_\tau}$. By hyp. 4 and [Def. 5], the sum $\mathrm{S}_{[E];\,(DARS)} \overrightarrow{\varphi_\alpha}$ exists. Since $[F] \leq [E]$, $[F] \in G$, it follows, by [§ 2; 4], that $\mathrm{S}_{[F]} \overrightarrow{\varphi_\alpha}$ also exists. Applying [Theor. 5.2], we get the thesis.

6.2. - If 1. E_1, E_2 are measurable sets of traces, 2. $E_1 \frown$

$\frown E_2 = \Theta$, 3. $S_{E_1 \cup E_2} \overrightarrow{f_\tau}$ exists then

$$S_{E_1 \cup E_2} \overrightarrow{f_\tau} = S_{E_1} \overrightarrow{f_\tau} + S_{E_2} \overrightarrow{f_\tau}.$$

Proof. We apply [Def. 5], [§ 2; 7], and the fact that $[E_1 \cup E_2] = [E_1] + [E_2]$, [§ 3].

6.3. - If 1. E_1, E_2 are measurable sets of traces 2. $E_1 \frown E_2 = \Theta$, 3. $S_{E_1} \overrightarrow{f_\tau}$ and $S_{E_2} \overrightarrow{f_\tau}$ both exists, then $S_{E_1 \cup E_2} \overrightarrow{f_\tau}$ exists too, and

$$S_{E_1 \cup E_2} \overrightarrow{\varphi_\alpha} = S_{E_1} \overrightarrow{f_\tau} + S_{E_2} \overrightarrow{f_\tau}. \qquad \text{(For } (DARS)\text{-sums).}$$

Proof. We rely on [§ 2; 8].

6.4. - If 1. E, E_n are measurable sets of traces, $(n = 1, 2, \ldots)$, 2. $\mu E_n \to 0$, 3. $E_n \subseteq E$, 4. $S_E \overrightarrow{f_\tau}$ exists, then

$$\lim S_{E_n} \overrightarrow{f_\tau} = \overrightarrow{0}$$

in the V-topology.

Proof. We rely on [§ 2; 5.1] and on equality $\mu[E_n] = \mu E_n$, [§ 3].

6.5. - If 1. E is a measurable set of traces 2. $S_E \overrightarrow{f_\tau}$ exists. 3. $\alpha > 0$, then there exists $\beta > 0$, such that if $\mu(F) \leq \alpha$, where F is a measurable set of traces, then

$$\| S_F \overrightarrow{f_\tau} \| \leq \beta.$$

Proof. We rely on [§ 2; 9.2].

6.6. - If 1. E_n, F are measurable sets of traces, $(n = 1, 2, \ldots)$, 2. E_n are disjoint with one another, 3. $E_n \subseteq F$, 4. $S_F \overrightarrow{f_\tau}$ exists,

then, if we put $E \underset{df}{=} \bigcup_{n=1}^{\infty} E_n$, we get

$$S_E \overrightarrow{f_\tau} = \Sigma_{n=1}^{\infty} S_{E_n} \overrightarrow{f_\tau},$$

so the vector-valued function $S_G \overrightarrow{f_\tau}$ of the variable measurable set G of traces, with $G \subseteq F$, is denumerably additive.

Proof. We rely on [§ 2; 10, 10.1], and on the equality

$$[\bigcup_{n=1}^{\infty} E_n] = \Sigma_{n=1}^{\infty} [E_n], \text{ [§ 3].}$$

6.7. - If 1. E_n, F are measurable sets of traces 2. $E_n \subseteq F$,

$(n = 1, 2, \ldots)$, 3. $|E_n \cdot E|_\mu \to 0$, 4. $\mathbf{S}_F \overrightarrow{f_\tau}$ exists, then $\lim\limits_{n \to \infty} \mathbf{S}_{E_n} \overrightarrow{f^\tau} = \mathbf{S}_E \overrightarrow{f_\tau}$ in the V-topology.

Proof. We rely on [§ 2; 12].

6.8. - If $\mathbf{S}_E \overrightarrow{f_\tau}$ exists for a measurable set E of traces, then if $\mu F = 0$, we have

$$\mathbf{S}_F \overrightarrow{f_\tau} = \overrightarrow{O}.$$

6.8.1. - If $\mathbf{S}_E \overrightarrow{f_\tau}$ exists, and $E =^\mu F$, then

$$\mathbf{S}_E \overrightarrow{f_\tau} = \mathbf{S}_F \overrightarrow{f_\tau}.$$

6.9. - 1. $\mathbf{S}_W \overrightarrow{f_\tau}$ exists, (W is the set of all traces), 2. we put for every brick p: $\overrightarrow{K}(p) \underset{df}{=} \mathbf{S}_P \overrightarrow{f_\tau}$, where $[P] \underset{df}{=} p$, then for every measurable set E of traces we have

$$\mathbf{S}_E \overrightarrow{f_\tau} = \mathbf{S}_{[E]} \overrightarrow{K}(\dot{p}).$$

Proof. We rely on [§ 2; 13].

6.9.1. - **Remark.** In relation to [6.9], if we define the quasi-vector $\overrightarrow{k_\tau}$ by putting $\overrightarrow{k_\tau}(v) \underset{df}{=} \overrightarrow{K}(v)$ for every neighborhood of τ, we get $\mathbf{S}_E \overrightarrow{f_\tau} = \mathbf{S}_E \overrightarrow{k_\tau}$ for every measurable set E of traces. Hence the $(DARS)$-sums of quasi-vectors can be transformed into sums of regular quasi-vector-sets, [4.3].

6.10. - If f_τ is summable on a measurable set E of traces and λ is a number (real or complex depending on the character of V), then

$$\mathbf{S}_E \lambda \overrightarrow{f_\tau} = \lambda \mathbf{S}_E \overrightarrow{f_\tau}.$$

Proof. [§ 2; 4.1].

6.11. - If f_τ, g_τ are both summable on a measurable se E of traces, then

$$\mathbf{S}_E (\overrightarrow{f_\tau} \pm \overrightarrow{g_\tau}) = \mathbf{S}_E \overrightarrow{f_\tau} \pm \mathbf{S}_E \overrightarrow{g_\tau}.$$

Proof. We shall represent the set of quasi-vectors $\overrightarrow{f_\tau}$ as a function $\overrightarrow{f}(\tau, p)$ of two variables: τ and p, where p is a neighborhood of τ. Similarly for $\overrightarrow{g_\tau}$. By definition of the sum of two quasi-vectors [3.1], for the quasi-vector $h_\tau \overline{\overline{df}} \overrightarrow{f_\tau} + \overrightarrow{g_\tau}$ we have

$$(1) \qquad \overrightarrow{h}(\tau, p) = \overrightarrow{f}(\tau, p) + \overrightarrow{g}(\tau, p).$$

whenever p is a neighborhood of τ. Take a choice of a vector field $\overrightarrow{h}(p)$ generated by h_τ. It is determined by the choice of the function $\tau = \alpha(p)$ where p is a neighborhood of τ. Then the vector fields for $\overrightarrow{h_\tau}$, $\overrightarrow{f_\tau}$, $\overrightarrow{g_\tau}$ will be $\overrightarrow{h}(\alpha(\dot{p}), \dot{p})$, $\overrightarrow{f}(\alpha(\dot{p}), \dot{p})$, $\overrightarrow{g}(\alpha\, \dot{p}), \dot{p})$ and, by (1) we have

$$(2) \qquad \overrightarrow{h}(\alpha(\dot{p}), \dot{p}) = \overrightarrow{f}(\alpha(\dot{p}), \dot{p}) + \overrightarrow{g}(\alpha(\dot{p}), \dot{p}),$$

for all bricks p. We have

$$\mathrm{S}_E \overrightarrow{f_\tau} = \mathrm{S}_{[E]} \overrightarrow{f}(\alpha(\dot{p}), \dot{p}). \qquad \mathrm{S}_E \overrightarrow{g_\tau} = \mathrm{S}_{[E]} \overrightarrow{g}(\alpha(\dot{p}), \dot{p}).$$

Hence by (2) and [§ 2; 15]

$$(3) \qquad \mathrm{S}_E \overrightarrow{f_\tau} + \mathrm{S}_E \overrightarrow{g_\tau} = \mathrm{S}_{[E]} \overrightarrow{h}(\alpha(\dot{p}), \dot{p}).$$

Thus the sum on the right in (3) exists and has the same value for any choice of the vector-field generated by h_τ. Hence [Def. 5]

$$\mathrm{S}_E \overrightarrow{f_\tau} + \mathrm{S}_E \overrightarrow{g_\tau} = \mathrm{S}_E(\overrightarrow{f_\tau} + \overrightarrow{g_\tau}). \qquad \text{Q. E. D.}$$

7. - We shall deal with (*DARS*)-sums only. Let $\overrightarrow{f_x}$, $\overrightarrow{g_x}$ be two total sets of quasi-vectors, which are summable on W.

Def. We say that $\overrightarrow{f_x}$ *is equivalent to* g_x:

$$\overrightarrow{f_x} \approx \overrightarrow{g_x},$$

whenever for every measurable set E of traces we have $\mathrm{S}_E \overrightarrow{f_x} = \mathrm{S}_E \overrightarrow{g_x}$.

7.1. - We have for total, summable sets of quasi-vectors

1) If $\overrightarrow{f_x} \approx \overrightarrow{g_x}$, then $\overrightarrow{g_x} \approx \overrightarrow{f_x}$,

2) $\overrightarrow{f_x} \approx \overrightarrow{f_x}$,

3) if $\overrightarrow{f_x} \approx \overrightarrow{g_x}$, $\overrightarrow{g_x} \approx \overrightarrow{h_x}$, then $\overrightarrow{f_x} \approx \overrightarrow{h_x}$.

7.2. - If $\overrightarrow{f_x} \asymp \overrightarrow{g_x}$ and λ is a number, then $\lambda \overrightarrow{f_x} \asymp \lambda \overrightarrow{g_x}$.

7.3. - If $\overrightarrow{f_x} \asymp \overrightarrow{g_x}$, $\overrightarrow{f'_x} \asymp \overrightarrow{g'_x}$, then $\overrightarrow{f_x} + \overrightarrow{f'_x} \asymp \overrightarrow{g_x} + \overrightarrow{g'_x}$.

Proof. We rely on [6.11].

7.4. - From [6.9.1] it follows, that every total summable set of quasi-vectors is equivalent to a regular total summable set of quasi-vectors.

8. - An important case of the vector-space V is the space of real number and the space of complex numbers. The vector-fields, in these particular cases, will be termed *scalar fields* and quasi-vectors will be termed *quasi-numbers*.

The function $\mu(\dot{a})$ of the variable brick a is a scalar field, and if for all neighborhoods p of x we define $\mu_x \underset{df}{=} \mu(\dot{p})$, we get a real quasi-number. We shall call it *measure-quasi-number*. The total set μ_x is regular.

8.1. - Quasi numbers f_x, g_x can be multiplied, getting a new quasi-number $f_x \cdot g_x$ defined as the function $f_x(p) \cdot \cdot g_x(p)$ for every neighborhood p of x. Given a quasi vector $\overrightarrow{f_x}$ taken from a general Banach space V, and given a quasi-number a_x, we can multiply them, getting a quasi-vector in V: $a_x \cdot \overrightarrow{f_x}$, defined in a similar way, as above.

9. - If M_x is a total set of quasi numbers, summable on W, then there exists a complex-valued function $F(\dot{x})$ of the trace x variable in W, such that, for every measurable set A of traces, we have

$$\mathrm{S}_A M_x = \int_{A^{\cdot}} F(\dot{x}) d\mu.$$

The function F is μ-unique. The integral is Fréchetian. [§ 3: 10].

Proof. Put $K(A) \underset{df}{=} \mathrm{S}_A M_x$ for all measurable A. By theor. [6.6] the set-function $K(A)$ is denumerably additive. If

$\mu(A)=0$, we have $[A]=O$, since the measure μ is effective on G. Hence $K(\mathrm{O})=0$. Consequently, (7), $K(A)$ is μ-continuous. Hence, by a known theorem, (7), there exists a μ-unique function $F(x)$, defined μ-a. e. on W, such that

$$\int_A F(x)d\mu = \mathrm{S}_A\, f_x,$$

so the theorem is proved.

10. - **Def.** Let $f(\dot{x})$ be a complex-number-valued function of the trace x variable in W. Suppose this function is μ-Fréchet-summable on W. Hence the integral

$$\int_A f(x)d\mu$$

exists for every measurable subset of traces. Considering a brick p, and a measurable set P of traces, with $[P]=p$, the integral

$$\int_P f(x)dx$$

does not depend on P but on p only. Having fixed x for a moment, and considering all neighborhoods p of x, we put

$$\mathrm{val}_x\, f \underset{df}{=} \frac{1}{\mu(P)}\int_P f(\dot{x})d\mu\,.$$

We may call it: *The mean-value quasi-vector of f at x.* This is a quasi-number with support x. We also define:

$$\mathrm{val}_p\, f \underset{df}{=} \frac{1}{\mu(p)}\int_p fd\mu.$$

11. - We shall prove the theorem: Under circumstances [1], if $f(\dot{x})$ is Fréchet-summable on W,
then the total set of quasi-numbers $\mu_x \cdot \mathrm{val}_x f$ is summable

on W, and we have

$$\mathbf{S}_A \operatorname{val}_x f \cdot \mu_x = \int_A f(x) d\mu$$

for every measurable set A.

Proof. Put $k_x \underset{df}{=} \mu_x \cdot \operatorname{val}_x f$. This quasi-number is defined as the function

$$k_x(p) \underset{df}{=} \mu(p) \cdot \frac{1}{\mu(p)} \int_P f(x) d\mu$$

defined for all neighborhoods p of x where $[P] = p$. Hence

$$k_x(p) = \int_P f(x) d\mu.$$

Now, let $P = \{p_1, p_2, \ldots\}$ be a complex. Since the somata p_i are disjoint, we get

$$\text{(1)} \qquad \mathbf{S}_{p'} k_x = \int_{P'} f(x) d\mu,$$

where $[P'] = \operatorname{som} P$. Now let A be any measurable set of traces, and let $\{P_n\}$ be any sequence of complexes with $|P_n, A|_\mu \to 0$. If we denote by P'_n the set of traces with $[P'_n] = \operatorname{som} P_n$, we get

$$\lim_{n \to \infty} \int_{P_n'} f(x) d\mu = \int_A f(x) d\mu.$$

Hence, by (1), the sequence $\{\mathbf{S}_{P_n'} k_x\}$ converges to

$$\int_A f(x) d\mu.$$

On the other hand this sequence converges to $\mathbf{S}_E k_x$, [6.7].

Consequently

$$\mathbf{S}_E k_x = \int_A f(x) d\mu. \qquad \text{Q. E. D.}$$

11.1. - **Remark.** Notice that the quasi-number-set $g_x \underset{df}{=} \mathrm{val}_x f$ is regular, but it may be not summable.

Ex. Let $f = \mathrm{const} = 1$, we get $\mathrm{val}_x f(p) = 1$ for all p. Hence if a complex P has n bricks, the number $g(P) = n$, so it does not tend to any limit.

12. - We shall be in circumstances [1] till the end of the [§ 4] and shall consider only $(DARS)$-summation. We admit (Hyp. S).

13. - If $f(x)$, $g(x)$ are Fréchet-summable on W, then

$$\mu_x \,\mathrm{val}_x (f + g) \simeq\simeq \mu_x \,\mathrm{val}_x f + \mu_x \,\mathrm{val}_x g,$$

[Def. 7].

Proof. Since f and g are Fréchet-summable on W, so is $f + g$. By [11], we have for every measurable set A of traces

$$\text{(1)} \qquad \mathrm{S}_A \,\mathrm{val}_x f \cdot \mu_x = \int_A f(x) d\mu,$$

$$\text{(2)} \qquad \mathrm{S}_A \,\mathrm{val}_x g \cdot \mu_x = \int_A g(x) d\mu,$$

and

$$\text{(3)} \qquad \mathrm{S}_A \,\mathrm{val}_x (f + g) \mu_x = \int_A (f + g) d\mu$$

Since the sum of the right-hand integrals in (1) and (2) equals that of (3), it follows that the same is for the left-hand sums, so the theorem is proved.

14. - Suppose [1]. If $f(x)$ is Fréchet-summable on W, and λ is a number, then

$$\mu_x \,\mathrm{val}_x (\lambda f) \simeq\simeq \lambda \,\mu_x \,\mathrm{val}_x f.$$

Proof. Similar to the forgoing one, based on [11].

15. - **Theorem.** If $f(x)$ is Fréchet-square-summable on W, then

$$\mu_x \operatorname{val}_x |f|^2 \approx\approx \mu_x |\operatorname{val}_x f|^2.$$

The proof of this theorem will require few auxiliary theorems and steps. In what follows, till the end of this § 4., we shall agree to denote by the same letter a measurable set of traces and its coat: this for simplifying the exposition. There will be an ambiguity up to null-sets of traces. These null-sets, however, do not matter: see [6.8.1.].

Notice that $\mu_x \operatorname{val}_x |f|^2$ is summable, [11], because $|f|^2$ is Fréchet-summable. The summability of the right-hand side expression in the thesis shall be proved.

15.a. - First of all we shall prove the theorem [15] for the functions $f(\dot{x})$ defined as follows: there is a brick $c \neq O$ such that $f(x) = 1$ for $x \in c$ and $f(x) = 0$ for $x \in \operatorname{co} c$.

Let A be a measurable set of traces. Consider a completely distinguished sequence P_n of complexes for A, and select any subsequence $\{P_{k(n)}\}$ of it. By [§ 1; 21.6] this is also a completely distinguished sequence for A. Applying [§ 1; 21.13] find a subsequence $\{P_{kl(n)}\}$ of it and a sequence $\{Q_n\}$ of complexes for co A such that som $Q_n \cdot$ som $P_{kl(n)} = O$ and where $\{P_{kl(n)} \cup Q_n\}$ is a completely distinguished sequence for I. Put

$$R_n \underset{df}{=} P_{kl(n)} \cup Q_n, \qquad (n = 1, 2, \ldots).$$

The complex $P_{kl(n)}$ is a partial complex of R_n.

Denote by g_x, h_x the quasi-numbers $\mu_x \operatorname{val}_x |f|^2$ and $\mu_x |\operatorname{val}_x f|^2$ respectively. For any brick p we have

$$g(p) = \int_{pc} d\mu = \mu(pc), \qquad h(p) = \frac{|\mu(pc)|^2}{\mu(p)}.$$

Hence

$$g(p) - h(p) = \frac{\mu(pc) \cdot \mu(p \operatorname{co} c)}{\mu(p)}.$$

Consider the bricks of $P_{kl(n)}$. Denote by $a_{n1}, a_{n2}, \ldots; b_{n1},$

$b_{n2}, \ldots; e_{n1}, e_{n2}, \ldots$ those of them for which $a_{nk} \leq c$; $b_{nk} \cdot \cdot c = O$; $e_{nk}c \neq O$, $e^{nk} \operatorname{co} c \neq O$ respectively.

We have

$$g(P_{kl(n)}) = \Sigma_k g(a_{nk}) + \Sigma_k g(e_{nk}),$$

and

$$h(P_{kl(n)}) = \Sigma_k h(a_{nk}) + \Sigma_k h(e_{nk}),$$

because $g(b_{nk}) = 0$.

Now

$$g(a_{nk}) = \mu(a_{nk}c) = \mu(a_{nk}), \qquad h(a_{nk}) = \mu(a_{nk}).$$

It follows that

$$(0) \qquad g(P_{kl(n)}) - h(P_{kl(n)}) = \Sigma_k \frac{\mu(e_{nk} \cdot c)\mu(e_{nk} \operatorname{co} c)}{\mu(e_{nk})}.$$

We recall that e_{nk} are all those bricks in $P_{kl(n)}$ for which

$$e_{nk} \cdot c \neq O, \qquad e_{nk} \operatorname{co} c \neq O.$$

Since $\{R_n\}$ is a completely distinguished sequence for I, there exists a partial complex S_n of R_n, such that $\{S_n\}$ is completely distinguished for som c.

We have $|c \cdot S_n| \to 0$. Hence $|\operatorname{co} S, T_n| \to 0$ where $T_n \underset{df}{=} R_n \sim S_n$.

We have

$$(1) \qquad S_n \frown T_n = \mathbf{O}, \qquad S_n \smile T_n = R_n.$$

By [§ 1; 21.9], T_n is a completely distinguished sequence for $\operatorname{co} c$.

Consider all bricks $d_{n1}, d_{n2} \ldots$ of R_n for which $d_{ni} \cdot c \neq O$, $d_{ni} \cdot \operatorname{co} c \neq O$.

They make up two classes: one composed of those bricks which are in S_n; denote then by

$$(2) \qquad d'_{n1}, d'_{n}, \ldots,$$

and the second composed of those bricks, which are in T_n: denote them by

$$(3) \qquad d''_{n1}, d''_{n2}, \ldots.$$

The classes (2), (3) are disjoint and make up the set d_{n1}, d_{n2}, ... of bricks. By [§ 1; 21.14], $\mu(\Sigma d''_{nk} \operatorname{co} c) \to 0$ and $\mu(\Sigma d''_{nk}\, c) \to 0$. Now a brick e_{nk} either belongs to (2) or to (3). Hence all e_{nk} can be devided into two disjoint classes $e'_{n1}, e'_{n2}, \ldots$, and $e''_{n1}, e''_{n2}, \ldots$, where we have

$$\mu(\Sigma e'_{n1} \operatorname{co} c) \to 0, \ \mu(\Sigma e''_{n1}\, c) \to 0.$$

Having this, resume the formula (0):

$$g(P_{kl(n)}) - h(P_{kl(n)}) = \Sigma_k \frac{\mu(e_{nk}\, c) \cdot \mu(e_{nk} \operatorname{co} c)}{\mu(e_{nk})} =$$

$$= \Sigma_k \frac{\mu(e'_{nk}\, c)\, \mu(e'_{nk} \operatorname{co} c)}{\mu(e'_{nk})} + \Sigma_k \frac{\mu(e''_{nk}\, c)\, \mu(e''_{nk} \operatorname{co} c)}{\mu(''_{nk})} \leq$$

$$\lessapprox \Sigma_k \mu(e' \operatorname{co} c) + \Sigma_k \mu(e''_{nk} c) \to 0.$$

Since, as we know, [11], the quasi vector set $\mu_x \operatorname{val}_x |f|^2$ is summable, it follows that $\Sigma_k h(a_{nk})$ tends to $\mathbf{S}_A \mu_x \operatorname{val}_x |f|^2$. Thus from every partial sequence $\{P_{k(n)}\}$ another one can be extracted $\{P_{lk(n)}\}$ such that $\lim h(P_{lk(n)}) = \mathbf{S}_A\, \mu_x \operatorname{val}_x |f|^2$. Consequently

$$\lim h(P_n) = \mathbf{S}_A \mu_x \operatorname{val}_x |f|^2.$$

Hence

$$\mathbf{S}_A \mu_x |\operatorname{val}_x f|^2 = \mathbf{S}_A \mu_x \operatorname{val}_x |f|^2$$

for every measurable set A of traces. Thus we get

$$\mu_x |\operatorname{val}_x f|^2 \approx \mu_x \operatorname{val}_x |f|^2. \qquad \text{Q. E. D.}$$

15.b. - If $f(x)$ is the function as in [15a] and we put $g(x) = \lambda f(x)$ where λ is a number, then the thesis holds for $g(x)$.

15.c. - **Lemma.** If 1. $[c] \neq O$ is a figure, 2. $f(x)$, $g(x)$ are μ–square summable on W, 3. $f(x) = 0$ for $x \in c$, $g(x) = 0$ for $x \in \operatorname{co} c = W - c$, 4. A is a measurable set of traces,

5. $P_n = \{p_{n1}, p_{n2}, \ldots\}$, $(n = 1, 2, \ldots)$, is a completely distinguished sequence for $[A]$, then

$$A_n \underset{df}{=} \Sigma_k \frac{\int\limits_{p_{kn}} f d\mu \cdot \int\limits_{p_{kn}} g d\mu}{\mu(p_{nk})} \to 0$$

for $n \to \infty$, (p_{nk} are considered as sets of traces).

Proof. We take over the notations from the preceding proof. $\{P_{(kn)}\}$ is subsequence of $\{P_n\}$. We build $R_n \underset{df}{=} P_{kl(n)} \cup \cup Q_n$.

We find $\{S_n\}$ and $\{T_n\}$ with $|c, S_n| \to 0$, $|\mathrm{co}\, c, T_n| \to 0$. We get the bricks a_{nk}, b_{nk} where $a_{nk} \leq p$, $b_{nk} \cdot p = O$, and e'_{nk}, e''_{nk} which all belong to $P_{kl(n)}$. We have

$$\mu(\Sigma e'_{nk}\, \mathrm{co}\, c) \to 0, \qquad \mu(\Sigma e''_{nk} c) \to 0.$$

Now

$$A_n = \Sigma_k \frac{\int\limits_{e'_{nk}} f d\mu \cdot \int\limits_{e'_{nk}} g d\mu}{\mu(e'_{nk})} + \Sigma_k \frac{\int\limits_{e''_{nk}} f d\mu \cdot \int\limits_{e''_{nk}} g d\mu}{\mu(''_{nk})}.$$

Denote the first term by B', the second by B''. We have

$$|B'| \leq \Sigma_k \frac{\sqrt{\mu(e'_{nk}) \int\limits_{e'_{nk}} |f|^2 d\mu} \cdot \sqrt{\mu(e'_{nk}) \int\limits_{e'_{nk}} |g|^2 d\mu}}{\mu(e'_{nk})} =$$

$$= \Sigma_k \sqrt{\int\limits_{e'_{nk}} |f|^2 d\mu} \cdot \sqrt{\int\limits_{e'_{nk}} |g|^2 d\mu}.$$

Applying once more the Cauchy–Schwarz–inequality, we get

$$|B'| \leq \sqrt{\Sigma_k \int\limits_{e'_{nk}} |f|^2 d\mu} \cdot \sqrt{\Sigma_k \int\limits_{e'_{nk}} |g|^2 d\mu}.$$

Since $f(x) = 0$ for $x \in c$ and $g(x) = 0$ for $x \in \mathrm{co}\, c$, we get

$$|B'| \leq \sqrt{\Sigma_k \int\limits_{e'_{nk} \mathrm{co}\, c} |f|^2 d\mu} \cdot \sqrt{\Sigma_k \int\limits_{e'_{nk} c} |g|^2 d\mu}.$$

Since $\mu(\Sigma e'_{nk} \operatorname{co} c \mid \to 0$, we get

$$| B' | \leq \sqrt{\int\limits_{\Sigma_k e'_{nk} \operatorname{co} e} | f |^2 d\mu} \cdot \sqrt{\int\limits_{W} | g |^2 d\mu} \to 0.$$

In a similar way we get $|B''| \to 0$. It follows $A_n \to 0$. Q.E.D.

15.d. - Lemma. If 1. f(x), $g(x)$ are μ-square summable on W, 2. c is a set whose coat $c = [c] \in F$, 3. $\mu_x \mid \operatorname{val}_x f |^2 \lessapprox \lessapprox \mu_c \operatorname{val}_x | f |^2$, $\mu_x \operatorname{val}_x g |^2 \lessapprox \mu_x \operatorname{val}_c | g |^2$, 4. $f(x) = 0$ for $x \in \in \operatorname{co} c$, $g(x) = 0$ for $x \in c$, 5. $h(x) \underset{df}{=} f(x) + g(x)$, then

$$\mu_x | \operatorname{val}_x h |^2 \lessapprox \mu_x \operatorname{val}_x | h |^2.$$

Proof. $| h |^2 = (\bar{f} + \bar{g})(f + g) = \bar{f}f + \bar{g}f + \bar{f}g + \bar{g}g = | f |^2 + + | g |^2 + \bar{f}g + \bar{g}f$. Hence

$$(1) \qquad \operatorname{val}_x | h |^2 \lessapprox \operatorname{val}_x | f |^2 + \operatorname{val}_x | g |^2 + \operatorname{val}_x | \bar{f}g | + + \operatorname{val}_x | \bar{g}f | \lessapprox \operatorname{val}_x | f |^2 + \operatorname{val}_x | g |^2$$

because $\bar{f}g = \bar{g}f = 0$.

The quasi-number $\operatorname{val}_x (x + g)$ is the function $\frac{1}{\mu(p)} \int (f + + g) d\mu$, hence $| \operatorname{val}_x (f + g) |^2$ is the function

$$\frac{1}{\mu(p)^2} \cdot \int\limits_p (\overline{f + g}) d\mu \cdot \int\limits_p (f + g) d\mu = \frac{1}{\mu(p)^2} \left\{ | f d\mu |^2 + \int\limits_p g d\mu \ ^2 + + \int\limits_p \bar{f} d\mu \cdot \int\limits_p g d\mu + \int\limits_p f d\mu \cdot \int\limits_p \bar{g} d\mu \right\}.$$

Hence

$$(2) \qquad |\operatorname{val}_x h|^2 = |\operatorname{val}_x f|^2 + |\operatorname{val}_x g|^2 + \operatorname{val}_x \bar{f} \cdot \operatorname{val}_x g + \operatorname{val}_x f \cdot \operatorname{val}_x \bar{g}.$$

From (1) and (2) we get

$$\mu_x \operatorname{val}_x | h |^2 - \mu_x | \operatorname{val}_x h |^2 \approx - \mu_x \operatorname{val}_x \bar{f} \operatorname{val}_x g - \mu_x \operatorname{val}_x f \cdot \operatorname{val}_x \bar{g}.$$

If we apply the [Lemma 15c.], we get

$$\mu_x \operatorname{val}_x | h |^2 \approx \mu_x | \operatorname{val}_x h |^2. \qquad \text{Q. E. D.}$$

15.e. - If $f(x)$ is a step-function, then

$$\mu_x \,|\, \mathrm{val}_x f \,|^2 \approxeq \mu_x \,\mathrm{val}_x \,|\, f \,|^2.$$

Proof. [15.a], [15.b] and [15.d].

15.f. - **Lemma.** If $f(x)$ is square-summable, then there exists an infinite sequence of step-functions $f_1(x)$, $f_2(x)$, ... which tends in μ-square-mean to $f(x)$. This is known from the general theory of Fréchet's integrals.

15.g. If

1. $f_1(x), f_2(x), \dots, f_n(x), \dots$ are all μ-square summable on W,
2. $f(x)$ is the square-mean-limit of $\{f_n(x)\}$ in W,
3. for every n we have $\mu_x \,\mathrm{val}_x \,|\, f_n \,|^2 \approxeq \mu_x \,|\, \mathrm{val}_x f_n \,|^2$,

then $\mu_x \,\mathrm{val}_x \,|\, f \,|^2 \approxeq \mu_x \,|\, \mathrm{val}_x f \,|^2$.

Proof. Suppose that for all functions $f_1(x), f_2(x), \dots, f_n(x), \dots$ which we suppose μ-square-summable in W, we have

$$\mu_x \,\mathrm{val}_x \,|\, f_n \,|^2 \approxeq \mu_x \,|\, \mathrm{val}_x f_n \,|^2, \tag{0}$$

Suppose that $\lim f_n(x) = f(x)$. We shall prove that

$$\mu_x \,\mathrm{val}_x \,|\, f \,|^2 \approxeq \mu_x \,|\, \mathrm{val}_x f \,|^2.$$

Put

$$A_n \underset{df}{=} \int_p |\, f_n \,|^2 d\mu, \quad A \underset{df}{=} \int_p |\, f \,|^2 d\mu,$$

$$B_n \underset{df}{=} \frac{1}{\mu(p)} \left| \int_p f_n d\mu \right|^2, \quad B \underset{df}{=} \frac{1}{\mu(p)} \left| \int_p f d\mu \right|^2,$$

where $p \neq O$ is a brick. We have

$$A_n - A = \int_p |\, f_n \,|^2 d\mu - \int_p |\, f \,|^2 d\mu = \int_p (f_n \bar{f}_n - f\bar{f}) d\mu =$$

$$= \int_p (f_n \bar{f}_n - f_n \bar{f} + f_n \bar{f} - f\bar{f}) d\mu =$$

$$= \int_p f_n(\bar{f}_n - \bar{f}) d\mu + \int_p \bar{f}(f_n - f) d\mu.$$

Hence

$$(1)\qquad |A_n - A| \leq \sqrt{\int_p |f_n|^2 d\mu} \cdot \sqrt{\int_p |\bar{f}_n - \bar{f}|^2 d\mu} +$$

$$+ \sqrt{\int_p |\bar{f}|^2 d\mu} \cdot \sqrt{\int_p |f_n - f|^2 d\mu} \leq \sqrt{\int_p |\bar{f}_n - \bar{f}|^2 d\mu} \cdot$$

$$\cdot \left[\sqrt{\int_p |f_n|^2 d\mu} + \sqrt{\int_p |f|^2 d\mu}\right] \cdot$$

We have $\mu(p)(B_n - B) = |\int_p f_n d\mu|^2 - |\int_p f d\mu|^2 = \int_p f_n d\mu \cdot \int_p \bar{f}_n d\mu -$
$- \int_p f d\mu \cdot \int_p \bar{f} d\mu = \int_p f_n d\mu \cdot [\int_p (\bar{f}_n - \bar{f}) d\mu] + \int_p \bar{f} d\mu \cdot [\int_p (f_n - f) d\mu].$

Hence

$$(2)\qquad |B_n - B| \leq \sqrt{\int_p |f_n|^2 d\mu} \cdot \sqrt{\int_p |\bar{f}_n - \bar{f}|^2 d\mu} +$$

$$+ \sqrt{\int_p |\bar{f}|^2 d\mu} \cdot \sqrt{\int_p |f_n - f|^2 d\mu} \leq$$

$$\leq \left[\sqrt{\int_p |f_n|^2 d\mu} + \sqrt{\int_p |f|^2 d\mu}\right] \cdot \sqrt{\int_p |f_n - f|^2 d\mu}.$$

We have the same estimate in (1) and (2). Take any measurable set E of traces.

Now let $P_\alpha = \{p_{\alpha 1}, p_{\alpha 2}, \dots\}$, $(\alpha = 1, 2, \dots)$ by a distinguished sequence for $[E]$. The relation (0) says that

$$(2.1)\qquad \mathbf{S}_E\, \mu_x \operatorname{val}_x |f|^2 = \mathbf{S}_E\, \mu_x |\operatorname{val}_x f_n|^2.$$

The left sum is the limit of the sequence

$$M_{n\alpha} = \Sigma_k \mu(p_{\alpha k}) \cdot \frac{1}{\mu(p_{\alpha k})} \int_{p_{\alpha k}} |f_n|^2 d\mu = \Sigma_k \int_{p_{\alpha k}} |f_n|^2 d\mu$$

for $n \to \infty$, while the right one is the limit of the sequence

$$N_{\cdot\alpha} = \Sigma_k \mu(p_{\alpha k}) \cdot \left| \frac{1}{\mu(p_{\alpha k})} \int_{p_{\alpha k}} f_n d\mu \right|^2 = \Sigma_k \frac{1}{\mu(p_{\alpha k})} \left| \int_{p_{\alpha k}} f_n d\mu \right|^2$$

for $n \to \infty$. Consider the sums

$$M_\alpha \overline{\overline{df}} \Sigma_k \int_{p_{\alpha k}} |f|^2 d\mu \qquad N_\alpha \overline{\overline{df}} \Sigma_k \frac{1}{\mu(p_{\alpha k})} \left| \int_{p_{\alpha k}} f d\mu \right|^2.$$

Taking (1) and (2) into account, we have

$$\left.\begin{array}{l} |M_{n\alpha} - M_\alpha| \\ |N_{n\alpha} - N_\alpha| \end{array}\right\} \leq \Sigma_k \sqrt{\int_{p_{\alpha k}} |f_n - f|^2 d\mu} \cdot$$

$$\cdot \left[\sqrt{\int_{p_{\alpha k}} |f_n|^2 d\mu} + \sqrt{\int_{p_{\alpha k}} |f|^2 d\mu}\right].$$

Applying once more the Cauchy-Schwarz lemma, we get

$$\left.\begin{array}{l} |M_{n\alpha} - M_\alpha| \\ |N_{n\alpha} - N_\alpha| \end{array}\right\} \leq \sqrt{\Sigma_k \int_{p_{\alpha k}} |f_n - f|^2 d\mu} \cdot$$

$$\cdot \sqrt{\Sigma_k \left(\sqrt{\int_{p_{\alpha k}} |f_n|^2 d\mu} + \sqrt{\int_{p_{\alpha k}} |f|^2 d\mu}\right)}.$$

Now since in general, for non negative numbers, x, y, we we have

$$(x + y)^2 = x^2 + 2xy + y^2 \leq x^2 + y^2 + (x^2 + y^2) = 2(x^2 + y^2),$$

we get

$$\left.\begin{array}{l} |M_{n\alpha} - M_\alpha| \\ |N_{n\alpha} - N_\alpha| \end{array}\right\} \leq \sqrt{\int_W |f_n - f|^2 d\mu} \cdot \sqrt{2\left(\int_W |f_n|^2 d\mu + \int_W |f|^2 d\mu\right)}.$$

Since the sequence $\left\{\int_W |f_n|^2 d\mu\right\}$ is bounded, because $\{f_n\}$ is convergent, there exists $K > 0$, such that

$$\left.\begin{array}{l} |M_{n\alpha} - M_\alpha| \\ |N_{n\alpha} - N_\alpha| \end{array}\right\} \leq K \sqrt{\int_W |f_n - f|^2 d\mu}. \tag{3}$$

We see that this estimate does not depend on α.

Having that, we shall prove that

$$(4) \qquad \lim_{n\to\infty} \mathrm{S}_E\, \mu_x \operatorname{val}_x |f_n|^2 = \mathrm{S}_E\, \mu_x \operatorname{val}_x |f|^2.$$

The sums in (4) exists by virtue of [11]. We have

$$\lim_{n\to\infty} M_{n\alpha} = \mathrm{S}_E\, \mu_x \operatorname{val}_x |f_n|^2$$

and

$$(4.1) \qquad \lim_{n\to\infty} M_\alpha = \mathrm{S}_E\, \mu_x \operatorname{val}_x |f|^2.$$

Hence

$$\lim_{\alpha\to\infty} |M_{n\alpha} - M_\alpha| = |\mathrm{S}_E\, \mu_x \operatorname{val}_x |f_n|^2 - \mathrm{S}_E\, \mu_x \operatorname{val}_x |f|^2| .$$

By (3) we get, when $\alpha \to \infty$,

$$|\mathrm{S}_E\, \mu_x \operatorname{val}_x |f_n|^2 - \mathrm{S}_E\, \mu_x \operatorname{val}_x |f|^2| \leq K \sqrt{\int |f_n - f|^2 d\mu}.$$

Hence, for $n \to \infty$ we get

$$\lim_{n\to\infty} |\mathrm{S}_E\, \mu_x \operatorname{val}_x |f_n|^2 - \mathrm{S}_E\, \mu_x \operatorname{val}_x |f|^2| = 0.$$

which gives

$$\lim \mathrm{S}_E\, \mu_x \operatorname{val}_x |f_n|^2 = \mathrm{S}_E\, \mu_x \operatorname{val}_x |f|^2;$$

so (4) is proved.

Now we shall prove that $\lim\limits_{n\to\infty} N_\alpha$ exists. Suppose it does not. We can find $\delta > 0$ and an infinite sequence of indices $\alpha_1' < \alpha_2'' < \alpha_1' < \alpha_2'' < \ldots$ such that $|N_{\alpha'_m} - N_{\alpha''_m}| \geq \delta$ for all $m = 1, 2, \ldots$ Now we have in general

$$\begin{aligned} |N_{n\alpha} - N_{n\beta}| &\geq |N_\alpha - N_\beta| - |(N_{n\alpha} - N_\alpha) + (N_\beta - N_{n\beta})| \geq \\ &\geq |N_\alpha - N_\beta| - |N_{n\alpha} - N_\alpha| - |N_\beta - N_{n\beta}| . \end{aligned}$$

Hence by (3),

$$\begin{aligned} |N_{n,\alpha'_m} - N_{n,\alpha''_m}| &\geq |N_{\alpha'_m} - N_{\alpha''_m}| - \\ - |N_{n,\alpha'_m} - N_{\alpha'_m}| &- |N_{n,\alpha''_m} - N_{\alpha''_m}| \geq \\ &\geq \delta - 2K \sqrt{\int_W |f_n - f|^2 d\mu}. \end{aligned}$$

If we choose n_0 so as to have

$$2K\sqrt{\int_W |f_n - f|^2\, d\mu} < \frac{\delta}{2},$$

we shall get

$$|N_{n_0, \alpha'_m} - N_{n_0, \alpha''_m}| \geq \frac{\delta}{2}$$

for all m. Consequently the sequence

$$N_{n_0, \alpha'_1},\ N_{n_0, \alpha''_1},\ N_{n_0, \alpha'_2},\ N_{n_0, \alpha''_2},\ \dots$$

does not converge. Hence $\lim\limits_{n\to\infty} N_{n_0, \alpha}$ does not exists, which contradicts the fact that

$$\lim_{\alpha\to\infty} N_{n_0, \alpha} = \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f_n|^2.$$

Thus we have proved that the sum $\mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f|^2$ exists, and we have

(5) $$\lim_{\alpha\to\infty} N_n = \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f|^2;$$

we also have, by (4.1),

(6) $$\lim_{\alpha\to\infty} M_\alpha = \mathbf{S}_E\, \mu_x \operatorname{val}_x |f|^2.$$

Having that, consider the following circumstances:

$$\lim_{n\to\infty} N_{n\alpha} = \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f_n|^2, \qquad \lim_{n\to\infty} N_\alpha = \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f|^2,$$

$$|N_{n\alpha} - N_\alpha| \leq K\sqrt{\int_W |f_n - f|^2 d\mu}.$$

It follows that

$$|\mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f_n|^2 - \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f|^2| \leq K\sqrt{\int_W |f_n - f|^2 d\mu}.$$

Consequently, for $n \to \infty$ we get

$$\lim_{n\to\infty} \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f_n|^2 = \mathbf{S}_E\, \mu_x\, |\operatorname{val}_x f|^2.$$

On the other hand we have, by (4),

$$\lim_{n \to \infty} \mathbf{S}_E\, \mu_x \operatorname{val}_x |f_n|^2 = \mathbf{S}_E\, \mu_x \operatorname{val}_x |f|^2.$$

If we take account of (2.1), i. e.

$$\mathbf{S}_E\, \mu_x |\operatorname{val}_x f_n|^2 = \mathbf{S}_E\, \mu_x \operatorname{val}_x |f_n|^2,$$

we get when $n \to \infty$:

$$\mathbf{S}_E\, \mu_x \operatorname{val}_x |f|^2 = \mathbf{S}_E\, \mu_x |\operatorname{val}_x f|^2 \text{ i.e. } \mu_x \operatorname{val}_x |f|^2 \approx\!\!\approx \mu_x |\operatorname{val}_x f|^2.$$

15.h - The items [15 $a - g$] imply the theorem [15], so it is proved.

16. - **Theorem.** If $f(\dot{x})$, $g(\dot{x})$ are μ-square-summable complex-number-valued functions on W, then

$$\mu_x \operatorname{val}_x (f \cdot g) \approx\!\!\approx \mu_x \operatorname{val}_x f \cdot \operatorname{val}_x g.$$

Proof. We have the formula for numbers:

$$\bar{a}b = \frac{1}{2}\overline{(a+b)}(a+b) - \frac{i}{2}\overline{(a+bi)}(a+bi) + \frac{i-1}{2}\bar{a}a + \frac{i-1}{2}\bar{b}b.$$

Hence we have

$$\bar{f}g = \frac{1}{2}\overline{(f+g)}(f+g) - \frac{i}{2}\overline{(f+ig)}(f+ig) + \frac{i-1}{2}\bar{f}f + \frac{i-1}{2}\bar{g}g.$$

Hence

$$(1) \qquad \int_P \bar{f} g\, d\mu = \frac{1}{2}\int_P |f+g|^2 d\mu - \frac{i}{2}\int_P |f+ig|^2 d\mu +$$
$$+ \frac{i-1}{2}\int_P |f|^2 d\mu + \frac{i-1}{2}\int_P |g|^2 d\mu.$$

Putting

$$a \overline{\overline{\scriptstyle df}} \int_P f\, d\mu, \quad b \overline{\overline{\scriptstyle df}} \int_P g\, d\mu$$

we also have

$$(2)\quad \overline{\int_p f d\mu} \cdot \int_p g d\mu = \frac{1}{2}\left|\int_p f d\mu + \int_p g d\mu\right|^2 - \frac{i}{2}\left|\int_p f d\mu + i \int_p g d\mu\right|^2 +$$

$$+ \frac{i-1}{2}\left|\int_p f d\mu\right|^2 + \frac{i-1}{2}\left|\int_p g d\mu\right|^2.$$

From (1) it follows

$$(3)\qquad \mathrm{val}_x(\bar{f}g) = \frac{1}{2}\mathrm{val}_x|f+g|^2 - \frac{i}{2}\mathrm{val}_x|f+ig|^2 +$$

$$+ \frac{i-1}{2}\mathrm{val}_x|f|^2 + \frac{i-1}{2}\mathrm{val}_x|g|^2.$$

and from (2) we get

$$(4)\quad \overline{\mathrm{val}_x f} \cdot \mathrm{val}_x g = \frac{1}{2}|\mathrm{val}_x(f+g)|^2 - \frac{i}{2}|\mathrm{val}_x(f+ig)|^2 +$$

$$+ \frac{i-1}{2}|\mathrm{val}_x f|^2 + \frac{i-1}{2}|\mathrm{val}_x g|^2.$$

By theor. [15.5] we have

$$\mu_x \mathrm{val}_x|f|^2 \approx \mu_x|\mathrm{val}_x f|^2,$$

$$\mu_x \mathrm{val}_x|g|^2 \approx \mu_x|\mathrm{val}_x g|^2,$$

$$\mu_x \mathrm{val}_x|f+g|^2 \approx \mu_x|\mathrm{val}_x(f+g)|^2,$$

$$\mu_x \mathrm{val}_x|f+ig|^2 \approx \mu_x|\mathrm{val}_x(f+ig)|^2.$$

Hence, by (3) and (4) we get

$$\mu_x \mathrm{val}_x(\bar{f}g) \approx \mu_x \overline{\mathrm{val}_x f} \cdot \mathrm{val}_x g.$$

Since f, g are arbitrary square-summable functions, the theorem is proved.

§ 5. - Summation of quasi-vectors in the separable and complete Hilbert-Hermite-space.

1. - In this § 5 we shall apply the theories, developped in preceding sections, to tribes of subspaces in the separable and complete Hilbert-Hermite space $\boldsymbol{H}$. We refer to Preliminaries concerning terminology and notions to be now used. Let G be a denumerably additive tribe of (closed) subspaces E, F, ... of H, with H as unit 1 and the space composed of the single vector $\overrightarrow{O}$, as zero, O. The ordering of the tribe is the inclusion of spaces, and co E denotes the ortho-complement of E in H. The relation $E \cdot F = O$ implies « E orthogonal to F ». All spaces of G are compatible with one another. Let $\boldsymbol{F}$ be a finitely additive tribe of spaces which is a finitely genuine strict subtribe of G. Let $\boldsymbol{B}$ be a base of $\boldsymbol{F}$, satisfying the conditions (Hyp. Ad), [§ 1; 3] and the hypotheses (Hyp. I) and (Hyp. II) of [§ 3]. We suppose that G is the smallest denumerably additive extension of $\boldsymbol{F}$. There always exists en effective, denumerably additive measure μ on G, ($\mu \geq 0$). G is not only the Lebesgue's-covering extension of $\boldsymbol{F}$, but also it coincides with the borelian extension of $\boldsymbol{F}$. Thus we are in the conditions [§ 1; 12, Hyp. $L\mu$] and [§ 1; 14]. In our paper (14), p. 21-22, we have proved that the μ-topology on the tribe G is separable. Consequently, in our case, (Hyp. S), [§ 1; 21.1] is satisfied. Having this, we can apply the theory of measurability of sets of traces [§ 3], and use all kinds of summations of quasi-vectors [§ 4].

2. - We like to make remarks concerning (Hyp. Ad), (Hyp. I) and (Hyp. II). There are important cases where these hypotheses are satisfied. We are going to define them.

2.1. - First, let us define some auxiliary tribes whose somata are Lebesgue-measurable sets of ordinary complex numbers. We consider the plane $\boldsymbol{P}'$ of complex numbers, provided with a cartesian system of coordinates x, y. By the *rectangle*

$$R(\alpha_1, \alpha_2; \beta_1, \beta_2),$$

where

$$-\infty \leq \alpha_1, \alpha_2 \leq +\infty, -\infty \leq \beta_1, \beta_2 \leq +\infty,$$

we shall understand the set points (x, y), (complex numbers),

$$\{(x, y) \mid \alpha_1 < x \leq \alpha_2, \beta_1 < y \leq \beta_2\}.$$

The rectangles will be termed *bricks*.

We define F' as the collection of all finite unions of bricks. If we consider the relation of inclusion $\subseteq$ of sets as the ordering relation, F' will be organized into a finitely additive Boolean tribe with unit P' and with the empty set $\emptyset$ as O. The collection B' of all rectangles is a base of F'. The condition (Hyp. *Ad*) is satisfied. Even more, (Hyp. *Af*), [§ 1; 3] holds true.

2.2. - Concerning B'-traces in F', (see [§3]), we have the following situation. If (x, y) is a point on the plane, then there are four different traces attached to it with representatives

$$R\left(x-\frac{1}{n}, x; y-\frac{1}{n}, y\right), \quad R\left(x-\frac{1}{n}, x; y, y+\frac{1}{n}\right),$$

$$R\left(x, x+\frac{1}{n}; y-\frac{1}{n}, y\right), \quad R\left(x, x+\frac{1}{n}; y, y+\frac{1}{n}\right), \quad (n=1, 2, \ldots)$$

respectively. The point (x, y) will be termed *vertex of these traces*. In addition to that there are eight « side-traces » at infinity, with representatives, e. g.

$$R\left(-\infty, -n; y-\frac{1}{n}, y\right), \qquad (n=1,2,\ldots)$$

and four « corner-traces » at infinity, with representatives, e. g. $R(-\infty, -n; n, +\infty)$. These are all existing traces. This can be proved by considering ultrafilters and two-valued measures on F', (29); (see also [§ 3; 2.3] footnote).

2.3. - If we consider F' modulo an ideal, the traces will be essentially the same, though some ones may not exists, because some rectangles seemingly eligible for yielding them, may belong to the ideal.

2.4. - The following theorem is important:

If 1. $\mu(f)$ is a finitely additive (and finite), non negative measure on F', 2. G' is the collection of all borelian subsets of the plane P', then the following are equivalent:

I. the measure μ can be extended over G' so as to obtain a denumerably additive measure on G',

II. 1°. If

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq \dots \qquad \text{tends to } \alpha_0,$$

$$\beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq \dots \qquad \text{tends to } \beta_0,$$

$$(-\infty \leq \alpha_n,\ \alpha_0,\ \beta_n,\ \beta_0 \leq +\infty), \quad (n = 1, 2, \dots),$$

then

$$\mu R(-\infty, \alpha_n; -\infty, \beta_n) \to \mu R(-\infty, \alpha_0; -\infty, \beta_0);$$

2°. if

$$\alpha_1 \leq \alpha_2 \leq \dots \to +\infty, \quad \beta_1 \leq \beta_2 \leq \dots \to +\infty,$$

then

$$\mu R(-\infty, \alpha_n; -\infty, \beta_n) \to \mu(P').$$

The proof relies essentially on the fact that the plane P' is locally compact.

2.5. - Put, in general,

$$Q(\alpha, \beta) \underset{df}{=} R(-\infty, \alpha; -\infty, \beta) \text{ for } -\infty \leq \alpha, \beta \leq +\infty.$$

We call these rectangles: ***plane-quarters***. Let $\mathfrak{S}$ be a correspondence which attaches to every plane-quarter Q a closed subspace of H with the conditions:

I°. If Q_1, Q_2 are two plane-quarters, then $\mathfrak{S}(Q_1)$, $\mathfrak{S}(Q_2)$ are compatible,

II°. $\mathfrak{S}(\emptyset) = (\vec{O}) = O$, $\mathfrak{S}(P') = H = I$), III°. if $Q_1 \subseteq Q_2$, then $\mathcal{S}(Q_1) \leq \mathcal{S}(Q_2)$.

This correspondence can be extended in a unique way to another one, also denoted by $\mathcal{S}$, which attaches to every figure $f \in \boldsymbol{F}'$ a space $\mathcal{S}(f)$, the resulting correspondence having the properties: $\mathcal{S}(f \cap g) = \mathcal{S}(f) \cdot \mathcal{S}(g)$, $\mathcal{S}(f \cup g) = \mathcal{S}(f) + \mathcal{S}(g)$. $\mathcal{S}$ is a homomorphism from $\boldsymbol{F}'$ onto a tribe $\boldsymbol{F}$ of spaces. The set of all figures f for which $\mathcal{S}f = O$ is an ideal in $\boldsymbol{F}'$.

2.6. - Now, in order that $\mathcal{S}$ can be extended to all borelian sets of the plane, the following condition is necessary and sufficient:

1) If $\alpha_1 \geq \alpha_2 \geq \ldots \to \alpha_0$, $\beta_1 \geq \beta_2 \geq \ldots \to \beta_0$, then

$$\Pi_{n=1}^{\infty} \mathcal{S}(Q(\alpha_n \beta_n)) = \mathcal{S}(Q(\alpha_0 \beta_0));$$

2) If $\alpha_1 \leq \alpha_2 \leq \ldots \to +\infty$, $\alpha_1 \leq \beta_2 \leq \ldots \to +\infty$, then

$$\Sigma_{n=1}^{\infty} \mathcal{S}(Q(\alpha_n \beta_n)) = I = \boldsymbol{H}.$$

Let us remark that this situation is present if we consider a normal maximal operator in $\boldsymbol{H}$, (16), and consider its spectral scale. Usually in the spectral theory projectors are used. We prefer to consider the spaces themselves rather than the corresponding projectors, (22), (26), (11).

2.7. - The extended correspondence $\mathcal{S}$ yields a denumerably additive tribe $\boldsymbol{G} = \mathcal{S}(\boldsymbol{G}')$ of spaces. The tribe $\boldsymbol{F} = \mathcal{S}(\boldsymbol{F}')$ is its finitely genuine strict subtribe, and the $\mathcal{S}$-correspondents of rectangles of $\boldsymbol{P}'$ constitute a base $\boldsymbol{B}$ for $\boldsymbol{F}$, so $\mathcal{S}R$ may be called *bricks* of $\boldsymbol{F}$, (« space-rectangle-bricks »). We define space-traces, as in the general theory of traces by means of these bricks. Let us remark that it is not true that to every $\boldsymbol{B}'$-trace in $\boldsymbol{F}$ there corresponds through $\mathcal{S}$ a space-trace; indeed, if $a'_1 \geq a'_2 \geq \ldots$ is a representative of a trace x on the plane, the spaces $\mathcal{S}(a'_n)$ may be all $= O$, so they do not yield any space trace.

Now, in the above construction one can prove, (considering circumstances on the plane), that for the space-traces the hypotheses I and II hold true, so we can apply the theory of measurability of sets of space-traces.

2.8. - **Remark.** A similar construction of tribes of spaces with a base can be obtained, if we consider the straight line instead of a plane, and instead of rectangles we consider half open-intervals. This construction would be related to the spectral scale of a selfadjoint operator. The construction by means of half open arcs on the unit circle will correspond to the spectral theory of unitary operators.

2.9. - The tribe $\mathcal{S}G'$ may be saturated or not. Whatever will be the case it admits an effective denumerably additive measure. Any tribe of spaces can be saturated by a suitable adjunction of spaces, (see Prelimin.).

3. - In the sequel we shall pay special attention to tribes of spaces obtained trough the above rectangle or half-open segment-construction, but in the general discussion which we shall soon start in [5], we shall consider the general situation as specified in [1]. We shall consider any kind (D') of summation, [§ 4].

4. - Let us admit that the tribe G of spaces is saturated, (see Prelim.). Then there exists an isomorphic mapping of the space $\boldsymbol{H}$ onto the space of some measurable, complex-valued fonctions of the variable trace, (14). This mapping $\mathcal{G}^{-1}$ is obtained as follows:

Since G is saturated, there exists a generating vector $\vec{\omega}$ of $\boldsymbol{H}$ with respect to G. Choose $\vec{\omega}$ and define, on G, the measure μ:

$$\mu(a) \underset{df}{=} \| \operatorname{Proj}_a \vec{\omega} \|^2 \text{ for all } a \in G.$$

This measure is denumerably additive, non negative and effective. It induces a denumerably additive measure for all measurable sets of traces, also denoted by μ. We shall consider μ-square-summable functions, [§ 3], of the variable trace τ, defined almost μ-everywhere on the set W of all traces. These functions will be considered modulo μ-null-sets. They are a kind of « functionoids », (30), (31), (32).

To define the said isomorphic mapping $\mathcal{G}^{-1}$, first consider step-functions. Given a step-function $\xi(\tau)$, there exists a finite number of mutually disjoint, measurable sets $E_1, \ldots, E_n$, with $\cup_{k=1}^n E_k = W$, and there exist complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\xi(\tau) = \Sigma_{k=1}^n \Omega_{E_k}(\tau)\lambda_k, \tag{1}$$

where $\Omega_{E_k}(\tau)$ is the characteristic function of the set, E_k i. e., $\Omega = 0$ for $\tau \in E_k$ and $\Omega = 1$ for $\tau \in E_k$. Denote by $\mathcal{G}$ the correspondence which attaches to (1) the « step »-vector

$$\overrightarrow{X} \underset{df}{=} \Sigma_{k=1}^n \operatorname{Proj}_{e_k} \overrightarrow{\omega} \cdot \lambda_k \tag{2}$$

where $e_k = [E_k]$. This correspondence does not depend on the way of representing $\xi(\tau)$ by the formula (1), it preserves operations of addition, multiplication by a number and the scalar product

$$(\xi_1, \xi_2) \underset{df}{=} \int_W \overline{\xi_1(\tau)}\xi_2(\tau) d\mu.$$

Since $\overrightarrow{\omega}$ is a generating vector, every vector $Y \in \boldsymbol{H}$ can be approximated in the $\boldsymbol{H}$-topology by step-vectors (2). Hence the correspondence $\mathcal{G}$ can be extended to all μ-square-summable functions. The extended correspondence, also denoted by $\mathcal{G}$ is an isometric, and homeomorphic isomorphism from the set of μ-square-summable functions taken modulo null-set onto the space $\boldsymbol{H}$.

5. - **Def.** Under circumstances specified in [1], a quasi-vector $\overrightarrow{\varphi}(p)$ with support x will be termed *normal* whenever $\overrightarrow{\varphi}(p) \leq p$ for all $p \in v(x)$. (See [§ 4; Def. 3]).

5.1. - **Theor.** Under circumstances [1] if

1. A is a measurable set of traces,

2. $\overrightarrow{f_x}$ is a set of quasi-vectors (D')-summable on A, (see [§ 4], and [1]),

3. The set is normal [Def. 5],

then the set of quasi-numbers $\|\overrightarrow{f_x}\|^2$ is also summable on A and

$$\mathrm{S}_A \|\overrightarrow{f_x}\|^2 = \| \mathrm{S}_A \overrightarrow{f_x} \|^2.$$

Proof. Take any choice of a vector-field, (see [§ 4; 4.4]), say $\overrightarrow{f}(p)$, generated by f_x. Let

$$P_n = \{ p_{n1}, p_{n2}, \ldots \}, \qquad (n = 1, 2, \ldots),$$

be any sequence of complexes distinguished for $[A]$. We have

$$\overrightarrow{\varphi_n} \underset{df}{=} \Sigma_i \overrightarrow{f}(p_{ni}) \rightarrow \mathrm{S}_A \overrightarrow{f_x} \tag{1}$$

in the H-topology. Now, since the spaces $p_{n1}, p_{n2}, \ldots$ are mutually orthogonal and since $\overrightarrow{f}(p_{ni}) \in p_{ni}$, we have

$$\|\overrightarrow{\varphi_n}\|^2 = \Sigma_i \|\overrightarrow{f}(p_{ni})\|^2. \tag{2}$$

From (1) it follows that

$$\|\overrightarrow{\varphi_n}\|^2 \rightarrow \| \mathrm{S}_A \overrightarrow{f_x} \|^2. \tag{3}$$

Hence, from (2) it follows that the sequence

$$\{ \Sigma_i \|\overrightarrow{f}(p_{ni})\|^2 \}, \qquad (n = 1, 2, \ldots)$$

possesses a limit, viz (3). Hence, the limit does not depend on the choice of the selected field $\overrightarrow{f}(p)$. Hence this limit is $\mathrm{S}_A \|\overrightarrow{f_x}\|^2$. Consequently

$$\mathrm{S}_A \|\overrightarrow{f_x}\|^2 = \mathrm{S}_A \overrightarrow{f_x} \|^2. \qquad \text{Q. E. D.}$$

6. - Theorem. Under circumstances [1] if

1. $\overrightarrow{f_x}$ is a total set of quasi-vectors, (D')-summable on W,

2. the set is normal [Def. 5],

then there exists $M > 0$, such that

$$(1) \qquad \| \mathrm{S}_A \vec{f_x} \|^2 = \mathrm{S}_A \| \vec{f_x} \|^2 < M$$

for all measurable sets A of traces.

Proof. By theor. [5.1] the sums (1) exist and we have

$$K(A) \underset{df}{=} \| \mathrm{S}_A \vec{f_x} \|^2 = \mathrm{S}_A \| \vec{f_x} \|^2.$$

Now, by theor. [§ 4; 6.6], the function $K(A)$ of the variable measurable set A is denumerably additive. Since $K(A) \geq 0$, it follows, by the known theorem, (e. g. (7)), on denumerably additive non negative measures on a denumerably additive tribe, that $K(A)$ is bounded, so the theorem is proved.

7. - **Theorem.** Under circumstances [1] if

1. $\vec{f_x}$ and $\vec{g_x}$ are sets of quasi-vectors (D')-summable on A,

2. A is a measurable set of traces,

3. The sets $\vec{f_x}$ and $\vec{g_x}$ are normal [Def. 5],

then the set of quasi-numbers (scalar product) $(\vec{f_x}, \vec{g_x})$, (see [§ 4; 3.1]) is also summable on A, and we have

$$\mathrm{S}_A(\vec{f_x}, \vec{g_x}) = (\mathrm{S}_A \vec{f_x}, \mathrm{S}_A \vec{g_x}).$$

Proof. Let us choose a selection of vector-fields, one for $\vec{f_x}$ another for $\vec{g_x}$. Denote them by $\vec{f}(\dot{p})$, $\vec{g}(\dot{p})$ respectively. They belong both to the same choice $x = \alpha(p)$ of traces covered by p. (Indeed, we can operate only on quasi-vectors having the same support).

Let $P_n = \{ p_{n1}, p_{n2}, \dots \}$ be a distinguished sequence of complexes for $[A]$. Since $p_{n1}, p_{n2}, \dots$ are mutually orthogonal spaces, and since

$$\vec{f}(p_{nk}) \in p_{nk}, \ \vec{g}(p_{nk}) \in p_{nk},$$

it follows that

$$(1) \qquad (\Sigma_k \vec{f}(p_{nk}), \ \Sigma_k \vec{g}(p_{nk})) = \Sigma_k(\vec{f}(p_{nk}), \ \vec{g}(p_{nk})).$$

We shall prove the existence of the sum

$$\mathrm{S}_A(\vec{f}_x,\ \vec{g}_x).$$

We have

$$(2)\quad (\vec{f}_x,\ \vec{g}_x) = \frac{1}{2}(\vec{f}_x + \vec{g}_x,\ \vec{f}_x + \vec{g}_x) - \frac{i}{2}(\vec{f}_x + i\vec{g}_x,\ \vec{f}_x + i\vec{g}) +$$
$$+ \frac{i-1}{2}(\vec{f}_x,\ \vec{f}_x) + \frac{i-1}{2}(\vec{g}_x,\ \vec{g}_x).$$

By [§ 4; 6.10] and [§ 4; 6.9] the sums of the quasi-vectors $\vec{f}_x + \vec{g}_x$, $\vec{f}_x + i\vec{g}_x$ are (D')-summable. Hence by [5.1], the (D')-sums

$$\mathrm{S}_A\|\vec{f}_x + \vec{g}_x\|^2,\ \mathrm{S}_A\|\vec{f}_x + i\vec{g}_x\|^2,\ \mathrm{S}_A\|\vec{f}_x\|^2,\ \mathrm{S}_A\|\vec{g}_x\|^2$$

also exists, and hence, by [§ 4; 6.10] the (D')-sum

$$\mathrm{S}_A(\vec{f}_x,\ \vec{g}_x)$$

exists. We have

$$\Sigma_k(\vec{f}(p_{nk}),\ \vec{g}(p_{nk})) \rightarrow \mathrm{S}_A(\vec{f}_x,\ \vec{g}_x).$$

Since

$$\Sigma_k \vec{f}(p_{nk}) \rightarrow \mathrm{S}_A \vec{f}_x$$

and

$$\Sigma_k \vec{g}(p_{nk}) \rightarrow \mathrm{S}_A \vec{g}_x,$$

it follows from (1) that

$$\mathrm{S}_A(\vec{f}_x,\ \vec{g}_x) = (\mathrm{S}_A \vec{f}_x,\ \mathrm{S}_A \vec{g}_x). \qquad \text{Q. E. D.}$$

8. - Under circumstances [1] if $\vec{f}_x$, $\vec{g}_x$ are quasi-vectors sets, both (D')-summable on a measurable set A, and they are normal, then

$$|\mathrm{S}_A(\vec{f}_x,\ \vec{g}_x)| \leq \|\mathrm{S}_A \vec{f}_x\| \cdot \|\mathrm{S}_A \vec{g}_x\|.$$

Proof. This follows from [7]. Indeed we have

$$\mathrm{S}_A(\overrightarrow{f_x},\ \overrightarrow{g_x}) = (\mathrm{S}_A\,\overrightarrow{f_x},\ \mathrm{S}_A\,\overrightarrow{g_x})$$

and the Cauchy-Schwarz inequality completes the proof.

9. Under circumstances [1] if

1. $\overrightarrow{f_x}$, $\overrightarrow{g_x}$ are quasi-vector sets, both (D')-summable on a measurable set A of traces,

2. f_x, g_x are normal,

then $\|\mathrm{S}_A(\overrightarrow{f_x}+\overrightarrow{g_x})\| \leq \|\mathrm{S}_A\,\overrightarrow{f_x}\| + \|\mathrm{S}_A\,\overrightarrow{g_x}\|$.

Proof. We have, [§ 4; 6.10],

$$\mathrm{S}_A(\overrightarrow{f_x}+\overrightarrow{g_x}) = \mathrm{S}_A\,\overrightarrow{f_x} + \mathrm{S}_A\,\overrightarrow{g_x}.$$

Hence

$$\|\mathrm{S}_A(\overrightarrow{f_x}+\overrightarrow{g_x})\|^2 = (\mathrm{S}_A\,\overrightarrow{f_x}+\mathrm{S}_A\,\overrightarrow{g_x},\ \mathrm{S}_A\,\overrightarrow{f_x}+\mathrm{S}_A\,\overrightarrow{g_x}) =$$

$$= \|\mathrm{S}_A\,\overrightarrow{f_x}\|^2 + (\mathrm{S}_A\,\overrightarrow{f_x},\ \mathrm{S}_A\,\overrightarrow{g_x}) + (\mathrm{S}_A\,\overrightarrow{g_x},\ \mathrm{S}_A\,\overrightarrow{f_x}) + \|\mathrm{S}_A\,\overrightarrow{g_x}\|^2.$$

Hence

$$\|\mathrm{S}_A(\overrightarrow{f_x}+\overrightarrow{g_x})\|^2 \leq \|\mathrm{S}_A\,\overrightarrow{f_x}\|^2 + 2\,|(\mathrm{S}_A\,\overrightarrow{f_x},\ \mathrm{S}_A\,\overrightarrow{g_x})| + \|\mathrm{S}_A\,\overrightarrow{g_x}\|^2.$$

Taking [8] into account, we get

$$\|\mathrm{S}_A(\overrightarrow{f_x}+\overrightarrow{g_x})\|^2 \leq (\|\mathrm{S}_A\,\overrightarrow{f_x}\| + \|\mathrm{S}_A\,\overrightarrow{g_x}\|)^2,$$

which completes the proof.

10. - Lemma. If

1. the sequence $a_1, a_2, \ldots, a_n, \ldots \in G$ of spaces μ-tends to a,

2. $\overrightarrow{\xi_n} \in a_n$,

3. $\lim \overrightarrow{\xi_n} = \overrightarrow{\xi}$,

then $\overrightarrow{\xi} \in a$.

10a. - Proof. Since $\lim^{\mu} a_n = a$, we can extract, by [§ 1, 12.2], from $\{a_n\}$ a partial sequence $\{a_{k(n)}\}$ such that

$a = \Pi_{s=1}^{\infty} b_s$, where $b_s \underset{df}{=} \Sigma_{n=s}^{\infty} a_{k(n)}$. We have $b_s \geq b_{s+1}$. Consider the sequence $\{\overrightarrow{\xi_{k(n)}}\}$. We have $\overrightarrow{\xi_{k(s)}} \in a_{k(s)}$; hence $\overrightarrow{\xi_{k(s)}} \in b_s$ for $s = 1, 2, \ldots$. We have

(2) $$\overrightarrow{\xi_{k(s)}} = \mathrm{Proj}_a \overrightarrow{\xi_{k(s)}} + \mathrm{Proj}_{co\,a} \overrightarrow{\xi_{k(s)}}.$$

Since

$$\mathrm{Proj}_{co\,a} \overrightarrow{\xi_{k(s)}} = \mathrm{Proj}_{co\,a} \mathrm{Proj}_{b_s} \overrightarrow{\xi_{k(s)}},$$

we have

$$\overrightarrow{\eta_s} \underset{df}{=} \mathrm{Proj}_{co\,a} \overrightarrow{\xi_{k(s)}} = \mathrm{Proj}_{b_s - a} \overrightarrow{\xi_{k(s)}} \in b_s - a.$$

Since projecting is a continuous operation, we have

$$\lim_{s \to \infty} \mathrm{Proj}_a \overrightarrow{\xi_{k(s)}} = \mathrm{Proj}_a [\lim_{s \to \infty} \overrightarrow{\xi_{k(s)}}];$$

hence

$$\lim_{s \to \infty} \mathrm{Proj}_a \overrightarrow{\xi_{k(s)}} = \mathrm{Proj}_a \vec{\xi}.$$

From (2) we get by the passage to limit:

(3) $$\vec{\xi} = \mathrm{Proj}_a \vec{\xi} + \lim \overrightarrow{\eta_s}, \text{ where } \overrightarrow{\eta_s} \in b_s - a.$$

Hence

(3.1) $$\lim \overrightarrow{\eta_s} \quad \text{exists.}$$

If we put

$$c_s \underset{df}{=} b_s - a,$$

we have $c_s \geq c_{s+1}$, and

$$\Pi_{s=1}^{\infty} c_s = O, \ \overrightarrow{\eta_s} \in c_s.$$

To prove the Lemma it is sufficient to prove that

$$\lim_{s \to \infty} \overrightarrow{\eta_s} = \vec{O}. \qquad \text{(This because of (3)).}$$

10.b - Supposing that this be not true, we have by (3.1),

$$\vec{\eta} \underset{df}{=} \lim_{s \to \infty} \overrightarrow{\eta_s} \neq \vec{O}.$$

To disprove that supposition we shall apply the representation in [4], which is valid for saturated tribes. Now, our tribe G may be not saturated, but it can always be extended (by adjunction of an at most denumerable number of spaces), so as to get a saturated tribe G_1. Take an effective measure μ_1 on G_1. The topology on G generated by μ_1 will coincide with that one generated by μ. This follows from [§ 1; 1 21].

Having this, we can operate in G_1 instead of G.

Let us consider the μ_1-square summable functions of the variable trace x:

$$H(x), \ H_s(x),$$

which are images of $\vec{\eta}$ and $\vec{\eta}_s$ respectively. Let $E_1, E_2, \ldots, E_s, \ldots$ be μ_1-measurable sets of traces with supports $c_1, c_2, \ldots, c_s, \ldots$ respectively. We can admit that

$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_s \supseteq \ldots$$

(because if not, we can replace E_s by $E'_s \underset{df}{=} E_1 \cdot E_2 \ldots E_s$, whose support is $c_1 \cdot c_2 \ldots c_s = c_s$). We have

$$\mu_1(E_n) = \mu_1(c_n) \to 0. \tag{3.2}$$

The functions $H_s(x)$ can be chosen so as to have

$$H_s(x) = 0 \text{ for } x \in \operatorname{co} E_s. \tag{4}$$

Indeed, $\vec{\eta}_s \in c_s$; hence $\operatorname{Proj}_{\operatorname{co} c_n} \vec{\eta} = \vec{0}$. Since $\vec{\eta} \neq \vec{0}$, there exists a set E of positive measure such that $H(x) \neq 0$. Hence, there exists a measurable set F and $\alpha > 0$ such that $\mu_1(F) > 0$ and $|H(x)| \geq \alpha$ for all $x \in F$. We have

$$|H_s(x) - H(s)|^2 \, dx \to 0.$$

Since $H_s(x)$ tends in μ-square means to $H(x)$, there exists a partial sequence $H_{l(s)}(x)$ which tends almost μ-everywhere to $H(x)$. By the theorem of Jegoroff, given $\varepsilon > 0$, there exists a set G such that $\mu_1(G) > \mu_1(1) - \varepsilon$, and where $H_{l(s)}$ converges uniformly to $H(x)$.

Hence, if $\varepsilon > 0$ is sufficiently small, we can find a subset F' of F with $\mu_1(F') > 0$ where $H_{l(s)}(x)$ converges uniformly to $H(x)$. Hence for n sufficiently great

(5) $$|\, H_{l(s)}(x)\, | \geq \frac{\alpha}{2} \qquad \text{on } F' \text{ for } s = 1, 2, \dots$$

But, by (4), $H_s(x) = 0$ for $x \in \operatorname{co} E_s$: hence

(6) $$\mu_1 \{\, x \mid H_s(x) \neq 0 \,\} = \mu_1 E_s \to 0,$$

by (3.2). Since

$$\{\, x \mid H_s(x) \neq 0 \,\} \geq F' \qquad \text{for all } s = 1, 2, \dots,$$

we have

$$\mu_1 \{\, x \mid H_s(x) \neq 0 \,\} \geq \mu_1 F' > 0.$$

But by (6) $\mu_1 F' = 0$, which is a contradiction. The lemma is established.

11. - Theorem. Under circumstances [1] if 1. $\vec{f_x}$ is a sets of quasi-vectors in $\boldsymbol{H}$ with support A, 2. $\vec{f_x}$ is normal, 3. A is a measurable set of traces, 4. $\vec{f_x}$ is (D')-summable on A, then, $\mathrm{S}_A \vec{f_x} \in [A]$ where $[A]$ is the coat of A.

Proof. Let $\{P_n\}$ be a distinguished sequence of complexes for $[A]$. Let

$$P_n = \{\, p_{n1},\ p_{n2},\ \dots \}.$$

We have

$$\vec{f}(P_n) \to \mathrm{S}_A \vec{f_x},$$

and $|\, P_n,\ A \,| \to 0$; hence

(1) $$\lim \operatorname{som} P_n = [A].$$

Now $\vec{f}(p_{ni}) \in p_{ni}$, hence $\vec{f}(P_n) \in \operatorname{som} P_n$. By [Lemma 10], we get $\lim \vec{f}(P_n) \in [A]$. Hence

$$\mathrm{S}_A \vec{f_x} \in [A]. \qquad \text{Q. E. D.}$$

12. - **Theor.** Under circumstances [1] if

1. $\vec{f_x}$ is a total set of quasi-vectors in H,
2. $\vec{f_x}$ is normal,
3. $\vec{f_x}$ is (D')-summable on W,
4. A is a summable set of traces,

$$\text{then } \mathrm{S}_A\, f_x = \mathrm{Proj}_{[A]}\, \mathrm{S}_W f_x .$$

Proof. Putting $\mathrm{co}\, A \underset{df}{=} W - A$, we have

$$A \cup (\mathrm{co}\, A) = W,$$

co A is a measurable set of traces. By [§ 4; 6.2]

$$\mathrm{S}_W \vec{f_x} = \mathrm{S}_A \vec{f_x} + \mathrm{S}_{\mathrm{co}\, A} \vec{f_x} . \tag{1}$$

Now, by [11] we have

$$\mathrm{S}_A\, f_x \in [A], \quad \mathrm{S}_{\mathrm{co}\, A}\, f_x \in \mathrm{co}\, [A].$$

Hence $\mathrm{S}_A\, f_x$ is orthogonal to $\mathrm{S}_{\mathrm{co}\, A} f_x$. Taking in (1) the projection on the space $[A]$, we get

$$\mathrm{Proj}_{[A]}\, \mathrm{S}_W f_x = \mathrm{Proj}_{[A]}\, \mathrm{S}_A\, f_x + \mathrm{Proj}_{[A]}\, \mathrm{S}_{\mathrm{co}\, A}\, f_x .$$

The second term is the zero-vector. Since $\mathrm{S}_A\, f_x \in [A]$, we get

$$\mathrm{Proj}_{[A]}\, \mathrm{S}_W f_x = \mathrm{S}_A\, f_x . \qquad \text{Q. E. D.}$$

§ 6. - General orthogonal system of coordonates in the separable and complete Hilbert-Hermite-space.

1. - We admit the hypotheses of [§ 5; 1]. Thus F, G, are tribes of subspaces of H, and B is a base of F, satisfyng all conditions required for the theory of measura-

bility of sets of B-traces and for applying the (*DARS*) summations of sets of quasi-vectors.

1.1. - We suppose that the tribe G is saturated and we select a generating vector $\overrightarrow{\omega}$ of H with respect to G, (see Preliminaries). The effective measure on G will be defined by

$$\mu(a) = \| \mathrm{Proj}_a \overrightarrow{\omega} \|^2 .$$

1.2. - **Def.** The set B (yields by extension through F) a saturated tribe G and $\overrightarrow{\omega}$ will determine a system of reference) $[B, \overrightarrow{\omega}]$ for vectors in H. We shall call it *frame* (*or system*) *of orthogonal coordinates, in* H, (14), (23), (24), (25), (26), (11), (22).

2. - **Def.** We introduce the following important notions, related to the given frame of coordinates. Let $\overrightarrow{X} \in H$, and let β be a B-trace in F. By the *β-component of* $\overrightarrow{X}$ we shall understand the quasi-vector $\overrightarrow{x_\beta}$, with support β, defined by $X(p) \underset{df}{=} \mathrm{Proj}_p \overrightarrow{X}$ for all neighborhoods p of β. By the *β-component-density of* $\overrightarrow{X}$ we shall understand the quasi-vector x^*_β with support β, defined by

$$\overrightarrow{x^*}(p) \underset{df}{=} \frac{\mathrm{Proj}_p \overrightarrow{X}}{\mu(p)}$$

for all neighborhoods p of β.

By the *β-coordinate of* $\overrightarrow{X}$, we shall understand the quasi-number x_β with support β, defined by $x(p) \underset{df}{=} (\mathrm{Proj}_p \overrightarrow{\omega}, \overrightarrow{X})$ for all neighborhoods p of β. By the *β-coordinate-density of* $\overrightarrow{X}$ we shall understand the quasi-number x^*_β with support β defined by

$$X^*(p) \underset{df}{=} \frac{(\mathrm{Proj}_p \overrightarrow{\omega}, \overrightarrow{X})}{\mu(p)}$$

for all neighborhoods p of β.

3. - We have $\mathrm{Proj}_p \overrightarrow{X} = \overrightarrow{x^*}(p) \cdot \mu(p)$, hence [§ 4; 8.1], [§ 4; 8],

$$\overrightarrow{x_\beta} = \overrightarrow{x^*_\beta} \cdot \mu_\beta . \tag{1}$$

We have $(\mathrm{Proj}_p \vec{\omega}, \vec{X}) = x^*(p) \cdot \mu(p)$; hence

$$(2) \qquad x_\beta = x^*_\beta \cdot \mu_\beta .$$

4. - Since $(\mathrm{Proj}_p \vec{\omega}, \vec{X}) = (\vec{\omega}, \mathrm{Proj}_p \vec{X})$, we have $(\vec{\omega}_\beta, \vec{X}) = (\vec{\omega}, \vec{X}_\beta) = x_\beta$.

5. - **Theor.** The total set of quasi-vectors $\vec{x}_\beta$ is regular, and normal, [§ 5; 5]. The same is for $\vec{x}^*$.

$$\text{Indeed } \vec{x}_\beta(p) = \mathrm{Proj}_p \vec{X} \in p, \ \vec{x}^*_\beta(p) \in p,$$

which proves the normality. Since $\mathrm{Proj}_p \vec{X}$ does not depend on the choice of the trace β where $\beta \in p$, the set of all β-components and also the set of β-component densities is regular, [§ 4; 4.3].

5.1. - **Theor.** The total sets of quasi-numbers x_β and x^*_β are also regular. We shall consider $(DARS)$-summations only, though some theorems are also true for any kind (D') of summation.

6. - **Theor.** The total set of β-components is (D')-summable on W.

Proof. It suffice to prove that the vector-field $\mathrm{Proj}_p \vec{X}$ defined for all bricks p is (D')-summable on I. Let $P_n = \{p_{n1}, p_{n2}, \dots\}$ be a distinguished sequence of complexes for I. We have $\Sigma_k \mathrm{Proj}_{p_{nk}} \vec{X} = \mathrm{Proj}_{\mathrm{som} P_n} \vec{X}$, because all space p_{nk} are mutually orthogonal, and since $\mathrm{Proj}_{p_{nk}} \vec{X} \in p_{nk}$. Now, since $| P_n, I ||_\mu \to 0$ i. e. $\mu(\mathrm{som}\ P_n) \to \mu(I)$; we get $\lim \mathrm{Proj}_{\mathrm{som} P_n} \vec{X} = \vec{X}$, [§ 5; 10. This proves the summability of $\vec{x}_\beta$ on W.

6.1. - If E is a measurable set of traces, then considering $(DARS)$-summation, we have

$$\mathrm{S}_E \vec{x}_\beta = \mathrm{Proj}_{[E]} \vec{X}, \ \mathrm{S}_W \vec{x}_\beta = X.$$

Proof. From [6], and by virtue of [§ 4; 6.1] it follows that $\overrightarrow{x_\beta}$ is (*DARS*)-summable on *E*. By theor. [§ 5; 12] we have

$$\mathbf{S}_E \overrightarrow{x_\beta} = \mathrm{Proj}_{[E]} \mathbf{S}_1 \overrightarrow{x_\beta} = \mathrm{Proj}_{[E]} \overrightarrow{X}, \qquad \text{Q. E. D.}$$

7. - The total set of x_β is (*DARS*)-summable on *W*.

Proof. It suffices to prove that the vector-field $x_\beta(p)$ is summable on *I*. Let $P_n = \{p_{n1}, p_{n2}, \ldots\}$ be a completely distinguished sequence for *I*. We have

$$\Sigma_k x_\beta(p_{nk}) = \Sigma_k(\overrightarrow{\omega}, \mathrm{Proj}_{nk} \overrightarrow{X}) = (\overrightarrow{\omega}, \mathrm{Proj}_{\mathrm{som} P_n} \overrightarrow{X}) \longrightarrow (\overrightarrow{\omega}, \overrightarrow{X}).$$

The summability follows.

7.1. - We have for every measurable set *E* of traces: if $\overrightarrow{X} \in \boldsymbol{H}$, then

$$\mathbf{S}_E x_\beta = (\overrightarrow{\omega}, \mathrm{Proj}_E \overrightarrow{X}), \ \mathbf{S}_W x_\beta = (\overrightarrow{\omega}, \overrightarrow{X}).$$

Proof. The set x_β is summable on *E*, (by [7]). Taking a completely distinguished sequence $\{P_n\}$ for *E*, we obtain, as in the proof of [7],

$$\Sigma_k(\overrightarrow{\omega}, \mathrm{Proj}_{p_{nk}} \overrightarrow{X}) = \Sigma_k(\overrightarrow{\omega}, \mathrm{Proj}_{\mathrm{som} P_n} \overrightarrow{X}) \longrightarrow (\overrightarrow{\omega}, \mathrm{Proj}_E \overrightarrow{X})$$

8. - We have

$$\mathbf{S}_E \overrightarrow{x_\beta}\, \mu_\beta^* = \mathbf{S}_E \overrightarrow{x_\beta^*}\, \mu_\beta = \mathbf{S}_E \overrightarrow{x_\beta} = \mathrm{Proj}_{[E]} \overrightarrow{X},$$

$$\mathbf{S}_E x_\beta \mu_\beta^* = \mathbf{S}_E x_\beta^* \mu_\beta = \mathbf{S}_E x_\beta = (\overrightarrow{\omega}, \mathrm{Proj}_E \overrightarrow{X})$$

for every measurable set *E* and every $\overrightarrow{X} \in \boldsymbol{H}$. (The quasi-number μ_β^* is defined by $\mu^*(p) = 1$ for all neighborhoods p of β).

9. - If $\overrightarrow{X} \in \boldsymbol{H}$, λ is a complex number, then

$$\overrightarrow{(\lambda \overrightarrow{X})_\beta} = \lambda \overrightarrow{x_\beta}, \ \overrightarrow{(\lambda \overrightarrow{X})_\beta^*} = \lambda \overrightarrow{x_\beta^*},$$

$$(\lambda \overrightarrow{X})_\beta = \lambda x_\beta, \ (\lambda \overrightarrow{X})_\beta^* = \lambda x_\beta^*.$$

10. - If $\overrightarrow{X} \in \boldsymbol{H}$, $\overrightarrow{Y} \in \boldsymbol{H}$, then

$$(\overrightarrow{X} \pm \overrightarrow{Y})_\beta = \overrightarrow{x_\beta} \pm \overrightarrow{y_\beta}, \ (\overrightarrow{X} \pm \overrightarrow{Y})_\beta^* = \overrightarrow{x_\beta^*} \pm \overrightarrow{y_\beta^*}.$$

Putting $\overrightarrow{Z} \underset{df}{=} \overrightarrow{X} \pm \overrightarrow{Y}$, we have

$$z_\beta = x_\beta \pm y_\beta, \quad z_\beta^* = x_\beta^* \pm y_\beta^* .$$

11. - If $\overrightarrow{X}, \overrightarrow{Y} \in \boldsymbol{H}$, then

$$(\overrightarrow{X}, \overrightarrow{y_\beta}) = (\overrightarrow{x_\beta}, \overrightarrow{Y}) = (\overrightarrow{x_\beta}, \overrightarrow{y_\beta}).$$

Proof. We have

$$(\mathrm{Proj}_p \overrightarrow{X}, \overrightarrow{Y}) = (\overrightarrow{X}, \mathrm{Proj}_p \overrightarrow{Y}) = (\overrightarrow{X}, \mathrm{Proj}_p \mathrm{Proj}_p \overrightarrow{Y}) =$$

$$= (\mathrm{Proj}_p \overrightarrow{X}, \mathrm{Proj}_p \overrightarrow{Y}).$$

13. - Our next purpose will be a proof of the formula $\overrightarrow{x_\beta} \approx\!\!\!\approx \overrightarrow{\omega_\beta} \cdot x_\beta^*$, [§ 4; 7]. It will be proved by steps expressed in few lemmas.

13.1. - **Lemma**. If $\overrightarrow{X} = \mathrm{Proj}_a \overrightarrow{\omega}$, and a is a brick, then $\overrightarrow{x_\beta} \approx\!\!\!\approx \overrightarrow{\omega_\beta} \cdot x_\beta^*$.

13.1a. - **Proof**. To simplify formulas, denote by $\overrightarrow{\omega_E}$ the vector $\mathrm{Proj}_E \overrightarrow{\omega}$ for any $E \in G$, use the same letter for a measurable set of traces and its coat, and write $|E|$ instead of μE. We have

$$\overrightarrow{x_\beta} = \mathrm{Proj}_{\dot p} \overrightarrow{X}, \quad \overrightarrow{\omega_\beta} x_\beta^* = \mathrm{Proj}_{\dot p} \overrightarrow{\omega} \cdot \frac{(\mathrm{Proj}_{\dot p} \overrightarrow{\omega}, \overrightarrow{X})}{\mu(\dot p)};$$

hence

$$\overrightarrow{x_\beta} = \mathrm{Proj}_{\dot p}\, \mathrm{Proj}_a \overrightarrow{\omega} = \mathrm{Proj}_{\dot p a} \overrightarrow{\omega} = \overrightarrow{\omega_{\dot p a}},$$

$$\overrightarrow{\omega_\beta} x_\beta^* = \overrightarrow{\omega_{\dot p}} \frac{(\overrightarrow{\omega_{\dot p}}, \overrightarrow{\omega_a})}{|\dot p|} = \overrightarrow{\omega_{\dot p}} \cdot \frac{|\dot p a|}{|\dot p|},$$

where p varies over all neighborhoods of β. Hence

$$\overrightarrow{x_\beta} - \overrightarrow{\omega_\beta} x_\beta^* = \overrightarrow{\omega_{\dot p a}} - \overrightarrow{\omega_{\dot p}} \frac{|\dot p a|}{|\dot p|}. \tag{1}$$

13.1b - Take any brick $q \neq 0$. We have

$$\vec{A}(q) \underset{df}{=} \vec{\omega}_{qa} - \vec{\omega}_q \cdot \frac{|qa|}{|q|} = \vec{\omega}_{qa} - (\vec{\omega}_{qa} + \vec{\omega}_{q\,coa}) \cdot \frac{|qa|}{|q|} =$$

$$= \vec{\omega}_{qa} - \vec{\omega}_{qa} \frac{|qa|}{|q|} - \vec{\omega}_{q\,coa} \cdot \frac{|qa|}{|q|} = \vec{\omega}_{qa}\left(1 - \frac{|qa|}{|q|}\right) -$$

$$- \vec{\omega}_{q\,coa} \cdot \frac{|qa|}{|q|} = \vec{\omega}_{qa} \frac{|q\,coa|}{|q|} - \vec{\omega}_{q\,coa} \cdot \frac{|qa|}{|q|}.$$

Hence

$$\vec{A}(q) \in qa + q\,coa = q. \tag{1.1}$$

Since the spaces $q\,co\,a$, qa are orthogonal, we get

$$\|\vec{A}(q)\|^2 = |qa| \cdot \frac{|q\,coa|^2}{|q|^2} + |q\,coa| \cdot \frac{|qa|^2}{|q|^2} = \tag{2}$$

$$= \frac{|qa| \cdot |q\,coa|}{|q|^2}(|q\,coa| + |qa|), \text{ i. e. } \|\vec{A}(q)\|^2 = \frac{|qa| \cdot |q\,coa|}{|q|},$$

valid for any brick $q \neq 0$.

13.1c. - Now let E be a measurable set of traces and $\{P_n\}$ a completely distinguished sequence for E. We shall use arguments similar to those in the proof of [§ 4,15]; they rely on [§ 1; 21.6, 21.13, 21.9, 21.14]. Take a subsequence $\{P_{k(n)}\}$ of $\{P_n\}$, and get a completely distinguished sequence $P_{kl(n)} \cup Q_n$ of I. Considering a partial complex $\{R_n\}$ of $P_{kl(n)} \cup Q_n$ such that $\{R_n\}$ is a completely distinguished sequence for q, take the bricks $e'_{n1}, e'_{n2}, \ldots$ with $\mu(\Sigma_k e'_{nk}\,coa) \to 0$ and also the bricks $e''_{n1}, e''_{n2}, \ldots$ with $\mu(\Sigma_k e''_{nk}\,a) \to 0$. The bricks belong to $P_{kl(n)}$. We have, putting

$$\vec{A}_\beta \underset{df}{=} \vec{x}_\beta - \vec{\omega}\, x^*_\beta,$$

$$\|\vec{A}(P_{kl(n)})\|^2 = \Sigma_j \|\vec{A}(e'_{nj})\|^2 + \Sigma_j \|\vec{A}(e''_{nj})\|^2.$$

Hence, by (2),

$$\|\vec{A}(P_{kl(n)}\|^2 = \Sigma_j \frac{|e'_{nj}\,a| \cdot |e'_{nj}\,\mathrm{coa}|}{|e'_{nj}|} + \Sigma_j \frac{|e''_{nj}\,a| \cdot |e''_{nj}\,\mathrm{coa}|}{|e''_{nj}|} \leq \tag{3}$$

$$\leq \Sigma_j |e'_{nj}\,\mathrm{coa}| + \Sigma_j |e''_{nj}\,a| = \mu(\Sigma_j e'_{nj}\,\mathrm{coa}) + \mu(\Sigma_j e''_{nj}\,a) \to 0.$$

Thus from every partial sequence $\{Q_n\}$ of $\{P_n\}$, another partial sequence $\{Q_{k(n)}\}$ can be extracted with [3].

Hence $\|\overrightarrow{A}(P_n)\|^2 \longrightarrow 0$, which gives $\overrightarrow{A}(P_n) \longrightarrow \overrightarrow{0}$ i. e. the quasi-vector $\overrightarrow{x_\beta} - \overrightarrow{\omega_\beta}\, x_\beta^*$ is summable over any measurable set E of traces, and we have

$$\mathrm{S}_E(\overrightarrow{x_\beta} - \overrightarrow{\omega_\beta}\, x_\beta^*) = \overrightarrow{0}.$$

Since $\overrightarrow{x_\beta}$ is summable on E, [6], it follows, by [§4; 6-4], that

$$\overrightarrow{x_\beta} - (\overrightarrow{x_\beta} - \overrightarrow{\omega_\beta}\, x_\beta^*) = \overrightarrow{\omega_\beta}\, x_\beta^*$$

is also summable on E, and we have

$$\mathrm{S}_E \overrightarrow{x_\beta} = \mathrm{S}_E \overrightarrow{\omega_\beta}\, x_\beta^*$$

for all summable E. Hence, by [§ 4; Def. 7]

$$\overrightarrow{x_\beta} \approxeq \overrightarrow{\omega_\beta} \cdot x_\beta^* \qquad \text{Q. E. D.}$$

13.2. - **Lemma.** If λ is a complex number, $\overrightarrow{x} = \lambda\, \mathrm{Proj}_a \overrightarrow{\omega}$, where a is a brick, we have $\overrightarrow{x_\beta} \approxeq \overrightarrow{\omega_\beta} \cdot x_\beta^*$.

13.3. - Now we notice that if $\overrightarrow{X}, \overrightarrow{Y} \in \boldsymbol{H}$, $\overrightarrow{Z} \underset{df}{=} \xi\overrightarrow{X} + \eta\overrightarrow{Y}$, and $\overrightarrow{x_\beta} \approxeq \overrightarrow{\omega_\beta} \cdot x_\beta^*$, $\overrightarrow{y_\beta} \approxeq \overrightarrow{\omega_\beta} \cdot y_\beta^*$, then $z_\beta^* \approxeq \overrightarrow{\omega_\beta} \cdot z_\beta^*$.

Proof. We have for a measurable set E:

$$\mathrm{S}_E \overrightarrow{x_\beta} = \mathrm{S}_E \overrightarrow{\omega_\beta}\, x_\beta^*, \quad \mathrm{S}_E \overrightarrow{y_\beta} = \mathrm{S}_E \overrightarrow{\omega_\beta}\, y_\beta^*;$$

hence

$$\mathrm{S}_E \overrightarrow{z_\beta} = \mathrm{S}_E (\overrightarrow{\omega_\beta} \cdot \xi x_\beta^* + \overrightarrow{\omega_\beta} \cdot \eta y_\beta^*) = \mathrm{S}_E \overrightarrow{\omega_\beta}\, z_\beta^*,$$

which completes the proof.

13.4. It follows that the theorem is true for any "step" vector $\Sigma_{i=1}^{n} \lambda_i\, \mathrm{Proj}_{a_i} \overrightarrow{\omega}, (n \geq I)$, where $a_1, a_2, \ldots a_n$ are disjoint bricks with $\Sigma_i a_i = I$ and λ_i complex numbers.

13.5. Now we shall prove that if $\|\overrightarrow{X_k}\|$, $(k = 1, 2, \ldots)$ are step functions as above, and if $\overrightarrow{X_k} \longrightarrow \overrightarrow{X}$, then the theorem [13] holds true for $\overrightarrow{X}$. It will follow that the theorem [13] is true in general.

Indeed, since $\overrightarrow{\omega}$ is a generating vector and G is the Lebesgue's covering extension of B, therefore for every $\overrightarrow{X}$ there exists a sequence of step-vectors $\overrightarrow{X}_1$, $\overrightarrow{X}_2, \ldots$ which tend in the H-topology to $\overrightarrow{X}$. Take $\varepsilon > 0$. Find k with $\|\overrightarrow{X} - \overrightarrow{X}_k\| < \varepsilon$. Put for any brick p

$$\overrightarrow{\Phi}(p) \underset{df}{=} \operatorname{Proj}_p \overrightarrow{X}, \quad \overrightarrow{\Psi}(p) \underset{df}{=} \operatorname{Proj}_p \overrightarrow{\omega} \frac{(\operatorname{Proj}_p \overrightarrow{\omega}, \overrightarrow{X})}{\mu(p)},$$

$$\overrightarrow{\varphi}_k(p) \underset{df}{=} \operatorname{Proj}_p \overrightarrow{X}_k, \quad \overrightarrow{\psi}_k(p) \underset{df}{=} \operatorname{Proj}_p \overrightarrow{\omega} \frac{(\operatorname{Proj}_p \overrightarrow{\omega}, \overrightarrow{X}_k)}{\mu(p)}.$$

First we shall prove some inequalities. Let

$$P_n = \{ p_{n1}, p_{n2}, \ldots . p_{ni}, \ldots \}, \qquad (n = 1, 2, \ldots),$$

be a completely distinguished sequence for a given measurable set E of traces. We have

$$\Sigma_i(\overrightarrow{\Phi}(p_{ni}) - \overrightarrow{\varphi}_k(p_{ni})) = \Sigma_i(\operatorname{Proj}_{p_{ni}} \overrightarrow{X} - \operatorname{Proj}_{p_{ni}} \overrightarrow{X}_k) =$$

$$= \Sigma_i \operatorname{Proj}_{p_{ni}}(\overrightarrow{X} - \overrightarrow{X}_k), \text{ and}$$

since the spaces p_{ni} are disjoint, the expression

$$= \operatorname{Proj}_{\Sigma_i p_{ni}}(\overrightarrow{X} - \overrightarrow{X}_k).$$

Hence

$$\| \Sigma_i (\overrightarrow{\Phi}(p_{ni}) - \overrightarrow{\varphi}(p_{ni})) \| \leq \| \overrightarrow{X} - \overrightarrow{X}_k \| < \varepsilon. \tag{1}$$

Hence

$$\Sigma_i \| \overrightarrow{\Phi}(p_{ni}) - \overrightarrow{\varphi}(p_{ni}) \|^2 < \varepsilon^2.$$

On the other hand we have

$$\| \Sigma_i \{ \overrightarrow{\Psi}(p_{ni}) - \overrightarrow{\psi}_k(p_{ni}) \} \|^2 = \| \Sigma_i \frac{\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}}{\mu(p_{ni})} [(\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}, \overrightarrow{X}) -$$

$$- (\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}, \overrightarrow{X}_k)] \|^2 = \| \Sigma_i \frac{\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}}{\mu(p_{ni})} \cdot (\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}, \overrightarrow{X} - \overrightarrow{X}_k) \|^2 \leq$$

$$\leq \| \Sigma_i \frac{\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}}{\mu(p_{ni})} (\operatorname{Proj}_{p_{ni}} \overrightarrow{\omega}, \operatorname{Proj}_{p_{ni}} (\overrightarrow{X} - \overrightarrow{X}_k)) \|^2 \leq$$

$$\leq \Sigma_i \left\| \frac{\mathrm{Proj}_{p_{ni}} \overrightarrow{\omega}}{\mu(p_{ni})^2} \cdot (\mathrm{Proj}_{p_{ni}} \overrightarrow{\omega},\ \mathrm{Proj}_{p_{ni}} (\overrightarrow{X} - \overrightarrow{X_k})) \right\|^2 =$$

$$= \Sigma_i \frac{\| \mathrm{Proj}_{p_{ni}} \overrightarrow{\omega} \|^2}{\mu(p_{ni})^2} \cdot |(\mathrm{Proj}_{p_{ni}} \overrightarrow{\omega},\ \mathrm{Proj}_{p_{ni}} (\overrightarrow{X} - \overrightarrow{X_k}))|^2,$$

by CAUCHY-SCHWARZ inequality:

$$\leq \Sigma_i \frac{\mu(p_{ni})}{\mu(p_{ni})^2} \cdot \| \mathrm{Proj}_{p_{ni}} \overrightarrow{\omega} \|^2 \cdot \| \mathrm{Proj}_{p_{ni}} (\overrightarrow{X} - \overrightarrow{X_k}) \|^2 =$$

$$= \Sigma_i \frac{\mu(p_{ni})}{\mu(p_{ni})^2} \cdot \mu(p_{ni}) \cdot \| \mathrm{Proj}_{p_{ni}} (\overrightarrow{X} - \overrightarrow{X_k}) \|^2 =$$

$$= \Sigma_i \| \mathrm{Proj}_{p_{ni}} (\overrightarrow{X} - \overrightarrow{X_k}) \|^2 \leq \Sigma_i \| X - X_k \|^2 \leq \varepsilon^2 .$$

Hence we have proved that

$$\Sigma_i \| \overrightarrow{\Psi} (p_{ni}) - \overrightarrow{\psi_k} (p_{ni}) \|^2 \leq \varepsilon^2 \text{ is true for all } n = 1, 2, \ldots \quad (2)$$

We have

$$\Sigma_i \| \overrightarrow{\Phi} p(_{ni}) - \overrightarrow{\Psi} (q_{ni}) \|^2 = \Sigma_i \| \{ \overrightarrow{\Phi} (p_{ni}) - \overrightarrow{\varphi_k} (p_{ni}) \} +$$

$$+ \{ \overrightarrow{\varphi_k} (p_{ni}) - \overrightarrow{\psi} (p_{ni}) \} + \{ \overrightarrow{\psi} (p_{ni}) - \overrightarrow{\Psi} (p_{ni}) \} \|^2 \leq$$

$$\leq 4 \Sigma_i \| \overrightarrow{\Phi} (p_{ni}) - \overrightarrow{\varphi_k} (p_{ni}) \|^2 + 4 \Sigma_i \| \overrightarrow{\varphi_k} (p_{ni}) - \overrightarrow{\psi_k} (p_{ni}) \|^2 +$$

$$+ 4 \Sigma_i \| \overrightarrow{\psi} (p_{ni}) - \overrightarrow{\Psi} (p_{ni}) \|^2 .$$

Hence, by (1) and (2)

$$\Sigma_i \| \overrightarrow{\Phi} (p_{ni}) - \Psi (p_{ni}) \|^2 \leq 8 \varepsilon^2 + 4 \Sigma_i \| \overrightarrow{\varphi_k} (p_{ni}) - \overrightarrow{\psi_k} (p_{ni}) \|^2$$

for all $n = 1, 2, \ldots$ Now we know, [13.4], that

$$\overrightarrow{x_{k\beta}} \approx \overrightarrow{\omega_\beta} \cdot x^*_{k\beta}$$

for $k = 1, 2, \ldots$ It follows that, given $\varepsilon > 0$, we have

$$\Sigma_i \| \overrightarrow{\varphi_k} (p_{ni}) - \overrightarrow{\psi_k} (p_{ni}) \|^2 < \varepsilon^2$$

for sufficiently great n.

Consequently for every $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ we have

$$\Sigma_i \| \overrightarrow{\Phi} (p_{ni}) - \overrightarrow{\Psi} (p_{ni}) \|^2 \leq 12 \varepsilon^2 .$$

It follows that

$$\|\overrightarrow{\Phi}(P_n) - \overrightarrow{\Psi}(P_n)\|^2 \longrightarrow 0,$$

for $n \longrightarrow \infty$.

Now, $\lim\limits_{n \longrightarrow \infty} \overrightarrow{\Phi}(P_n)$ exists and equals $\mathbf{S}_E \overrightarrow{x_p}$.

Hence $\lim\limits_{n \longrightarrow \infty} \overrightarrow{\Psi}(P_n)$ exists too. Hence $\mathbf{S}_E\, \omega_\beta \cdot \overrightarrow{x_\beta^*}$ exists and equals $\mathbf{S}_E \overrightarrow{x_\beta}$. Thus we have proved the theorem.

13.6. - **Theorem**. If $\overrightarrow{X}$ is any vector in $\boldsymbol{H}$, then

$$\overrightarrow{x_\beta} \approx\!\!\approx \overrightarrow{\omega_\beta} \cdot x_\beta^*.$$

§ 7. - Dirac's Delta-Function.

This § 7 is devoted to a mathematically precise theory of the δ-function, (see Preliminaries). We shall introduce even more general notions having some properties of the δ-function. Our theory is based on the general topics which were developped in the preceding sections.

1. - **Def.** We admit the hypotheses stated in [§ 4; 1 and 2], concerning the tribes $\boldsymbol{G}$, $\boldsymbol{F}$, the basis $\boldsymbol{B}$ and the linear vector space $\boldsymbol{V}$.

Let x_0, y_0 be two traces; then by a *quasi-vector with support* (x_0, y_0) we shall understand any function $\overrightarrow{f}(\dot{p}, \dot{q})$ with values taken from V and defined for all neighborhoods p of the trace x_0, [§ 3; Def. 10], and for all neighborhoods q of y_0. We shall write $\overrightarrow{f_{x_0, y_0}}$, or $\overrightarrow{f}(x_0, y_0)$.

1.1. - **Def.** We shall consider sets of quasi-vectors $\overrightarrow{f_{x, y}}$, where x varies in a measurable set E of traces, and y varies in a measurable set F of traces. We shall call the couple (E, F) the *support of the set of quasi-vectors.*

1.2 We have some modifications of these notion:

If x_0 is a trace and q a brick. we can consider the vector valued function $\overrightarrow{f}(x_0, \dot{q})$ which attaches to every neighborhood p of x_0 a vector of $\boldsymbol{V}$. We can vary x_0 over a set of traces and q over a set of bricks, getting a kind of sets of quasi-vectors.

1.3. - Given a quasi-vector $\overrightarrow{\dot{f}}(x_0, y_0)$, we shall write it $\overrightarrow{f_{x_0}}(\dot{q})$, $\overrightarrow{f_{y_0}}(\dot{p})$, $\overrightarrow{f}(\dot{p}, \dot{q})$ according to whether we like to emphasize the variable neighborhood q of y, the variable neighborhood p of x_0, or both variable neighborhoods respectively.

2. - **Def.** The following notion of summation will be important: We say that the *set of quasi-vectors* $\overrightarrow{f_{x_0}}, {}_y$ *with support* (x_0, E), (where x_0, y are traces, and y varies in E) *is summable on* E *with respect to* y whenever for every neighborhood p of x_0, the set of quasi-vectors $\overrightarrow{f_y}(p)$ is summable on E with respect to y, i.e., when $\underset{y \in E}{\mathrm{S}} \overrightarrow{f_y}(p)$ exists for every neighborhood p of x_0.

In the case of summability, we get a quasi-vector

$$\overrightarrow{g_{x_0}} \underset{df}{=} g(\dot{p}) \underset{df}{=} \underset{y \in E}{\mathrm{S}} \overrightarrow{f_y}(\dot{p})$$

with support x_0.

3. - **Def.** We are introducing the number-valued function $\Delta(\dot{p}, \dot{q})$ of variable non null bricks p, q, defining it by:

$$(1) \qquad \Delta(p, q) \underset{df}{=} \begin{cases} 1 & \text{whenever } \; p \cdot q \neq 0\,, \\ 0 & \quad \text{»} \qquad\; p \cdot q = 0\,. \end{cases}$$

This function generates the following ones:

If τ is a trace, then $\Delta(\tau, q)$ is the quasi number $\Delta(\dot{p}, q)$ whit support τ, defined for all neighborhoods p of τ, by (1). It depends on the parameter q. Similarly $\Delta(p, \dot{q})$ will be denoted by $\Delta(p, \xi)$ whenever q varies over all neighborhoods of ξ. By $\Delta(\tau, \xi)$, where τ, ξ are two traces, we shall understand the function $\Delta(\dot{p}, \dot{q})$ defined for all neighborhoods p of τ and for all neighborhoods q of ξ.

4. - We shall take over the topic of [§ 6] to have a system of coordinates in the H.H. — space $\boldsymbol{H}$. Thus $\boldsymbol{G}$ is supposed to be a saturated tribe of spaces.

4.1. - **Lemma.** For any vector $\overrightarrow{X} \in \boldsymbol{H}$ and any spaces

$a, b \in G$ we have

$$\| \mathrm{Proj}_a \vec{X} - \mathrm{Proj}_b \vec{X} \|^2 = \| \mathrm{Proj}_{a-b} \vec{X} \|^2 + \| \mathrm{Proj}_{b-a} \vec{X} \|^2,$$

(see [(14), p 21]).

4.2. - Lemma. If $a_n \in G$, $n = 1, 2, \dots$, $\vec{X} \in \boldsymbol{H}$ and $\mu(a) \longrightarrow 0$, then $\mathrm{Proj}_{a_n} \vec{X} \longrightarrow \vec{0}$ in the $\boldsymbol{H}$ — topology.

Proof. The number valued function $\| \mathrm{Proj}_a \vec{X} \|^2$ of the variable $a \in G$ is denumerably additive and continuous in the μ–topology in G. Hence $\| \mathrm{Proj}_{a_n} \vec{X} \|^2 \longrightarrow 0$ which gives $\mathrm{Proj}_{a_n} \vec{X} \longrightarrow \vec{0}$ in the $\boldsymbol{H}$–topology.

4.3. - Lemma. If a_n, $a \in G$, $\vec{X} \in \boldsymbol{H}$, $a_n \xrightarrow{\mu} a$ in the μ–topology in G, then $\mathrm{Proj}_{a_n} \vec{X} \longrightarrow \mathrm{Proj}_a \vec{X}$ in the $\boldsymbol{H}$–topology.

Proof. Since $a_n \xrightarrow{\mu} a$, we have $\mu(a_n - a) + \mu(a - a_n) \longrightarrow 0$. Hence, by [4.2], $\| \mathrm{Proj}_{a_n - a} \vec{X} \|^2 \longrightarrow 0$ and $\| \mathrm{Proj}_{a - a_n} \vec{X} \|^2 \longrightarrow 0$. Hence, by [4.1], we get $\| \mathrm{Proj}_{a_n} \vec{X} - \mathrm{Proj}_a \vec{X} \|^2 \longrightarrow 0$; hence $\mathrm{Proj}_{a_n} \vec{X} - \mathrm{Proj}_a \vec{X} \longrightarrow \vec{0}$, which gives the thesis.

5. - Def. Let $Q_n = \{ q_{n1}, q_{n2}, \dots \}$, $(n = 1, 2, \dots)$, be a completely distinguished sequence of complexes for I. Given a brick $p \neq 0$, let $p_{n1}, p_{n2}, \dots$ be all those bricks q_{nk} for which $q_{nk} \leq p$. We get a complex $\{ p_{n1}, p_{n2}, \dots \}$, which may be empty or not. Now, if for every $p \neq 0$ we have $\lim_{n \to \infty} \mu(\sum_i p_{ni}) = \mu(p)$, we shall call $\{ Q_n \}$ a *special sequence for* I.

5.1. - Remark. We do not know whether from all admitted hypotheses it follows that there exists at least one completely distinguished and special sequence $\{ Q_n \}$ for I.

5.2. - Hyp. We shall admit the following *hypothesis*: There exists at least one completely distinguished special sequence of complexes for I.

5.3. - In the case where the base $\boldsymbol{B}$ is composed of spaces which correspond to half — open rectangles or half — open segments, (see [§ 5; 2.1–2.9]). the hypothesis [5.2] is satisfied.

5.4. - We shall consider summations of total quasi-vector-set defined by means of special sequences. This means that, given a total set of quasi-vectors $\overrightarrow{f_\tau}$, we say that $\overrightarrow{f_\tau}$ is "*specially*,. *summable over* W" whenever for every completely distinguished and special sequence $\{Q_n\} = \{q_{n1}, q_{n2}, \ldots\}$, the limit $\lim\limits_{n\to\infty} \Sigma_k \overrightarrow{f}(q_{nk})$ exists and has the same value. The limit will be denoted by $\underset{W}{\mathrm{S}^\bullet} \overrightarrow{f_\tau}$ and called "*special*,, *sum*.

5.5. - If Hyp. 5.2 holds true and $\overrightarrow{f_\tau}$ is $(DRAS)$-summable, then $\underset{W}{\mathrm{S}^\bullet} \overrightarrow{f_\tau}$ exists too. The converse does not seem to be true.

5.6. - **Lemma.** If 1. $\{Q_n\}$ is a completely distinguished and special sequence of complexes for I,

2. f is a figure, $(f \in F)$, $f \neq 0$,

3. $f_{n1}, f_{n2}, \ldots$ are all bricks of Q_n with $f_{nk} \leq f$,

then we have $\lim\limits_{n\to\infty} \mu(\Sigma_k f_{nk}) = \mu(f)$.

Proof. First we shall prove the lemma under hypothesis that f is a finite sum of disjoint bricks. Let

$$f = a_1 + \ldots + a_s, \quad (s \geq 2).$$

Denote by $q_{n1}^{(k)}, q_{n2}^{(k)}, \ldots$ all different bricks of Q_n, which are contained in a_k, $(k = 1, \ldots, s)$. We have $\lim\limits_{n\to\infty} \mu(\Sigma_j q_{nj}^{(k)}) = \mu(a_k)$.

Since $a_1, \ldots, a_s$ are disjoint, and consequently also the bricks of Q_n which are inside of them, we get

$$(1) \qquad \lim_{n\to\infty} \mu(\Sigma_{k=1}^{s} \Sigma_j q_{nj}^{(k)}) = \mu(\Sigma_{k=1}^{s} a_k) = \mu(f).$$

The bricks $q_{nj}^{(k)}$ for fixed n are certainly bricks which are inside of f. If there are some other supplementary bricks of Q_n which are inside f, we have $\Sigma_{k=1}^{s} \Sigma_j q_{nj}^{(k)} \leq \Sigma_k f_{nk} \leq f$, where f_{nk} are all bricks of Q_n which are inside f; hence $\lim\limits_{n\to\infty} \mu(\Sigma_k f_{kn}) = \mu(f)$.

Having that, let us go over to the general figure f. By [§ 1; 3.6] we have $f = a_1 + a_2 + \dots$ where a_j are disjoint bricks.

Take $\varepsilon > 0$, and find s such that $0 \leq \mu(f) - \Sigma_{j=1}^{s} \mu(a_j) \leq \varepsilon \dots (1)$

Put $f_s \underset{df}{=} a_1 + \dots + a_s$. For such a figure the theorem has been proved. Let $Q_n = \{ q_{n1}, q_{n2}, \dots \}$ be a special sequence for 1. Denote by $q'_{n1}, q'_{n2}, \dots$ all those bricks q_{nk} of Q_n which are inside f_s. We have $\lim\limits_{n \to \infty} \Sigma_i \mu(q'_{ni}) = \mu f_s$.

There exists M such that for all $n \geq M$ we have

$0 \leq \mu(f) - \mu(f_s) \leq \varepsilon$. We get $0 \leq \mu(f) - \Sigma_i \mu(q'_{ni}) \leq 2\varepsilon$ for $n \geq M \dots$ (2)

Now, if there are bricks in Q_n differing from q'_{ni} which are inside f, their addition will not spoil the inequality (1), so we get

$$0 \leq \mu(f) - \Sigma_k \mu(f_{nk}) \leq 2\varepsilon \tag{3}$$

where f_{nk} are all bricks of Q_n which are inside f. The inequality (3) is valid for all $n \geq M$. This completes the proof of the lemma.

5.7. - Lemma. If 1. $\{ Q_n \}$ is a completely distinguished and special sequence of complexes for 1, 2. f is a figure. 3. $e_{n1}, e_{n2}, \dots$ are all bricks of Q_n for which

$$e_{nk} \cdot f \neq 0, \quad e_{nk} \cdot \mathrm{co}\, f \neq 0,$$

then we have $\lim\limits_{n \to \infty} \mu(\Sigma_k e_{nk}) = 0$.

Proof. Let $a_{n1}, a_{n2}, \dots$ be all bricks of Q_n for which $a_{nk} \leq p$, and let $b_{n1}, b_{n2}, \dots$ be all bricks of Q_n for which $b_{nk} \leq \mathrm{co}\, p$. By [Lemma 5.6] we have

$$\lim_{n \to \infty} \mu(\Sigma_k a_{nk}) = \mu(p), \qquad \lim_{n \to \infty} \mu(\Sigma_k b_{nk}) = \mu(\mathrm{co}\, p).$$

Since $\lim\limits_{n \to \infty} \mu(\Sigma_k a_{nk} + \Sigma_k b_{nk} + \Sigma_k e_{nk}) = \mu(1)$, it follows that $\mu(p) + \mu(\mathrm{co}\, p) + \lim \mu(\Sigma_k e_{nk}) = \mu(1)$, which completes the proof.

6. - Theor. If 1. Hypothesis [5.2] is admited, 2. $p \neq 0$ is a brick-space, 3. $\vec{X} \in \boldsymbol{H}$, then considering special summations, [5.4], we have

$$\text{Proj}_p \overrightarrow{X} = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta(p, \beta)\, \overrightarrow{x_\beta}, \text{ [Def. 3]}, \tag{1}$$

(where W is the set of all traces and $\overrightarrow{x_\beta}$ is the β-component of $\overrightarrow{X}$, [§ 6; 2]).

6a. - **Proof.** We shall schedule our argument so as to put in evidence the reason of admitting [Hyp. 5.2].

To simplify print, we shall use the alternative symbol $\text{Proj}(p)\overrightarrow{X}$ for $\text{Proj}_p\overrightarrow{X}$. Let $Q_n = \{q_{n1}, q_{n2}, \dots\}$ be a completely distinguished and special sequence of complexes for I. Take the partial complex R_n of Q_n with $|R_n, p|_\mu \to 0$. By [§ 1; 21.9] R_n is a completely distinguished sequence for p.

Consider the complex $S_n \underset{df}{=} Q_n \backsim R_n$ i. e. the complex complementary to R_n in Q_n. We have $|\text{som}\, Q_n - \text{som}\, R_n, I - p|_\mu \to 0$, [§ 1; 5.14], i. e. $|S_n, \text{co}\, p|_\mu \to 0$. Hence, by [§ 1; 21.9]. S_n is a completely distinguished sequence for $\text{co}\, p$.

Denote by a_i', b_i', e_i' those bricks of R_n for which $a_i' \leq p$; $b_i' \cdot p = 0$; $e_i' \cdot p \neq 0$, $e_i' \cdot \text{co}\, p \neq 0$ respectively, and denote by a_i'', b_i'', e_i'' those bricks of S_n for which $a_i'' \leq p$; $b_i'' \cdot p = 0$; $e_i'' \cdot p \neq 0$, $e_i'' \cdot \text{co}\, p \neq 0$ respectively.

We have, by [§ 1; 21.14],

$$|\Sigma a_i' + \Sigma e_i' p,\ p| \to 0,\quad \mu(\Sigma e_i' \,\text{co}\, p) \to 0, \tag{2}$$

$$|\Sigma b_i'' + \Sigma e_i'' \,\text{co}\, p,\ \text{co}\, p| \to 0,\quad \mu(\Sigma e_i'' p) \to 0 \tag{3}$$

for $n \to \infty$.

6b. - We also have

$$\mu(\Sigma b_i') \to 0,\quad \mu(\Sigma a_i'') \to 0. \tag{4}$$

Indeed, from (2) we get, by the help of

$$|\Sigma e_i' \,\text{co}\, p;\ 0|_\mu \to 0,\quad |\Sigma a_i' + \Sigma e_i' p + \Sigma e_i' \,\text{co}\, p;\ p| \to 0, \text{ i. e.},$$

$$|\Sigma a_i' + \Sigma e_i';\ p| \to 0.$$

Since, on the other hand we have

$$|R_n, p| \to 0, \text{ i. e.}, |\Sigma a_i' + \Sigma b_i' + \Sigma e_i';\ p| \to 0,$$

we get, by subtraction, relying on [§ 1; 5.14],

$$|\Sigma b_i',\ O| \longrightarrow 0, \text{ i. e.,} \quad \mu(\Sigma b_i') \longrightarrow 0.$$

Similarly we prove the second relation in (4).

6c. - We shall build sums which approximate the expression (1) in [6]. Put

$$\overrightarrow{A_n} \underset{df}{=} \Sigma_k \Delta(p,\ q_{nk}) \operatorname{Proj}(q_{nk}) \overrightarrow{X}. \tag{5}$$

The bricks a_i', b_i', e_i', a_i'', b_i'', e_i'' constituite the whole complex Q_n, and they are disjoint. In (5) all terms, where $q_{nk} \cdot p = O$, disappear, and for the other we have $\Delta = 1$.

Thus

$$\overrightarrow{A_n} = \Sigma_i \operatorname{Proj}(a_i') \overrightarrow{X} + \Sigma_i \operatorname{Proj}(a_i'') \overrightarrow{X} + \Sigma_i \operatorname{Proj}(e_i') \overrightarrow{X} + \Sigma_i \operatorname{Proj}(e_i'') \overrightarrow{X}.$$

Since the brick-spaces are orthogonal to one another, we get

$$(6)\ \overrightarrow{A_n} = \operatorname{Proj}(\Sigma_i a_i' + \Sigma_i e_i' p) \overrightarrow{X} + \operatorname{Proj}(\Sigma_i a_i'') \overrightarrow{X} + \operatorname{Proj}(\Sigma_i e_i' \operatorname{co}p) \overrightarrow{X} +$$
$$+ \operatorname{Proj}(\Sigma_i e_i'' p) \overrightarrow{X} + \operatorname{Proj}(\Sigma_i b_i'') \overrightarrow{X} - \operatorname{Proj}(\Sigma_i b_i'') \overrightarrow{X}.$$

In (6) the first term tends to $\operatorname{Proj}_p \overrightarrow{X}$ because of (2) and by virtue of [Lemma 4.3], the second therm tends to $\overrightarrow{O}$, because of (4) and [Lemma 4.2]; the third term tends to $\overrightarrow{O}$ because of (2). Concerning the last three terms in (6) their sum can be written as

$$\operatorname{Proj}(\Sigma_i b_i'' + \Sigma e_i'' p) \overrightarrow{X} - \operatorname{Proj}(\Sigma_i b_i'') \overrightarrow{X}.$$

Here the first term tends to $\operatorname{Proj}(\operatorname{co}p) \overrightarrow{X}$, by (3) and [Lemme 4.3]. Hence a necessary and sufficient condition that $\overrightarrow{A_n}$ tends to a limit is that $\operatorname{Proj}(\Sigma_i b_i'') \overrightarrow{X}$ tends to a limit.

6d. - Till now we did not use at all the condition that the sequence $\{Q_n\}$ is special. We used only the fact that it is completely distinguished. From [6c] we get

$$\lim_{n \to \infty} \overrightarrow{A_n} = \overrightarrow{X} - \lim_{n \to \infty} \operatorname{Proj}(\Sigma_i b_i'') \overrightarrow{X},$$

whenever at least one of these limits exists.

This shows the role of [Hyp. 5.2] which we have admitted in the wording of our theorem. Since, by [Lem. 5.6] $|\Sigma_i b_i''; \text{cop}| \to 0$ for $n \to \infty$, it follows, by [Lemma 4.3]:

$$\text{Proj}(\Sigma_i b_i')\vec{X} \to \text{Proj}(\text{cop})\vec{X}.$$

Hence we get

$$\lim \vec{A}_n = \vec{X} - \text{Proj}_{(\text{cop})}\vec{X} = \text{Proj}_p \vec{X}, \text{ Q.E.D.}$$

6.1. - Theor. If 1. Hypothesis [5.2] is admitted, 2. $X \in \boldsymbol{H}$, then considering special summations, [5.4], we have

$$\vec{x}_\alpha = \underset{\beta \in W}{\mathrm{S}^\bullet} \Delta(\alpha, \beta)\vec{x}_\beta,$$

where $\vec{x}_\alpha$ is the α-component of $\vec{X}$, [§ 6; 2].

Proof. Follows from [6].

6.2. - Remark. We do not know whether the formula of [Theor. 6.1] is true if we do not use special summations, (see [Rem. 5.1]).

7. - Theor. If 1. Hypothesis [5.2] is admitted, 2. $p \neq 0$ is a brick-space, 3. $\vec{\omega}$ is a generating vector of the space $\boldsymbol{H}$ with respect to the saturated tribe G, [§ 5], [§ 6], 4. $\vec{X} \in \boldsymbol{H}$, then, considering special sums, we have

$$(\text{Proj}_p \vec{\omega}. \vec{X}) = \underset{\beta \in W}{\mathrm{S}^\bullet} \Delta(p, \beta) x_\beta,$$

where x_β is the β-coordinate of $\vec{X}$, [§ 6; 2].

Proof. The theorem can be proved just by the method used in the proof of [Theor. 6]. We shall give a simpler proof.

Let $Q_n = \{q_{n1}, q_{n2}, \ldots\}$ be a special completely distinguished sequence of complexes for *1*. Let a_i, b_i, e_i be those bricks of Q_n for which $a_i \leq p$; $a_i \cdot p = 0$; $e_i \cdot p \neq 0$, $e_i \cdot \text{cop} \neq 0$ respectively. Put

$$A_n \underset{df}{=} \Sigma_k \Delta(p, q_{nk}) \cdot (\text{Proj}(q_{nk})\vec{\omega}, \vec{X}).$$

We have

$$A_n = \Sigma_i (\text{Proj}_{a_i}\vec{\omega}, \vec{X}) + \Sigma_i (\text{Proj}_{e_i}\vec{\omega}, \vec{X}).$$

Since the bricks are orthogonal spaces and since projections are hermitian operators, we get

$$A_n = (\vec{\omega}, \operatorname{Proj}(\Sigma_i a_i)\vec{X}) + (\vec{\omega}, \operatorname{Proj}(\Sigma_i e_i)\vec{X}).$$

Now, since by [Lemma 5.7], $\mu(\Sigma_i e_i) \to 0$ for $n \to \infty$, we get by [Lemma 4.2], $\operatorname{Proj}(\Sigma_i e_i)\vec{X} \to \vec{0}$.

Hence

$$(\vec{\omega}, \operatorname{Proj}(\Sigma_i e_i)\vec{X}) \to (\vec{\omega}, \vec{0}) = 0.$$

Consequently

$$\lim A_n = (\vec{\omega}; \operatorname{Proj}_p \vec{X}),$$

because $|\Sigma_i a_i;\ p|_\mu \to 0$, which gives

$$\operatorname{Proj}(\Sigma_i a_i)\vec{X} \to \operatorname{Proj}_p \vec{X}.$$

The theorem is established.

7.1. - Theor. If 1. Hypothesis [5.2] is admitted, 2. $\vec{X} \in \boldsymbol{H}$, then, for special sum we get

$$x_\alpha = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta(\alpha, \beta) x_\beta,$$

where x_α is the α-coordinate of $\vec{X}$.

Proof. This follows from [7].

8. - **Def.** We define the number-valued function $\Delta'(\dot{p}, \dot{q})$ of the variable bricks p, q, both $\neq 0$, as follows:

$$\Delta'(p, q) \underset{df}{=} \begin{cases} 0 \text{ whenever } p \cdot q = 0, \\ \dfrac{1}{\max(\mu(p), \mu(q))} \quad \text{whenever } p \cdot q \neq 0. \end{cases}$$

8.1. - Theorem. If

1. Hypothesis [5.2] is admitted,
2. $p \neq 0$ is a brick space,
3. $\vec{X} \in \boldsymbol{H}$,

then, using special summation, we have

$$\frac{\operatorname{Proj}_p \vec{X}}{\mu(p)} = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta'(p, \beta) \vec{x}_\beta, \tag{1}$$

where $\vec{x}_\beta$ is the β-component of $\vec{X}$.

Proof. Consider a completely distinguished, special sequence $Q_n \underset{df}{=} \{ q_{n1}, q_{n2}, \ldots \}$ for 1. Denote by a_i, b_i, e_i the bricks of Q_n for which we have $a_i \leq p$; $b_i \cdot p = 0$; $e_i p \neq 0$, $e_i \operatorname{co} p \neq 0$ respectively. For the sum $\vec{A}_n \underset{df}{=} \Sigma_k \Delta'(p, q_{nk}) \operatorname{Proj}(q_{nk}) \vec{X}$ which approximates (1), we have:

$$\vec{A}_n = \Sigma_i \Delta'(p, a_i) \operatorname{Proj}(a_i) \vec{X} + \Sigma_i \Delta'(p, e_i) \operatorname{Proj}(e_i) \vec{X}.$$

Since $a_i \leq p$, we have $\Delta'(p, a_i) = \dfrac{1}{\mu(p)}$. Since $\mu(\Sigma e_i) \to 0$, (by [Lemma 5.7]) we have for sufficiently great n: $\mu(e_i) \leq \mu(p)$ for all i; hence $\Delta'(p, e_i) = \dfrac{1}{\mu(p)}$. Thus we get

$$\vec{A}_n \cdot \mu(p) = \operatorname{Proj}(\Sigma_i a_i) \vec{X} + \operatorname{Proj}(\Sigma_i e_i) \vec{X}.$$

Now $|\Sigma_i a_i, p| \to 0$ and $\mu(\Sigma_i e_i) \to 0$. Hence $\mu(p) \cdot \lim \vec{A}_n = \operatorname{Proj}_p \vec{X}$, so the theorem is proved.

8.2. - **Theorem.** If

1. Hypothesis [5.2] is admitted, 2. $\vec{X} \in \boldsymbol{H}$, then for special summation we have

$$\vec{x}^*_\alpha = \underset{\beta \varepsilon W}{S^\bullet} \Delta'(\alpha, \beta) \vec{x}_\beta, \quad \text{[Def. 8]},$$

where $\vec{x}^*_\alpha$ is the α-component-density of $\vec{X}$, and $\vec{x}_\beta$ the β-component of $\vec{X}$, [§ 6;2].

Proof. Relying on [8.1].

9. - **Theor.** If

1. Hypothesis [5.2) is admitted,
2. $p \neq 0$ is a space brick,
3. $\vec{X} \in \boldsymbol{H}$,

then for special summation we have

$$\frac{(\operatorname{Proj}_p \vec{\omega}, \vec{X})}{\mu(p)} = \underset{\beta \varepsilon W}{S^\bullet} \Delta'(p, \beta) \cdot x_\beta, \quad \text{[Def. 8]}.$$

Proof. Similar to that of [Theor. 8.1].

9.1. - Theor. If

1. Hypothesis [5.2] is admitted,
2. $\overrightarrow{X} \in \boldsymbol{H}$,

then for special summation we have

$$x_\alpha^* = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta'(\alpha, \beta) \cdot x_\beta ,$$

where x_α^* is the α-coordinate-density of $\overrightarrow{X}$, [§ 6; 2].

Proof. Follows from [9].

9.2. - The formulas in [8.2] and [9.1] can be written

$$\overrightarrow{x_\alpha^*} = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta'(\alpha, \beta) \overrightarrow{x_\beta^*} \cdot \mu_\beta , \quad x_\alpha^* = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta'(\alpha, \beta) \cdot x_\beta^* \cdot \mu_\beta ,$$

which have the same shape as Dirac's formula.

Proof. This follows from the equalities;

$$\overrightarrow{x_\beta} = \overrightarrow{x_\beta^*} \cdot \mu_\beta , \quad x_\beta = x_\beta^* \cdot \mu_\beta , \quad [\S\ 6;\ 3].$$

10. - Lemma. If $E \in G$, E does not contain any atom, P_n is a (DR)-distinguished sequence for E, then $\mathfrak{N}(P_n) \to 0$.

Proof. Suppose that the thesis is not true. Then there exists a partial sequence $P_{s(n)}$ and $\eta > 0$ such that $\mathfrak{N}(P_{s(n)}) > \eta$.

Put $Q_n \underset{df}{=} P_{s(n)}$; $\{Q_n\}$ is also a (DR)-distinguished sequence for E.

Put $Q_n = \{ q_{n1}, q_{n2}, \ldots \}$. Since

$$\mathfrak{N}(Q_n) = \max \{ \mu(q_{n1}), \mu(q_{n2}), \ldots \},$$

there exists $k(n)$ with $\mu(q_{n, k(n)}) \geq \eta$. Now, since $\{Q_n\}$ is a (DR)-distinguished sequence for E, it follows that

$$\mathfrak{N}_R(Q_n) \to 0 \quad \text{i.e.} \quad \max_k \{ \mu(q_{nk} - \beta) \} \to 0,$$

where $\beta = A_1 + A_2 + \ldots$, the sum of all atoms of G. Hence $\mu(q_{n, k(n)} - \beta) \to 0$. Since $q_{n, k(n)} = (q_{n, k(n)} - \beta) + q_{n, k(n)}\beta$, there exists n_0 such that for all $n \geq n_0$,

$$\mu(\beta \cdot q_{nk(n)}) \geq \frac{\eta}{2} \cdot \tag{1}$$

Let m_0 be such that

(2) $$\sum_{n>m_0} \mu A_n < \frac{\eta}{4}.$$

We have from (1):

$$\mu[q_{nk(n)} \Sigma_{n\leq m_0} A_n + q_{nk(n)} \Sigma_{n>m_0} A_n] \geq \frac{\eta}{2}.$$

Since the two terms are disjoint, we have

$$\mu[q_{nk(n)} \cdot \Sigma_{n\leq m_0} A_n] + \mu[q_{nk(n)} \cdot \Sigma_{n>m_0} A_n] \geq \frac{\eta}{2}.$$

Here the second term is $< \frac{\eta}{4}$. Hence we have

$$\mu[q_{nk(n)} \cdot \Sigma_{n\leq m_0} A_n] \geq \frac{\eta}{4}$$

for all $n \geq n_0$.

Hence at least one atom among $A_1 \dots A_{m_0}$ must be contained in $q_{nk(n)}$, for all $n \geq n_0$.

Consequently there exists a partial sequence $l(n)$ and an atom A_i, $(i \leq m)$, such that $A_i \leq q_{l(n),\, k(l(n))}$ for $n = 1, 2, \dots$.

Since $|E, Q_{l(n)}| \longrightarrow 0$, we have $|E.A_i, Q_{l(n)} A_i| \longrightarrow 0$, $|E.A_i, A_i| \longrightarrow 0$, i.e. $|0, A_i| \longrightarrow 0$ which is impossible.

11. - **Def.** We define the function

$$\Delta''(p, q) = \begin{cases} 0 \text{ whenever } & p \cdot q = 0, \\ \dfrac{1}{\mu(p) + \mu(q)} & \text{whenever } p \cdot q \neq 0. \end{cases}$$

This function generates the functions $\Delta''(\alpha, q)$, $\Delta''(p, \beta)$, $\Delta''(\alpha, \beta)$ where α, β are traces, (see [3]). We shall see that $\Delta''(\alpha, \beta)$ also has some properties of the δ-function.

12. - **Theorem.** If

1. Hypothesis [5.2] is admitted,
2. G does not contain any atom,
3. $p \neq 0$ is a brick-space,
4. $\overrightarrow{X} \in H$,

then, using special summations, we have

$$\frac{\mathrm{Proj}_p \overrightarrow{X}}{\mu(p)} = \underset{\beta \in W}{\mathbf{S}^{\bullet}} \Delta''(p, \beta)\overrightarrow{x_\beta}. \tag{1}$$

Proof. Let $Q_n = \{q_{n1}, q_{n2}, \dots\}$ be a completely distinguished and special sequence of complexes for I. Denote by a_i, b_i, e_i the bricks of Q_n with $a_i \leq p$; $b_i \cdot p = 0$; $e_i \cdot p \neq 0$, $e_i \operatorname{co} p \neq 0$ respectively. Consider the sum yielding (1):

$$\overrightarrow{A_n} \underset{af}{=} \Sigma_k \Delta''(p, q_{nk}) \cdot \mathrm{Proj}_{q_{nk}} \overrightarrow{X}.$$

We have

$$\overrightarrow{A_n} = \Sigma_i \frac{1}{\mu(p)} \mathrm{Proj}_{a_i} \overrightarrow{X} + \Sigma_i \frac{1}{\mu(p)} \mathrm{Proj}_{e_i} \overrightarrow{X} + \Sigma_i \left[\frac{1}{\mu(p) + \mu(a_i)} - \right.$$
$$\left. - \frac{1}{\mu(p)}\right] \mathrm{Proj}_{a_i} \overrightarrow{X} + \Sigma_i \left[\frac{1}{\mu(p) + \mu(e_i)} - \frac{1}{\mu(p)}\right] \mathrm{Proj}_{e_i} \overrightarrow{X}. \tag{2}$$

The sum of two first terms in (2) is

$$\frac{1}{\mu(p)} \mathrm{Proj}\,(\Sigma_i a_i + \Sigma_i e_i)\overrightarrow{X}$$

and tends to $\frac{1}{\mu(p)} \mathrm{Proj}_p \overrightarrow{X}$, if $n \to \infty$. This follows from that $|\Sigma_i a_i, p|_\mu \to 0$ and $\mu \Sigma_i e_i \to 0$, [5.7]. Concerning the two last terms in (2), they are composed of expressions having the form:

$$\left[\frac{1}{\mu(p) + \mu(c)} - \frac{1}{\mu(p)}\right] \mathrm{Proj}_c \overrightarrow{X}, \tag{3}$$

where c is a brick. We have

$$\left\| \left[\frac{1}{\mu(p) + \mu(c)} - \frac{1}{\mu(p)}\right] \mathrm{Proj}_c \overrightarrow{X} \right\|^2 \leq \frac{\mu(c)^2}{\mu(p)^4} \| \mathrm{Proj}_c \overrightarrow{X} \|^2.$$

Hence the square of the norm of the sum of the last terms in (2) does not exceed the number

$$\frac{1}{\mu(p)^4} \Sigma_i \mu(a_i)^2 \| \mathrm{Proj}_{a_i} \overrightarrow{X} \|^2 + \frac{1}{\mu(p)^4} \Sigma_i \mu(e_i)^2 \cdot \| \mathrm{Proj}_{e_i} \overrightarrow{X} \|^2. \tag{4}$$

Take $\varepsilon > 0$. By Lemma [10], we have for sufficiently great n the inequality $\mathfrak{M}(Q_n) \leq \varepsilon$; hence $\mu(a_i) \leq \varepsilon$, $\mu(e_i) \leq \varepsilon$.

Thus the expression (4) does not exceed, (for those n), the number

$$\frac{\varepsilon^2}{\mu(p)^4}\{\Sigma_i \| \mathrm{Proj}_{a_i} \overrightarrow{X} \|^2 + \Sigma_i \| \mathrm{Proj}_{e_i} \overrightarrow{X} \|^2 \} =$$

$$= \frac{\varepsilon^2}{\mu(p)^4} \| \mathrm{Proj}\,(\Sigma_i\, a_i + \Sigma_i\, e_i) \overrightarrow{X} \|^2 \leq \frac{\varepsilon^2}{\mu(p)^4} \cdot \| \overrightarrow{X} \|^2.$$

Consequently the sum of two last terms in (2) tends to $\overrightarrow{0}$, so

$$\lim \overrightarrow{A_n} = \frac{1}{\mu(p)} \mathrm{Proj}_p \overrightarrow{X},$$

what completes the proof.

12.1. - If

1. Hypothesis [5.2] is admitted,
2. G does not contain atoms,
3. $\overrightarrow{X} \in \boldsymbol{H}$,

then for special summation we have:

$$\overrightarrow{x_\alpha^*} = \underset{\beta \varepsilon W}{\mathrm{S}^\bullet} \Delta''(\alpha,\ \beta) \overrightarrow{x_\beta} = \underset{\beta \varepsilon W}{\mathrm{S}^\bullet} \Delta''(\alpha,\ \beta) \overrightarrow{x_\beta^*}\, \mu_\beta .$$

Proof. The theorem follows from [12].

13. - **Theorem.** If

1. Hypothesis [5.2] is admitted,
2. G does not contain atoms,
3. $p \neq 0$ is a brick,
4. $\overrightarrow{X} \in \boldsymbol{H}$,

then, using special summation, we have, [§ 6],

$$\frac{(\mathrm{Proj}_p \overrightarrow{\omega},\ \overrightarrow{X})}{\mu(p)} = \underset{\beta \varepsilon W}{\mathrm{S}^\bullet} \Delta''(p,\ \beta) \cdot x_\beta .$$

Proof. Similar to that of [Theor. 12].

13.1. - **Theor.** If

1. Hypothesis [5.2] is admitted,
2. G does not contain atoms,
3. $\overrightarrow{X} \in \boldsymbol{H}$,

then, using special summation, we have

$$x_\alpha^* = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta''(\alpha, \beta)\, x_\beta = \underset{\beta \in W}{\mathbf{S}^\bullet} \Delta''(\alpha, \beta)\, x_\beta^*\, \mu_\beta . \tag{1}$$

(Which resembles the known Dirac's formula).

Proof. This follows from [13].

13.2. Remark. The formula (1) in [Theor. 13.1] may be not true if G has an atom. E.g. Take the one-dimensional H.H.-space. The corresponding G is composed of two somata only, viz. 0 and 1. There exists only one trace which is heavy. In the right hand side expression in (1), we get

$$\Delta''(1,\ 1) = \frac{1}{2\mu(1)},$$

so the formula (1) is not true.

14. - Consider the mapping $\mathcal{G}^{-1}$ of the H.H.-space $\boldsymbol{H}$ onto the space $\boldsymbol{H}'$ of μ-square summable functions of the variable trace, as explained in [§ 5;4] and **(14)**.

Let the $\mathcal{G}^{-1}$ — image of the vector $\overrightarrow{X}$ be the function $\overrightarrow{f} \underset{df}{=} f(\alpha)$. The system of coordinates in $\boldsymbol{H}$ goes over to an analogous system of coordinates in the space $\boldsymbol{H}'$. The generating vector $\overrightarrow{\omega}$ goes over into the characteristic function $\Omega(x)$ of W i. e. $\Omega(\alpha) = 1$ for all α. Then

$$(\mathrm{Proj}_p \overrightarrow{\omega},\ \overrightarrow{X}) = \int \Omega_P(\alpha) f(\alpha)\, d\mu = \int_P f(\alpha)\, d\mu,$$

where P is such a set of traces, that the set of μ–square summable functions vanishing outside P is just the $\mathcal{G}^{-1}$ — image of the space–brick p. It follows that, for the $\mathcal{G}^{-1}$ —

corresponding system of coordinates in H' we have:

$$f_\alpha = \frac{\int_{\dot P} f(\alpha)\, d\mu}{\mu(\dot P)} = \mathrm{val}_\alpha f, \text{ (see [§ 4 ; Def 10]).}$$

Thus the formula (1) in [13.1] will become

$$\mathrm{val}_\alpha f = \underset{\beta \in W}{\mathbf{S}^{\bullet}} \Delta''(\alpha, \beta)\, \mathrm{val}_\beta f\, \mu_\beta,$$

and a similar formula will be obtained from [9.2], by using the Δ'-function.

15. - Lemma. Under circumstances of [§ 3;1] we have the following: If

1. $\{p_n\}$ is a representative of the trace τ,
2. q is a brick,
3. $p_n \cdot q \neq 0$ for $n = 1, 2, \dots$,

then q is a neighborhood of τ. [§ 3; Def 10].

Proof. Suppose that the thesis is not true; hence q does not cover the trace τ. Hence, whatever the representative $a_1 \geq a_2 \geq \dots$ of τ may be, we always have $a_1 \cdot \mathrm{co}\, q \neq 0$. Since $p_k \geq p_{k+1} \geq \dots$ is a representative of τ, $(k = 1, 2 \dots)$, it follows:

(1) $$p_k \cdot \mathrm{co}\, p \neq 0 \text{ for } k = 1, 2, \dots$$

We have

(2) $$p_1 q \geq p_2 q \geq \dots, \text{ all} \neq 0.$$

Since $p_1 \geq p_2 \geq \dots$ is a minimal sequence and

$$\{p_1 q, p_2 q, \dots\} \leq \{p_1, p_2, \dots\}, \text{ (see [§ 3; 2.1])},$$

it follows that either

$$\{p_1 q, p_2 q, \dots\} \backsim \{0, 0, \dots\} \text{ or}$$

$$\{p_1 q, p_2 q, \dots\} \backsim \{p_1, p_2, \dots\}.$$

The first alternative is impossible, because it would imply $p_n q = 0$ for sufficiently great n, [§ 3; 10.2], so it would lead to contradiction with (2). Hence $\{p_n q\} \backsim \{p_n\}$. Consequently, for every n we can find m such that $p_m \leq p_n q$; hence $p_m \leq q$ which contradicts (1). The contradiction thus obtained proves that q is a neighborhood of τ.

15.1. - Lemma. Under circumstances of [§ 3; 1] if q is not a neighborhood of the trace τ, then there exists a neighborhood p of τ with $p \cdot q = 0$.

Proof. Suppose the thesis is not true. Then for every neighborhood p of τ we have $p \cdot q \neq 0$. Hence, if we take a representative $\{p_1 \geq p_2 \geq \dots\}$ of τ, we get $p_k q \neq 0$, $(k = 1. 2, \dots)$. Hence, by the forgoing lemma [15], we see that q is a neighborhood of τ, which is a contradiction.

16. - In this §, number [5], we have defined " special completely distinguished sequences of complexes for 1 ", and have used them in some theorems. To have useful consequences of them in the form of their modifications, we admit the following general definition, which however, will be later used only in the case where G is a tribe of spaces in *H.H.*

Def. Let $\{Q_n\}$ be a completely distinguished and special sequence for I and s a figure $\neq 0$. Consider all bricks a_{nk}, e_{nk} of Q_n such that $a_{nk} \leq s$; $e_{nk} \cdot s \neq 0$, $e_{nk} \cdot \mathrm{co}\, s \neq 0$. We shall consider the two partial complexes $\{a_{n1}, a_{n2}, \dots\}$ and $\{a_{n1}, a_{n2}, \dots, e_{n1}, e_{n2}, \dots\}$ of Q_n. The firstt will be termed *inner Q_n-coat of* s and denoted by $\mathrm{int}(Q_n)s$, the second will be termed *outer Q_n-coat of s*, and denoted by $\mathrm{ext}\,(Q_n)s$. Now take any partial complex T_n of Q_n, such that $\mathrm{int}\,(Q_n)s \subseteq \subseteq T_n \subseteq \mathrm{ext}(Q_n)s$. If we do that for all n, we get a sequence $\{T_n\}$ of complexes. Any sequence $\{T_n\}$ obtained in the above way will be termed ***« special sequence for s, induced by*** $\{Q_n\}$ **»**. Of course we have $\mathrm{som\,int}\,(Q_n)s \leq s \leq \mathrm{som\,ext}\,(Q_n)s$, and

$$\mu[s - \mathrm{som\,int}(Q_n)s] \to 0 \quad \mu[\mathrm{som\,ext}\,(Q_n)s - s] \to 0 \quad \text{for } n \to \infty.$$

Hence we get $|s, T_n|_\mu \to 0$ for $n \to \infty$, so $\{T_n\}$ is a completely distinguished sequence for s, [§ 1; 21.14].

16.1. - **Def.** Continuing the topic [16], let us consider a regular and total set of quasi-vectors $\overrightarrow{f_x}$ in the vector-space V. If the sum

$$\overrightarrow{f}(T_n) = \Sigma_k \overrightarrow{f}(t_{nk}), \quad \text{where} \quad T_n = \{ t_{n1}, t_{n2}, \dots \}$$

is a special sequence for s induced by $\{ Q_n \}$, tends to a limit which does not depend neither on the choice of $\{ T_n \}$ for a given $\{ Q_m \}$, nor on the choice of $\{ Q_n \}$, we shall say that $\overrightarrow{f_x}$ is *«specially summable on the figure s»*, and the limit mentioned above will be termed *« special sum of $\overrightarrow{f_x}$ on the figure s »*, and denoted by

$$\mathbf{S}_s^{\bullet} \overrightarrow{f_x}.$$

16.2. - The following theorem is valid. Consider circumstances of [§ 4; 1, 2] and admit [Hyp. 5.2].

Theorem. If the special sum $\mathbf{S}_I^{\bullet} \overrightarrow{f_x}$ exists, and s is a figure, then

$$\mathbf{S}_s^{\bullet} \overrightarrow{f_x} \quad \text{also exists.} \tag{1}$$

Proof. Suppose that (1) does not exist. This means that there exists a completely distinguished and special sequence of complexes $\{ Q_n \}$ for I, such that if we consider

$$\text{int}(Q_n)\, s \quad \text{and} \quad \text{ext}(Q_n)\, s,$$

we can find T_n, as in [16], such that

$$\text{int}(Q_n)\, s \subseteq T_n \subseteq \text{ext}(Q_n)\, s, \tag{2}$$

and where $\overrightarrow{f}(T_n)$ does not tend to any limit. Having that, we can find indices $n_1' < n_1'' < n_2' < n_2'' < \dots$ such that

$$\| f(T_{n_k''}) - f(T_{n_k'}) \| \geq a \text{ for } k = 1, 2, \dots, \tag{3}$$

and for some positive number a.

$$\text{Put } P_n^{(1)} \underset{df}{=} \text{int}(Q_n)\, s, \; P_n^{(2)'} \underset{df}{=} T_{n_k'},$$

$$P_n^{(2)''} \underset{df}{=} T_{n_k''}, \; P_n^{(3)} = \text{ext}(Q_n)\, s, \; P_n^{(4)} = \text{int}(Q_n)(\text{co}\, P_n^{(3)}).$$

We shall see that the complexes

(4) $$P_1^{(2)\prime} \cup P_1^{(4)},\ P_1^{(2)\prime\prime} \cup P_1^{(4)},\ P_2^{(2)\prime} \cup P_2^{(4)},\ P_2^{(2)\prime\prime} \cup P_2^{(4)}, \dots$$

make up a completely distinguished and special sequence for *1*. To prove that notice that, since Q_n is completely distinguished and special for *1*, we have

$$\mu \operatorname{som}[P_n^{(3)} - P_n^{(1)}] \to 0 \text{ and } \mu \operatorname{som}[\operatorname{co} P_n^{(3)} - P_n^{(4)}] \to 0,$$
$$[5.6] \text{ and } [5.7].$$

Since

$$\mu \operatorname{som} P_n^{(3)} + \mu \operatorname{co} \operatorname{som} P_n^{(3)} = \mu(1),$$

it follows that the sum of all bricks of Q_n, which neither belong to $P_n^{(4)}$ nor to $P_n^{(1)}$, has the measure tending to 0 for

(5) $$n \to \infty.$$

Let q be a brick $\neq 0$. We need to prove that

$$\mu[\operatorname{som} \operatorname{int}(P_n^{(2)\prime} \cup P_n^{(4)})q] \to \mu q$$

and

$$\mu[\operatorname{som} \operatorname{int}(P_n^{(2)\prime\prime} \cup P_n^{(4)})q] \to \mu q.$$

Since

$$P_n^{(2)\prime} \cup P_n^{(4)} \subseteq Q_n \quad \text{and} \quad P_n^{(2)\prime\prime} \cup P_n^{(4)} \subseteq Q_n,$$

it follows that

$$\operatorname{int}(P_n^{(2)\prime} \cup P_n^{(4)})q \subseteq \operatorname{int}(Q_n)q \xrightarrow{n} q, \text{ and } \operatorname{int}(P_n^{(2)\prime\prime} \cup P_n^{(4)})q \subseteq \operatorname{int}(Q_n)q \xrightarrow{n} q.$$

Now the bricks of $\operatorname{int}(Q_n)q$ which do not belong to $P_n^{(2)\prime} \cup P_n^{(4)}$, are not contained in $P_n^{(1)}$, because $P_n^{(1)} \subseteq P_n^{(2)\prime}$, hence, by (5), the measure of the sum of bricks which do not belong to $P_n^{(2)\prime} \cup P_n^{(4)}$, has the measure tending to 0. Consequently

$$\mu \operatorname{som}(P_n^{(2)\prime} \cup P_n^{(4)})q \to 0.$$

Similarly we have

$$\mu \operatorname{som}(P_n^{(2)\prime\prime} \cup P_n^{(4)})q \to 0.$$

This proves that (4) is a completely distinguished and special sequence for 1. By hypothesis both sequences

$$\overrightarrow{f}(P_n^{(2)\prime} \cup P_1^{(4)}), \overrightarrow{f}(P_n^{(2)\prime\prime} \cup P_1^{(4)})$$

tend to the same limit. Hence

$$\overrightarrow{f}(P_n^{(2)\prime}) - \overrightarrow{f}(P_n^{(2)\prime\prime}) \text{ tends to } \overrightarrow{0},$$

which contradicts (3). The theorem is proved.

16.3. - Theorem. We admit the [Hyp. 5.2] and circumstances [§ 4; 1, 2]. Then if

1) $\overrightarrow{f_\tau}$ is a regular total set of V-quasi-vectors,

2) a, b, c are bricks, $c = a + b$, $a \cdot b = 0$,

3) the special sum $\mathbf{S}_c^\bullet \overrightarrow{f_\tau}$ exists,

then for special sums we have

$$\mathbf{S}_a^\bullet \overrightarrow{f_\tau} + \mathbf{S}_b^\bullet \overrightarrow{f_\tau} = \mathbf{S}_c^\bullet \overrightarrow{f_\tau}.$$

Proof. By [16.2] all these sums exists. Let $\{Q_n\}$ be a completely distinguished and special sequence of complexes for 1. Consider the complexes $\mathrm{int}(Q_n)a$, $\mathrm{int}(Q_n)b$ and $\mathrm{int}(Q_n)c$. We have $\mathrm{int}(Q_n)a \cup \mathrm{int}(Q_n)b \subseteq \mathrm{int}(Q_n)c$. We have

$$\mu \text{ som } \mathrm{int}(Q_n)a \longrightarrow \mu(a), \quad \mu \text{ som } \mathrm{int}(Q_n)b \longrightarrow \mu(b),$$

$$\mu \text{ som } \mathrm{int}(Q_n)c \longrightarrow \mu(c).$$

Hence if we put

$$P_n \underset{df}{=} \mathrm{int}(Q_n)c - [\mathrm{int}(Q_n)a \cup \mathrm{int}(Q_n)b]$$

we get

$$\lim \mu P_n = \mu(c), \qquad P_n \subseteq \mathrm{int}(Q_n)c.$$

Consider the sequence

$$(1) \qquad R_n \underset{df}{=} \mathrm{int}(Q_n)c - P_n = \mathrm{int}(Q_n)a \cup \mathrm{int}(Q_n)b.$$

This is a sequence induced for c by the sequence $Q_n - P_n$, which is a completely distinguished and special sequence for 1. The last can be proved by using a method given in the forgoing proof. We have

$$\lim \overrightarrow{f}(Q_n) = \mathbf{S}^{\bullet}_c \overrightarrow{f_\tau}, \quad \lim \overrightarrow{f}[\operatorname{int}(Q_n)a] = \mathbf{S}^{\bullet}_a \overrightarrow{f_\tau},$$

and

$$\lim \overrightarrow{f}[\operatorname{int}(Q_n)b] = \mathbf{S}^{\bullet}_b \overrightarrow{f_\tau}.$$

Thus, by (1), we get the thesis.

16.4. - Remark. We do not know whether the special sum $\mathbf{S}^{\bullet}_a \overrightarrow{f}$ can exist without existence of the sum $\mathbf{S}_{a(\mathrm{DARS})} \overrightarrow{f}$. Our conjecture is "Yes".

16.5. - If

1. $\{Q_n\}$ is a completely distinguished and special sequence for 1.

2. s is a brick $\neq 0$,

3. $T_n \underset{df}{=} \operatorname{int}(Q_n)s$, $n = 1, 2, \ldots$,

then $\{T_n\}$ is a completely distinguished and special sequence for the tribe $s \rceil G$, (restriction to s) in the sense of [Def. 5].

17. - Having that, we are going to get some useful modifications of various forgoing theorems which involve Δ-functions. To simplify wording we shall use the same letter for a measurable set of traces and for its coat.

17.1. - Theorem. If

1. [Hyp. 5.2] is admitted.

2. p is a brick,

3. s is a figure whith $p \leq s$.

4. $\overrightarrow{X} \in \boldsymbol{H}$, then

$$\operatorname{Proj}_p \overrightarrow{X} = \underset{\beta \subseteq s}{\mathbf{S}^{\bullet}} \Delta(p, \beta) \overrightarrow{x_\beta}, \text{ [Def. 3]}.$$

Proof. Since $\underset{b \subseteq W}{\mathbf{S}^{\bullet}} \Delta(p, \beta) \overrightarrow{x_\beta}$ exists, [6]. therefore, by [16.3],

$$\underset{\beta \subseteq W}{\mathbf{S}^{\bullet}} \Delta(p, \beta) \overrightarrow{x_\beta} = \underset{\beta \subseteq s}{\mathbf{S}^{\bullet}} + \underset{\beta \subseteq \operatorname{co} s}{\mathbf{S}^{\bullet}}. \tag{1}$$

To evaluate $\underset{\beta \subseteq \operatorname{co} s}{\mathbf{S}^{\bullet}}$ take a completely distinguished and special sequence $\{Q_n\}$ for 1 and consider $\mathrm{int}(Q_n)$ co s. We have for its bricks b_{nk}

$$\Sigma_k \Delta(p,\ b_{nk})\ \mathrm{Proj}\,(b_{nk})\overrightarrow{X} = \overrightarrow{0}. \tag{2}$$

Since $\mathrm{int}(Q_n)\,\mathrm{co}\,s$ is a special sequence, induced by $\{Q_n\}$, [Def. 16], the sum (2) tends to $\underset{\beta \leq \mathrm{co}\,s}{\mathbf{S}^\bullet}$. Now, $\underset{\beta \leq \mathrm{co}\,s}{\mathbf{S}^\bullet} \Delta(p,\ \beta)\overrightarrow{x_\beta} = \overrightarrow{0}$, because $\Delta(p,\ b_{nk}) = 0$ for all k. Consequently, by [6] and (1): $\mathrm{Proj}_p \overrightarrow{X} = \underset{\beta \leq s}{\mathbf{S}^\bullet}$, so the theorem is proved.

17.2. - In a similar way considering $\mathrm{int}\,(Q_n)$ co s, we can prove similar variants of the theorems [6.1], [7], [7.1]. [8.1], [8.2], [9], [9.1]. [9.2], [12], [12.1], [13], [13.1], [14]. In the changed theorems we have the additional hypothesis $p \leq s$, where s is a figure, or in theorems involving the trace α we have the hypothesis $\alpha \in s$, and take account of [Lemma [15] and [15.1]]. $\underset{\beta \leq W}{\mathbf{S}^\bullet}$ is replaced by $\underset{\beta \leq s}{\mathbf{S}^\bullet}$.

We shall refer to these theorems by giving the number e.g. [7.1] and adding [17.2].

18. - Te following remark considers the influence exercised on various summation formulas, by the change of measure. If $\underset{\tau \leq W}{\mathbf{S}^\bullet} \overrightarrow{f_\tau} = \underset{\tau \leq s}{\mathbf{S}^\bullet} \overrightarrow{f_\tau}$ where s is a figure, we see that the change of the measure μ outside s will not influence the summation-formula.

19. - A similar remark can be made on functions which vanish outside a given brick: this in relation to topics [§ 4; 15 etc.].

20. - **Theorem.** If

1. G has no atoms,

2. Hyp. [5.2] is admitted.

3. $\overrightarrow{f_\tau}$, $\overrightarrow{g_\tau}$ are total sets of regular and normal quasi-vectors,

4. $\overrightarrow{f_\tau} \approx\simeq \overrightarrow{g_\tau}$ in the sense of [§ 4; 7],

5. $p \neq 0$ is a brick,

6. $\underset{W}{\mathrm{S}} \Delta''(p, \tau)\overrightarrow{f_\tau}$ exists, [11]

then $\underset{W}{\mathrm{S}} \Delta''(p, \tau)\overrightarrow{g_\tau}$ exists too and equals (6).

Proof. Hypothesis 3 means that for every measurable set E of traces we have in the $(DARS)$-summation [Def. § 4; 5],

$$(0) \qquad \mathrm{S}_E \overrightarrow{f_\tau} = \mathrm{S}_E \overrightarrow{g_\tau}.$$

Consider a completely distinguished and special sequence $Q_n = \{q_{n1}, q_{n2}, \dots\}$ of complexes for I. Put $\overrightarrow{h_\tau} \underset{df}{=} \overrightarrow{f_\tau} - \overrightarrow{g_\tau}$. Consider the sum

$$\overrightarrow{A_n} \underset{df}{=} \Sigma_k \Delta''(p, q_{nk}) \cdot \overrightarrow{h(q_{nk})}.$$

Denoting by c_k the bricks of $\operatorname{int}(Q_n)p$, [16], we get

$$\overrightarrow{A_n} = \Sigma_k \frac{1}{\mu(p) + \mu(c_k)} \cdot \overrightarrow{h(c_k)},$$

$$(1) \qquad \overrightarrow{A_n} = \Sigma_k \frac{1}{\mu(p)} \overrightarrow{h(c_k)} + \Sigma_k \left(\frac{1}{\mu(p) + \mu(c_k)} - \frac{1}{\mu(p)}\right) \overrightarrow{h(c_k)}.$$

We have

$$\left\| \Sigma_k \left(\frac{1}{\mu(p) + \mu(c_k)} - \frac{1}{\mu(p)}\right) \cdot \overrightarrow{h(c_k)} \right\|^2 \leq \Sigma_k \frac{\mu(c_k)^2}{(\mu(p))^4} \left\| \overrightarrow{h(c_k)} \right\|^2.$$

Since G has no atoms' we have $\mathfrak{N}(Q_n) \to 0$. Hence for sufficiently great index n we get $\mu(c_k) \leq \varepsilon$, where ε is any positive number given in advance. We get

$$(2) \qquad \left\| \Sigma_k \left(\frac{1}{\mu(p) + \mu(c_k)} - \frac{1}{\mu(p)}\right) \overrightarrow{h(c_k)} \right\|^2 \leq \frac{\varepsilon^2}{(\mu(p))^4} \cdot \left\| \Sigma_k \overrightarrow{h(c_k)} \right\|^2.$$

As $\overrightarrow{f_\tau}$ and $\overrightarrow{g_\tau}$ are summable, it follows that $\overrightarrow{h_\tau} = \overrightarrow{f_\tau} - \overrightarrow{g_\tau}$ is summable.

Hence, [§ 5; 5.1], $\|\overrightarrow{h_\tau}\|^2$ is summable. Since

$$\lim \left\| \Sigma_k \overrightarrow{h(c_k)} \right\|^2 \to \mathrm{S}_p \|\overrightarrow{h_\tau}\|^2,$$

it follows that $\|\Sigma_k \overrightarrow{h(c_k)}\|^2$ is bounded. Thus there exists $M > 0$ such that $\|\Sigma_k \overrightarrow{h(c_k)}\|^2 \leq M$. It follows that the expression on the left in (2) does not exceed $\frac{\varepsilon^2}{(\mu(p))^4} \cdot M$, so it tends to 0 for $\varepsilon \to 0$. The first term in (1) is

$$\frac{1}{\mu(p)} \Sigma_k \overrightarrow{h}(c_k),$$

and tends to

$$\frac{1}{\mu(p)} \mathrm{S}_p \overrightarrow{h_\tau} = \overrightarrow{0}, \text{ by } (0).$$

Consequently $\overrightarrow{A_n} \to \overrightarrow{0}$, which proves the theorem.

21. - **Def.** Denote by c_α the quasi-number defined by $f(p) \underset{df}{=} c =$ constant for all neighborhoods of α. We can write $c_\alpha = c(\dot{p})$ where p is a neighborhood of α.

21.1. - **Theorem.** If G does not admit atoms, we have

$$(1) \qquad 1_\alpha = \underset{\beta \varepsilon W}{\mathrm{S}^\bullet} \Delta''(\alpha, \beta)_\beta$$

Proof. We have for every $\overrightarrow{X} \in H$, [13.1],

$$x_\alpha^* = \underset{\beta \varepsilon W}{\mathrm{S}^\bullet} \Delta''(\alpha, \beta)\, x_\beta^*\, \mu_\beta .$$

Applying this formula to the generating vector $\overrightarrow{\omega}$, we get

$$\omega_\alpha^* = \underset{\beta \varepsilon W}{\mathrm{S}^\bullet} \Delta''(\alpha, \beta)\, \omega_\beta^*\, \mu_\beta .$$

Now

$$\omega_\alpha^* = \frac{(\mathrm{Proj}_{\dot{p}} \overrightarrow{\omega}, \overrightarrow{\omega})}{\mu(\dot{p})} = \frac{\| \mathrm{Proj}_{\dot{p}} \overrightarrow{\omega} \|^2}{\mu(\dot{p})} = 1_\alpha .$$

Since

$$\Delta''(\alpha, \beta) \cdot 1_\beta = \Delta''(\dot{p}, \dot{q}) \cdot 1 = \Delta''(\dot{p}, \dot{q}) = \Delta''(\alpha, \beta),$$

we get the formula (1).

22. - We have defined three functions Δ, Δ', Δ'' which have some properties of the Dirac's-delta function under very general conditions. We shall terminate this paper with the study of the genuine δ-function in relation to ordinary functions of a real variable and Lebesguean measure.

22.1. - To do this we shall consider the space $\boldsymbol{H}$ of all complex-valued Lebesgue-square-summable functions $f(\dot{x})$ defined almost everywhere in the half open interval (A, B) which, for simplicity we admit $(0, 1)$. The governing equality, $\doteq$, will be that which is induced by the ideal of sets of measure 0. We shall have the alternative notation $\vec{f}$ for $f(x)$. The scalar product $(\vec{f}, \vec{g})$ is defined as

$$\int_0^1 \overline{f(x)}\, g(x)\, dx,$$

so $\boldsymbol{H}$ is a separable and complete, infinite dimensional $H.H.$-space.

Consider the Lebesgue-measurable subsets of $(0, 1)$ considered modulo sets of measure 0, so we have $E \doteq F$ whenever meas $(E - F) +$ meas $(F - E) = 0$. The collection of these sets, with ordering defined by

$$E \dot{\leq} F \cdot \underset{df}{=} \cdot \text{meas}\,(E - F) = 0,$$

is organized into a Boolean denumerably additive tribe $\boldsymbol{g}$ with effective measure. Consider the collection $\boldsymbol{b}$ of all subsets

$$(\alpha, \beta) \underset{df}{=} \{ x \mid \alpha < x \leq \beta \} \text{ of } (0, 1),$$

where $0 \leq \alpha$, $\beta \leq 1$, and denote by $\boldsymbol{f}$ the set of all finite unions of sets of $\boldsymbol{b}$. We see that $\boldsymbol{f}$ is a finitely additive tribe and $\boldsymbol{b}$ its base [§ 1]. The tribe $\boldsymbol{g}$ is a finitely genuine extension of $\boldsymbol{f}$ through the isomorphism from $\boldsymbol{f}$ into $\boldsymbol{g}$, which attaches to every set (α, β) the sets $(\alpha, \beta) + E_1 - E_2$ where meas $E_1 =$ meas $E_2 = 0$. The hypothesis (Hyp. A_f) is satisfied; hence, a fortiori, (Hyp A_d), $\boldsymbol{g}$ constitutes the borelian extension of $\boldsymbol{f}$ within $\boldsymbol{g}$. The tribe $\boldsymbol{g}$ is also the Lebesgue-covering extension [§ 1; 9] of $\boldsymbol{f}$ within $\boldsymbol{g}$, where the measure on $\boldsymbol{f}$ is euclidian (hence Lebesguean). Thus [§ 1; Hyp. 12] is satisfied. The hypothesis [§ 1; Hyp S] of measure-separability of g [§ 1; 21.1] is satisfied. Consequently there exists a completely distinguished sequence of complexes for the soma $1 \overline{\overline{df}} \langle 0, 1\rangle$, [§ 1; Def. 21.3].

If we consider the partitions of $\langle 0, 1\rangle$ into n equal half-open-segments, $(n = 1, 2, ...)$, we get a completely distinguished and special sequence for 1, [Def. 5]; thus [Hyp. 5.2] is satisfied. Since g has no atoms, the distinguished sequences of type (D), (DA), (DR), (DAR) for a measurable set E coincide, [§ 2].

The base b in f gives rise to traces, [§ 3], (see also [§ 5; 2.8]). To every point x where $0 < x < 1$ there correspond two different traces, one x^+ with representative $\langle x, 1+\varepsilon_n\rangle$ where $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\varepsilon_n \geq \varepsilon_{n+1}$, $(n = 1, 2, ...)$,

$$\langle x, x + \varepsilon_n\rangle \subseteq \langle 0, 1\rangle,$$

and another one x^- with representative $\langle x - \varepsilon, x\rangle$. At $x = 0$ we have only one trace 0^+ and at $x = 1$ only onetrace 1^-. The point x will be termed *vertex of* x^+ *and of* x^-. We shall write $x = \text{vert } x^+ = \text{vert } x^-$.

The hypotheses (Hyp. I) and (Hyp II), [§ 3; 6], are satisfied, so the whole theory of measurability of sets of traces, [§ 3], is valid. One can prove that if α is a measurable set of traces in g, then α is measure-equivalent to some set of traces which always contains with x^+ also x^-, and with x^- also x^+, and whose measure equals the measure of the set of all its vertices.

22.2. - Let us define the correspondence $\mathfrak{M}$ as follows. If E is a measurable subset of $\langle 0,1\rangle$, consider all square-summable functions $f(\dot{x})$ defined a.e. on $\langle 0,1\rangle$ such that $f(x) = 0$ on a set E' where $E' \doteq \text{co}\, E$. Denote by e the collection of all those functions. Now the correspondence $\mathfrak{M}$ is defined as that which attaches e to E. The correspondence $\mathfrak{M}$ is invariant in its domain with respect to the equality $\doteq$ of sets. The set e is a closed subspace of the Hilbert-Hermite space $\boldsymbol{H}$. The correspondence $\mathfrak{M}$ is an ordering- and operation-isomorphism from g into a denumerably additive tribe G of subspaces. $\mathfrak{M}$ transforms sets into spaces.

Put $\boldsymbol{F} \underset{df}{=} \mathfrak{M} f$, $\boldsymbol{B} \underset{df}{=} \mathfrak{M}\, \boldsymbol{b}$, and define on G the measure by $\mu e \underset{df}{=}$ meas E. Thus we see that all circumstances in g have their image in G. The brick-spaces are the $\mathfrak{M}$-images of half-open intervals (α, β).

22.3. - The tribe G is saturated. The function

$$\overrightarrow{\omega} \underset{df}{=} \omega(\dot{x}) \underset{df}{=} \text{const.} = 1$$

is a generating vector of $\boldsymbol{H}$ with respect to G, so we can use the whole theory of quasi-vectors as developped in [§ 5] and the system of coordinates, as defined in [§ 6]. The generating vector $\overrightarrow{\omega}$ and the saturated tribe generate an isomorphic correspondence $\mathcal{G}^{-1}$ [§ 5; 4] which transforms the vectors $\overrightarrow{f}(\dot{x})$ of the space of $\boldsymbol{H}$ into square-summable functions $F(\tau)$ of the variable trace. Now, we can prove that $F(x^{+}) = F(x^{-}) = f(x)$ for almost all x, so we may for functions use the symbol **f** instead of $\boldsymbol{F}$. We can also always suppose that $f(x) = f(x^{+}) = f(x^{-})$ for every x, since null sets will not matter

23. - The set $\boldsymbol{B}$ of bricks, the saturated tribe G of spaces and the generating vector $\overrightarrow{\omega}$ constitute a system of coordinates in $\boldsymbol{H}$, [§ 6]. Let $\overrightarrow{F} \underset{df}{=} f(\dot{x})$ be a vector, Φ a space-trace, P its variable neighborhood. Put $\varphi \underset{df}{=} \mathfrak{M}^{-1}\Phi$ and $p = \mathfrak{M}^{-1}P$. The representatives of Φ are descending sequences of spaces P and the representatives of φ a descending sequences of half-open segments p, [§ 3]. Instead of dealing with spaces we prefer to describe space circumstances by the $\mathfrak{M}^{-1}$ - corresponding items in relation to the real axis. Indeed, the space $\boldsymbol{H}$ and its $\mathcal{G}^{-1}$ image [§ 5, 4] are isomorphic and isometric, so we may «identify» - the corresponding items - just for the sake of simplicity.

23.1. - The $\overrightarrow{\Phi}$ - component of F is, [§ 6; Def. 2], the quasi-vector

$$\overrightarrow{F_{\Phi}} \underset{df}{=} \text{Proj}_{P}\, \overrightarrow{F}.$$

We may represent it as the function $f_p(\dot{x})$ depending on p

and defined by

$$f_p(x) \underset{df}{=} \begin{cases} 0 \text{ for } x \in \operatorname{co} p \\ f(x) \text{ for } (x) \in p \end{cases},$$

It can be denoted by $\overrightarrow{f_\varphi}$, and it looks like an infinitesimal piece of the function taken on p at the trace φ, and com pleted a.e. oustide of p by the values 0. The Φ - component density of $\overrightarrow{F}$ is, [§ 6, Def. 2], the quasi-vector

$$\overrightarrow{F^*_\Phi} \underset{df}{=} \frac{\operatorname{Proj}_{\dot{P}} \overrightarrow{F}}{\mu(\dot{P})}, \quad \text{where } \mu(P) = \text{meas } p.$$

It may be represented by «infinitesimal» function-pieces:

$$f^*_p(\dot{x}) \underset{df}{=} \begin{cases} 0 \text{ for } x \in \operatorname{co} p, \\ \dfrac{f(x)}{\mu(p)} \text{ for } x \in p. \text{ It can be denoted by } \overrightarrow{f^*_\varphi}. \end{cases}$$

The Φ - component of $\overrightarrow{F}$ is, [§ 6; Def. 2], the quasi-number

$$F_\Phi \underset{df}{=} (\overrightarrow{\omega_{\dot{P}}}\,;\ \overrightarrow{F}), \text{ (scalar product).}$$

Since to the vector $\overrightarrow{\omega}$ there corresponds the a.e. constant function $\Omega(x) = 1$, and since to the vector $\overrightarrow{\omega_P}$ there correponds the characteristic function $\Omega_p(x)$ of the interval p, we have

$$(\overrightarrow{\omega_P},\ F) = \int_0^1 \overline{\Omega_p\ (x)}\, f(x)\ dx = \int_p f(x)\, dx,$$

so F_Φ can be represented by the quasi-number

$$f^*_\varphi \underset{df}{=} \int_{\dot{p}} f(x)\, dx,$$

defined for all half-open segments p, which cover the trace φ.

The Φ – component-density of $\overrightarrow{F}$ is, [§ 6; Def. 2], the quasi-number

$$F^*_\Phi \underset{df}{=} \frac{\overline{(\overrightarrow{\omega_{\dot{P}}},\ \overrightarrow{F})}}{\mu(\dot{P})}.$$

Hence it can be represented by the quasi-number

$$f_\varphi^* \overline{\overline{df}}\, f^*(\dot{p}) = \frac{1}{\text{meas } \dot{p}} \int_{\dot{p}} f(x)\, dx = \text{val}_\varphi f, \quad [\S\,4;\ \text{Def. } 10], \text{ i.e.}$$

by the locally taken mean-value of $f(x)$.

23.2. - If $f(x)$ is continuous at the point x, and we consider the two traces x_0^+, x_0^- (with vertex x_0), [22.1], and if (a, b) is a variable segment which covers one, or another of this traces, with $\lim\,(b - a) = 0$, we get $\lim\, f^*((a, b)) = f(x_0)$.

23.3. - The theorem [6; 13.b] gives

$$\overrightarrow{F_\Phi} \approx\approx \overrightarrow{\omega_\Phi} \cdot F_\Phi^*,$$

which can be written as

$$\overrightarrow{f_\varphi} \approx\approx \overrightarrow{\Omega_\varphi} \cdot \text{val}_\varphi f = \Omega\,(\dot{p})\ \frac{\int_{\dot{p}} f dx}{\text{meas } \dot{p}}.$$

The quasi-vector $\overrightarrow{\Omega_\varphi}$ may be called *characteristic «function» of the trace* φ. If the function $f(x)$ is continuous at x_0, (compare [23.2]), $\overrightarrow{f_\varphi}$ can be intuitively conceived as $f(x_0)\ \Omega\ (\dot{p})$.

24. - We shall need operations on sets. In general, if E, F are segments, and $\Gamma(x, y)$ is a real-valued function of the real variables x, y, we define

$\Gamma\,(E, F)$ as the set $\{\Gamma(x, y) \mid x \in E,\ y \in F\}$.

Thus e.g. we have $E - F = \{x - y \mid x \in E,\ y \in F\}$,

$a.E = \{ax \mid x \in E\}$ for any number a,

$E^2 = \{x^2 \mid x \in E\}$.

We define:

$$E - a \overline{\overline{df}} \{x - a \mid x \in E\}, \quad -E \overline{\overline{df}} \{-x \mid x \in E\}.$$

Similarly we define $f(E) \overline{\overline{df}} \{f(x) \mid x \in E\}$.

24.1. - Concerning the « difference » $E - F$ of two sets E, F, the following are equivalent:

$$\text{I. } 0 \in E - F, \quad \text{II. } E \cap F \neq \Theta.$$

Indeed, let I. There exist $x \in E$, $y \in F$ with $0 = x - y$. Taking such x and y, we get $x = y$; hence $E \cap F \neq \Theta$.

Let II. There exists $x \in E \cap F$. Since $x - x = 0$, we get $0 \in E - F$.

24.2. - The «difference» $(a, b) - (c, d)$ is always an open segment. The difference of any two intervals is an interval.

24.3. - We have for the Lebesgue's measure
meas $E =$ meas $[E + a]$ for any number a.

24.4 - For intervals E, F of any kind we have meas $(E - F) =$ meas $E +$ meas F.

Proof. Let us close the intervals. This will not affect the measure. Let $a < b$, $c < d$ be the extremities of E and F respectively.

If $a \leq x \leq b$, $c \leq y \leq d$, we get $a - d \leq x - y \leq b - c$. Hence meas $(E - F) = (b - c) - (a - d) = (b - a) + (d - c) =$ $=$ meas $E +$ meas F.

24.5. - If $k > 0$, then $(a, b)k = (ka, kb)$ and $(a, b)(-k) =$ $= (-kb, -ka)$.

24.6. - If $0 \leq a < b$, then $(a, b)^2 = (a^2, b^2)$.

If $a < b \leq 0$, then $(a, b)^2 = (b^2, a^2)$.

If $a < 0 < b$, then $(a, b)^2$ is an interval with left extremity 0, its length is max (a^2, b^2).

24.7. - The notion of the number-valued function $f(E)$, or $g(E, F)$, where E, F are intervals half closed to the right induces the notion of symbols $f(\alpha)$, $g(\alpha, F)$, $g(E, \beta)$, $g(\alpha, \beta)$, where α, β are traces. Thus $f(\alpha)$ means the quasi-number $f(\dot{p})$

where p varies over all neighborhoods of α. Similarly $g(\alpha, \beta)$ is the function $g(\dot{p}, \dot{q})$ where p varies over all neighborhoods of α and q over all neighborhoods of β.

25. - Def. We define the function $\delta(E, F)$ for any intervals E, F as follows

$\delta(E, F) \underset{df}{=} 0$ whenever $E \cap F = \emptyset$,

$$\delta(E, F) \underset{df}{=} \frac{I}{\text{meas }(E - F)} = \frac{I}{\text{meas } E + \text{meas } F} \text{whenever } E \cap F \neq \emptyset.$$

This function will be proved to be a good version of Dirac's δ — function. Notice that we consider our δ — function as a function of two variables. The function generates the functions $\delta(\alpha, F)$, $\delta(E, \beta)$, $\delta(\alpha, \beta)$ where α, β are traces. (see [1.2]).

E. g. $\delta(-\varphi, p)$ will mean, by definition, the quasi-number $\delta(-q, p)$ where q varies over all neighborhoods of the trace φ. Thus if $q = (a, b)$ then we take $\delta((-b, -a), p)$ where (a, b) is the variable neighborhood of φ.

25.0. - Remark. Concerning — φ, we do not need to define it. Its use, as given above, will be meaningful only.

25.1. - We have $\delta(E, F) = \delta(F, E)$, hence $\delta(\alpha, \beta) = \delta(\beta, \alpha)$.

25.1.1. $\delta(E, F) = \delta(-E, -F)$.

25.2. - Remark. The function $\delta(E. F)$ of variable intervals can be considered as depending only on $E - F$. Indeed, let $E - F = E_1 - F_1$. If $E \cap F \neq \emptyset$, then, [24.1], $0 \in E - F$; hence $0 \in E_1 - F$ and then, [24.1], $E_1 \cap F_1 \neq \emptyset$. Conversely, if $E_1 \cap F_1 \neq \emptyset$, then $E \cap F \neq \emptyset$. Consequently $\delta(E, F) = \delta(E_1, F_1)$.

25.3. - $\delta(E, F)$ has the translation property, i. e. if a is a number, then $\delta(E + a, F + a) = \delta(E, F)$.

Proof. If $E \cap F \neq \emptyset$, then $0 \in E - F$, [24.1], hence there exists x with $x \in E$, $x \in F$. Hence $x + a \in E + a$, $x + a \in F + a$. Consequently $x + a \in (E + a) \cap (F + a)$, which gives $(E + a) \cap (F + a) \neq \emptyset$. Similarly we prove that if $(E + a) \cap (F + a) \neq \emptyset$, we get $E \cap F \neq \mathbf{O}$. Thus $\delta(E, F) = 0$ is equivalent to $\delta E(+a, F+a) = 0$. On the other hand we have, the measure being Lebesguean, meas$E$ = meas$(a+E)$, meas F = meas $(a+F)$.

Hence $\dfrac{1}{\text{meas}\, E + \text{meas} F} \doteq \dfrac{1}{\text{meas}\,(E+a) + \text{meas}\,(F+a)}$.

25.4. - We have $b^{+} - a = (b-a)^{+}$,

$$b^{-} - a = (b-a)^{-}.$$

25.5. We have for any two traces $a^{\pm}$, $b^{\pm}$

$$\delta(a^{\pm}, b^{\pm}) = \delta(a^{\pm} - a,\ b^{\pm} - a) = \delta(0^{\pm}, (b-a)^{\pm}) \text{ and}$$
$$\delta(a^{\pm}, b^{\mp}) = \delta(a^{\pm} - a,\ b^{\mp} - a) = \delta(0^{\pm}, (b-a)^{\mp}).$$

26. - **Theor.** If $f(\dot{x})$ is a square-summable function on $(0, 1)$, then

$$\text{val}_{\alpha} f(\dot{x}) = \underset{\beta \varepsilon W}{\mathbf{S}^{\bullet}} \delta(\alpha, \beta)\ \text{val}_{\beta} f(\dot{x})\, |\beta|$$

where $|\beta|$ denotes the quasi-number of the measure (denoted in [§ 4; 8] as μ_{β}).

The theorem follows from [14]. Indeed $\Delta''(\alpha, \beta) = \delta(\alpha, \beta)$ and we can take [24.4] into account.

26.0. - **Remark.** The theorem [26] can be considered as a corrolary of the proof of [Theor. 12], which is more general. Now if we consider the proof, we can notice that, in our case with the function δ, in the expressions $\delta(q_{ik}, p)$ the interval p can be replaced by the interval with the same extremities, but closed on the left. There will be only at most two intervals q_{ik} which have a single point in common with the changed p, and their presence or absence will not influence the limit.

Another remark is that considering the above sums, we can drop those q_{ik} which have no common point with p, so the summation may be carried out even even on suitable subsets of W instead on W itself. Later we shall apply this remark in proofs. We shall consider functions defined in a sufficiently great interval $(-\lambda, +\lambda)$.

26.1. - Remark. The theor. [26] constitutes the main theorem on the Dirac's δ-function. It is the main source of other ones. The theorem can be put into another form:

$$\mathrm{val}_{a^\pm} f(\dot{x}) = \underset{\beta \varepsilon W}{\mathbf{S}^\bullet} \delta(0^\pm;\ \beta - a)\ \mathrm{val}_\beta f(x) \,|\, \beta \,| =$$

$$= \underset{\beta \varepsilon W}{\mathbf{S}^\bullet} \delta(\beta - a;\ 0^\pm)\ \mathrm{val}_\beta f(\dot{x}) \,|\, \beta \,|.$$

26.2. - Remark. We have many theorems having the shape of [26.1], e.g. From [12.1] we get by [23.3] and [20]:

$$\overrightarrow{f_\alpha^*} = \underset{\beta \varepsilon W}{\mathbf{S}^\bullet} \delta(\alpha,\ \beta) \overrightarrow{f_\beta} \,|\, \beta \,| = \underset{\beta \varepsilon W}{\mathbf{S}^\bullet} \delta(\alpha,\ \beta) \overrightarrow{\Omega_\beta}\ \mathrm{val}_\beta f \,|\, \beta \,|.$$

27. - To give precise statement of some theorem on δ-function we need the following notion, (see [27.1] Def.), of *Dirac-equivalence for quasi-numbers.* First we define:

Def. Let $\overrightarrow{f_\alpha}$, $\overrightarrow{g_\alpha}$ be two quasi-vectors (quasi-numbers) with support α.

We say that $\overrightarrow{f_\alpha} \overset{\lim}{=} \overrightarrow{g_\alpha}$ (they are *equal in "limit"*) whenever $\lim \overrightarrow{f(p)} = \lim \overrightarrow{g(p)}$ for meas $p \longrightarrow 0$ and p covering the trace α.

27.1. - Def. Let $\overrightarrow{A}(\varphi \cdot \alpha_0)$, $\overrightarrow{B}(\varphi, \alpha_0)$ be two functions where φ is a variable trace and α_0 a constant trace. We say that $\overrightarrow{A}(\varphi, \alpha_0) \overset{D}{=} \overrightarrow{B}(\varphi,\ \alpha_0)$, «*A* is *Dirac-equal to B*» whenever for every continuous function $h(x)$ in $\langle -\lambda + \lambda \rangle$ we have

$$\mathbf{S}^\bullet_{\varphi \epsilon W}\ \overrightarrow{A}(\varphi,\ \alpha_0)\ \mathrm{val}_\varphi h(\dot{x}) \,|\, \varphi \,| \overset{\lim}{=} \mathbf{S}^\bullet_{\varphi \epsilon W}\ \overrightarrow{B}(\varphi,\ \alpha_0)\ \mathrm{val}_\varphi h(\dot{x}) \,|\, \varphi \,|,$$

Both sides are quasi-vectors with support α_0.

Having these notions we are going to prove some Dirac's formulas on δ - function.

28. - Theor. For traces φ we have

$$\delta(-\varphi,\ 0+) \overset{D}{=} \delta(\varphi,\ 0^+),\ \delta(-\varphi,\ 0^-) \overset{D}{=} \delta(\varphi,\ 0^-).$$

Proof. The sum

$$\mathbf{S}^{\bullet}_{\dot\varphi \in W}\, \delta(-\varphi,\ 0^+)\ \mathrm{val}_{\dot\varphi}\, h \,|\, \varphi \,|$$

is understood as quasi-number with support 0^+: $A(\dot p) \overline{\overline{_{df}}}\ \mathbf{S}^{\bullet}_{\dot\varphi \in W}\, \delta(-\varphi,\ \dot p)\ \mathrm{val}_{\varphi}\ h \,|\, \varphi \,|$, which in turn, for a given p, is the limit of, [28.1.1],

$$\Sigma_k \delta(-a_{nk},\ p)\ \mathrm{val}_{a_{nk}}\ h \cdot |\, a_{nk} \,|,$$

where $Q_n \overline{\overline{_{df}}} \{ a_{n1}, a_{n2}, \}$ is a completely distinguished and special sequence for *1.* It equals, [25.1.1],

$$(1) \qquad \Sigma_k \delta(-p,\ a_{nk})\ \mathrm{val}\ a_{nk}\, h \cdot |\, a_{nk} \,|.$$

Let $p = (\alpha,\ \beta)$. Then $-p = (-\beta,\ -\alpha)$. Now among a_{nk}

there exists one and at most only one, say $(a'_{nk},\ a''_{nk})$ such that $a''_{nk} = -\beta$ and at most one only such that $a'_{nk} = -\alpha$.

If we drop these two intervals from every complex Q_n, we get another sequence $\{ Q'_n \}$ which is also completely distinguished and special for *1.* The dropped terms yield a contribution tending to 0. By remark [26.0] if we replace in (1) $-p$ by $(-\beta, -\alpha)$ and drop the interval mentioned, we get a sum which tends to

$$\mathbf{S}^{\bullet}_{\dot\varphi \in W}\, \delta(\varphi,\ 0^-)\ \mathrm{val}_{\dot\varphi} h \,|\, \varphi \,| = \mathrm{val}_{0^-} h.$$

$$\text{Thus } \mathbf{S}^{\bullet}_{\dot\varphi \in W}\, \delta(-\varphi,\ 0^+)\ \mathrm{val}_{\dot\varphi}\ h \,|\, \varphi \,| = \mathrm{val}_{0^-} h.$$

On the other hand

$$\mathbf{S}^{\bullet}_{\dot\varphi \in W} \delta(\varphi,\ 0^+)\ \mathrm{val}_{\dot\varphi}\ h \,|\, \varphi \,| = \mathrm{val}_{0^+} h.$$

Since h is continuos, we have, [23.2], $\mathrm{val}_{0^-}\ h \overset{\lim}{=} \mathrm{val}_{0^+}\ h.$ Thus we have proved that

$$\delta(-\varphi,\ 0^+) \overset{D}{=} \delta(\varphi,\ 0^+), \text{ Q.E.D.}$$

Similarly we can prove the second thesis.

29. - Theor. $\delta(\alpha,\ 0\pm)\cdot \mathrm{val}_\alpha\, \dot{x} \overset{D}{=} 0_\alpha$.

Proof. Let $h(x)$ be a continuous function in $(-\lambda,\ +\lambda)$. Consider

$J = \mathbf{S}^\bullet\, \delta(\alpha,\ 0\pm)\ \mathrm{val}_\alpha\, \dot{x} \cdot \mathrm{val}_\alpha h(\dot{x})\, |\, \alpha\, |$. Since by [§ 4; 16]

$\mathrm{val}_\alpha\, \dot{x} \cdot \mathrm{val}_\alpha\, h(\dot{x})\, |\, \alpha\, | \approx \mathrm{val}_\alpha(\dot{x}h(\dot{x}))\, |\, \alpha\, |$, we get, by [20],

$J = \mathbf{S}^\bullet \delta(\alpha,\ 0\pm)\ \mathrm{val}_\alpha\, (\alpha h(x))$. Hence, by [26],

$$(1) \qquad J = \mathrm{val}_{0\pm}\ (\dot{x}h(\dot{x})).$$

On the other hand $\mathbf{S}^\bullet 0_\alpha\ \mathrm{val}_\alpha\, h(\dot{x})\, |\, \alpha\, | = 0_\alpha$. By continuity of $h(x)$ we get

$$\lim_{\text{meas } p \to 0} \frac{\int_p x h(x) dx}{\text{meas } p} = 0 \text{ and } \lim 0(p) = 0, \text{ (see [21]). Hence the}$$

theorem is proved.

30. - Theorem. If $k > 0$, then

$$\delta(k\beta,\ 0\pm) \overset{D}{=} \frac{1}{k}\, \delta(\beta,\ 0\pm).$$

Proof. Let p be a neighborhood of $0\pm$ The sum $\mathbf{S}^\bullet_{\beta \in W}\, \delta(k\beta),\ p)\ \mathrm{val}_\beta f(x) \cdot |\, \beta\, |$ is the limit of the expression

$$(1) \qquad A_n \underset{df}{=} \Sigma_i\, \delta(k \cdot q_{ni},\ p) \int_{q_{ni}} f(x) dx,$$

where $Q_n = \{q_{n1},\ q_{n2}, \ldots\}$ is a completely distinguished and special sequence for I.

Put $kx = y$; we have

$$\int\limits_{q_{ni}} f(x)dx = \frac{1}{k}\int\limits_{p_{ni}} f\left(\frac{y}{k}\right) dy,$$

where $p_{ni} = k \cdot q_{ni}$.

Hence $A_n = \frac{1}{k}\, \Sigma_i\, \delta(p_{ni},\ p) \int\limits_{p_{ni}} f\left(\frac{y}{k}\right) dy$, which tends to

$$(2) \qquad \frac{1}{k}\, \mathbf{S}^{\bullet}\, \delta(\beta,\ p)\, \mathrm{val}_{\beta}\, g(\dot{y}), \text{ where } g(y) = f\left(\frac{y}{k}\right).$$

(2) is the quasi-number:

$$\frac{1}{k}\, \mathbf{S}^{\bullet}\, \delta(\beta,\ 0\pm)\, \mathrm{val}_{\beta}\, g(\dot{y})\, |\, \beta\, | = \mathrm{val}_{0\pm}\, g(\dot{y}) = \mathrm{val}_{0\pm}\, f\left(\frac{\dot{x}}{k}\right).$$

For continuous functions f we have
$\lim\ \mathrm{val}_{0\pm} f\left(\frac{\dot{x}}{k}\right) = \lim\ \mathrm{val}_{0\pm} f(\dot{x}) = f(0)$. Hence we can write

$$\delta(k\beta,\ 0\pm) \stackrel{D}{=} \frac{1}{k}\, \delta(\beta,\ 0\pm), \text{ Q. E. D.}$$

32. - **Theorem.** If α is a trace, a, b numbers, then

$$\mathbf{S}^{\bullet}_{\alpha\epsilon W}\, \delta(\alpha - a,\ b\pm)\, \mathrm{val}_{\alpha}\, f(\dot{x})\, |\alpha| =$$

$$= \mathrm{val}_{(a+b\pm)}\, f(\dot{x}) = \mathrm{val}_{b\pm}\, f(\dot{x}+a) = \mathrm{val}_{a\pm}\, f(\dot{x}+\mathrm{b}).$$

Proof. We have by [25.5] and [25.4]:
$\delta(\alpha - a,\ b\pm) = \delta(\alpha,\ b\pm + a) = \delta(\alpha,\ (b + a)\pm)$. Hence our theorem is equivalent to
$\mathbf{S}^{\bullet}_{\alpha\epsilon W}\, \delta(\alpha,\ (b + a)\pm)\, \mathrm{val}_{\alpha}\, f(\dot{x})\, |\, \alpha\, | = \mathrm{val}_{(\alpha+b\pm)}\, f(\dot{x})$. This however follows from [26]. Let $(b',\ b'')$ be the variable neighborhood

of $b^{\pm}$; then $(a+b', a+b'')$ is the variable neighborhood of $(a+b)^{\pm}$. The quasi-number $\mathrm{val}_{(a+b)^{\pm}} f$ is defined by

$$(1) \qquad \frac{1}{b''-b'} \int_{a+b'}^{\overset{\bullet}{a+b''}} f(x)dx.$$

Put $x = y + a$. Then (1) equals $\int_{b'}^{b''} f(y+a)\, dy$ which defines

$\mathrm{val}_{b^{\pm}} f(\dot{x}+a)$.

33. · Theorem. If $a > 0$ and τ is a variable trace, then

$$\delta(\tau^2 - a^2, 0^{\pm}) \overset{D}{=} \frac{1}{2a} [\delta(\tau - a, 0^{\pm}) + \delta(\tau + a, 0^{\pm})].$$

Proof. The expression $\delta(\tau^2 - a^2, 0^{\pm})$ is defined, [24 and 1.2], as a number-valued function of two intervals p and q where p is a variable neighborhood of $0^{\pm}$ and q a variable neighborhood of τ. Consider the sum

$$(1) \qquad \mathbf{S}^{\bullet}_{\tau \in W} \delta(\tau^2 - a^2, 0^{\pm}) \,\mathrm{val}_{\tau} f(\dot{x}) \cdot |\tau|,$$

where $f(x)$ is a continuous function. (1) is defined as the quasi-number $A(\dot{p})$ with support $0^{\pm}$:

$$A(p) = \mathbf{S}_{\tau \in W} \delta(\tau^2 - a^2, p) \,\mathrm{val}_{\tau} f \cdot |\tau|.$$

Now $A(p)$ is the limit of the following sum

$$(2) \qquad \Sigma_k \delta(q_{nk}^2 - a^2, p) \,\mathrm{val}_{q_{nk}} f \cdot |q_{nk}|,$$

where $Q_n = \{q_{n1}, q_{n2}, \dots\}$ is a completely distinguished and special sequence of complexes for *1*.

The intervals q_{nk} are disjoint, so there exists at most one interval, — denote it by $(z'z'')$, — such that $z' < 0 < z''$.

All other intervals q_{nk} can be devided into two classes; the first will contain all those (x'_k, x''_k) for which

$$0 \leq x'_k < x''_k;$$

to the second all those (y'_k, y''_k) for which $y'_k < y''_k \leq 0$. The sum (2) can be written as

$$\Sigma_k \delta((x'_k, x''_k)^2 - a^2, p) \cdot \mathrm{val}_{(x'_k, x''_k)} f \cdot (x'' - x' +$$

$$+ \delta((z', z'')^2 - a^2, p) \cdot \mathrm{val}_{(z', z'')} f \cdot (z'' - z') +$$

$$(2.1) \quad + \Sigma_k \delta((y'_k, y''_k)^2 - a^2, p) \cdot \mathrm{val}_{(y'_k, y''_k)} f \cdot (y''_k - y'_k).$$

Hence, by [24.6], the expression equals

$$(3) \quad \Sigma_k \delta((x''^2_k - a^2, x'^2_k - a^2), p) \int_{x'_k}^{x''_k} f(x) dx +$$

$$+ \delta((0, \max(z', z'')), p) \cdot \int_{z'}^{z''} f(x) dx +$$

$$+ \Sigma_k ((\delta y'^2_k - a^2, y''^2_k - a^2), p) \cdot \int_{y'_k}^{y''_k} f(x) dx.$$

We shall transform the first term of (3) by changing the variable x as follows.

Put $u \underset{df}{=} x^2 - a^2$. We have $x^2 = a^2 + u$ and $2x dx = du$. Put $u'_k \underset{df}{=} x'^2_k - a^2$, $u''_k \underset{df}{=} x''^2_k - a^2$. We have, since all x'_k, x''_k are non positive,

$$x'_k = -\sqrt{u'_k + a^2}, \quad x''_k = -\sqrt{u''_k + a^2}.$$

Hence

$$(4) \qquad \int_{x'_k}^{x''_k} f(x)dx = -\int_{u'_k}^{u''_k} f(-\sqrt{a^2+u}) \cdot \frac{du}{-2\sqrt{u+a^2}}.$$

The formula (4) holds true, even if $u''_k = 0$. Consequently the first term in (3) can be written:

$$(5) \qquad -\Sigma_k \, \delta((u''_k, u'_k), p) \cdot \int_{u'_k}^{u''_k} \frac{f(-\sqrt{a^2+u})}{2\sqrt{a^2+u}} \, du =$$

$$= \Sigma_k \, \delta((u''_k, u'_k), p) \cdot \mathrm{val}_{(u''_k, u'_k)} \, g_-(u),$$

where

$$(6) \qquad g_-(u) \underset{df}{=} \frac{f(-\sqrt{a^2+u})}{2\sqrt{a^2+u}} \quad \text{for } u > -a^2.$$

Concerning (5) let us remark that the transformation

$$x = -\sqrt{u+a^2}, \; x^2 - a^2 = u,$$

is one to one and monotonic non increasing, transforming the interval

$$(7) \qquad -\infty < x \leq 0 \text{ into } -\infty < u \leq -a^2.$$

If we take meas $p < \frac{a}{4}$, the interval p will belong to the set (7).

In the sum (5) only those terms may not disappear for which $(u''_k \, u'_k) \cdot p \neq O$.

Hence we can confine the sum (5) only to terms for which

$$\frac{-a^2}{2} < u''_k < u'_k < \frac{a^2}{2}.$$

In the interval

$$\left\langle \frac{-a^2}{2}, \frac{+a^2}{2} \right\rangle$$

the function $g_-(u)$ is continuous.

If we take account of remark [26.0], we can change in (5)

$$\delta(u_k'', u_k'), p) \text{ into } \delta(\langle u_k'', u_k' \rangle, p).$$

Hence, by [26], the limit of the sum (5) is the quasi--number $\mathrm{val}_{0\pm}\, g_-(\dot{u})$.

Thus we have proved, that the first term in (2.1) tends to

$$\mathrm{val}_{0\pm}\, g_-(\dot{u}), \tag{8}$$

where $g_-(u)$ is defined in (6).

Now, consider the third term in (2.1), hence in (3). We shall use the transformation $v = x^2 - a^2$, $x = \sqrt{v + a^2}$, ... (9) since all y'_k, y''_k are non negative. The transformation (9) is $1 \to 1$ and monotone non decreasing. It transforms the interval $0 \leq x < \infty$ into $-a^2 \leq v < \infty$. We put $v'_k \underset{df}{=} y_k^2 - a^2$, $v''_k \underset{df}{=} y_k''^2 - a^2$ and we have $y'_k = \sqrt{v'_k + a^2}$, $y''_k = \sqrt{v''_k + a^2}$.

Hence

$$\int_{y'_k}^{y''_k} f(x)dx = \int_{v'_k}^{v''_k} \frac{f(\sqrt{a^2 + v})}{2\sqrt{a^2 + v}}\, dv.$$

Hence the third term in (3) equals

$$\Sigma_k \delta(\langle v'_k, v''_k \rangle, p) \int_{v'_k}^{v''_k} \frac{f(\sqrt{a^2 + v})}{2\sqrt{a^2 + v}}\, dv. \tag{10}$$

(11) If we put $g_+(v) \underset{df}{=} \dfrac{f(\sqrt{a^2 + v})}{2\sqrt{a^2 + v}}$,

we get, by argument similar to the above ones, that (10) tends to

$$\text{(12)} \qquad \mathrm{val}_{0\mp}\, g_{+}(u).$$

Since the functions g_{+} and g_{-} are both continuous at 0, the limit value of $\mathrm{val}_{0\mp}\, g_{+}$ equal to $g_{+}(0) = \frac{f(a)}{2a}$, because $a > 0$.

Similarly we have the limit value of $\mathrm{val}_{0\mp} g_{-}$ equal to

$$g_{-}(0) = \frac{f(-a)}{2a}.$$

There remains the second term in (3) and (2.1) to be considered:

$$\delta((0, \max(z'\ z'')) - a^2,\ p) \cdot \int_{z'}^{z''} f(x)\, dx \text{ where } z' < 0 < z''.$$

For p chosen in $\left\langle -\frac{a^2}{4}, \frac{a^2}{4} \right\rangle$ and sufficiently small intervals q_{nk}, the intervals p and $(0, \max(z' z''))$ do not overlap, so the term considered vanishes. Thus this term does not contribute anything to the limit. We have proved that

$$\mathbf{S}^{\bullet}_{I_4}\, w\, \delta(\tau^2 - a^2,\ 0\pm)\ \mathrm{val}_\tau f(\dot{x})\, |\, \tau\, |$$

has the limit value $\frac{f(a) + f(-a)}{2a}$

Now we have from [32] that

$$\mathbf{S}^{\bullet}\, \delta(\tau - a,\ 0\pm)\ \mathrm{val}_\tau f(\dot{x})\, |\, \tau\, | = \mathrm{val}_{0\mp} f(\dot{x} + a) \overset{\lim}{=} f(a) \text{ and}$$

$$\mathbf{S}^{\bullet} \delta(\tau + a,\ 0\pm)\ \mathrm{val}_\tau f(\dot{x})\, |\, \tau\, | = \mathrm{val}_{0\mp} f(\dot{x} - a) \overset{\lim}{=} f(-a).$$

Consequently

$$\mathbf{S}^{\bullet} \left[\frac{\delta(\tau - a,\ 0\pm) + \delta(\tau + a,\ 0\pm)}{2a} \right] \mathrm{val}\, f(\dot{x})\, |\, \tau\, | \overset{\lim}{=} \frac{f(a) + f(-a)}{2},$$

which completes the proof.

34. Theor. If β is a variable trace and α_0 a fixed trace, then for every continuous function $f(x)$ we have

$$\mathrm{val}_{\alpha_0} f \cdot \delta(\alpha_0, \beta) \stackrel{D}{=} \mathrm{val}_\beta f \cdot \delta(\alpha_0, \beta).$$

Proof. Take a continuous function $h(x)$. We have

$$\mathbf{S}^{\bullet}_{\beta\varepsilon W} \mathrm{val}_\beta h \cdot \mathrm{val}_{\alpha_0} f \delta(\alpha_0 \beta) \, |\beta| = \mathrm{val}_{\alpha_0} f \cdot \mathbf{S}^{\bullet}_{\beta\varepsilon W} \mathrm{val}_\beta h \cdot \delta(\alpha_0 \beta) \, |\beta| =,$$

(1) by [26], $= \mathrm{val}_{\alpha_0} f \cdot \mathrm{val}_{\alpha_0} h.$

$$\mathbf{S}^{\bullet}_{\beta\varepsilon W} \mathrm{val}_\beta h \cdot \mathrm{val}_\beta f \cdot \delta(\alpha_0 \beta) \, |\beta| =,$$

by [§ 4; 16],

$$= \mathbf{S}^{\bullet}_{\beta\varepsilon W} \mathrm{val}_\beta (hf) \cdot \delta(\alpha_0 \beta) \, |\beta| =, \qquad [26],$$

$$= \mathrm{val}_\alpha (hf).$$

Since f, h are continuous, we have for a variable neighborhood p of α_0 with meas $p \longrightarrow 0$: the limit value of $\mathrm{val}_p f$ is $f(\mathrm{vert}\,\alpha)$, that of $\mathrm{val}_p h$ is $h(\mathrm{vert}\,\alpha_0)$ and that of $\mathrm{val}_p(fh)$ is $f(\mathrm{vert}\,\alpha_0) \cdot h(\mathrm{vert}\,\alpha_0)$, so the theorem is proved.

34.1 - Remark. We can state the theorem as follows:

$$\mathrm{val}_\alpha f \cdot \delta(\alpha_0 - \mathrm{vert}\ \beta, 0\pm) \stackrel{D}{=} \mathrm{val}_\beta f \cdot \delta(\alpha_0 - \mathrm{vert}\ \beta, 0\pm)$$

which looks like the Dirac's formula:

$$f(a)\delta(a - b) = f(b)\delta(a - b).$$

35. - Theorem. If γ_0 is a fixed trace and α a variable trace, then

$$\mathbf{S}^{\bullet}_{\beta\varepsilon W} \delta(\alpha, \beta) \, \delta(\beta, \gamma_0) \, |\beta| \stackrel{D}{=} \delta(\alpha, \gamma_0).$$

Proof. We shall confine ourselves to a sketch of the proof. Let $h(x)$ be a continuos function. We shall compare the expression

(1) $$\mathbf{S}^{\bullet}_{\alpha\varepsilon W} (\mathbf{S}^{\bullet}_{\beta\varepsilon W} \delta(\alpha, \beta) \, \delta(\beta, \gamma_0) \, |\beta| \ \mathrm{val}_\alpha h . |\alpha|$$

(2) with $\mathbf{S}^{\bullet}_{\alpha\varepsilon W} \delta(\alpha, \gamma_0) \ \mathrm{val}_\alpha h \, |\alpha|$.

Since $\mathbf{S}^{\bullet}_{\beta\varepsilon W} \delta(\alpha, \beta) \, \delta(\beta, \dot{p}) \, |\beta|$, where p is a neighborhood of γ_0, is a function $F(\alpha, \gamma_0)$, the expression (1) has the form:

$$\mathbf{S}^{\bullet}_{\alpha \varepsilon W} F(\alpha, \gamma_0) \operatorname{val}_\alpha h \cdot |\alpha| .$$

Hence, by [§ 4], it is a quasi-number with support γ_0, so we evaluate the sum by taking a neighborhood Γ of γ_0 and consider the sum

(3) $$\mathbf{S}^{\bullet}_{\alpha \varepsilon W} F(\alpha, \Gamma) \operatorname{val}_\alpha h |\alpha|,$$

which in turn is the limit of

(4) $$F_n(\Gamma) \underset{df}{=} \Sigma_k F(\alpha_{nk}, \Gamma) \operatorname{val}_{\alpha_{nk}} h \cdot |\alpha_{nk}|,$$

where $A_n \underset{df}{=} \{ \alpha_{n1}, \alpha_{n2}, \dots \}$ is a completely distinguished and special sequence of complexes for *1*. (4) is a function of Γ.

(5) Now $$F(\alpha_{nk}, \Gamma) = \mathbf{S}^{\bullet}_{\beta \varepsilon W} \delta(\alpha_{nk} \beta) \, \delta(\beta, \Gamma) |\beta|;$$

hence (4) can be written:

(6) $$F_n(\Gamma) = \Sigma_k [\mathbf{S}^{\bullet}_{\beta \varepsilon W} \delta(\alpha_{nk} \beta) \, \delta(\beta, \Gamma) \cdot |\beta|] \operatorname{val}_{\alpha_{nk}} h \cdot |\alpha_{nk}|.$$

(7) The sum $$\mathbf{S}^{\bullet}_{\beta \varepsilon W} \delta(\alpha_{nk}, \beta) \, \delta(\beta, \Gamma) \cdot |\beta|$$

is just the sum of a total set of quasi-numbers, so it is a limit of the sum:

(8) $$G_{n, m}(\Gamma) \underset{df}{=} \Sigma_j \delta(\alpha_{nk} \beta_{mj}) \, \delta(\beta_{mj} \, \Gamma) \cdot |\beta_{mj}|,$$

where $B_m \underset{df}{=} \{ \beta_{m1}, \beta_{m2}, \dots \}$ is a completely distinguished and special sequence of complexes for *1*. We recall that G has no atoms, hence for any special sequence of complexes, the net number tends to 0. It follows that if the intervals Γ and α_{nk} after closure are disjoint, then for sufficiently great m, all terms in (8) will vanish and hence the sum (7) will be equal 0.

Hence the sum (6) can be restricted to only those nk for which $\overline{\Gamma} \cap \overline{\alpha}_{nk} \neq \mathbf{0}$. Hence it can be restricted to ext

$(A_n)\,\overline{\Gamma}$. (see [Def. 16]). If $\overline{\Gamma} \cap \overline{\alpha}_{nk} \neq \mathbf{0}$, the corresponding term in (8) will be

$$\frac{1}{|\alpha_{nk}| + |\beta_{mj}|} \cdot \frac{1}{|\beta_{mj}| + |\Gamma|} \cdot |\beta_{mj}|, \tag{9}$$

and the summation in (8) can be taken over all β_{nj} where $\beta_{nj} \cap \Gamma \neq \mathbf{0}$ and at the same time $\beta_{mj} \cap \alpha_{xk} \neq \mathbf{0}$.

The sum of these terms in (9) can be written:

$$\Sigma_j |\beta_{mj}| \cdot \frac{1}{|\alpha_{nk}|} \cdot \frac{1}{|\Gamma|} + \Sigma_j |\beta_{nj}| \left\{ \frac{1}{|\alpha_{nk}| + |\beta_{mj}|} \cdot \right.$$
$$\left. \cdot \frac{1}{|\beta_{mj}| + |\Gamma|} - \frac{1}{|\alpha_{nk}| \cdot |\Gamma|} \right\}.$$

To the given n the factor in braces is small for sufficiently great m, and $\Sigma_j |\beta_{mj}|$ will exceed meas $(\Gamma \cap \alpha_{nk})$ by a small quantity for sufficiently great m, so the last term will be small.

Hence (7), i.e. the limit of the expression (8) will be

$$\frac{|\alpha_{nk}| + \varepsilon_n}{|\Gamma|\ |\alpha_{nk}|} \quad \text{where } \varepsilon_n \longrightarrow 0.$$

It follows that the expression (6), i.e.

$$F_n(\Gamma) = \Sigma_k \frac{|\alpha_{nk}| + \varepsilon_n}{|\Gamma| \cdot |\alpha_{nk}|} \operatorname{val}_{\alpha_{nk}} h\, |\alpha_{nk}|,$$

where the summation is extended over all α_{nk} with $\overline{\alpha}_{nk} \cap \overline{\Gamma} \neq \mathbf{0}$,

It differs but a little from

$$\Sigma_k \frac{1}{|\Gamma|} \int\limits_{\alpha_{nk}} h, \text{ hence from } \frac{\int_\Gamma h}{\Gamma} = \operatorname{val}_{\gamma_0} h. \tag{10}$$

Concerning (δ, γ_0) we have

$$\mathbf{S}_{\alpha \in W}\, \delta(\alpha, \gamma_0) \operatorname{val}_\alpha h\, |\alpha| = \operatorname{val}_{\gamma_0} h, \tag{11}$$

From (10) and (11) the theorem follows.

36. - Remark. We have proved several theorems concerning the function $\delta(\alpha, \beta)$, where α, β are traces. We believe that the Dirac's delta function should be defined as a function of two variables, since it is like an integral-equation-kernel. We believe that our δ-function should replace the genuine δ-function introduced by Dirac, (33), p. 58 and 60:

1. $\int \delta(x)\, dx = 1$

1′. $\mathrm{S}^{\bullet}_{\beta}\delta(\alpha, \beta) \mid \beta \mid = 1_{\alpha}$.

2. $f(x) = \int \delta(x - y) f(y)\, dy$.

2′. $\mathrm{val}_{\alpha} f(\dot{x}) =$
$= \mathrm{S}^{\bullet}_{\beta}\delta(\alpha, \beta)\, \mathrm{val}_{\beta} f(\dot{x}) \mid \beta \mid$.

3. $\delta(-x) = \delta(x)$.

3′. $\delta(-\varphi, 0\pm) \overset{D}{=} \delta(\varphi, 0\pm)$.

4. $x\delta(x) = 0$.

4′. $\delta(\alpha, 0\pm)\, \mathrm{val}_{\alpha}\, \dot{x} \overset{D}{=} 0_{\alpha}$.

5. $\delta(ax) = \frac{1}{a}\, \delta(x)$, $(a > 0)$.

5′. $\delta(k\beta, 0\pm) \overset{D}{=} \frac{1}{k}\, \delta(\beta, 0\pm) (k>0)$.

6. $\delta(x^2 - a^2) = \frac{1}{2a} [\delta(x - a) +$
$+ \delta(x + a)]$, $(a > 0)$.

6′. $\delta(\tau^2 - a^2, 0\pm) \overset{D}{=} \frac{1}{2a} [\delta(\tau -$
$- a, 0\pm) + \delta(\tau + a, 0\pm)]$,
$(a > 0)$.

7. $\int \delta(a - x)\, dx\, \delta(x - b) =$
$= \delta(a - b)$.

7′. $\mathrm{S}^{\bullet}_{\beta}\delta(\alpha\beta) \mid \beta \mid \delta(\beta, \gamma_0) \overset{D}{=} \delta(\alpha, \gamma_0)$.

8. $f(x)\, \delta(x - a) = f(a)\, \delta(x - a)$.

8′. $\mathrm{val}_{\alpha_0} f(\dot{x}) \delta(\alpha_0 \beta) \overset{D}{=} \mathrm{val}_{\beta} f(\dot{x}) \delta(\alpha_0 \beta)$.

In addition to that, $\delta(\alpha, \beta)$ behaves like a function of the difference of variables, since $\delta(\alpha, \beta)$ has the translation property [25.3].

Concerning the equality $\overset{D}{=}$, it seems to be in agreement with Dirac's remark (33) p. 60: «The meaning of any of these equation is that its two sides give equivalent results as factors of an integral».

Concerning $\mathrm{val}_x f(\dot{x})$ — this is an «ideal» average value which physicist approach by taking the average values from measurment made with more and more precise istruments.

37. - We like to remark that various statements on the modified, genuine function $\delta(\alpha, \beta)$ can be generalized to the case where bricks are half open rectangles–or even half open hypercubes in n-dimensional space with Lebesgue'an measure admitted.

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