# REGULAR FLOWS OF VISCOELASTIC FLUIDS AND THE INCOMPRESSIBLE LIMIT

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***Résumé.*** — We consider compressible viscoelastic fluids obeying the Oldroyd constitutive law. Local existence and uniqueness of flows are proven by a classical fixed point argument. Some global properties of the solutions are also derived. In particular, we obtain some a priori estimates which are uniform in the Mach number and prove global existence of weakly compressible fluid flows. We show that weakly compressible flows with *well-prepared* initial data converge to incompressible ones when the Mach number goes to zero.

## 1. Introduction

The mathematical analysis of flows at vanishing Mach number begins with the works of Klainerman and Majda [**6**]. The incompressible limit of the isentropic Navier-Stokes equations has been justified for weak solutions by Desjardins and Grenier [**4**], Lions and Masmoudi [**8**], Bresch, Desjardins and Gérard-Varet [**2**]. For viscous gas, global well-posedness in critical spaces has been established by Danchin [**3**]. In the case of well-prepared initial data and regular solutions, Bessaih [**1**] has studied the compressible Navier-Stokes limit. Lei, in [**7**], has proven the local and global existence of classical solutions for an Olydroyd-B system in a torus when the initial data are sufficiently small.

In this presentation[1], we show the global existence of solutions for weakly compressible viscoelastic fluid flows in the case of the Oldroyd-B model in a regular bounded domain in $\mathbb{R}^3$. We also study the convergence of the weakly compressible model to the incompressible one, for *well-prepared* initial data, when the Mach number tends to zero.

## 2. The Modeling

**2.1. Unsteady Flows of Compressible Viscoelastic Fluids. —** Consider unsteady flows of viscoelastic fluids in a bounded domain $\Omega^*$ of $\mathbb{R}^3$ with a regular boundary $\Gamma^*$. The system, obtained from the laws of conservation of momentum, and of mass, and from the constitutive equation of the fluid, reads as follows (see for instance [**9**]) :

[1]based on a study completed in collaboration with C. GUILLOPÉ and R. TALHOUK [**5**].

in $Q^*_{T^*} = (0, T^*) \times \Omega^*$,

(1)

$$\begin{cases} \rho^* \left( \dfrac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* \right) &= \rho^* \mathbf{f}^* + \eta_s(\Delta^* \mathbf{u}^* + \nabla^* \mathrm{div}^* \mathbf{u}^*) - \nabla^* p^* + \mathbf{div}^* \tau^*, \\ \dfrac{\partial \rho^*}{\partial t^*} + \mathrm{div}^*(\rho^* \mathbf{u}^*) &= 0, \\ \tau^* + \lambda \dfrac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} &= 2\eta_e \mathbf{D}^*[\mathbf{u}^*]. \end{cases}$$

The *-variables are the dimensional ones in the domain of the flow $\Omega^*$, and $T^* > 0$ is a dimensional time. The unknowns are the velocities $\mathbf{u}^*$, the density $\rho^*$, the pressure $p^*$, and the symmetric tensor of constraints $\tau^*$. $\eta = \eta_s + \eta_e$ is the total viscosity of the fluid and $\lambda > 0$ is a relaxation time. The objective derivative of the tensor $\tau^*$ is given by $\dfrac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} = \left( \dfrac{\partial}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \right) \tau^* + \tau^* \mathbf{W}^* - \mathbf{W}^* \tau^* - a(\mathbf{D}^* \tau^* + \tau^* \mathbf{D}^*)$, where $\mathbf{W}^* = \dfrac{1}{2}(\nabla^* \mathbf{u}^* - \nabla^{*T} \mathbf{u}^*)$ and $\mathbf{D}^* = \mathbf{D}^*[\mathbf{u}^*] = \dfrac{1}{2}(\nabla^* \mathbf{u}^* + \nabla^{*T} \mathbf{u}^*)$ are, respectively, the rate of rotation and the rate of deformation tensors. $a$ is a real parameter in $[-1, 1]$.

**2.2. Well-Prepared Initial Conditions. —** We first define the Mach number $\varepsilon$ as being the ratio of the typical velocity of the fluid $U_0$ to the speed of sound $\left( \frac{dp^*}{d\rho^*}(\overline{\rho}_0^*) \right)^{1/2}$ in the same fluid at the same state. We divide the density $\rho^* = \rho^{*\varepsilon}$ into two parts : a *constant* one $\overline{\rho}_0^*$, independent of the Mach number $\varepsilon$, and a remainder, which is small for small $\varepsilon'$s, say $\rho^{*\varepsilon} = \overline{\rho}_0^* + \varepsilon^2 \sigma^{*\varepsilon}$. We also suppose that the initial conditions $\rho_0^{*\varepsilon}$, $\mathbf{u}_0^{*\varepsilon}$ and $\tau_0^{*\varepsilon}$ are *well-prepared*, which means that they take a similar form, say $\rho_0^{*\varepsilon} = \overline{\rho}_0^* + \varepsilon^2 \sigma_0^{*\varepsilon}$, $\mathbf{u}_0^{*\varepsilon} = \mathbf{v}_0^* + \mathbf{v}_0^{*\varepsilon}$, with $\mathrm{div}\, \mathbf{v}_0^* = 0$, $\tau_0^{*\varepsilon} = \mathbf{S}_0^* + \mathbf{S}_0^{*\varepsilon}$, where $\mathbf{v}_0^*$ and $\mathbf{S}_0^*$ are, respectively, a vector and a symmetric tensor, both independent of $\varepsilon$. Assuming that $p^* = p^*(\rho^*)$ is regular, say class $\mathcal{C}^3$ at least, we introduce the function $w^*$ defined by $w^*(\sigma^*) = \frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) - \frac{dp^*}{d\rho^*}(\overline{\rho}_0^*)$. System (1) can be written as follows : in $Q^*_{T^*}$,

$$(2) \quad \begin{cases} (\overline{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \dfrac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* \right) + (U_0)^2 \nabla^* \sigma^* = (\overline{\rho}_0^* + \varepsilon^2 \sigma^*)\mathbf{f}^* + \mathbf{div}^* \tau^* \\ \qquad + \eta_s(\Delta^* \mathbf{u}^* + \nabla^* \mathrm{div}^* \mathbf{u}^*) - \varepsilon^2 w^*(\sigma^*) \nabla^* \sigma^*, \\ \varepsilon^2 \dfrac{\partial \sigma^*}{\partial t^*} + \overline{\rho}_0^* \mathrm{div}^* \mathbf{u}^* + \varepsilon^2 \mathrm{div}^*(\sigma^* \mathbf{u}^*) = 0, \\ \tau^* + \lambda \dfrac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} = 2\eta_e \mathbf{D}^*[\mathbf{u}^*]. \end{cases}$$

**2.3. Dimensionless Variables. —** We introduce dimensionless variables and three non-dimensional numbers : a number $\alpha$ similar to the Reynolds number for incompressible flows, the Weissenberg number We, and a retardation number $\omega$ related to the viscosities of the fluid, precisely $\alpha = \frac{\overline{\rho}_0^* U_0 L_0}{\eta}$, $\quad \mathrm{We} = \frac{\lambda U_0}{L_0}$, $\quad \omega = 1 - \frac{\eta_s}{\eta}$. We also define the non-dimensional function $w(\sigma) = \alpha \left\{ \frac{dp}{d\rho}(\alpha + \varepsilon^2 \sigma) - \frac{dp}{d\rho}(\alpha) \right\}$. Using the notation $\mathbf{u}' = \dfrac{\partial \mathbf{u}}{\partial t}$, $\sigma' = \dfrac{\partial \sigma}{\partial t}$ and $\tau' = \dfrac{\partial \tau}{\partial t}$, as well as $A = -(\Delta + \nabla \mathrm{div}\,)$, we may rewrite System (2) as follows : in $Q_T$,

$$
(3)\qquad \begin{cases} \alpha\Big(\mathbf{u}' + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) + (1-\omega)A\mathbf{u} + \nabla\sigma &= \mathbf{F}(\mathbf{u},\sigma,\tau) + \mathbf{div}\,\tau, \\ \sigma' + (\mathbf{u}\cdot\nabla)\sigma + \sigma \mathrm{div}\,\mathbf{u} &= -\varepsilon^2\alpha\,\mathrm{div}\,\mathbf{u}, \\ \tau + \mathrm{We}\Big(\tau' + (\mathbf{u}\cdot\nabla)\tau + \mathbf{g}(\nabla\mathbf{u},\tau)\Big) &= 2\omega\mathbf{D}[\mathbf{u}], \end{cases}
$$

with $\mathbf{F}(\mathbf{u},\sigma,\tau) = \alpha\mathbf{f} + \frac{(1-\omega)\varepsilon^2\sigma}{\alpha+\varepsilon^2\sigma}A\mathbf{u} + \frac{\varepsilon^2(\sigma - w(\sigma))}{\alpha+\varepsilon^2\sigma}\nabla\sigma - \frac{\varepsilon^2\sigma}{\alpha+\varepsilon^2\sigma}\mathbf{div}\,\tau$. System (3) is completed by a homogeneous condition on the boundary, $\mathbf{u} = 0$ on $\Sigma_T = (0,T)\times\Gamma$, and by three initial conditions, $\mathbf{u}(0) = \mathbf{u}_0$, $\sigma(0) = \sigma_0$, $\tau(0) = \tau_0$ in $\Omega$.

## 3. The Main Results

We assume that the data $\mathbf{u}_0$, $\sigma_0$ and $\tau_0$ are regular enough and satisfy

$$
\mathfrak{m} \le \alpha + \varepsilon^2\sigma_0 \le \mathfrak{M}, \quad \int_\Omega \sigma_0\,dx = 0,
$$

for some given constants $\mathfrak{m} > 0$ and $\mathfrak{M}$. Moreover the function $p = p(\rho)$ satisfies the condition $\sup_{[\frac{\alpha}{4},3\alpha]}(|p''| + |p'''|) \le \frac{C}{\varepsilon^2}$, for some constant C independent of $\epsilon$.

***Theorem 3.1***. — (Existence and uniqueness of a global solution) *Assume that $\Omega \subset \mathbb{R}^3$ is a domain of class $\mathcal{C}^4$, $\mathbf{f} \in L^\infty(\mathbb{R}_+;\mathbf{H}^1(\Omega))$, and $\mathbf{f}' \in L^\infty(\mathbb{R}_+;\mathbf{L}^2(\Omega))$. There exist three positive constants $\mathfrak{b}_1$, $\mathfrak{b}_2$ and $\omega_0$ ($0 < \omega_0 < 1$), independent of $\varepsilon$, such that if $0 < \omega \le \omega_0$, $|\mathbf{u}_0|_2^2 + \varepsilon^2|\sigma_0|_2^2 + |\tau_0|_2^2 \le \mathfrak{b}_1$, and $[\mathbf{f}]^2_{\infty,1,\infty} + [\mathbf{f}']^2_{\infty,0,\infty} \le \mathfrak{b}_2$, then System (3) admits a unique solution $(\mathbf{u},\sigma,\tau)$, satisfying*

$$
\begin{gathered}
\mathbf{u} \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^3(\Omega)) \cap \mathcal{C}_b(\mathbb{R}_+;\mathbf{H}^2(\Omega)),\ \ \mathbf{u}' \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^1(\Omega)) \cap \mathcal{C}_b(\mathbb{R}_+;\mathbf{L}^2(\Omega)),\\
\sigma \in \mathcal{C}_b(\mathbb{R}_+;H^2(\Omega)),\ \ \sigma' \in \mathcal{C}_b(\mathbb{R}_+;H^1(\Omega)),\ \ \tau \in \mathcal{C}_b(\mathbb{R}_+;\mathbf{H}^2(\Omega)),\ \ \tau' \in \mathcal{C}_b(\mathbb{R}_+;\mathbf{H}^1(\Omega)),
\end{gathered}
$$

*and*

$$
\frac{\alpha}{2} \le \alpha + \varepsilon^2\sigma \le \frac{3\alpha}{2}, \quad \text{in } \overline{\Omega}\times\mathbb{R}_+.
$$

*Moreover, for all $t \in \mathbb{R}_+$,*

$$
|\mathbf{u}(t)|_1^2 + \varepsilon^2|\mathbf{u}(t)|_2^2 + \varepsilon^2|\sigma(t)|_2^2 + |\tau(t)|_2^2 \le \mathfrak{b}_3,\ \ |\mathbf{u}'(t)|^2 + \varepsilon^2|\sigma'(t)|^2 + |\tau'(t)|^2 \le \mathfrak{b}_4,
$$

*where $\mathfrak{b}_3$ and $\mathfrak{b}_4$ are some positive constants independent of $\varepsilon$.*

*If, in addition, $\mathbf{f} \in L^2(\mathbb{R}_+;\mathbf{H}^1(\Omega))$, and $\mathbf{f}' \in L^2(\mathbb{R}_+;\mathbf{L}^2(\Omega))$, then the solution $(\mathbf{u},\sigma,\tau)$ of (3) also satisfies*

$$
\mathbf{u} \in L^2(\mathbb{R}_+;\mathbf{H}^3(\Omega)),\ \ \mathbf{u}' \in L^2(\mathbb{R}_+;\mathbf{H}^1(\Omega)),
$$

$$
\sigma \in L^2(\mathbb{R}_+;H^2(\Omega)),\ \ \sigma' \in L^2(\mathbb{R}_+;H^1(\Omega)),\ \ \tau \in L^2(\mathbb{R}_+;\mathbf{H}^2(\Omega)),\ \ \tau' \in L^2(\mathbb{R}_+;\mathbf{H}^1(\Omega)).
$$

The second result concerns the convergence of weakly compressible flows towards incompressible ones when $\varepsilon$ tends to zero. We now make the $\varepsilon$-dependence explicit, that is denote the solution $(\mathbf{u}, \sigma, \tau)$ of Problem (3) by $(\mathbf{u}^\varepsilon, \sigma^\varepsilon, \tau^\varepsilon)$. Let $(\mathbf{v}, \pi, \mathbf{S})$ be the solution of the incompressible model, where $\rho \equiv \alpha$,

$$(4) \qquad \begin{cases} \alpha\Big(\mathbf{v}' + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) - (1-\omega)\Delta\mathbf{v} + \nabla\pi = \alpha\mathbf{f} + \mathbf{div}\ \mathbf{S}, & \\ \operatorname{div}\mathbf{v} = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \mathbf{S} + \mathrm{We}\Big(\mathbf{S}' + (\mathbf{v}\cdot\nabla)\mathbf{S} + \mathbf{g}(\nabla\mathbf{v}, \mathbf{S})\Big) = 2\omega\mathbf{D}[\mathbf{v}], & \\ \mathbf{v} = 0, & \text{on } \mathbb{R}_+ \times \Gamma, \\ \mathbf{v}(0) = \mathbf{v}_0, & \text{in } \Omega, \\ \mathbf{S}(0) = \mathbf{S}_0, & \text{in } \Omega. \end{cases}$$

***Theorem 3.2.*** — *Let $0 \leq \mathfrak{r} < 1$. We suppose that $(\mathbf{u}_0^\varepsilon, \tau_0^\varepsilon)$ converges towards $(\mathbf{v}_0, \mathbf{S}_0)$ in $\mathbf{H}^{1+\mathfrak{r}}(\Omega)\times\mathbf{H}^{1+\mathfrak{r}}(\Omega)$, as $\varepsilon$ tends to zero. Under the assumptions of Theorems 3.1 , the solution of Problem (3) tends towards the solution of Problem (4) as $\varepsilon$ tend to zero. More precisely, for all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon$ in $(0, \varepsilon_0]$ and $\mathrm{T} > 0$,*

$$|\mathbf{u}^\varepsilon - \mathbf{v}|_{\mathcal{C}([0,\mathrm{T}];\mathbf{H}^{\mathfrak{r}}(\Omega))} + \left\|\varepsilon^2\sigma^\varepsilon\right\|_{\mathcal{C}([0,\mathrm{T}];\mathrm{H}^{1+\mathfrak{r}}(\Omega))} + |\tau^\varepsilon - \mathbf{S}|_{\mathcal{C}([0,\mathrm{T}];\mathbf{H}^{1+\mathfrak{r}}(\Omega))} < \delta.$$

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