AN ALGORITHM FOR SOLVING MULTIPLE OBJECTIVE INTEGER LINEAR PROGRAMMING PROBLEM

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Abstract. In the present paper a complete procedure for solving Multiple Objective Integer Linear Programming Problems is presented. The algorithm can be regarded as a corrected form and an alternative to the method that was proposed by Gupta and Malhotra. A numerical illustration is given to show that this latter can miss some efficient solutions. Whereas, the algorithm stated bellow determines all efficient solutions without missing any one.

Keywords. Multiple objective programming, integer linear programming.

1. Introduction

Multiple Integer Linear Programming problems have diverse applications such as agricultural planning, financial analysis of a firm, travelling salesman problems, Markovian replacement problems, the cutting stock problems, and portfolio selection problems.

It has been studied by several authors (*e.g.* Gupta and Malhotra [10], Klein [13], Bitran [7], Zionts [22], Abbas and Moulaï [1], etc.).

The main focus of this paper, is to find all the efficient points using the cutting plane technique and simple pivoting procedures. In the following section some theoretical tools are presented to prove the convergence of the proposed method followed by the algorithm, then lasted by a numerical illustration, conclusions and comments with short comparative indications to Gupta and Malhotra algorithm.

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2. THEORETICAL TOOLS

The multiple objective integer linear programming problem (**MOILP**) can be formulated as follows

$$(P) \begin{cases} \text{Max } Z_1 & = C^1 X \\ \text{Max } Z_2 & = C^2 X \\ \quad \cdots \\ \text{Max } Z_p & = C^p X \\ \text{subject to:} & X \in F = \{X \in \mathbb{R}^n / AX = b, X \text{ positive integer}\} \end{cases}$$

$A \in \mathbb{R}^{m \times n}$; $b' \in \mathbb{R}^m$; $C^i \in \mathbb{R}^n$ for all $i \in I' = \{1, 2, ..., p\}$.

Definition 2.1. A point $X^0 \in F$ is an efficient solution if there is no $X \in F$ such that : $Z_i(X) \geq Z_i(X^0)$ for all $i \in I'$ and $Z_i(X) > Z_i(X^0)$ for at least one $i \in I'$. Otherwise, X^0 is not efficient and $(Z_1(X^0), Z_2(X^0), ..., Z_p(X^0))$ is said to be a dominated p-tuple.

The problem $(P1)$ is defined as follows:

$$(P1) \begin{cases} \text{Max } Z_1 & = C^1 X \\ \text{subject to:} & X \in F. \end{cases}$$

The relaxed problem is:

$$(P_R) \begin{cases} \text{Max } Z_1 & = C^1 X \\ \text{subject to:} & X \in F. \end{cases}$$

$$F_R = \{X \in \mathbb{R}^n / AX = b, X \text{ is positive}\} \cdot$$

Definition 2.2.   An optimal solution X^0 of the problem $(P1)$ is said to be unique if there does not exist $X^1 \in F$ such that $X^1 \neq X^0$ with $Z_1(X^0) = Z_1(X^1)$. Otherwise, the problem $(P1)$ has got at least one alternate solution over he region F.

Notations

- X_1^*: is the optimal solution of the problem $(P1)$. This is obtained by applying Gomory cuts over the problem (P_R).
- $Z_1^1 = Z_1^*$: is the optimal solution obtained in $(P1)$.
- $(Z_1^1, Z_2^1, ..., Z_p^1)$: is the first non-dominated p-tuple, where: $Z_i^1 = Z_i(X_1^*)$ for $i = 2, 3, ..., p$.

For $k \geq 1$:

- X_k: is the efficient solution underlying $(Z_1^k, Z_2^k, ..., Z_p^k)$;
- B_k: basis associated with X_k;
- a_{kj}: activity vector of x_{kj} with respect to the truncated region;
- $y_{kj} = (B_k)^{-1} a_{kj}$;
- $I_k = \{j / a_{kj} \in B_k\}$;
- $N_k = \{j / a_{kj} \notin B_k\}$;

- $\Gamma_k = \left\{ \begin{array}{c} j \in N_k \,/\, z_{1j} - c_j^1 > 0 \text{ and } z_{ij} - c_j^i > 0 \text{ for at} \\ \text{least one } i \in \{2, 3, ..., p\} \end{array} \right\}$;

- $E_{j_k} = \left\{ (x_i) \in \mathbb{R}^n \,/\, \begin{array}{l} x_i = x_{ki} - \theta_{j_k} y_{k, ij_k} \\ x_{j_k} = \theta_{j_k} \\ x_\alpha = 0 \quad \text{for all } \alpha \in N_k \backslash \{j_k\} \end{array} \right\}.$ $\qquad (*)$

Where θ_{j_k} is the integer value between 0 and $\min\limits_{i \in I_k} \left\{ \dfrac{x_{ki}}{y_{k, ij_k}} \,;\, y_{k, ij_k} > 0 \right\}$

and θ_{j_k}, $\theta_{j_k} \times y_{k, ij_k}$ are integers.

Theorem 2.1. All integer feasible solutions of problem $(P1)$ alternate to X_1 on edge E_{j_1}, $j_1 \in \Gamma_1$ in the region F (or truncated region of F) emanating from it lie in the open half space $\sum\limits_{j \in N_1 \backslash \{j_1\}} x_j < 1$.

Proof. We have $AX_1 = b$ (X_1 is a feasible solution of $(P1)$). $AX_1 = \sum\limits_{i \in I_1} a_{1i} x_{1i} = b$.

For some $j_1 \in \Gamma_1$, $\sum\limits_{i \in I_1} a_{1i} x_{1i} - \theta_{j_1} a_{1j_1} + \theta_{j_1} a_{1j_1} = b$, where θ_{j_1} is a nonzero positive scalar. $\sum\limits_{i \in I_1} a_{1i} x_{1i} - \theta_{j_1} \left(\sum\limits_{i \in I_1} a_{1i} y_{1, ij_1} \right) + \theta_{j_1} a_{1j_1} = b$; $\sum\limits_{i \in I_1} a_{1i} (x_{1i} - \theta_{j_1} y_{1, ij_1}) + \theta_{j_1} a_{1j_1} = b$.

For $0 \leq \theta_{j_1} \leq \min\limits_{i \in I_1} \left\{ \dfrac{x_{1i}}{y_{1, ij_1}} \,;\, y_{1, ij_1} > 0 \right\}$, X_2 is defined as follows:

$$X_2 = \left\{ \begin{array}{lll} x_{2i} & = & x_{1i} - \theta_{j_1} y_{1, ij_1} \, i \in I_1 \\ x_{2j_1} & = & \theta_{j_1} \\ x_{2\alpha} & = & 0 \text{ for all } \alpha \in N_1 \backslash \{j_1\} \end{array} \right\} ; \text{which is a new}$$

integer feasible solution of $(P1)$ provided θ_{j_1} and $\theta_{j_1} \times y_{1, ij_1}$ are positive integers. We show now that $Z_1(X_2) = Z_1(X_1)$.

$$\begin{aligned} Z_1(X_2) &= \sum_{i \in I_2} c_{1i} x_{2i} = \sum_{i \in I_1} (x_{1i} - \theta_{j_1} y_{1, ij_1}) c_{1i} + c_{1j_1} \theta_{j_1} \\ &= \sum_{i \in I_1} c_{1i} x_{1i} - \sum_{i \in I_1} c_{1i} \theta_{j_1} y_{1, ij_1} + c_{1j_1} \theta_{j_1} \\ &= \sum_{i \in I_1} c_{1i} x_{1i} - \theta_{j_1} \sum_{i \in I_1} c_{1i} y_{1, ij_1} + c_{1j_1} \theta_{j_1} \\ &= \sum_{i \in I_1} c_{1i} x_{1i} - \theta_{j_1} \left(\sum_{i \in I_1} c_{1i} y_{1, ij_1} - c_{1j_1} \right) \\ &= \sum_{i \in I_1} c_{1i} x_{1i} = Z_1(X_1) \end{aligned}$$

since $\sum\limits_{i \in I_1} c_{1i} y_{1, ij_1} - c_{1j_1} = 0$.

Therefore, X_2 is an integer feasible solution of (P_1), alternate to X_1 lying on an edge

$$E_{j_1} = \left\{ (x_i) \in \mathbb{R}^{(|I_1|+|N_1|)} / \begin{array}{rcl} x_{2i} & = & x_{1i} - \theta_{j_1} y_{1,ij_1} \;\; i \in I_1 \\ x_{2j_1} & = & \theta_{j_1} \\ x_{2\alpha} & = & 0 \text{ for all } \alpha \in N_1 \backslash \{j_1\} \end{array} \right\}.$$

We have, $\displaystyle\sum_{j \in N_1 \backslash \{j_1\}} x_{2j} < 1$; since $x_{2j} = 0$ for all $j \in N_1 \backslash \{j_1\}$. Thus, the point X_2 lies in the open half space $\displaystyle\sum_{j \in N_1 \backslash \{j_1\}} x_j < 1$.

Corollary 2.1. An integer feasible solution of problem $(P1)$ not on edge E_{j_1}, $j_1 \in \Gamma_1$ in the truncated region F throughout the best solution X_1 of $(P1)$ lies in $\displaystyle\sum_{j \in N_1 \backslash \{j_1\}} x_j \geq 1$.

Proof. Let $\widetilde{X} = (\widetilde{x}_j)_{i \in I_1 \cup N_1}$ be an integer feasible solution of problem $(P1)$ not on an edge E_{j_1} such that: $\displaystyle\sum_{j \in N_1 \backslash \{j_1\}} \widetilde{x}_j < 1$. Then $\widetilde{x}_j = 0$ for all $j \in N_1 \backslash \{j_1\}$.

- $\widetilde{x}_{j_1} > \mathbf{0}$, $\left(\widetilde{X} \text{ is alternate to } X_1 \right)$ then, $\widetilde{X}$ must lie in the direction of vector a_{1j_1}, $j_1 \in \Gamma_1$.

  * $\widetilde{x}_{j_1} > \displaystyle\min_{i \in I_1} \left\{ \frac{x_{1i}}{y_{1,ij_1}}; \; y_{k,ij_1} > 0 \right\} = \frac{x_{1q}}{y_{1,ij_1}}$ (for example), then $\widetilde{x}_q = \widetilde{x}_{1q} - \widetilde{x}_{j_1} y_{1,qj_1} < 0$ which implies infeasibility of $\widetilde{X}_1$ contradicting to the hypothesis.

  * $\widetilde{x}_{j_1}$ is a positive integer such that: $\widetilde{x}_{j_1} \leq \displaystyle\min_{i \in I_1} \left\{ \frac{x_{1i}}{y_{1,ij_1}} \; ; \; y_{k,ij_1} > 0 \right\}$, this also implies that $\widetilde{X}$ lies on the edge E_{j_1}, which is not true.

  Hence, $\widetilde{x}_j > 0$ for at least one $j \in N_1 \backslash \{j_1\}$.

- $\widetilde{x}_{j_1} = \mathbf{0}$, $\left(\widetilde{X} \text{ is not alternate to } X_1 \right)$ then, the index set of non-basic variables in the optimal table corresponding to $\widetilde{X}$ is the same as N_1 since $N_1 = N_1 \backslash \{j_1\} \cup \{j_1\}$, therefore the index set of basic variables in the optimal table corresponding to $\widetilde{X}$ is the same as B. Thus $\widetilde{X} = X_1$ which is not the case.

Hence $\widetilde{x}_j > 0$ for at least one $j \in N_1 \backslash \{j_1\}$. It is shown now, that $\widetilde{x}_j > 0$ for at least one $j \in N_1 \backslash \{j_1\}$. Therefore, $\widetilde{X}$ lies in the closed half space $\displaystyle\sum_{j \in N_1 \backslash \{j_1\}} x_j \geq 1$.

3. THE ALGORITHM

The purpose of the algorithm is to determine all efficient solutions of the problem (P). Starting with an initial efficient solution derived from solving $(P1)$,

Dantzig cuts and Gomory cuts (if necessary) are used in sequence after having scanned the edges E_k. The output of the algorithm gives all efficient solutions existing in the bounded set F. without missing any one. This will be justified later.

Step 1. *Solve the problem* $(P1)$. *Instead of* $(P1)$ *one can constructs one of the problems* $(Pi \; ; \; i = 2, 3, ..., p)$ *and solve it.*

- If the optimal solution is unique, then record the first non-dominated p-tuples $\left(Z_1^1, Z_2^1, ..., Z_p^1\right)$ and put it in the set of non-dominated solutions **Opt**$_0$. Go to Step 2.
- If the optimal solution of the problem $(P1)$ is not unique.

Find all alternate solutions to X_1^*; for each solution evaluate the criterions and eliminate all dominated p-tuples to obtain **Opt**$_0$. Consider the p-tuple $(Z_1, Z_2, ...Z_p)$ as the first non-dominated p-tuple such that Z_2 is the best among all second criterions in **Opt**$_0$. If there exists a tie, the p-tuple considered will have the best value of the third criterion with respect to all p-tuples in **Opt**$_0$ and the process continue whenever a tie appears. Rename the first efficient p-tuple found as $\left(Z_1^1, Z_2^1, ...Z_p^1\right)$. Then, go to Step 2.

Step 2. Let $k = 1$.

2.1. Construct the set Γ_k.

2.1.1. If $\Gamma_k = \emptyset$ then, choose any $j_k \in N_k$ and go to (2.2).

2.1.2. Else, choose any $j_k \in \Gamma_k$. Go to (2.2).

2.2. Calculate $\theta_{kj_k} = \underset{i \in I_k}{\text{Min}} \left\{ \frac{x_{ki}}{y_{k,ij_k}} \; ; \; y_{k,ij_k} > 0 \right\}$.

Let $\theta_{kj_k}^0$ be the integer part of θ_{kj_k}.

  - If $\theta_{kj_k}^0 < 1$ then ignore it and go to (2.3).

  - If not, for $\theta = 1, 2, ..., \theta_{kj_k}^0$; calculate all integer feasible solutions lying

    on the edge E_{j_k} using $\begin{cases} x_i = x_{ki} - \theta \times y_{k,ij_k} \\ x_{j_k} = \theta \\ x_\alpha = 0 \quad \text{for all } \alpha \in N_k \backslash \{j_k\} . \end{cases}$

    Evaluate all the criterions at each of these solutions and add only non-dominated criterions in the set **Opt**$_{k-1}$ to obtain the new set **Opt**$_k$. Go to (2.3).

2.3. Truncate the solution by $\underset{j \in N_k \backslash \{j_k\}}{\sum} x_j \geq 1$.

  Apply the dual simplex method to obtain positive feasible solution X_{k+1}. Evaluate all the criterions at this solution and add the p-tuple corresponding into **Opt**$_k$ to obtain the new set **Opt**$_{k+1}$ if it is not dominated, else ignore it. Go to Step. 3.

Step 3. *If* X_{k+1} *obtained in* (2.3) *is integer positive infeasible solution (lying outside F) then* **stop**. *Otherwise put* $k = k + 1$ *and return to* (2.1).

Proposition 1. The algorithm above converges in a finite number of steps.
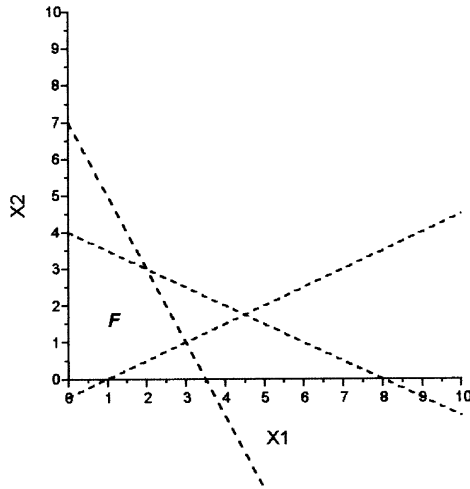
FIGURE 1.

Proof. The feasible region F is being truncated at each step by respected application of the cuts $\sum\limits_{j \in N_k \backslash \{j_k\}} x_j \geq 1$ and Gomory cut is used whenever needed.
The entire region F is scanned such that, an edge or a point once deleted can not reappear again. Therefore, the algorithm converges in a finite number of steps.

4. NUMERICAL ILLUSTRATION

$$\text{Max } Z_1 = x_1 - 3x_2$$
$$\text{Max } Z_2 = x_1 + 3x_2$$
$$\text{Subject to}$$

Consider the following example

$$x_1 + 2x_2 \leq 8$$
$$2x_1 + x_2 \leq 7$$
$$x_1 - 2x_2 \leq 1$$
$$x_1, x_2 \geq 0 \text{ and integers}$$

The region F is shown in the figure above.
First $(P1)$ is being solved.

TABLE 1.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5
x_3	7	0	4	1	0	-1
x_4	5	0	5	0	1	-2
x_1	1	1	-2	0	0	1
$Z_j^{C^1} - C_j^1$	1	0	1	0	0	1
$Z_j^{C^2} - C_j^2$	1	0	-5	0	0	1

The solution is unique and it is given in the table above: $X_1 = X_1^* = (1,0)$ and the first non-dominated couple $\left(Z_1^1, Z_2^1\right) = (1,1)$.

$$\mathbf{Opt}_0 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

$k = 1$,

We have: $I_1 = \{1, 3, 4\}$, $N_1 = \{2, 5\}$.

$$\Gamma_1 = \left\{ \begin{array}{c} j \in N_1 \,/\, z_{1j} - c_j^1 > 0 \text{ and } z_{ij} - c_j^i > 0 \text{ for at} \\ \text{least one } i \in \{2\} \end{array} \right\} = \{5\}.$$

$j_1 = 5$; $\theta_{15} = \text{Min}\left\{\frac{1}{1}\right\} = 1$; $\theta_{15}^0 = 1$.

The solution on the edge E_5 is: $\begin{cases} x_3^1 = 7 - (-1) = 8 \\ x_4^1 = 5 - (-2) = 7 \\ x_1^1 = 1 - (1) = 0 \\ x_5^1 = 1 \\ x_2^1 = 0. \end{cases}$

Thus, $Z^{1,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is dominated.

Truncate by $x_2 \geq 1$ (see Fig. 2 bellow) and $-x_2 + x_6 = -1$ is to be added at the bottom of the previous table.

TABLE 2.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6
x_3	7	0	4	1	0	-1	0
x_4	5	0	5	0	1	-2	0
x_1	1	1	-2	0	0	1	0
x_6	-1	0	-1	0	0	0	1
$Z_j^{C^1} - C_j^1$	1	0	1	0	0	1	0
$Z_j^{C^2} - C_j^2$	1	0	-5	0	0	1	0
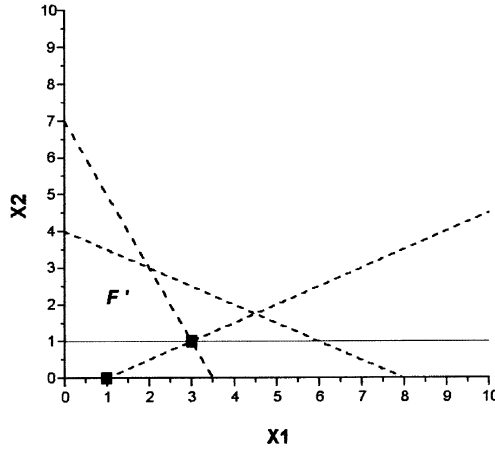
FIGURE 2.

Applying the dual simplex method we obtain:

TABLE 3.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6
x_3	3	0	0	1	0	-1	4
x_4	0	0	0	0	1	-2	5
x_1	3	1	0	0	0	1	-2
x_2	1	0	1	0	0	0	-1
$Z_j^{C^1} - C_j^1$	0	0	0	0	0	1	1
$Z_j^{C^2} - C_j^2$	6	0	0	0	0	1	-5

The solution is $X_2 = (3,1)$ yields to $\left(Z_1^2, Z_2^2\right) = (0,6)$ which is non-dominated with regards to the previous.

Thus $\mathbf{Opt}_2 = \mathbf{Opt}_1 \cup \left\{ \begin{pmatrix} 0 \\ 6 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix} \right\}$.

$k = 2$;

$I_2 = \{1, 2, 3, 4\}$, $N_2 = \{5, 6\}$.

$\Gamma_2 = \{5\}$ and $\theta_{25}^0 = 3$.

Calculate the solutions on the edge E_5

$\theta_2 = 1$; $\{x_3^1 = 4, x_4^1 = 2, x_1^1 = 2, x_2^1 = 1, x_5^1 = 1, x_6^1 = 0\}$

$\Rightarrow Z^{2,1} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ which is dominated.

$\theta_2 = 2$; $\{x_3^2 = 5, x_4^2 = 4, x_1^2 = 1, x_2^2 = 1, x_5^2 = 1, x_6^2 = 0\}$

$\Rightarrow Z^{2,2} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ which is dominated.
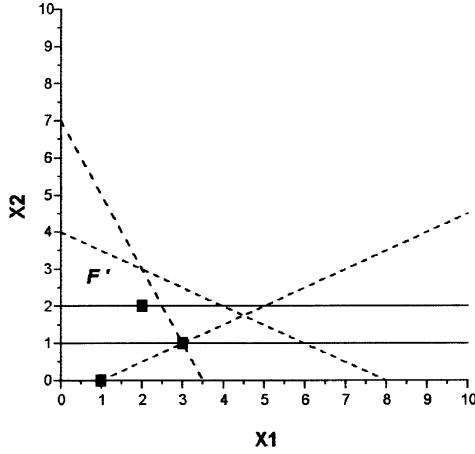
FIGURE 3.

$\theta_2 = 3$; $\{x_3^3 = 6\,,\, x_4^3 = 6\,,\, x_1^3 = 0\,,\, x_2^3 = 1\,,\, x_5^3 = 1\,,\, x_6^3 = 0\}$

$\Rightarrow Z^{2,3} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ which is dominated.

The edge E_5 is dominated for all $j \in \Gamma_2$. Thus, according to Gupta and Malhotra the procedure is terminated. But, the remaining efficient solutions are not yet detected.

Within the algorithm proposed, the cut $\sum\limits_{j \in N_2 \setminus \{5\}} x_j \geq 1$ is being performed.

Then, $x_6 \geq 1$ (equivalently $x_2 \geq 2$ – see Fig. 3) is added to the previous tableau and Gomory cuts and dual simplex method are used to obtain the following table:

TABLE 4.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_3	2	0	0	1	0	0	0	2	$-1/2$
x_5	3	0	0	0	0	1	0	-2	$-1/2$
x_1	2	1	0	0	0	0	0	0	$1/2$
x_2	2	0	1	0	0	0	0	-1	0
x_6	1	0	0	0	0	0	1	-1	0
x_4	1	0	0	0	1	0	0	1	-1
$Z_j^{C^1} - C_j^1$	-4	0	0	0	0	0	0	3	$1/2$
$Z_j^{C^2} - C_j^2$	8	0	0	0	0	0	0	-3	$1/2$

The solution $X_3 = (2,2)$ yields non dominated couple $(Z_1^3, Z_2^3) = (-4, 8)$.

$\mathbf{Opt}_3 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix}, \begin{pmatrix} -4 \\ 8 \end{pmatrix} \right\}.$

$k = 3;$

$I_3 = \{1, 2, 3, 4, 5, 6\}; \; N_3 = \{7, 8\}.$

$\Gamma_3 = \{8\} \Rightarrow \theta_{38}^0 = 4.$

$\theta_3 = 1;$ the solution is infeasible.

$\theta_3 = 2; \left\{ \begin{array}{c} x_3^3 = 3, \; x_5^3 = 4, \; x_1^3 = 1, \; x_2^3 = 2, \; x_6^3 = 1, \; x_4^3 = 3, \\ x_8^3 = 2, \; x_7^3 = 0 \end{array} \right\}$

yielding to $Z^{3,1} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ dominated.

$\theta_3 = 3;$ the solution is infeasible.

$\theta_3 = 4; \left\{ \begin{array}{c} x_3^3 = 4, \; x_5^3 = 5, \; x_1^3 = 0, \; x_2^3 = 2, \; x_6^3 = 1, \; x_4^3 = 5, \\ x_8^3 = 4, \; x_7^3 = 0 \end{array} \right\}$

yielding to $Z^{3,2} = \begin{pmatrix} -6 \\ 6 \end{pmatrix}$ which is dominated.

Add the cut $x_7 \geq 1$ ($x_2 \geq 3$ as shown in Fig. 4) and apply the dual simplex to obtain the following results.

TABLE 5.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_3	0	0	0	1	0	0	0	0	1/2	2
x_5	5	0	0	0	0	1	0	0	$-1/2$	-2
x_1	2	1	0	0	0	0	0	0	1/2	0
x_2	3	0	1	0	0	0	0	0	0	-1
x_6	2	0	0	0	0	0	1	0	0	-1
x_4	0	0	0	0	1	0	0	0	-1	1
x_7	1	0	0	0	0	0	0	1	0	-1
$Z_j^{C^1} - C_j^1$	-7	0	0	0	0	0	0	0	1/2	3
$Z_j^{C^2} - C_j^2$	11	0	0	0	0	0	0	0	1/2	-3

The solution $X_4 = (2, 3)$ yields to $(Z_1^4, Z_2^4) = (-7, 11)$ which is non-dominated.

$\mathbf{Opt}_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix}, \begin{pmatrix} -4 \\ 8 \end{pmatrix}, \begin{pmatrix} -7 \\ 11 \end{pmatrix} \right\}.$

$k = 4;$

$I_4 = \{1, 2, 3, 4, 5, 6, 7\}; \; N_4 = \{8, 9\}.$

$\Gamma_4 = \{8\} \Rightarrow \theta_{48}^0 = 4.$

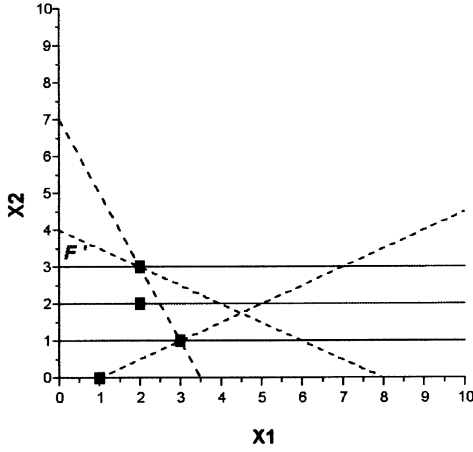$\theta_4 = 1;$ the solution is infeasible.

FIGURE 4.

$\theta_4 = 2$; $\left\{ \begin{array}{c} x_3^4 = 1\,,\ x_5^4 = 6,\ x_1^4 = 1\,,\ x_2^4 = 3\,,\ x_6^4 = 2\,,\ x_4^4 = 2, \\ x_7^4 = 1, x_8^4 = 2, x_9^4 = 0 \end{array} \right\}$

yielding to $Z^{4,1} = \begin{pmatrix} -8 \\ 11 \end{pmatrix}$ dominated.

$\theta_4 = 3$; the solution is infeasible.

$\theta_4 = 4$; $\left\{ \begin{array}{c} x_3^4 = 2\,,\ x_5^4 = 7,\ x_1^4 = 0\,,\ x_2^4 = 3\,,\ x_6^4 = 2\,,\ x_4^4 = 4, \\ x_7^4 = 1, x_8^4 = 4, x_9^4 = 0 \end{array} \right\}$

yielding to $Z^{4,2} = \begin{pmatrix} -9 \\ 9 \end{pmatrix}$ which is dominated.

Add the cut $x_9 \geq 1$ ($x_2 \geq 4$ as shown in Fig. 5) to the Table 5 and apply the dual simplex method.

TABLE 6.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_8	4	0	0	-2	0	0	0	0	1	0	0
x_5	9	0	0	-1	0	1	0	0	0	0	-2
x_1	0	1	0	1	0	0	0	0	0	0	0
x_2	4	0	1	0	0	0	0	0	0	0	-1
x_6	3	0	0	0	0	0	1	0	0	0	-1
x_4	1	0	0	-2	1	0	0	0	0	0	-1
x_7	2	0	0	0	0	0	0	1	0	0	-1
x_9	1	0	0	0	0	0	0	0	0	1	-1
$Z_j^{C^1} - C_j^1$	-12	0	0	1	0	0	0	0	0	0	3
$Z_j^{C^2} - C_j^2$	12	0	0	1	0	0	0	0	0	0	-3

FIGURE 5.

The solution $X_5 = (0, 4)$ yields to $Z_5 = \begin{pmatrix} -12 \\ 12 \end{pmatrix}$ which is non-dominated.

$\mathbf{Opt}_5 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix}, \begin{pmatrix} -4 \\ 8 \end{pmatrix}, \begin{pmatrix} -7 \\ 11 \end{pmatrix}, \begin{pmatrix} -12 \\ 12 \end{pmatrix} \right\}$.

$\Gamma_5 = \{3\} \Rightarrow \theta_{53}^0 = 0 < 1$. Ignore it.

Cut with $x_{10} \geq 1$ (equivalently $x_2 \geq 5$) which lies outside the region F. If we apply one iteration of the dual simplex technique we achieve to an infeasible solution (see the table bellow) $X_6 = (0, 5) \notin F$.

The algorithm terminates with the set of all non-dominated $\mathbf{Opt}_4$ and the set of the efficient solutions (square dots in Fig. 5) is given by: $Eff = \{(1, 0), (3, 1),$ $(2, 2), \quad (2, 3), (0, 4)\}$.

TABLE 7.

Basis	value of basic variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_8	4	0	0	-2	0	0	0	0	1	0	0	0
x_5	11	0	0	-1	0	1	0	0	0	0	0	-2
x_1	0	1	0	1	0	0	0	0	0	0	0	0
x_2	5	0	1	0	0	0	0	0	0	0	0	-1
x_6	4	0	0	0	0	0	1	0	0	0	0	-1
x_4	2	0	0	-2	1	0	0	0	0	0	0	-1
x_7	3	0	0	0	0	0	0	1	0	0	0	-1
x_9	2	0	0	0	0	0	0	0	0	1	0	-1
x_{10}	1	0	0	0	0	0	0	0	0	0	1	-1
$Z_j^{C^1} - C_j^1$	-15	0	0	1	0	0	0	0	0	0	0	3
$Z_j^{C^2} - C_j^2$	15	0	0	1	0	0	0	0	0	0	0	-3

5. Conclusion and comments

In this paper, we have proposed an algorithm that can be regarded as a modified and corrected version of Gupta and Malhotra's procedure. The test that they proposed in the terminal step ($\Gamma = \emptyset$ and $z_{1j} - c_j^1 > 0$ or $\Gamma_n \neq \emptyset$ and for all $j \in \Gamma_n$ yields dominated edge) can be held and not yet all the efficient solutions are detected. As an alternative, the Γ_n set is modified and the cuts of type $\sum_{j \in N_k \setminus \{j_k\}} X_j \geq 1$ are carried out until obtaining an infeasible solution outside the feasible region F.

In addition, solving the problem (P) is not an obvious extension from a continuous case (without integer constrained) and the fundamental principal the so called Geoffrion's theorem is no longer valid in presence of discrete variables, which makes this class of problems very difficult as was stated by Teghem Jr. [17].

Thus, using metods introduced for continous variables, such Ecker's and Kouada can be missleading when applied to problems in the presence of discrete variables.

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