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A SUMMARY OF BLOCK REPLACEMENT POLICIES (*)

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Abstract. — *It is of practical importance to consider a block replacement policy in which a unit is replaced at times kT ($k=1, 2, \ldots$) and at failure. In this paper we propose three block replacement policies in which a unit is replaced at kT and a failed unit* (i) *is replaced,* (ii) *remains failure, and* (iii) *undergoes a minimal repair, respectively. These would give useful results to determine which kinds of policy should be adopted. Models with discounting are further considered. Several useful extended and modified models are given as remarks.*

Résumé. — *Il est important en pratique de considérer une politique de remplacement par blocs où une unité est remplacée aux instants kT ($k=1, 2, \ldots$) et en cas de panne. Dans cet article, nous proposons trois politiques de remplacement par blocs dans lesquelles une unité est remplacée en kT et une unité en panne :* (1) *est remplacée;* (2) *demeure en panne;* (3) *subit une réparation minime, respectivement. Ces considérations donneraient des résultats utiles pour déterminer quelles sortes de politiques devraient être adoptées. Des modèles avec amortissement sont ensuite considérés. Plusieurs extensions et modifications des modèles sont données en remarques.*

1. INTRODUCTION

Consider block replacement policies in which all units are replaced periodically at times kT ($k=1, 2, \ldots$) independent of the ages of units in use. These policies are commonly used with complex electronic systems such as digital computers, and electrical parts such as light bulbs and vacum tubes.

Barlow and Proschan [1] compared block replacement with age replacement, and studied various replacement policies. Some of their results are introduced in this paper. After that, Marathe and Nair [8] and Jain and Nair [6] defined the n-stage block replacement policy and compared it with other replacement policies. Bhat [2] suggested that a failed unit could be replaced by one of used units under some conditions, which have been replaced earlier at times kT. Schweitzer [12] compared policies of block replacement and replacement only of individual failures for hyperexponentially and uniformly distributed failure times. Tilquin and Cléroux [13] introduced the adjustment costs, in addition to replacement costs, which are increasing with the age of a unit. Holland and McLean [5] gave a

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practical procedure for a replacement policy (Policy III in this paper) to parts of equipments, as examples of large motors and small electrical parts.

In this paper, we summarize the results of the following block replacement policies:

(i) A failed unit is replaced instantaneously at failure;

(ii) A failed unit remains failure until the next planned replacement;

(iii) A failed unit undergoes a minimal repair.

Further, we consider the above replacements with continuous discounting and gave the total expected costs for each policy. Examples are presented when the failure time is a gamma density with a shape parameter 2.

As remarks, we consider extended and modified models of block replacement which could be more realistic than the above models. For instance, a model combined (i) and (iii) is introduced.

2. BLOCK REPLACEMENT POLICIES

A unit is replaced at times kT ($k=1, 2, \ldots$) independent of the age of a unit. Assume that each unit has a failure time distribution $F(t)$ with finite mean $1/\lambda$. Then, we consider the following three block replacement policies which could be useful in practical fields.

(i) *Policy I*

A failed unit is discovered instantaneously and replaced by a new unit. Then, the expected cost rate is

$$C_1(T)=\frac{c_1 M(T)+c_2}{T}, \tag{1}$$

where $M(t)$ the expected number of failed units during the interval $(0, t]$; c_1 = cost of replacement for a failed unit; c_2 = cost of planned replacement.

We seek an optimum planned replacement time T_1^* ($0<T_1^*\leqq\infty$) which minimizes the expected cost $C_1(T)$. We assume that $M(t)$ is differential and define $m(t)\equiv dM(t)/dt$. Then, differentiating $C_1(T)$ with respect to T and setting it equal to zero, we have

$$Tm(T)-\int_0^T m(t)\,dt=\frac{c_2}{c_1}. \tag{2}$$

This equation is a necessary condition that there exists a finite T_1^*, and in this case, the resulting value of the expected cost rate $C_1(T_1^*)$ is

$$C_1(T_1^*)=c_1 m(T_1^*). \tag{3}$$

Let σ^2 be variance of the failure time distribution $F(t)$. Then, from Cox [3], p. 119, there exists a large T such that $C_1(T) < C_1(\infty)(=\lambda c_1)$ if

$$\frac{c_2}{c_1} < \frac{1}{2}(1-\lambda^2\sigma^2),$$

and $\lambda^2\sigma^2$ is not small.

Further, Cox [3] considered a modification of Policy I in which a failure occurs just before one of the planned replacement times, it may postpone replacing a failed unit until the next planned replacement. That is, if a failure occurs in an interval $(kT-T_d, kT)$, the replacement is not made until the time kT, and the unit will be down for the time interval. Suppose that the cost suffered for unit failure in this interval is proportional to the down time, i. e., let $a+bt$ be the cost of the time t elapsed between failure and its detection. Then, the expected cost rate is

$$\hat{C}_1(T) = \frac{c_1 M(T-T_d) + c_2 + \int_{T-T_d}^{T} [a+b(T-u)]\,\overline{F}(T-u)\,dM(u)}{T} \tag{4}$$

where $\overline{F}(t) \equiv 1-F(t)$.

(ii) *Policy II*

In the first policy, we have assumed that a failed unit is detected instantaneously and its replacement is also made instantaneously. In this policy, we assume that failure is discovered only at times kT $(k=1, 2, \ldots)$.

The unit is always replaced at times kT but is not replaced at failure, and hence, the unit remains failure for the time interval from the occurrence of failure to its detection. Then, the expected cost rate is

$$C_2(T) = \frac{c_1 \int_0^T F(t)\,dt + c_2}{T}, \tag{5}$$

where c_1 = cost of the time elapsed between failure and its detection per unit of time; c_2 = cost of planned replacement.

Differentiating $C_2(T)$ with respect to T and setting it equal to zero, we have

$$TF(T) - \int_0^T F(t)\,dt = \frac{c_2}{c_1}. \tag{6}$$

Thus, if $1/\lambda > c_2/c_1$ then there exists an optimum time T_2^* uniquely which satisfies (6), and in this case, the expected cost rate is

$$C_2(T_2^*) = c_1 F(T_2^*). \tag{7}$$

Further, we introduce not only the loss costs c_1 and c_2 but also a net earning of the working unit. Then, the expected earning rate is

$$\hat{C}_2(T)=\frac{e_0\int_0^T \bar{F}(t)\,dt-c_1\int_0^T F(t)\,dt-c_2}{T}, \tag{8}$$

where e_0 = net earning cost per unit of time made by the production of working unit, where $e_0 > c_1$.

The corresponding equations of (6) and (7) for this model are rewritten as, respectively,

$$TF(T)-\int_0^T F(t)\,dt=\frac{c_2}{e_0-c_1}, \tag{9}$$

$$\hat{C}_2(T_2^*)=(e_0-c_1)F(T_2^*). \tag{10}$$

(iii) *Policy III*

It is assumed that we make a minimal repair when the unit fails and the failure rate is not disturbed by each repair. Then, the expected cost rate is, from [1]:

$$C_3(T)=\frac{c_1\int_0^T r(t)\,dt+c_2}{T}, \tag{11}$$

where $r(t)$ = the failure rate of the failure time distribution $F(t)$, i.e., $r(t)\equiv f(t)/\bar{F}(t)$, where f is a density of F; c_1 = cost of minimal repair; c_2 = cost of planned replacement.

Differentiating $C_3(T)$ with respect to T and setting it equal to zero, we have

$$Tr(T)-\int_0^T r(t)\,dt=\frac{c_2}{c_1}. \tag{12}$$

Suppose that $r(t)$ is monotonely increasing. Then, if $\int_0^\infty t\,dr(t) > c_2/c_1$ then there exists a T_3^* uniquely which satisfies (12), and the expected cost rate is

$$C_3(T_3^*)=c_1\,r(T_3^*). \tag{13}$$

It is further shown that $C_3(T)\geqq C_1(T)$ if the respective costs for each policy are the same.

Next, consider the same policy for a used unit. A unit is replaced at times kT by a used unit with age of x. The failure rate of the used unit with age of x after time duration t becomes $r(t+x)$. Then, the expected cost rate is

$$\hat{C}_3(T;\, x)=\frac{c_1\displaystyle\int_0^T r(t+x)\,dt+c_2}{T},\tag{14}$$

and the corresponding equations for (12) and (13) become

$$Tr(T+x)-\int_0^T r(t+x)\,dt=\frac{c_2}{c_1},\tag{15}$$

$$\hat{C}_3(T_3^*;\, x)=c_1\, r(T_3^*+x).\tag{16}$$

Actually, the cost c_2 of replacement for a used unit could be cheaper than that for a new unit, however, the number of failures could be greater than that for a new unit. We compare the expected cost rates for a used unit and a new unit and can determine which kind of units should be used.

3. EXAMPLES

Suppose that the failure time distribution is a gamma distribution with a shape parameter 2, i. e.:

$$f(t)=a^2\, te^{-at},$$

$$r(t)=\frac{a^2\, t}{1+at},$$

$$m(t)=\frac{a}{2}(1-e^{-2at}).$$

Then, we derive the following optimum planned replacement times T_i^* which minimize the expected cost rates $C_i(T)$ for $i=1, 2, 3$.

For Policy I, we have, from (2):

$$\frac{1}{4}(1-e^{-2aT}-2\,aTe^{-2aT})=\frac{c_2}{c_1}.\tag{17}$$

Thus, if $c_1/c_2 \geqq 1/4$, then we should make no planned replacement, i. e., a unit is replaced only at failure. If $c_1/c_2<1/4$, then there exists a T_1^* uniquely which satisfies (17) and the resulting expected cost rate is

$$C_1(T_1^*)=\frac{c_1\, a}{2}(1-e^{-2aT_1^*}).$$

For Policy II, if $\int_0^\infty tf(t)\,dt = 2/a \leqq c_2/c_1$, then we should make no planned replacement. If $2/a > c_2/c_1$, then there exists a T_2^* uniquely which satisfies

$$\frac{1}{a}[2 - e^{-aT}(2 + 2aT + a^2T^2)] = \frac{c_2}{c_1},$$

and in this case, the expected cost rate is

$$C_2(T_2^*) = c_1[1 - (1 + aT_2^*)e^{-aT_2^*}].$$

For Policy III, there exists T_3^* uniquely which satisfies

$$\log(1 + aT) - \frac{aT}{1 + aT} = \frac{c_2}{c_1},$$

and the expected cost rate is

$$C_3(T_3^*) = c_1 \frac{a^2 T_3^*}{1 + aT_3^*}.$$

4. BLOCK REPLACEMENT WITH DISCOUNTING

When we adopt the total expected cost as an appropriate objective function for an infinite time span, we should evaluate values of all future costs by using a discount rate. We apply the continuous discounting to the costs at the times when these costs occur actually.

Let $C_i(\alpha; T)$ be the total expected cost rate for Policy i when the unit is replaced at times kT $(k = 1, 2, \ldots)$. Then, the expected cost between replacements is the same, except for a discount rate, and hence, the total expected cost is equal to the sum of the discounted costs incurred between individual replacements. Let $\tilde{C}_i(\alpha; T)$ be the expected cost of one replacement cycle from 0 to T. Then, we easily have the following renewal-type equation:

$$C_i(\alpha; T) = \tilde{C}_i(\alpha; T) + e^{-\alpha T} C_i(\alpha; T),$$

i.e.:

$$C_i(\alpha; T) = \tilde{C}_i(\alpha; T)/(1 - e^{-\alpha T}). \tag{18}$$

Thus, we have the following results for each policy:

For Policy I:

$$C_1(\alpha; T) = \frac{c_1 \int_0^T e^{-\alpha t}\,dM(t) + c_2 e^{-\alpha T}}{1 - e^{-\alpha T}}, \tag{19}$$

$$\frac{1-e^{-\alpha T}}{\alpha} m(T)-\int_0^T e^{-\alpha t} m(t)\,dt=\frac{c_2}{c_1}, \tag{20}$$

$$C_1(\alpha;\ T_1^*)=\frac{c_1}{\alpha} m(T_1^*)-c_2. \tag{21}$$

For Policy II:

$$C_2(\alpha;\ T)=\frac{c_1\int_0^T e^{-\alpha t} F(t)\,dt+c_2 e^{-\alpha T}}{1-e^{-\alpha T}}, \tag{22}$$

$$\frac{1-e^{-\alpha T}}{\alpha} F(T)-\int_0^T e^{-\alpha t} F(t)\,dt=\frac{c_2}{c_1}, \tag{23}$$

$$C_2(\alpha;\ T_2^*)=\frac{c_1}{\alpha} F(T_2^*)-c_2. \tag{24}$$

For Policy III:

$$C_3(\alpha;\ T)=\frac{c_1\int_0^T e^{-\alpha t} r(t)\,dt+c_2 e^{-\alpha T}}{1-e^{-\alpha T}}, \tag{25}$$

$$\frac{1-e^{-\alpha T}}{\alpha} r(T)-\int_0^T e^{-\alpha t} r(t)\,dt=\frac{c_2}{c_1}, \tag{26}$$

$$C_3(\alpha;\ T_3^*)=\frac{c_1}{\alpha} r(T_3^*)-c_2. \tag{27}$$

Note that $\lim_{\alpha \to 0} \alpha C_i(\alpha;\ T)=C_i(T)$ which is the expected cost rate with no discounting.

5. REMARKS

(i) Morimura [9] introduced the following modification of block replacement where a unit is replaced at the *k-th* failure and the $(k-1)$-th previous failures are corrected with minimal repair. He obtained the expected cost rate

$$C(k)=\frac{(k-1)c_1+c_2}{\sum_{j=0}^{k-1}\int_0^\infty ([R(t)]^j/j\,!)\,e^{-R(t)}\,dt} \qquad (k=1,\ 2,\ \ldots), \tag{28}$$

where $R(t)=\int_0^t r(u)\,du$. Further, if $c_2>c_1$ and the failure rate is monotonely increasing, there exists k^* which is uniquely given by obtaining a minimum k such that

$$\frac{\sum_{j=0}^{k-1}\left[\int_0^\infty ([R(t)]^j/j!)\,e^{-R(t)}\,dt-\int_0^\infty ([R(t)]^k/k!)\,e^{-R(t)}\,dt\right]}{\int_0^\infty ([R(t)]^k/k!)\,e^{-R(t)}\,dt}>\frac{c_2-c_1}{c_1}. \tag{29}$$

(ii) Consider a system with n identical units which operate independently each other. Assume that all together are replaced at times $kT\,(k=1, 2, \ldots)$ and each unit is replaced immediately upon failure. Then, the expected cost rate is

$$C_1(T;\,n)=\frac{c_1\,nM(T)+c_2}{T}, \tag{30}$$

where c_1 = cost of replacement for one failed unit; c_2 = cost of planned replacement for all units.

If we make a minimal repair for each failed unit, then the expected cost rate is

$$C_3(T;\,n)=\frac{c_1\,n\int_0^T r(t)\,dt+c_2}{T}. \tag{31}$$

Finally, assume that any unit is replaced only at times kT and a system failure occurs when all units have failed. Then, the expected cost rate is:

$$C_2(T;\,n)=\frac{c_1\int_0^T F^{(n)}(t)\,dt+c_2}{T}, \tag{32}$$

where c_1 = cost of the time interval for system failure per unit of time; $F^{(n)}(t)$ = the n-th convolution of the failure time distribution $F(t)$ with itself.

(iii) We consider the inspection policy [1], p. 107, which is similar to block replacement policy. A unit is checked at times $kT\,(k=1, 2, \ldots)$ and a failed unit is detected only by checking. The time required for checking is negligible and the failure rate of a unit remains undisturbed by any inspection. Then, the expected cost rate is

$$C_4(t)=\frac{\sum_{k=1}^{\infty}\int_{(k-1)T}^{kT}[c_2\,k+c_1(kT-t)]\,dF(t)}{\sum_{k=1}^{\infty}\int_{(k-1)T}^{kT}kT\,dF(t)}$$

$$=\frac{c_1\left[T-(1/\lambda)\Big/\sum_{k=0}^{\infty}\bar{F}(kT)\right]+c_2}{T}, \tag{33}$$

where c_1 =cost of the time elapsed between failure and its detection by checking per unit of time; c_2 =cost of checking.

Differentiating (33) with respect to T and putting it equal to zero, we have

$$(c_1/\lambda)\, T \frac{\sum_{k=0}^{\infty} kf(kT)}{\sum_{k=0}^{\infty} \overline{F}(kT)} + c_2 \sum_{k=0}^{\infty} \overline{F}(kT) = -\frac{c_1}{\lambda}. \tag{34}$$

(iv) In the Policy I, it is sometimes useless to replace a failed unit by a new one just before the planned replacement time, but we can not leave as it is until the planned replacement is made. To overcome this, we can combine Policy I and Policy III together. That is, a failed unit is replaced by a new one during $(0,\ T_0]$ and undergoes a minimal repair during $(T_0,\ T)$ for $0 \leqq T_0 \leqq T$. Then, the total expected cost of minimal repairs during $(T_0,\ T)$ is

$$c_1^* \left\{ \int_0^{T_0} \left[\int_{T_0}^{T} r(t-u)\, dt \right] d_u Pr[\delta(T_0) \leqq T_0 - u] \right\},$$

where $\delta(t)$=an age of an operating unit at time t; c_1^*=cost of minimal repair.

Thus, from [1], p. 58, we have

$$C_5(T;\ T_0) = \frac{\left(c_1 M(T_0) + c_2 + c_1^* \left\{ \overline{F}(T_0) \int_{T_0}^{T} r(t)\, dt + \int_0^{T_0} \left[\int_{T_0}^{T} r(t-u)\, dt \right] \overline{F}(T_0 - u)\, dM(u) \right\} \right)}{T}. \tag{35}$$

Evidently

$$C_1(T) = C_5(T;\ T),$$

and

$$C_3(T) = C_5(T;\ 0).$$

It is further shown that all expected cost rates $C_1(T)$, $C_3(T)$, and $C_5(T;\ T_0)$ are equal when failure times of units are exponential.

6. CONCLUSION

We have summarized the three block replacement policies in which a unit is replaced at times kT $(k=1,\ 2,\ \ldots)$. Further, we have extended and modified block replacement models as remarks. These could be applicable to practical fields.

In general, the results are summarized as follows: The expected cost rate is

$$C_i(T)=\frac{c_1\int_0^T \varphi(t)\,dt+c_2}{T}, \tag{36}$$

where $\varphi(t)$ is $m(t)$, $F(t)$, and $r(t)$ for Policy i, respectively. Differentiating $C_i(T)$ with respect to T and setting it equal to zero:

$$T\varphi(T)-\int_0^T \varphi(t)\,dt=\frac{c_2}{c_1}. \tag{37}$$

If there exists a T_i^* which satisfies (37), then the expected cost rate is

$$C_i(T_i^*)=c_1\,\varphi(T_i^*). \tag{38}$$

For a discount case

$$C_i(\alpha;\ T)=\frac{c_1\int_0^T e^{-\alpha t}\varphi(t)\,dt+c_2\,e^{-\alpha T}}{1-e^{-\alpha T}}, \tag{39}$$

$$\frac{1-e^{-\alpha T}}{\alpha}\,\varphi(T)-\int_0^T e^{-\alpha t}\varphi(t)\,dt=\frac{c_2}{c_1}, \tag{40}$$

$$C_i(\alpha;\ T_i^*)=\frac{c_1}{\alpha}\,\varphi(T_i^*)-c_2. \tag{41}$$

It has assumed that the time required to make a replacement or to make a repair is negligible. It has further assumed that at any time there is an unlimitted supply of units available for replacement. The results obtained here could be theoretically extended and be modified in such cases.

We have discussed only the block replacement policies. Other replacement policies are referred to [1, 7, 10, 11].

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