FUNCTIONAL INTEGRATION : A new perpective

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Les quelques pages suivantes sont une introduction aux travaux récents de Pierre Cartier et Cécile DeWitt-Morette. Le lecteur/la lectrice en trouvera un exposé dans :

- *Journal of Mathematical Physics 36 (1995) p. 2237- 2312, "A new perspective on functional integration"*
- *deux exposés dans Functional integration, Basics and Applications, Eds. C. DeWitt-Morette, P. Cartier, A. Folacci (Plenum Press New York, 1997)*
  - i) *"A rigourous mathematical foundation of functional integration", pp 1-50*
  - ii) *"Physics on and near caustics" pp. 51-66*

Les bibliographies de ces articles donnent les références nécessaires aux travaux antérieurs. Le cours de P. Cartier fait pendant le trimestre (automne 1997) du Centre Emile Borel sera disponible dans la série « Cours avancés »de la Société Mathématique de France.

Functional integration is a natural concept : the domains of integration are function spaces ; the identification and properties of function spaces are one of the great achievements of the twentieth century mathematics.

A new perspective on integration over finite dimensional domains of integration is necessary for constructing a coherent theory of functional integration. A simple and often quoted example proposed by P. Cartier will show the shortcomings of ordinary integration and suggest a new approach. Let

$$I_D(a) := \int_{\mathbf{R}^D} exp(-\frac{\pi}{a}|x|^2)dx = a^{\frac{D}{2}}$$

$$I_D(a) = \begin{cases} 0 & \text{for } 0 < a < 1 \\ 1 & \text{for } a = 1 \\ \infty & \text{for } a > 1 \end{cases}$$

ID (a) fails to be continuous as should reasonably be desired. A way out of this difficulty is to introduce a scale invariant volume element (invariant under the change of unit length).

$$\mathcal{D}_a x := \frac{dx^1}{sqrta}\frac{dx^2}{sqrta}\dots\frac{dx^D}{sqrta} \text{ (no physical dimension)}$$

and to define implicitely $D_a x$ by a dimensionless expression

$$\int_{\mathbf{R}^D} exp(-\frac{\pi}{a}|x|^2 - 2\pi i(x', x)) = exp(-\pi a|x'|^2).$$

*. Notes remises par à l'issue de sa conférence "Nouvelles perspectives sur l'intégrale de Feynman" du 6 Décembre 1997.

This expression can be generalized from R^D to an infinite dimensional Banach space X readily :

$$\int_X \mathcal{D}_{s,Q,W} x exp(-\frac{\pi}{s}Q(x) - 2\pi i(x',x)) = exp(-\pi s W(x')). \quad (1)$$

where

- Q is a quadratic form on X
- W is a quadratic form on the dual X' of X
- (x, x') for $x' \in X'$, $x \in X$ the dual pairing
- $Q(x) = (Dx, x)$ and $W(x') = (x', Gx')$
- Q and G are inverse of each other in the following sense :

$$DG = 1_{X'} \quad GD = 1_X$$

  Usually X is sufficiently restricted for a differential operator D on X to have a unique inverse, and we abreviate $\mathcal{D}_{s,Q,W}$ to $\mathcal{D}_{s,Q}$.
- $s \in 1, i$

$$\begin{array}{ll} \text{if } s = 1, & Q(x) > 0 \text{ for } x \neq 0 \\ \text{if } s = i, & Q(x) \text{ real, no other restriction.} \end{array}$$

  Henceforth, we abreviate $\mathcal{D}_{s,Q}$ to $\mathcal{D}_Q$ but we keep s in the integrand.

Given X, Q, W if necessary, and $\mathcal{D}_Q$, the next problem is to identify a suitable space $\mathcal{F}(X)$ of functionals on X integrable by $\mathcal{D}_Q$. Given $\mathcal{F}(X)$, the problem is to compute

$$I = \int_X \mathcal{D}_Q x F(x) \text{ for } F \in \mathcal{F}(X).$$

The domain of integration is the driving force in setting up and commuting the functional integral. An example :

$$\langle\beta|exp(-\frac{2\pi i}{h})(t_b - t_a)H|\alpha\rangle = \int_{X_{\alpha,\beta}} \mathcal{D}_{X_{\alpha,\beta}} x exp(\frac{2\pi i}{h}S(x, t_a, t_b))$$

$x \in X_{\alpha,\beta}$ is a map $x : [t_a, t_b] \to M^D$

- with D boundary conditions at t_a corresponding to α.
- with D boundary conditions at t_b corresponding to β.

The condition $S(x; t_a, t_b) < \infty$ determines the analytic properties of x. The action S offers a choice of quadratic forms Q.

Some techniques

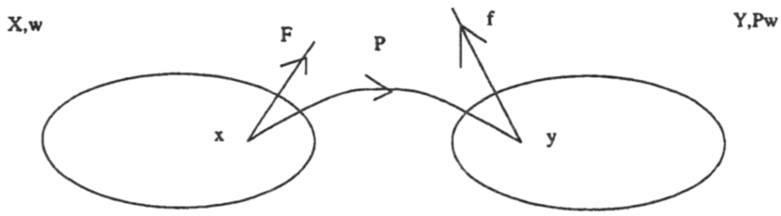
1. Choosing the defining equation of a scale invariant volume element $D_{X_{\alpha,\beta}} x$
   i) In a gaussian process, the quadratic form Q can be :

- a metric on $X_{\alpha,\beta}$ suggested by the metric on M^D,
- a metric on $X_{\alpha,\beta}$ invariant under the Cartan development map,
- a quadratic form. suggested by the action, e.g. its hessian.

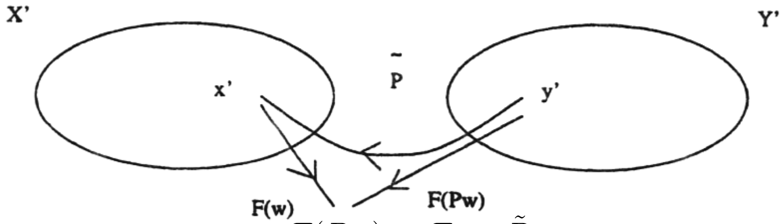
ii) See the references for choices of volume elements in a Poisson process.

2. Change of variable of integration

i) In a gaussian integral, set $dw(x) = \mathcal{D}_Q x exp(-\frac{\pi}{s}(x))$.



P linear continuous $F = f \circ P$



$\mathcal{F}(Pw) = \mathcal{F}w \circ \tilde{P}$
$\mathcal{F}w = exp(-\pi sW)$ (see equation (1))
$\int_X F(x)dw(x) = \int_Y f(y)dPw(y)$.

If Y is finite dimensional, then dPw is easily obtained from $\mathcal{F}(Pw)$. If Y is countable (mode decomposition $x(t) = \sum_k \xi^k \Psi_k(t)$ with $\{\Psi_k\}$ a basis for Y, and $\xi = \{\ldots, \xi^{k-1}, \xi^k, \xi^{k+1}, \ldots\} \in \Xi$) one can decompose the integral into an integral over a few selected $\{\xi^k\}_{k\in \text{finite set}}$ and an integral over the remaining finite set components of k.

ii) Let $dx^\alpha(t) = X_i^\alpha(x(t))dz^i(t)$ for $x, z \in L^{2,1}$.
Example 1 : $x^a(t)$ in arbitrary coordinates, $z^i(t)$ in cartesian coordinates.
Example 2 : x the Cartan development of z .
Example 3 : X a space of pointed paths on $M^D \to Y$ of pointed paths on $\mathbf{R}^D$ because X is contractible .
Example 4 : X a fibre bundle over M^D for a physical system on M^D.

Some uses of functional integrals solving P.D.E

i) Parabolic P.D.E : gaussian processes including processes on fibre bundles stated in terms of Lie derivatives.

ii) Hyperbolic P.D.E, Dirac equation : Poisson processes.

iii) Elliptic P.D.E, fixed energy problems : First exit time followed by time reparametrisation.

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