Diffusion processes and nonlinear parabolic equations

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From the very first steps of the development of the diffusion process theory its deep connections with PDE theory were acknowledged and fruitfully used. At the beginning PDE results were used to study properties of transition probabilities of diffusion processes. Later the results of stochastic differential equation (SDE) theory come to be very effective in deriving results in PDE. Actually SDE methods allow to derive a priori estimates of PDE solutions with constants independent of the dimension of the phase space and hence show the way to deal with infinite dimensional PDEs. In addition they were less sensitive to the degeneration of principal terms in PDE equations then classical methods of investigation.

Starting from the works by H. McKean [1] and M. Freidlin [2] it was revealed that probabilistic approach could be applied to investigation of a large class of quasilinear elliptic and parabolic equations as well. Later this approach was applied to study fully nonlinear PDE and systems of such equations (see [4]-[7]).

To give a more detailed description of the probabilistic approach let us introduce some notation. Let $(\Omega, \mathcal{F}, P)$ denote a complete probability space, $H_+ \subset H \subset H_-$ be a Gelfand triple of Hilbert spaces , $dim H = n \le \infty$. Denote by $w(t) \in H_-$ a Wiener process defined on Ω and associated with this triple, $a(t,x) \in H, A(t,x) \in L_{12}(H), x \in H, t \in [0,T]$ where $L_{12}(H)$ is a space of Hilbert-Schmidt operators with the norm $\sigma(A) = (\mathrm{Tr}[A^*A])^{\frac{1}{2}}$.

The connection between PDE and SDE theories is based on the fact that the transition probability $P(s,x,t,dy) = P(\xi(t) \in dy | \xi(s) = x), 0 \le s \le t \le T$ of the Markov process $\xi(t) \in H$ which solves

$$d\xi(t) = a(t,\xi(t))dt + A(t,\xi(t))dw, \xi(s) = x \tag{1}$$

gives rise to an evolution family $V(t,s)f(x) = Ef(\xi_{s,x}(t))$ where $\xi_{s,x}(t)$ is the solution to (1). The family $V(t,s)$ acts in the space $B(H)$ of bounded measurable functions on H and the restriction of its generator of to the space $C^2(H)$ of twice differentiable functions has the form

$$\mathcal{A}(s)f(x) = \frac{1}{2}\mathrm{Tr} f''(A(s,x), A(s,x)) + (f', a(s,x)).$$

Hence $u(s,x) = Ef(\xi_{s,x}(t))$ solves the Cauchy problem

$$\frac{\partial u}{\partial s} + \mathcal{A}(s) = 0, u(t,x) = f(x).$$

Here $f' = \frac{\partial f}{\partial x}, \mathrm{Tr} f''(A,A) = \sum_{i,j,k=1}^n f''_{x_i x_j} A_i^k A_j^k$.

Following this line consider a stochastic equation

$$d\xi = a(t, \xi(t), u(t, \xi(t)))dt + A(t, \xi(t), u(t, \xi(t)))dw, \xi(s) = x \tag{2}$$

where $u(t, x)$ is an unknown function and choose

$$u(s, x) = Ef(\xi_{s,x}(t)) \tag{3}$$

This idea which is due to McKean [1] and Freidlin [2] was developed later in [3], [4]. Under certain assumptions on a, A and f (see [3]) we can prove the existence and uniqueness of the solution to (2),(3) and check that $u(s, x)$ given by (3) is a (generally speaking) generalized solution of the Cauchy problem

$$\frac{\partial u}{\partial s} + \frac{1}{2}\mathrm{Tr}u''(A(s, x, u), A(s, x, u)) + (a(s, x, u), u') = 0, u(t, x) = f(x), t \geq s. \tag{4}$$

The next extension of the theory is connected with the so called multiplicative operator functionals of Markov processes and allows to apply the above results to systems of PDE and state smooth property of $u(s, x)$.

Let H_1 be another Hilbert space, $dim H_1 = m \leq \infty, c(s, x) \in L(H_1), C(s, x) \in L_{12}(H, L(H_1))$ where $L(H_1)$ is a space of bounded linear operators in H_1. Consider a system

$$d\xi = a(\xi(\theta), v(t - \theta, \xi(\theta)))d\theta + A(\xi(\theta), v(t - \theta, x(\theta)))dw, \tag{5}$$

$$d\eta = c(\xi(\theta), v(t - \theta, \xi(\theta)))\eta(\theta)d\theta + C(\xi(\theta), v(t - \theta, x(\theta)))(\eta(\theta), dw), \tag{6}$$

$$(h, v(t, x)) = E(\eta(t), f(\xi(t))), \xi(0) = x \in H, \eta(0) = h \in H_1. \tag{7}$$

Systems of this type were studied in [3],[4]. In particular there were stated conditions on coefficients and initial data to ensure the existence and uniqueness of solution to (5)-(7) and to prove that $v(t, x)$ determined by (7) is a generalized solution to

$$\frac{\partial v_l}{\partial t} = \frac{1}{2}\mathrm{Tr}v_l''(A, A) + (v_l', a) + \sum_{k=1}^{n}\sum_{q=1}^{m} C_{lq}^k(v_q', A^k) + \sum_{q=1}^{m} c_{lq}v_q, v_l(0, x) = f_1(x). \tag{8}$$

Notice that in general case we can prove only the existence of a local (in time) solution and state some conditions leading to the existence of a global solution.

Further development of the theory is connected with applying the whole machinery to the fully nonlinear parabolic equations and systems as well as treating all the problems in the framework of smooth manifolds and fibre bundles rather than linear spaces.

Actually the probabilistic approach described above could be directly applied to the investigation of the solution to the Cauchy problem for nonlinear equations and systems. One more extension allows to apply the above approach to investigation of the first initial boundary value problem in a smooth bounded region $G \subset \mathbb{R}^n$ for PDEs and systems [5],[6].

To deal with Dirichlet boundary conditions in elliptic case and first initial boundary value problem in parabolic case one could reduce the problem under investigation to the auxiliary Cauchy problem. This reduction is based on an observation due to Krylov [7] which allows to reduce the first initial boundary-value problem for a parabolic equation in a domain $G \subset \mathbb{R}^n$ with a smooth boundary $\partial G = \{x \in \mathbb{R}^n : \psi(x) = 0\}$ to an auxiliary Cauchy problem on a surface $G \subset \mathbb{R}^{n+4}$ determined by

$$U = \{y = (x, \hat{y}) \in \mathbb{R}^{n+4} : \psi(x) = \sum_{i=n+1}^{n+4} \hat{y}_i^2 > 0\}.$$

To explain how one could apply SDE theory to construct a solution to a Cauchy problem for a fully nonlinear PDE consider the problem

$$\frac{\partial u}{\partial t} + \Phi(t, x, u, u', u'') = 0, u(0, x) = f(x) \tag{9}$$

and reduce it to a system of quasilinear parabolic equations with respect to a function $v = (v_1, v_2, v_3, v_4)$ where $v_1 = u, v_2 = u', v_3 = u'', v_4 = u'''$. To this end let us differentiate (9) with respect to x variable and notice that the resulting system could be rewritten in the form (8) under some assumptions on the data in (9).

To be more precise assume that $\Phi(t, x, u, p, r)$ defined on $[0, T] \times R^n \times R^1 \times R^n \times R^{n^2}$ is at least three times continuously differentiable with respect to (x, u, p, r). Let in addition Φ'_r, be positive for all values of its arguments and $\Phi'_r = A * A$. Differentiating (9) three times one could see that the resulting system of equations could be rewritten in the form (8) where $a(t, x, v)$ denotes a factor of diagonal first order terms in it , $B(t, x, v)$ is a factor of nondiagonal first order terms and $c(t, x, v)$ is a factor of zero order terms. Let in addition $k(t, x)$ correspond to terms of the form $\Phi'_x(t, x, 0, 0, 0), ..., \Phi'''(t, x, 0, 0, 0)$.

Finally suppose that there exist positive constants C, C_1, C_2, ρ_1, positive bounded K and a constant ρ_0 such that

$$\|a(t,x,v)-a(t,x_1,v_1)\|^2+\sigma^2(A(t,x,v)-A(t,x_1,v_1)) \leq C[\|x-x_1\|^2+K(v,v_1)\|v-v_1\|^2],$$

$$\|a(t, x_1, v)\|^2 + \sigma^2(A(t, x, v)) \leq C_1[1 + \|x\|^2 + \|v\|^p],$$

$$\sigma^2(B(t, x, v)h) \leq C_2[1 + \|v\|^p], h \in \mathbb{R}^\alpha, \alpha = n + n^2 + n^3 + 1,$$

$$(c(t, x, v)h, h) \leq [\rho_0 + \rho_1\|v\|^p]\|h\|^2,$$

We say C.1 is valid if all above assumptions are satisfied.

Theorem 1. Assume that C.1 is valid. Then there exists a unique (local in time) solution to Cauchy problem (9). If in addition $\rho_0$is negative and have large enough absolute value then there exists a unique global solution to (9). Moreover there exists a Markov process $\xi(t) \in \mathbb{R}^n$ and a multiplicative functional of $\xi(t)$ generated by the process $\eta(t) \in \mathbb{R}^\alpha$ such that the solution to (9) admits the probabilistic representation in terms of $\xi(t)$ and $\eta(t)$.

In fact for a number of interesting fully nonlinear equations and systems the assumption that Φ'_r is nonnegative for all values of its arguments is too strong and should be modified. Eventually it is reasonable to demand Φ'_r to be nonegative at least on solutions of the problem or on some more or less good subsets of the space of functions valued in $R^n \times R^1 \times R^n \times R^{n^2}$. In this case the probabilistic approach could be used to derive the a priori estimates necessary to allow the application of a fixed point theorem. The corresponding results could be seen in the series of works by N.Krylov devoted to Bellman equation (see [7] and references there). Systems of fully nonlinear parabolic and elliptic equations on this way were studied in [8] .

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