

RAMIFICATION THEORY FOR VARIETIES OVER A LOCAL FIELD

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ABSTRACT

We define generalizations of classical invariants of wild ramification for coverings on a variety of arbitrary dimension over a local field. For an ℓ -adic sheaf, we define its Swan class as a 0-cycle class supported on the wild ramification locus. We prove a formula of Riemann-Roch type for the Swan conductor of cohomology together with its relative version, assuming that the local field is of mixed characteristic.

We also prove the integrality of the Swan class for curves over a local field as a generalization of the Hasse-Arf theorem. We derive a proof of a conjecture of Serre on the Artin character for a group action with an isolated fixed point on a regular local ring, assuming the dimension is 2.

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Introduction

0.1. *The goal of this paper.* — Let K be a complete discrete valuation field and \mathcal{O}_K be the valuation ring. We assume that the residue field F is perfect of characteristic $p > 0$. In this article, we generalize the classical ramification theory of extensions of K briefly recalled in 0.2 to the ramification theory for varieties over K as is described in 0.3–0.5 below. We also prove a conductor formula of Riemann-Roch type stated in 0.6.

We fix a prime number ℓ different from p . Let U be a separated smooth scheme of finite type over K and \mathcal{F} be a smooth ℓ -adic sheaf on U . The alternating sum $\mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathcal{F})$ of the Swan conductor is defined as an invariant measuring the wild ramification of the ℓ -adic representation $H_c^*(U_{\bar{K}}, \mathcal{F})$ of the absolute Galois group G_K of K . We define an element $\mathrm{Sw}_U \mathcal{F}$ (see 0.3–0.5 in the introduction and Definition 7.2.4 in the text) called the Swan class as a certain 0-cycle class supported on the closed fiber of a compactification of U over \mathcal{O}_K and prove a conductor formula

$$(0.1) \quad \mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathcal{F}) = \mathrm{rank} \mathcal{F} \cdot \mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathbf{Q}_\ell) + \deg \mathrm{Sw}_U \mathcal{F},$$

assuming that K is of characteristic 0 in Corollary 7.5.3. We also prove a relative version (see (0.6) below) of the conductor formula in Theorem 7.5.1.

The formula (0.1) is an arithmetic analogue of the higher dimensional generalization of the Grothendieck-Ogg-Shafarevich established in [27]. The term $\mathrm{Sw}_K H_c^*(U_{\bar{K}}, \mathbf{Q}_\ell)$ has been computed in the case where U is further assumed proper over K by the conductor formula of Bloch, proved under some mild assumption in [26]. In this paper, we will focus on a mixed characteristic case. Another approach in a geometric equal characteristic case is studied in [42].

0.2. *Invariants in classical ramification theory.* — We first recall the classical ramification theory. For a finite separable extension L of K , we have the following invariants of ramification in (i)–(iii) below, which are integers ≥ 0 . In (ii) and (iii), we assume that L/K is a Galois extension with Galois group G .

- (i) The different $D_{L/K}$ and the logarithmic different $D_{L/K}^{\log} = D_{L/K} - e_{L/K} + 1$, where $e_{L/K}$ is the ramification index of the extension L/K .

- (ii) The Lefschetz number $i(\sigma)$ and the logarithmic Lefschetz number $j(\sigma)$ for $\sigma \in G \setminus \{1\}$ defined as

$$i(\sigma) = \min\{\text{ord}_L(\sigma(a) - a) \mid a \in \mathcal{O}_L\},$$

$$j(\sigma) = \min\{\text{ord}_L(\sigma(a)/a - 1) \mid a \in L^\times\}.$$

- (iii) The Artin conductor $\text{Art}(\rho)$ and the Swan conductor $\text{Sw}(\rho)$ for a finite dimensional representation ρ of G over a field of characteristic 0. They are defined by

$$\text{Art}(\rho) = \frac{1}{e_{L/K}} \sum_{\sigma \in G - \{1\}} i(\sigma)(\dim(\rho) - \text{Tr}(\rho(\sigma))),$$

$$\text{Sw}(\rho) = \frac{1}{e_{L/K}} \sum_{\sigma \in G - \{1\}} j(\sigma)(\dim(\rho) - \text{Tr}(\rho(\sigma))).$$

The Hasse-Arf theorem asserts the highly non-trivial fact that these conductors are in fact integers.

These invariants are linked by several important formulas (see [39] for example). For example, in the case L/K is Galois with Galois group G , we have

$$D_{L/K} = \sum_{\sigma \in G \setminus \{1\}} i(\sigma), \quad D_{L/K}^{\log} = \sum_{\sigma \in G \setminus \{1\}} j(\sigma).$$

The invariants $D_{L/K}^{\log}$, $j(\sigma)$ and $\text{Sw}(\rho)$ are the parts of $D_{L/K}$, $i(\sigma)$ and $\text{Art}(\rho)$, respectively, which handles the wild ramification. We will focus on the wild ramification and introduce generalizations of $D_{L/K}^{\log}$, $j(\sigma)$, and $\text{Sw}(\rho)$.

0.3. Generalization. — In our generalization of ramification theory in [27] (resp. in this paper), in place of L/K in 0.2, we consider a finite étale morphism

$$f: V \rightarrow U$$

of non-singular algebraic varieties over a perfect field k of characteristic p (resp. over \mathbf{K}) and study the ramification of f along the boundary of compactifications of V and U over k (resp. $\mathcal{O}_{\mathbf{K}}$). We call the case over k the geometric case (geo) and the case over \mathbf{K} the arithmetic case (ari). For simplicity, in this introduction, we assume $\text{char } \mathbf{K} = 0$ in the case (ari). Although the main theme of this paper is the arithmetic case, we describe also the geometric case in 0.3 and 0.4 to compare.

In the case (geo) (resp. (ari)), for a proper scheme Y over k (resp. $\mathcal{O}_{\mathbf{K}}$) which contains V as a dense open subscheme, we define in Definition 2.4.1 the wild ramification locus $\Sigma_{V/U} Y$ of $f: V \rightarrow U$ on Y as a closed subset of Y . The wild ramification locus satisfies the following properties:

1. $V \cap \Sigma_{V/U} Y = \emptyset$.
2. If Y' is a proper scheme over k (resp. \mathcal{O}_K) containing V as a dense open subscheme and if $Y' \rightarrow Y$ is a morphism inducing the identity on V , then $\Sigma_{V/U} Y$ coincides with the image of $\Sigma_{V/U} Y'$.
3. In the case (ari) (recall that we assume $\text{char } K = 0$), $\Sigma_{V/U} Y$ is contained in the special fiber $Y \otimes_{\mathcal{O}_K} F$ of Y .

For a proper scheme X over k (resp. \mathcal{O}_K) which contains U as a dense open subscheme, we define the wild ramification locus $\Sigma_{V/U} X$ of f on X , which is a closed subset of X , to be the image $\bar{f}(\Sigma_{V/U} Y)$ for a morphism $f: Y \rightarrow X$ of compactifications extending $f: V \rightarrow U$. This also satisfies analogous properties corresponding to the above 1, 2, 3.

For a commutative ring R , let

$$F_0 G(\partial_{V/U} V)_R := \varprojlim_Y (F_0 G(\Sigma_{V/U} Y) \otimes_{\mathbf{Z}} R),$$

$$F_0 G(\partial_{V/U} U)_R := \varprojlim_X (F_0 G(\Sigma_{V/U} X) \otimes_{\mathbf{Z}} R)$$

where Y runs through proper integral schemes over k (resp. \mathcal{O}_K) containing V as a dense open subscheme and X runs through proper integral schemes over k (resp. \mathcal{O}_K) containing U as a dense open subscheme. Here $G(-)$ denotes the Grothendieck group of coherent sheaves, and $F_0 G(-)$ denotes the part generated by the classes of coherent sheaves with finite supports.

Let $Z(V/U)$ denote the free abelian group on the set of connected components of the complement $V \times_U V \setminus \Delta_V$ of the diagonal $\Delta_V \subset V \times_U V$. Note that since $f: V \rightarrow U$ is étale, Δ_V is open and closed in $V \times_U V$. The definition of generalizations of invariants of wild ramification is based on a homomorphism

$$(0.2) \quad Z(V/U) \rightarrow F_0 G(\partial_{V/U} V)_{\mathbf{Q}},$$

whose definition will be sketched in 0.4 below. The homomorphism (0.2) is called the localized intersection product with logarithmic diagonal and denoted by $(-, \Delta_V)^{\log}$ (resp. $((-, \Delta_V))^{\log}$) in the case (geo) (reps. (ari)). Though $V \times_U V \setminus \Delta_V$ does not intersect with the diagonal, the localized intersection with the log diagonal appears on the boundary of V in a compactification Y .

(i) We define

$$D_{V/U}^{\log} \in F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$$

by

$$D_{V/U}^{\log} = ([V \times_U V \setminus \Delta_V], \Delta_V)^{\log} \quad \text{in the case (geo),}$$

$$D_{V/U}^{\log} = ((([V \times_U V \setminus \Delta_V], \Delta_V))^{\log})^{\log} \quad \text{in the case (ari).}$$

- (ii) In the case $V \rightarrow U$ is a Galois covering with Galois group G , then for $\sigma \in G \setminus \{1\}$, we define

$$j(\sigma) \in F_0G(\partial_{V/U}V)_{\mathbf{Q}}$$

by

$$j(\sigma) = ([\Gamma_\sigma], \Delta_V)^{\log} \quad \text{in the case (geo),}$$

$$j(\sigma) = (([\Gamma_\sigma], \Delta_V))^{\log} \quad \text{in the case (ari),}$$

where Γ_σ is the graph of σ .

- (iii) For a finite dimensional representation ρ of G over a field of characteristic 0, we define the Swan class

$$\text{Sw}(\rho) = \frac{1}{\sharp(G)} \sum_{\sigma \in G - \{1\}} f_*(j(\sigma))(\dim(\rho) - \text{Tr}(\rho(\sigma))) \in F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_{p^\infty})}$$

where f_* is the push forward $F_0G(\partial_{V/U}V)_{\mathbf{Q}} \rightarrow F_0G(\partial_{V/U}U)_{\mathbf{Q}}$ and $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_n \mathbf{Q}(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n -th root of unity.

In (ii), we have $D_{V/U}^{\log} = \sum_{\sigma \in G \setminus \{1\}} j(\sigma)$ simply because $V \times_U V \setminus \Delta_V = \coprod_{\sigma \in G \setminus \{1\}} \Gamma_\sigma$. We expect that we can remove $\otimes \mathbf{Q}$ and $\otimes \mathbf{Q}(\zeta_{p^\infty})$ in the definitions of the above invariants in (i)–(iii).

To formulate a conductor formula given below, we define $\text{Sw}(\rho)$ also for a finite dimensional representation ρ of G over a field of characteristic ℓ by

$$\begin{aligned} \text{Sw}(\rho) &= \frac{1}{\sharp(G)} \sum_{\sigma \in G - \{1\}} f_*(j(\sigma))(\dim(\rho) - \text{Tr}^{\text{Br}}(\rho(\sigma))) \\ &\in F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_{p^\infty})} \end{aligned}$$

using the Brauer trace. The definition makes sense because we have $j(\sigma) = 0$ unless the order of σ is not a power of p .

The relation with classical ramification theory is as follows.

In the case (ari), assume $U = \text{Spec } K$, $V = \text{Spec } L$, $Y = \text{Spec } \mathcal{O}_L$. Then if L/K is wildly ramified (resp. at most tamely ramified), $\Sigma_{V/U}Y$ consists of the closed point of Y (resp. the empty set). If L/K is wildly ramified, we have $F_0G(\Sigma_{V/U}Y) = \mathbf{Z}$, and $D_{V/U}^{\log}$ and $j(\sigma)$ defined above recover the classical $D_{L/K}^{\log}$ and $j(\sigma)$, respectively. In the case (geo), assume that Y, V, U are smooth curves over k , and let K_0 (resp. L_0) be the function field of U (resp. V). Then $\Sigma_{V/U}Y$ consists of the places of L_0 where the extension L_0/K_0 is wildly ramified and $F_0G(\Sigma_{V/U}Y)$ is the direct sum of \mathbf{Z} indexed by these places. For $v \in \Sigma_{V/U}Y$, if u denotes the place of K_0 lying under v and if K (resp. L) denotes the completion of K_0

(resp. L_0) at u (resp. v), the v -components of $D_{V/U}^{\log}$ and $j(\sigma)$ defined above coincide with $D_{L/K}^{\log}$ and $j(\sigma)$ in the classical ramification theory for L/K , respectively.

The revolutionary idea that the invariant of ramification should be defined as a 0-cycle class on the ramification locus is due to S. Bloch [3].

0.4. *The definition of the localized intersection product with logarithmic diagonal.* — Let $V \rightarrow U$ be a finite étale morphism of smooth integral schemes over k (resp. over \mathbf{K}) in the case (geo) (resp. (ari)). We put $n = \dim U$ (reps. $n = \dim U_{\mathbf{K}} + 1$) in the case (geo) (resp. (ari)). The embedding theorem of Nagata and the theory of alteration of de Jong give us a Cartesian diagram of integral schemes over k (resp. $\mathcal{O}_{\mathbf{K}}$) in the case (geo) (resp. (ari))

$$(0.3) \quad \begin{array}{ccc} V & \xleftarrow{g} & W \\ \cap \downarrow & & \downarrow \cap \\ Y & \xleftarrow{\bar{g}} & Z \end{array}$$

where Y and Z are proper over k (resp. $\mathcal{O}_{\mathbf{K}}$) and satisfy the following properties: The vertical arrows are open immersions with dense images, the arrow $\bar{g}: Z \rightarrow Y$ is surjective and generically finite and Z is regular and contains W as the complement of a divisor with simple normal crossings.

In the case (geo) (resp. (ari)), we define the logarithmic self-product $(Z \times_k Z)^\sim$ (resp. $(Z \times_{\mathcal{O}_{\mathbf{K}}} Z)^\sim$) as a modification of the usual product $Z \times_k Z$ (resp. $Z \times_{\mathcal{O}_{\mathbf{K}}} Z$). Let P denote $(Z \times_k Z)^\sim$ (resp. $(Z \times_{\mathcal{O}_{\mathbf{K}}} Z)^\sim$). The diagonal map $Z \rightarrow Z \times_k Z$ (resp. $Z \rightarrow Z \times_{\mathcal{O}_{\mathbf{K}}} Z$) is canonically lifted to a closed immersion $Z \rightarrow P$ called the log diagonal map. The scheme P contains $W \times_k W$ (resp. $W \times_{\mathcal{O}_{\mathbf{K}}} W$) as an open subscheme. Let A be the closure of $W \times_U W \setminus W \times_V W$ in P and Σ be the intersection of A with the logarithmic diagonal Z in P .

We define the intersection product with the logarithmic diagonal Z in P as a homomorphism

$$(0.4) \quad G(A) \rightarrow G(\Sigma)$$

as follows. See Proposition 5.3.3 for detail. Regard \mathcal{O}_Z as an \mathcal{O}_P -module via the log diagonal. In the case (geo), the map (0.4) is defined as the usual intersection product with the class $[\mathcal{O}_Z]$ for a smooth scheme P . Namely, it maps the class of a coherent \mathcal{O}_P -module \mathcal{F} supported on A to the alternating sum:

$$[\mathcal{F}] \mapsto \sum_{i=0}^{\dim P} (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_Z)].$$

In the case (ari), it is defined as

$$[\mathcal{F}] \mapsto [\mathrm{Tor}_{2i}^{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_Z)] - [\mathrm{Tor}_{2i-1}^{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_Z)]$$

for sufficiently large integer i . In the case (ari), it is proved in [26] that the class $[\mathcal{T}or_j^{\mathcal{O}_p}(\mathcal{F}, \mathcal{O}_Z)] \in G(\Sigma)$ depends only on the parity of j for sufficiently large j .

The maps (0.4) for various diagrams (0.3) induce (0.2) as follows. Let F_\bullet denote the topological filtration on the Grothendieck group $G(-)$. We regard the free abelian group $Z(V/U)$ as the graded quotient $\mathrm{Gr}_n^F G(V \times_U V \setminus \Delta_V)$ by the canonical surjection defined by taking the length at the generic point of each connected component of $V \times_U V \setminus \Delta_V$. We prove in Proposition 4.3.5 that the homomorphisms $\mathrm{Gr}_\bullet^F G(A) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\Sigma)$ induced by (0.4) factor through the canonical surjection $\mathrm{Gr}_\bullet^F G(A) \rightarrow \mathrm{Gr}_\bullet^F G(W \times_U W \setminus W \times_V W)$ defined by the restriction if the following condition is satisfied:

(X) There exists a Cartesian diagram

$$\begin{array}{ccc} U & \xleftarrow{f \circ g} & W \\ \cap \downarrow & & \downarrow \cap \\ X & \longleftarrow & Z \end{array}$$

over k (resp. over \mathcal{O}_K) where X is a proper scheme over k (resp. over \mathcal{O}_K) containing U as the complement of a Cartier divisor.

Consequently, we obtain

$$\begin{aligned} Z(V/U) = \mathrm{Gr}_n^F G(V \times_U V \setminus \Delta_V) &\rightarrow \mathrm{Gr}_n^F G(W \times_U W \setminus W \times_V W) \\ &\rightarrow F_0 G(\Sigma) \end{aligned}$$

where the first arrow is the pull-back by $g \times g$ and the second arrow is induced by (0.4).

If further the condition

(Y) $\bar{g}(\Sigma) \subset \Sigma_{V/U} Y$

is satisfied, the composition

$$Z(V/U) \rightarrow F_0 G(\Sigma) \xrightarrow{\bar{g}_*} F_0 G(\Sigma_{V/U} Y) \otimes_{\mathbf{Z}} \mathbf{Q}$$

with the push-forward map \bar{g}_* divided by the generic degree $[Z : Y]$ of Z over Y is defined. We prove in Theorem 5.3.7 that such Z satisfying the conditions (X) and (Y) does exist and that the composition $Z(V/U) \rightarrow F_0 G(\Sigma_{V/U} Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ is independent of Z and forms an inverse system to define the required map (0.2).

0.5. *The Swan class of a constructible sheaf.* — In order to formulate a conductor formula of Riemann-Roch type in 0.6 below, we extend the definition of the Swan class to constructible sheaves. In the rest of introduction, we consider the arithmetic case (we assume $\mathrm{char} K = 0$).

For a separated scheme U of finite type over \mathbf{K} and for a commutative ring \mathbf{R} , let

$$F_0G(\partial_F U)_{\mathbf{R}} := \varprojlim_{\mathbf{X}} (F_0G(\mathbf{X} \otimes_{\mathcal{O}_{\mathbf{K}}} F) \otimes_{\mathbf{Z}} \mathbf{R})$$

where \mathbf{X} runs through proper schemes over $\mathcal{O}_{\mathbf{K}}$ which contain U as a dense open subscheme.

Let $f: V \rightarrow U$ be a finite étale morphism and let $\bar{Z}(V/U)$ denote the free abelian group on the set of connected components of $V \times_U V$. It is the direct sum of $Z(V/U)$ with the free abelian group of rank 1 generated by the class of Δ_V . The composition $((-, \Delta_V))^{\log}: Z(V/U) \rightarrow F_0G(\partial_{V/U} V)_{\mathbf{Q}} \rightarrow F_0G(\partial_F V)_{\mathbf{Q}}$ is naturally extended to

$$(0.5) \quad ((-, \Delta_V))^{\log}: \bar{Z}(V/U) \rightarrow F_0G(\partial_F V)_{\mathbf{Q}}.$$

To define the map (0.5), we proceed similarly as in 0.4. Namely, we consider a diagram (0.3) and, letting $A' \subset P$ denote the closure of $W \times_U W$, we define a homomorphism

$$\begin{aligned} G(A') &\rightarrow F_0G(Z_F); \\ [\mathcal{F}] &\mapsto [\mathcal{T}or_{2i}^{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_Z)] - [\mathcal{T}or_{2i-1}^{\mathcal{O}_P}(\mathcal{F}, \mathcal{O}_Z)] \quad (i \gg 0) \end{aligned}$$

similarly as (0.4). Then this induces (0.5) in the same way as (0.4) induces (0.2). In particular, $j(\sigma) \in F_0G(\partial_F V)_{\mathbf{Q}}$ is defined even for $\sigma = 1$ as $((\Delta_V, \Delta_V))^{\log}$.

We extend the definition of the Swan class of smooth sheaves sketched in 0.2 and 0.3 to constructible sheaves.

Proposition 1 (Proposition 7.4.2, Corollary 7.4.5). — *Assume $\text{char } \mathbf{K} = 0$. Then, there is a unique way to define*

$$\text{Sw}_U \mathcal{F}, \overline{\text{Sw}}_U \mathcal{F} \in F_0G(\partial_F U)_{\mathbf{Q}(\zeta_p^\infty)}$$

for any separated scheme U of finite type over \mathbf{K} and for any constructible $\bar{\mathbf{F}}_\ell$ -sheaf \mathcal{F} on U , satisfying the following conditions (1)–(3).

- (1) *Assume U is a non-singular variety and \mathcal{F} is locally constant. Let $f: V \rightarrow U$ be a finite étale Galois covering of U with Galois group G on which the pull-back $f^* \mathcal{F}$ is a constant sheaf. Then $\text{Sw}_U \mathcal{F}$ is the image of $\text{Sw}(\rho)$ in 0.3 for the $\bar{\mathbf{F}}_\ell$ -representation ρ of G corresponding to \mathcal{F} . We also have*

$$\begin{aligned} \overline{\text{Sw}}_U \mathcal{F} &= \text{Sw}_U \mathcal{F} - \text{rank } \mathcal{F} \cdot ((\Delta_U, \Delta_U))^{\log} \\ &= -\frac{1}{\sharp(G)} \sum_{\sigma \in G} f_* j(\sigma) \cdot \text{Tr}^{\text{Br}}(\rho(\sigma)). \end{aligned}$$

- (2) *For an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of constructible $\bar{\mathbf{F}}_\ell$ -sheaves on U , we have*

$$\text{Sw}_U \mathcal{F} = \text{Sw}_U \mathcal{F}' + \text{Sw}_U \mathcal{F}'', \quad \overline{\text{Sw}}_U \mathcal{F} = \overline{\text{Sw}}_U \mathcal{F}' + \overline{\text{Sw}}_U \mathcal{F}''.$$

(3) If $i : U' \rightarrow U$ is an immersion of schemes over \mathbf{K} , we have

$$\overline{\mathrm{Sw}}_U(i_*\mathcal{F}) = i_*\overline{\mathrm{Sw}}_{U'}\mathcal{F}$$

for a constructible $\overline{\mathbf{F}}_\ell$ -sheaf \mathcal{F} on U' . Here i_* on the right hand side is the canonical homomorphism $\mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U')_{\mathbf{Q}(\zeta_{\rho^\infty})} \rightarrow \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_{\rho^\infty})}$.

Here in (1), $((\Delta_U, \Delta_U))^{\mathrm{log}} \in \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}}$ is defined by (0.5) for $U = V$. The key ingredient of the proof of Proposition 1 is an excision formula Theorem 6.2.2.

For U as in Proposition 1 and for a constructible \mathbf{Q}_ℓ -sheaf \mathcal{F} on U , we define $\mathrm{Sw}_U\mathcal{F}, \overline{\mathrm{Sw}}_U\mathcal{F} \in \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_{\rho^\infty})}$ as those of the $\overline{\mathbf{F}}_\ell$ -sheaf which is obtained from \mathcal{F} by taking modulo ℓ . In the case U is regular and \mathcal{F} is smooth and trivialized by a finite étale Galois covering $V \rightarrow U$ with Galois group G , $\mathrm{Sw}_U\mathcal{F}$ is the image of $\mathrm{Sw}(\rho)$ in 0.3 where ρ is the representation of G over $\overline{\mathbf{Q}}_\ell$ corresponding to \mathcal{F} .

0.6. *The conductor formula.* — We prove the following conductor formula of Riemann-Roch type.

Theorem 2 (Theorem 7.5.1). — Assume $\mathrm{char} \mathbf{K} = 0$. Let $f : U \rightarrow V$ be a morphism of separated schemes of finite type over \mathbf{K} and let \mathcal{F} be a constructible $\overline{\mathbf{F}}_\ell$ -sheaf (resp. $\overline{\mathbf{Q}}_\ell$ -sheaf) on U . Then we have

$$(0.6) \quad \overline{\mathrm{Sw}}_V Rf_*\mathcal{F} = f_*\overline{\mathrm{Sw}}_U\mathcal{F}$$

where f_* on the right hand side is the canonical homomorphism $\mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_{\rho^\infty})} \rightarrow \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}V)_{\mathbf{Q}(\zeta_{\rho^\infty})}$.

In the case where \mathcal{F} is smooth and $V = \mathrm{Spec} \mathbf{K}$, the equality (0.6) specializes to the conductor formula (0.1) for the alternating sum of the Swan conductor (Corollary 7.5.3). It also gives

$$(0.7) \quad \mathrm{Sw}_{\mathbf{K}} H_c^*(U_{\overline{\mathbf{K}}}, \mathbf{Q}_\ell) = -\mathrm{deg}((\Delta_U, \Delta_U))^{\mathrm{log}},$$

which is a generalization of the conductor formula of Bloch [3] proved under some mild assumption in [26]. A special case of $\dim U_{\overline{\mathbf{K}}} = 1$ and $V = \mathrm{Spec} \mathbf{K}$ has been studied in [1]. A crucial ingredient in the proof of the equality (0.6) is a logarithmic variant Theorem 1.4.7 of the Lefschetz trace formula for open varieties.

0.7. *Integrality.* — As a generalization of the classical theorem of Hasse-Arf, we expect that the Swan class $\mathrm{Sw}_U\mathcal{F}$ should have no denominator, Conjecture 7.2.8. By a standard argument using Brauer induction, it is reduced to the rank one case. Theorem 8.3.7 comparing the Swan class $\mathrm{Sw}_U\mathcal{F}$ for a smooth sheaf \mathcal{F} of rank 1 with a cycle class $c_{\mathcal{F}}$, defined earlier by one of the authors, implies the following integrality.

Theorem 3 (Corollary 8.3.8.1). — Assume $\text{char } \mathbf{K} = 0$. Let U be a scheme of finite type over \mathbf{K} of dimension ≤ 1 and let \mathcal{F} be a constructible $\bar{\mathbf{F}}_\ell$ -sheaf (resp. $\bar{\mathbf{Q}}_\ell$ -sheaf) on U . Then $\text{Sw}_U \mathcal{F}$ belongs to the image of

$$F_0 G(\partial_F U)_{\mathbf{Z}} \rightarrow F_0 G(\partial_F U)_{\mathbf{Q}(\zeta_p^\infty)}.$$

From this integrality, we derive the two dimensional case of a conjecture of Serre [41], as is announced in [23]. In [41], Serre conjectures (see Conjecture 7.2.9) that the theory of Artin characters in the ramification theory of a discrete valuation ring can be generalized to any regular local ring A with a finite group of automorphisms under a condition of isolated fixed point. An equal characteristic case has been proved earlier in [28] and some special case has been proved in [2].

Theorem 4 (Corollary 8.3.8.2). — The conjecture of Serre [41] is true in the case $\dim(A) = 2$.

0.8. Organization of this paper. — We sketch the content of each section. The first three sections are preliminaries. In Section 1, after preparing general terminologies on semi-stable schemes, log products, etc., we prove a logarithmic Lefschetz trace formula, which is a crucial step in the proof of the formula (0.6). The trace formula is a sort of mixture of those proved in [26] and in [27]. In Section 2, we study the tame ramification of an étale morphism along the boundary, using log products. The purpose of studying tame ramification first is to define the wild ramification locus and to focus on it. We give criterions for tameness in terms of valuation rings, using the quasi-compactness of the limit of proper modifications. In Section 3, first we compute certain tor-sheaves, which is a crucial step in the proof of the excision formula. We also give some complement on the localized Chern classes and the excess intersection formula studied in [26] as a preliminary for the computation of the logarithmic different.

In Sections 4, 5 and 6, we define the invariants of wild ramification and establish their properties. First, in Section 4, we study the local structures of log products of schemes over $S = \text{Spec } \mathcal{O}_K$. In Section 5, we define the invariants and study its basic properties. Section 6 is technically the heart of the article. We prove the excision formula for the invariants. We also give a formula in some semi-stable case, which is a crucial step in the proof of the formula (0.6).

In Section 7, we define the Swan class and prove the formula (0.6). In Section 8, we compute the Swan class in the case of rank 1 and deduce the integrality of the Swan class and complete the proof of the conjecture of Serre in the case of dimension 2.

The logical structure of the proof of the formula (0.6) is summarized as follows. We deduce a formula Proposition 6.3.2 in some semi-stable case from the log Lefschetz trace formula Theorem 1.4.7. We prove a formula Propositions 7.3.4, 7.3.5 for stable curves using Proposition 6.3.2 and a compatibility with cospecialization map Proposition 1.6.2. We complete the proof of the formula (0.6) in Theorem 7.5.1 by deducing it from a special case Corollary 7.3.6, by devissage.

1. Log Lefschetz trace formula

We prove a logarithmic Lefschetz trace formula Theorem 1.4.7 for schemes over a discrete valuation ring and give a complement in Section 1.6. They play a crucial role in the proof of the conductor formula in Section 7.3. As preliminaries, we fix terminologies on semi-stable schemes, log blow-ups, log products and on log stalks in Sections 1.1, 1.2, 1.3, 1.5 respectively.

1.1. Semi-stable schemes and stable curves. — We fix some terminology on semi-stable schemes.

Definition 1.1.1. — Let $f: X \rightarrow S$ be a morphism of schemes and $r \geq 0$ be an integer.

1. We say that X is weakly strictly semi-stable of relative dimension r over S if the following condition is satisfied:

(1.1.1.1) For every point $x \in X$, there exist an open neighborhood $x \in U \subset X$, an affine open neighborhood $s = f(x) \in \text{Spec } R \subset S$, an integer $1 \leq q \leq r + 1$, an element $a \in R$ and an étale morphism

$$U \rightarrow \text{Spec } R[T_1, \dots, T_{r+1}]/(T_1 \cdots T_q - a)$$

over S .

If $S = \text{Spec } R$ for a discrete valuation ring R , we say a weakly strictly semi-stable scheme over S is strictly semi-stable if $a \in R$ in (1.1.1.1) is a uniformizer.

2. We say that X is weakly semi-stable of relative dimension r over S if, étale locally on X and on S , it is weakly strictly semi-stable of relative dimension r over S . Namely, if the following condition is satisfied:

(1.1.1.2) For every geometric point $\bar{x} \rightarrow X$, there exist étale neighborhoods $\bar{x} \rightarrow U \rightarrow X$ and $\bar{s} = f(\bar{x}) \rightarrow V \rightarrow S$ and a morphism $U \rightarrow V$ compatible with $X \rightarrow S$ and with $\bar{x} \rightarrow \bar{s}$ such that U is weakly strictly semi-stable of relative dimension r over V .

If $S = \text{Spec } R$ for a discrete valuation ring R , a scheme X over S is said to be semi-stable if, étale locally on X , it is strictly semi-stable over S .

If X is weakly semi-stable over S , the scheme X is flat over S and is smooth over S on a dense open subscheme of each fiber.

We show that, locally on X , the subscheme of S defined by a is well-defined and the subschemes of X defined by T_1, \dots, T_q are well-defined up to permutation.

Lemma 1.1.2. — Let $S = \text{Spec } R$ be an affine scheme and $f: X \rightarrow S$ be a scheme over S . Assume that X is étale over $R[T_1, \dots, T_{r+1}]/(T_1 \cdots T_q - a)$ for an element $a \in R$ and $q \geq 1$. Let x be a point of X where the morphism $f: X \rightarrow S$ is not smooth and $s = f(x)$.

1. The annihilator of the $\mathcal{O}_{S,s}$ -module $\Omega_{X/S,x}^{r+1}$ is generated by a .
2. Assume that there exist q irreducible components of the fiber X_s containing x . Then, the intersection $\text{Spec } \mathcal{O}_{X,x} \cap (X \times_S \text{Spec } R/(a))^{\text{sm}}$ with the smooth locus has q connected components. Their schematic closures in $\text{Spec } \mathcal{O}_{X,x}$ are defined by T_1, \dots, T_q .

Proof. — 1. An explicit computation in the case where $X = \text{Spec } R[T_1, \dots, T_{r+1}]/(T_1 \cdots T_q - a)$ shows that the $\mathcal{O}_{S,s}$ -module $\Omega_{X/S,x}^{r+1}$ is generated by one element and the annihilator is generated by $T_1 \cdots T_{i-1} T_{i+1} \cdots T_q$ for $i = 1, \dots, q$. The assertion follows from this easily.

2. The irreducible components of the fiber X_s containing x are defined by T_1, \dots, T_q . The connected components of the intersection $\text{Spec } \mathcal{O}_{X,x} \cap (X \times_S \text{Spec } R/(a))^{\text{sm}}$ are also defined by T_1, \dots, T_q . Thus the assertion follows. \square

In Definition 1.1.1.2, we may take $V = S$ in the condition (1.1.1.2) by Lemma 1.1.2.1.

Corollary 1.1.3. — *Let X be a weakly semi-stable scheme over a scheme S . Then, X is weakly strictly semi-stable over S if and only if, for every point s of S , each irreducible component of the fiber $X_s = X \times_S s$ is smooth over s .*

Proof. — If X is weakly strictly semi-stable, each irreducible component of the fiber $X_s = X \times_S s$ is clearly smooth over s for every point s of S . Let $x \in X$ be a point above $s \in S$. If $X \rightarrow S$ is smooth at x , it is weakly strictly semi-stable at x . Assume $f: X \rightarrow S$ is not smooth at x . Let \bar{x} be a geometric point above x . Then the irreducible components of the strict henselization $\text{Spec } \mathcal{O}_{X,\bar{x}}$ of the fiber are defined by T_1, \dots, T_q in the notation of Lemma 1.1.2.1. The pull-back of an irreducible component of the fiber X_s is the union of some of them. Hence, each irreducible component of the fiber $X_s = X \times_S s$ is smooth at x if and only if the ideals $(T_1), \dots, (T_q)$ are defined in $\mathcal{O}_{X,x}$. \square

We may modify a weakly semi-stable curve to a weakly strictly semi-stable curve, under an assumption. This construction will be used in the proof of Lemma 5.3.2 in the case (5.3.2.1a).

Lemma 1.1.4. — *Let X be a weakly semi-stable curve over a normal scheme S and let $E \subset X$ denote the closed subset consisting of the points where X is not smooth over S . Assume that X is smooth on a dense open subscheme of S and that the following condition is satisfied:*

- (1.1.4.1) *For every point $x \in E$ and $s = f(x)$, the element $a \in \mathcal{O}_{S,s}$ in Lemma 1.1.2 is a square up to a unit.*

Then, there exists a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I} = \mathcal{O}_X$ outside E and that the blow-up X' of X at \mathcal{I} is weakly strictly semi-stable over S .

Proof. — Let x be a point of E . Then, étale locally on a neighborhood of x , X is étale over the scheme defined by $T_1 T_2 - a$ and the ideal (a) is well-defined by Lemma 1.1.2.1. We put $a = b^2$. By the assumption that S is normal, the ideal (b) is also well-defined. Since the ideal (T_1, T_2) is the annihilator of $\Omega_{X/S}^1$ at x , the ideal $\mathcal{I} \subset \mathcal{O}_X$ étale locally defined by (T_1, T_2, b) is well-defined on X . Then, the blow-up $X' \rightarrow X$ by the ideal \mathcal{I} satisfies the condition by Corollary 1.1.3. \square

Definition 1.1.5. — *Let $f: X \rightarrow S$ be a weakly semi-stable scheme over S .*

1. *Let $D = D_1 + \cdots + D_n$ be the sum of Cartier divisors of X . Then, we say that D has simple normal crossings relatively to S if the following condition is satisfied:*

(1.1.5.1) *For every point $x \in X$, there exist an open neighborhood $x \in U \subset X$, a weakly semi-stable scheme Y over S and a smooth morphism $U \rightarrow \mathbf{A}_Y^m$ to the affine space with coordinate T_1, \dots, T_m such that, for each $i = 1, \dots, n$, the restriction $D_i \times_X U$ is either empty or defined by T_{j_i} for some $1 \leq j_i \leq m$. Further, for $1 \leq i < i' \leq n$ such that $D_i \times_X U$ and $D_{i'} \times_X U$ are non-empty, we have $j_i \neq j_{i'}$.*

2. *Let D be a Cartier divisor of X . Then, we say that D has normal crossings relatively to S if, étale locally on X , it has simple normal crossings relatively to S .*

If a Cartier divisor of X has normal crossings relatively to S , it is flat over S . If $D = D_1 + \cdots + D_n$ is a divisor with simple normal crossings relatively to S , for a subset $I \subset \{1, \dots, n\}$, the intersection $D_I = \bigcap_{i \in I} D_i$ is weakly strictly semi-stable over S . If X is smooth over S , the terminology on simple normal crossing divisors is the same as the usual one defined in [37, 2.1].

We recall the following fact on the tameness of the direct image for a proper semi-stable scheme.

Lemma 1.1.6. — *Let S be a regular noetherian scheme and $D \subset S$ be a divisor with normal crossings. Let $f: X \rightarrow S$ be a proper weakly semi-stable scheme such that the base change $X \times_S W \rightarrow W = S \setminus D$ is smooth and $E \subset X$ be a divisor with normal crossings relatively to S . We put $U = X \setminus E$ and $f_U: U \rightarrow S$ be the restriction of f .*

Then, for an integer $n \geq 1$ invertible on S , the higher direct image $R^q f_{U!} \mathbf{Z}/n\mathbf{Z}$ is locally constant on $W = S \setminus D$ and is tamely ramified along D for every $q \geq 0$.

Proof. — By the assumption that S is regular and D has normal crossings, it is reduced to the case where $S = \text{Spec } \mathcal{O}_K$ for a discrete valuation ring and D consists of the closed point s , by Abhyankhar's lemma [37, Proposition 5.2]. Let $j: U \rightarrow X$ denote the open immersion. Then, it suffices to show that the action of the inertia group $I = \text{Gal}(K^{\text{sep}}/K^{\text{ur}})$ on the sheaf $R^q \psi_{j!} \mathbf{Z}/n\mathbf{Z}$ of nearby cycles is tamely ramified. If $E = \emptyset$, it is proved in [35].

We show the general case. Since the assertion is étale local on X , we may assume X is weakly strictly semi-stable over S . Let $j: U \rightarrow X$ denote the open immersion and for

a finite set I of indices of irreducible components E_i of E , let $i_1: E_I \rightarrow X$ be the closed immersion of the intersections. Then, E_I are semi-stable over S and we have an exact sequence $0 \rightarrow j_! \mathbf{Z}/n\mathbf{Z} \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow \bigoplus_{\#I=1} i_{1*} \mathbf{Z}/n\mathbf{Z} \rightarrow \bigoplus_{\#I=2} i_{1*} \pi_* \mathbf{Z}/n\mathbf{Z} \rightarrow \cdots \rightarrow$. Using this, the assertion is reduced to the case $E = \emptyset$. \square

We recall the definition of a stable curve [8]. Let $f: X \rightarrow S$ be a proper weakly semi-stable scheme of relative dimension 1 over a scheme S and $(s_i)_{i=1, \dots, d}$ be a finite family of sections $s_i: S \rightarrow X$. Let $\omega_{X/S} = R^{-1}f^! \mathcal{O}_X$ be the relative dualizing sheaf. Then, we say a pair $(f: X \rightarrow S, (s_i)_{i=1, \dots, d})$ is a d pointed stable curve if the following condition is satisfied.

- The divisor $D = \sum_{i=1}^d s_i(S)$ has simple normal crossings relatively to S , the canonical map $\mathcal{O}_S \rightarrow f_* \mathcal{O}_X$ is an isomorphism and the invertible \mathcal{O}_X -module $\omega_{X/S}(D)$ is relatively ample.

If $(X, (s_i))$ is a pointed stable curve over S , the sections $s_i(S)$ do not meet each other and are contained in the locus where f is smooth. Further the \mathcal{O}_S -module $f_* \omega_{X/S}$ is locally free. The rank of $f_* \omega_{X/S}$ is called the genus of X . If $(X, (s_i))$ is a d pointed stable curve of genus g , we have $2g - 2 + d > 0$.

We recall some facts on the moduli of pointed stable curves, used in the proof of the conductor formula for a relative curve in Proposition 7.3.4 and Corollary 7.3.6. Let $\bar{S} = \bar{\mathcal{M}}_{g,d}$ be the moduli stack of d pointed stable curves of genus g . It is a proper smooth Deligne-Mumford stack over \mathbf{Z} [29] and the coarse moduli scheme $\bar{M}_{g,d}$ is a projective scheme [30]. Let $f: X \rightarrow \bar{S}$ be the universal family and $s_1, \dots, s_d: \bar{S} \rightarrow X$ be the universal sections. Let $S = \mathcal{M}_{g,d} \subset \bar{S}$ be the open substack where X is smooth. It is the complement of a divisor with normal crossings [29].

Let $n \geq 1$ be an integer. The n -torsion part $\text{Jac}_{X_{[1/n]}/S_{[1/n]}}[n] = R^1 f_{S_{[1/n]}*} \mu_n$ of the Jacobian is a locally constant sheaf of $\mathbf{Z}/n\mathbf{Z}$ -modules of rank $2g$ on $S_{[1/n]}$. Let $\mathcal{M}_{g,d,n}$ over $\mathcal{M}_{g,d}[1/n] = S_{[1/n]}$ be the moduli of an isomorphism $(\mathbf{Z}/n\mathbf{Z})^{2g} \rightarrow R^1 f_{S_{[1/n]}*} \mu_n$. If $n \geq 3$, then $\mathcal{M}_{g,d,n}$ is represented by a scheme $S_n = M_{g,d,n}$ smooth over $\mathbf{Z}[1/n]$. Further, the normalization $\bar{S}_n = \bar{\mathcal{M}}_{g,d,n}$ of $\bar{\mathcal{M}}_{g,d}[1/n]$ in $S_n = M_{g,d,n}$ is a projective scheme over $\mathbf{Z}[1/n]$ [7]. See also [6, 2.24].

1.2. Semi-stable schemes and log blow-up. — We briefly recall the log blow-up and apply it to give some constructions related to semi-stable schemes. For terminologies on log blow-up, we refer to [26, Section 4.2]. Let P be a finitely generated commutative integral saturated torsion free monoid, called a torsion free fs-monoid for short. In other words, the associated group P^{gp} is a finitely generated free abelian group and there exists a finitely many elements f_1, \dots, f_m of the dual group $P^{\text{gp}*} = \text{Hom}(P^{\text{gp}}, \mathbf{Z})$ such that P is identified with the submonoid $\{x \in P^{\text{gp}} \mid f_i(x) \geq 0 \text{ for } i = 1, \dots, m\} \subset P^{\text{gp}}$, see [34, Proposition 1.1]. We identify the dual monoid $P^* = \text{Hom}_{\text{monoid}}(P, \mathbf{N})$ with the submonoid $\{f \in P^{\text{gp}*} \mid f(x) \geq 0 \text{ for } x \in P\}$. If $P^\times = \{x \in P \mid x^{-1} \in P\}$ is trivial, the abelian group $P^{\text{gp}*}$

is generated by the submonoid P^* . Further in this case, P^* is the intersection of P^{gp^*} in $P^{\text{gp}^*} \otimes_{\mathbf{Z}} \mathbf{Q}$ with $\{a_1 f_1 + \cdots + a_m f_m \mid a_i \in \mathbf{Q}, a_i \geq 0\}$ (loc. cit.).

Let X be a log scheme and $f: P \rightarrow \Gamma(X, \mathcal{O}_X)$ be a chart. It defines a strict morphism $f^*: X \rightarrow \text{Spec } \mathbf{Z}[P]$ of log schemes. Recall that a morphism $X \rightarrow Y$ is strict if the log structure of X is the pull-back of that of Y . Let $V_\sigma \subset P^{\text{gp}^*} \otimes_{\mathbf{Z}} \mathbf{Q}$ be a \mathbf{Q} -linear subspace. Then the intersection $N_\sigma = P^* \cap V_\sigma$ is a finitely generated saturated submonoid. Let $P_\sigma \supset P$ be the finitely generated saturated monoid defined by $\{x \in P^{\text{gp}} \mid f(x) \geq 0 \text{ for } f \in N_\sigma\}$. Then, we define a scheme X_σ by $X_\sigma = X \times_{\text{Spec } \mathbf{Z}[P]} \text{Spec } \mathbf{Z}[P_\sigma]$. Let $g: P \rightarrow \Gamma(X, \mathcal{O}_X)$ be another chart such that there exists a morphism $u: P \rightarrow \Gamma(X, \mathcal{O}_X^\times)$ satisfying $g = f \cdot u$. Then, the schemes X_σ over X defined by f and by g are canonically isomorphic to each other.

Let Σ be a subdivision of the dual monoid P^* . Recall that a subdivision Σ consists of finite family of submonoids $N_\sigma = P^* \cap V_\sigma$ of the dual monoid P^* indexed by $\sigma \in \Sigma$. Recall also that Σ is regular means that the monoid $N_\sigma \subset P^*$ for every $\sigma \in \Sigma$ is isomorphic to \mathbf{N}^r for some $r \geq 0$ and hence P_σ is isomorphic to $\mathbf{N}^r \times \mathbf{Z}^{n-r}$ where n is the rank of P^{gp} . By patching the schemes X_σ over X , we obtain a scheme X_Σ over X . Recall that if Σ is a proper subdivision, the scheme X_Σ is proper over X . In this case, we call X_Σ a log blow-up of X .

Let S be a regular noetherian scheme and $D \subset S$ be a divisor with normal crossings. Let $j_W: W = S \setminus D \rightarrow S$ denote the open immersion and we regard S as a log scheme defined by the log structure $\mathcal{M}_S = \mathcal{O}_S \cap j_{W*} \mathcal{O}_W^\times$. We consider a weakly semi-stable scheme $f: X \rightarrow S$ and a divisor $E \subset X$ with normal crossings relatively to S such that the base change $X \times_S W \rightarrow W$ is smooth. Let $j_U: U = X \setminus (f^{-1}(D) \cup E) \rightarrow X$ denote the open immersion and we regard X as a log scheme defined by the log structure $\mathcal{M}_X = \mathcal{O}_X \cap j_{U*} \mathcal{O}_U^\times$. Then the map $f: X \rightarrow S$ is log smooth.

We construct proper modifications of weakly semi-stable schemes using log blow-ups. This will be used at the end of the proof of Theorem 1.4.7.

Lemma 1.2.1. — *Let \mathcal{O}_K be a discrete valuation ring and X be a weakly semi-stable scheme over $S = \text{Spec } \mathcal{O}_K$ with smooth generic fiber X_K . Then, there exists a proper modification $X' \rightarrow X$ such that $X'_K \rightarrow X_K$ is an isomorphism and that X' is semi-stable over S .*

Proof. — Let π be a prime element of \mathcal{O}_K . First, we consider the case where there exists an étale morphism $X \rightarrow \text{Spec } \mathcal{O}_K[T_1, \dots, T_{r+1}]/(T_1 \cdots T_q - \pi^e)$ for an integer $e \geq 1$. Let $P_{q,e}$ be the monoid $\mathbf{N}^q + \langle (\frac{1}{e}, \dots, \frac{1}{e}) \rangle \subset \mathbf{Q}^q$. The uniformizer π and the pull-backs of T_1, \dots, T_q define a morphism $X \rightarrow \text{Spec } \mathbf{Z}[P_{q,e}] = \text{Spec } \mathbf{Z}[T_1, \dots, T_q, S]/(T_1 \cdots T_q - S^e)$.

We identify the dual monoid $N_{q,e} = P_{q,e}^*$ with $\{(a_1, \dots, a_q) \in \mathbf{N}^q \mid a_1 + \cdots + a_q \equiv 0 \pmod{e}\}$. Let $B_{q,e} \subset N_{q,e}$ be the finite set $\{(a_1, \dots, a_q) \in \mathbf{N}^q \mid a_1 + \cdots + a_q = e\}$ and define $\Sigma_{q,e}$ by $\{\sigma \subset B_{q,e} \mid (a_1, \dots, a_q), (b_1, \dots, b_q) \in \sigma \text{ implies } |a_1 - b_1| + \cdots + |a_q - b_q| \leq 2\}$. For $\sigma \in \Sigma_{q,e}$, let N_σ denote the submonoid of $N_{q,e}$ generated by σ . Then, $\Sigma_{q,e}$ defines a regular proper subdivision of $N_{q,e}$ and we obtain a log blow-up $X_{\Sigma_{q,e}} \rightarrow X$.

We show that the scheme $X_{\Sigma_{q,e}}$ is semi-stable over \mathcal{O}_K . Let $f: \mathbf{N} \rightarrow \mathbf{P}_{q,e}$ be the map sending 1 to $\langle \frac{1}{e}, \dots, \frac{1}{e} \rangle$. Then, the dual map $f^*: \mathbf{N}_{q,e} \rightarrow \mathbf{N}$ sends $a \in \mathbf{N}_{q,e}$ to $(a_1 + \dots + a_q)/e$ and hence an arbitrary element of $\mathbf{B}_{q,e}$ to 1. A numbering on $\sigma \in \Sigma_{q,e}$ defines an isomorphism $\mathbf{N}^s \rightarrow \mathbf{N}_\sigma \subset \mathbf{N}_{q,e}$. Hence, the composition $\mathbf{N}^s \rightarrow \mathbf{N}$ of the restriction $f^*|_{\mathbf{N}_\sigma}: \mathbf{N}_\sigma \rightarrow \mathbf{N}$ with an isomorphism $\mathbf{N}^s \rightarrow \mathbf{N}_\sigma$ sends every member of the canonical basis of \mathbf{N}^s to 1. From this, it follows immediately that X_σ is semi-stable over \mathcal{O}_K for every $\sigma \in \Sigma_{q,e}$.

By Lemma 1.1.2, the exponent e and the divisors defined by T_1, \dots, T_q are well-defined étale locally up to permutation. Since the regular proper subdivision $\Sigma_{q,e}$ is invariant under permutations of q letters, the étale locally constructed log blow-ups $X_{\Sigma_{q,e}} \rightarrow X$ patch each other and define a semi-stable modification $X' \rightarrow X$ globally. \square

Next, we reformulate [6, Proposition 3.6] in our terminology. This together with Lemma 1.1.4 will be used in the proof of Lemma 5.3.2 in the case (5.3.2.1a).

Lemma 1.2.2. — *Let S be a regular noetherian scheme and $D \subset S$ be a divisor with simple normal crossings. Let $f: X \rightarrow S$ be a weakly strictly semi-stable curve such that the base change $X_W = X \times_S W \rightarrow W = S \setminus D$ is smooth.*

Then, there exists a proper modification $X' \rightarrow X$ such that $X'_W \rightarrow X_W$ is an isomorphism, that X' is regular and weakly strictly semi-stable over S and that $X' \times_S D$ is a divisor with simple normal crossings.

Proof. — First, we consider the case where the following data are given:

Let $\text{Spec } \mathbf{R} \subset S$ be an affine open subscheme, $s_1, \dots, s_n \in \mathbf{R}$ be elements defining irreducible components D_1, \dots, D_n of $D \cap \text{Spec } \mathbf{R}$, $d_1, \dots, d_n > 0$ be integers and let $X \rightarrow \text{Spec } \mathbf{R}[T_1, T_2]/(T_1 T_2 - s_1^{d_1} \cdots s_n^{d_n})$ be an étale morphism over S . Let $c: \{1, \dots, n\} \rightarrow \{1, 2\}$ be a function.

We define maps $\mathbf{N} \rightarrow \mathbf{N}^2, \mathbf{N} \rightarrow \mathbf{N}^n$ of monoids by $(1, 1)$ and (d_1, \dots, d_n) and consider the amalgamate sum $\mathbf{P} = \mathbf{N}^2 +_{\mathbf{N}} \mathbf{N}^n$. The dual $\mathbf{N} = \mathbf{P}^*$ is identified with $\{(a, b) \in \mathbf{N}^2 \times \mathbf{N}^n \mid a_1 + a_2 = d_1 b_1 + \dots + d_n b_n\}$. Let $e_1, \dots, e_n \in \mathbf{N}^n$ be the standard basis. For $i = 1, \dots, n$, we put $\mathbf{B}_i = \{(a, b) \in \mathbf{N} \mid b = e_i\}$. We identify $(1, i), (2, i) \in \mathbf{A} = \{1, 2\} \times \{1, \dots, n\}$ with $((d_i, 0), e_i), ((0, d_i), e_i) \in \mathbf{B}_i$ and regard \mathbf{A} as a subset of $\mathbf{B} = \bigsqcup_i \mathbf{B}_i \subset \mathbf{N}$. For each $j \in \{1, \dots, n\}$, let Σ_j be the finite set consisting of $\sigma \subset \mathbf{B}_j \cup \mathbf{A}$ satisfying the following conditions:

- If $(a, i) \in \sigma \cap \mathbf{A}$ for $i < j$, we have $a = c(i)$.
- If $((a_1, a_2), e_j), ((a'_1, a'_2), e_j) \in \sigma \cap \mathbf{B}_j$, we have $|a_1 - a'_1| \leq 1$.
- We have $\{a \in \{1, 2\} \mid (a, i) \in \sigma \cap \mathbf{A}, i > j\} \subsetneq \{1, 2\}$.

We put $\Sigma = \bigcup_{j=1}^n \Sigma_j$. For each $\sigma \in \Sigma$, the submonoid $\mathbf{N}_\sigma \subset \mathbf{N}$ generated by σ is isomorphic to \mathbf{N}^s for $s = \text{Card } \sigma \geq 0$. For $(a, b) \in \mathbf{N}$, if there exists an integer $1 \leq j \leq n$ not

satisfying the inequalities

$$a_1 \geq \sum_{i \leq j, c(i)=1} b_i d_i \quad \text{and} \quad a_2 \geq \sum_{i \leq j, c(i)=2} b_i d_i,$$

then, for the smallest such j , there exists $\sigma \in \Sigma_j$ such that $(a, b) \in N_\sigma$. If otherwise, we have $(a, b) \in N_C$ for $C = \{(c(i), i) \mid i \in \{1, \dots, n\}\} \in \Sigma_n$. Hence, Σ defines a regular proper subdivision of N .

The étale morphism $X \rightarrow \text{Spec } \mathbf{R}[T_1, T_2]/(T_1 T_2 - s_1^{d_1} \cdots s_n^{d_n})$ induces a morphism $X \rightarrow \text{Spec } \mathbf{Z}[T_1, T_2, S_1, \dots, S_n]/(T_1 T_2 - S_1^{d_1} \cdots S_n^{d_n}) = \text{Spec } \mathbf{Z}[P]$. Hence the log blow-up X_Σ is defined by the regular proper subdivision Σ .

We show that the scheme X_Σ satisfies the condition. We consider the dual $\pi^*: N \rightarrow \mathbf{N}^n$ of the canonical map $\pi: \mathbf{N}^n \rightarrow P = \mathbf{N}^2 +_{\mathbf{N}} \mathbf{N}^n$. Let e_1, \dots, e_s and e'_1, \dots, e'_n be standard bases of \mathbf{N}^s and of \mathbf{N}^n . Then, π^* maps the elements of B_i to $e'_i \in \mathbf{N}^n$. Let $\sigma \in \Sigma_i$ and take an isomorphism $\mathbf{N}^s \rightarrow N_\sigma$ to the submonoid generated by σ . We consider the composition $\varphi: \mathbf{N}^s \rightarrow \mathbf{N}^n$ with the restriction $N_\sigma \rightarrow \mathbf{N}^n$. Then, there exists a map $g: \{1, \dots, s\} \rightarrow \{1, \dots, n\}$ such that $\varphi(e_j) = e'_{g(j)}$ for $j = 1, \dots, s$. Further, for $i' = 1, \dots, n$, we have $\text{Card}(g^{-1}(i')) \leq 1$ for $i \neq i'$ and $\text{Card}(g^{-1}(i)) \leq 2$. Thus, we have either an étale map $X_\sigma \rightarrow \text{Spec } \mathbf{R}[T_1, T_2]/(T_1 T_2 - s_i)$ or an étale map $X_\sigma \rightarrow \text{Spec } \mathbf{R}[T]$. Hence, the log blow-up X_Σ is weakly strictly semi-stable over S and regular. Further $D \times_S X_\Sigma$ is a divisor with simple normal crossings.

We prove the general case. To patch the local construction above, we fix a numbering of irreducible components of $E = X \times_S D$. Let $\text{Spec } \mathbf{R} \subset S$ be an affine open and $V \rightarrow \text{Spec } \mathbf{R}[T_1, T_2]/(T_1 T_2 - s_1^{d_1} \cdots s_n^{d_n})$ be an étale map defined on an open subscheme V of X . We assume that each $V \times_S D_i$ has two irreducible components $E_{1,i}$ and $E_{2,i}$ defined by (T_1, s_i) and (T_2, s_i) respectively. We define a function $c: \{1, \dots, n\} \rightarrow \{1, 2\}$ by requiring that the index of the irreducible component $E_{c(i),i}$ is the smaller among $E_{1,i}$ and $E_{2,i}$ with respect to the fixed numbering of the irreducible components of E . By changing the numbering of D_1, \dots, D_n , we may assume that the indices of the sequence $E_{c(1),1}, \dots, E_{c(n),n}$ is increasing. With this numbering and the definition of c , it is easily seen that the log blow-ups V_Σ patch globally and define a modification $X' \rightarrow X$. \square

The following lemma will be used in the proof of Corollary 5.3.2 the case (5.3.2.1b) but *not* in the proof of the conductor formula.

Lemma 1.2.3. — *Let S be a regular noetherian scheme and $D \subset S$ be a divisor with simple normal crossings. Let $f: X \rightarrow S$ be a weakly strictly semi-stable scheme such that the base change $X_W = X \times_S W \rightarrow W = S \setminus D$ is smooth.*

For an irreducible component D_i of D , let I_i be the set of irreducible components of $X \times_S D_i$ and, for $x \in X$ and $s = f(x) \in S$, let I_x be the set of irreducible components of the fiber X_s containing x . We assume that the following condition is satisfied:

(1.2.3.1) *There exist a family of functions $\varphi_i: I_i \rightarrow \mathbf{N}$ and a total order on the finite set I_x for every $x \in \mathbf{X}$ satisfying the following condition: If $s = f(x) \in D_i$ and if the map $I_x \rightarrow I_i$ induced by the inclusion $\mathbf{X}_s \rightarrow \mathbf{X}_{D_i}$ is injective, then the composition $I_x \rightarrow \mathbf{N}$ with φ_i is injective and increasing.*

Then, there exists a proper modification $\mathbf{X}' \rightarrow \mathbf{X}$ such that $\mathbf{X}'_{\mathbf{W}} \rightarrow \mathbf{X}_{\mathbf{W}}$ is an isomorphism, that \mathbf{X}' is regular and weakly strictly semi-stable over \mathbf{S} and that $\mathbf{X}' \times_{\mathbf{S}} \mathbf{D}$ is a divisor with simple normal crossings.

Proof. — First, we consider the case where the irreducible components D_1, \dots, D_q of \mathbf{D} are defined by t_1, \dots, t_q , there exists a smooth map $\mathbf{X} \rightarrow \mathbf{S}[\mathbf{T}_1, \dots, \mathbf{T}_r]/(\mathbf{T}_1 \cdots \mathbf{T}_r - t_1^{m_1} \cdots t_q^{m_q})$ for integers $m_1, \dots, m_q \geq 0$ and the total order on I_x is induced by the natural order on $\{1, \dots, r\}$. We define morphisms $\mathbf{N} \rightarrow \mathbf{N}^r$ and $e: \mathbf{N} \rightarrow \mathbf{N}^q$ by $1 \mapsto (1, \dots, 1)$ and by $1 \mapsto (m_1, \dots, m_q)$ respectively. Let \mathbf{P} be the amalgamate sum $\mathbf{N}^r +_{\mathbf{N}} \mathbf{N}^q$ with respect to the morphisms above. We consider the map $\mathbf{P} \rightarrow \Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ of monoids defined by $\mathbf{T}_1, \dots, \mathbf{T}_r$ and t_1, \dots, t_q .

We define the dual morphisms $|\cdot|: \mathbf{N}^r \rightarrow \mathbf{N}$ and $m^*: \mathbf{N}^q \rightarrow \mathbf{N}$ by $|(a_1, \dots, a_r)| = a_1 + \cdots + a_r$ and $m^*(b_1, \dots, b_q) = m_1 b_1 + \cdots + m_q b_q$. Then, the dual monoid $\mathbf{N} = \mathbf{P}^*$ is identified with $\{(a, b) \in \mathbf{N}^r \times \mathbf{N}^q \mid |a| = m^*(b)\}$. We define a regular proper subdivision Σ of \mathbf{N} . Let \mathbf{V} be the finite set $\{(a, j) \in \mathbf{N}^r \times \{1, \dots, q\} \mid |a| = m_j\}$. We regard \mathbf{V} as a subset of \mathbf{N} by identifying $(a, j) \in \mathbf{V}$ with $f_{a,j} = (a, f_j) \in \mathbf{N}$ where f_1, \dots, f_q denote the canonical basis of \mathbf{N}^q . For a vector $a \in \mathbf{N}^r$, we put $\text{Supp}(a) = \{i \in \{1, \dots, r\} \mid a_i > 0\}$. For elements $(a, j), (a', j') \in \mathbf{V}$, we write $(a, j) \leq (a', j')$ if $\max \text{Supp}(a) \leq \min \text{Supp}(a')$ and $j \leq j'$. The relation \leq satisfies the anti-symmetry law and the transitivity law but not the reflexive law. By abuse of terminology, we say a subset $\sigma \subset \mathbf{V}$ is totally ordered if $(a, j), (a', j') \in \sigma$ implies either $(a, j) \leq (a', j')$, $(a', j') \leq (a, j)$ or $(a, j) = (a', j')$. We put $\Sigma = \{\sigma \subset \mathbf{V} \mid \sigma \text{ is totally ordered}\}$. For $\sigma \in \Sigma$, we consider the submonoid $\mathbf{N}_{\sigma} \subset \mathbf{N}$ generated by $f_{a,j}$ for $(a, j) \in \sigma$. For each $(a, b) \in \mathbf{N}$, one can easily find the minimum totally ordered subset $\sigma \in \Sigma$ satisfying $(a, b) \in \mathbf{N}_{\sigma}$. Thus, Σ defines a regular proper subdivision. Hence the log blow-up $\mathbf{X}' = \mathbf{X}_{\Sigma}$ is regular and $\mathbf{X}' \times_{\mathbf{S}} \mathbf{D}$ is a divisor with simple normal crossings.

By the assumption on the existence of the functions and the total orders, the log blow-ups constructed above patch globally to give the required \mathbf{X}' . \square

1.3. Log products and log blow-ups. — We fix some terminology and notation on log products, which will be constantly used throughout this paper. For the generality on log schemes, we refer to [22], [19], [26, Section 4]. In this paper, unless otherwise explicitly stated, a log structure means an fs-log structure defined Zariski locally. In particular, a log structure $\mathcal{M}_{\mathbf{X}}$ is a sheaf of commutative monoids on the Zariski site of a scheme \mathbf{X} endowed with a morphism of sheaf of monoids $\mathcal{M}_{\mathbf{X}} \rightarrow \mathcal{O}_{\mathbf{X}}$ where $\mathcal{O}_{\mathbf{X}}$ is regarded as a sheaf of monoids with respect to the multiplication. Further, Zariski locally on \mathbf{X} , the log structure $\mathcal{M}_{\mathbf{X}}$ admits a chart by an fs-monoid. For a log structure $\mathcal{M}_{\mathbf{X}}$, let $\bar{\mathcal{M}}_{\mathbf{X}}$ denote the quotient $\mathcal{M}_{\mathbf{X}}/\mathcal{O}_{\mathbf{X}}^{\times}$.

We recall some basic facts on log schemes from [22], [26, Section 4.3]. For morphisms $X \rightarrow S$ and $Y \rightarrow S$ of log schemes, the fiber product $X \times_S^{\log} Y$ is defined as a log scheme. Note that X, Y, S are assumed to be fs-log schemes and $X \times_S^{\log} Y$ is the fiber product in the category of fs-log schemes. We put log in the notation to indicate that the underlying scheme can be different from $X \times_S Y$ in the category is schemes. However, for example if at least one of the morphisms $X \rightarrow S$ and $Y \rightarrow S$ is strict, the underlying scheme is $X \times_S Y$. In such a case, we will drop log in the notation.

For a Cartier divisor D of a scheme X defined by the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$, the associated log structure is defined to be $\mathcal{M}_X = \bigcup_{n \in \mathbf{N}} \mathcal{I}som_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}_D^n)$ endowed with the injection $\mathcal{M}_X \rightarrow \mathcal{O}_X$ induced by the inclusions $\mathcal{I}_D^n \rightarrow \mathcal{O}_X$. For a finite family $\mathcal{D} = (D_i)_{i \in I}$ of Cartier divisors $D_i \subset X$, the associated log structure is defined by the amalgamated sum of those associated to D_i for $i \in I$ over \mathcal{O}_X^\times . It is the push-out of the log structures defined by D_i for $i \in I$. We have a canonical map $\mathbf{N}^I \rightarrow \Gamma(X, \bar{\mathcal{M}}_X)$ that can be lifted to a chart locally on X .

Let P be an fs-monoid and we consider two morphisms $P \rightarrow \Gamma(X, \bar{\mathcal{M}}_X)$ of monoids. Then, by applying [26, Proposition 4.2.3] to the surjection $P + P \rightarrow P$, we conclude that the functor sending a log scheme T to the set

$$\left\{ f: T \rightarrow X \mid \begin{array}{c} \text{the two compositions } P \rightarrow \Gamma(X, \bar{\mathcal{M}}_X) \xrightarrow{f^*} \Gamma(T, \bar{\mathcal{M}}_T) \\ \text{are equal to each other} \end{array} \right\}$$

of morphisms of log schemes is representable by a log étale scheme over X , that may be denoted by $X \times_{X, P}^{\log} X$. Locally on X , it is constructed as follows. Let \tilde{P} be the inverse image of P by the sum $P^{\text{gp}} \oplus P^{\text{gp}} \rightarrow P^{\text{gp}}$. Locally on X , we take liftings $P \rightarrow \Gamma(X, \mathcal{M}_X)$ of $P \rightarrow \Gamma(X, \bar{\mathcal{M}}_X)$ and let $X \rightarrow \text{Spec } \mathbf{Z}[P + P]$ be the induced morphism of log schemes. Then, $X \times_{X, P}^{\log} X$ is constructed as $X \times_{\text{Spec } \mathbf{Z}[P + P]}^{\log} \text{Spec } \mathbf{Z}[\tilde{P}]$.

We apply the construction in the following case. Let $X \rightarrow S$ and $Y \rightarrow S$ be morphisms of log schemes, P be an fs-monoid and $P \rightarrow \Gamma(X, \bar{\mathcal{M}}_X)$ and $P \rightarrow \Gamma(Y, \bar{\mathcal{M}}_Y)$ be morphisms of monoids. Then, they induces two morphisms $P \rightarrow \Gamma(X \times_S^{\log} Y, \bar{\mathcal{M}}_{X \times_S^{\log} Y})$. By applying the construction above, we define the log product $X \times_{S, P}^{\log} Y$. It represents the functor sending a log scheme T over S to the set

$$\left\{ (f: T \rightarrow X, g: T \rightarrow Y) \mid \begin{array}{ccc} P & \longrightarrow & \Gamma(X, \bar{\mathcal{M}}_X) \\ \downarrow & & \downarrow f^* \\ \Gamma(Y, \bar{\mathcal{M}}_Y) & \xrightarrow{g^*} & \Gamma(T, \bar{\mathcal{M}}_T) \end{array} \text{ is commutative} \right\}$$

of pairs of morphisms of log schemes over S .

Lemma 1.3.1. — *If $P \rightarrow \Gamma(X, \bar{\mathcal{M}}_X)$ and $P \rightarrow \Gamma(Y, \bar{\mathcal{M}}_Y)$ are locally lifted to charts, then the projections $X \times_{S, P}^{\log} Y \rightarrow X$ and $X \times_{S, P}^{\log} Y \rightarrow Y$ are strict morphisms of log schemes.*

Proof. — By the construction above, the monoid P also defines charts on $X \times_{S, P}^{\log} Y$. \square

We consider the following variant. Let $X \rightarrow S$ and $Y \rightarrow S$ be morphisms of log schemes, P and Q be fs-monoids and

$$\begin{array}{ccccc} P & \longleftarrow & Q & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(X, \bar{\mathcal{M}}_X) & \longleftarrow & \Gamma(S, \bar{\mathcal{M}}_S) & \longrightarrow & \Gamma(Y, \bar{\mathcal{M}}_Y) \end{array}$$

be a commutative diagram of morphisms of monoids. Then, we define the log product $X \times_{S, P/Q}^{\log} Y$ by the Cartesian diagram

$$(1.3.1.1) \quad \begin{array}{ccc} X \times_{S, P/Q}^{\log} Y & \longrightarrow & X \times_{S, P}^{\log} Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \times_{S, Q}^{\log} S \end{array}$$

where the bottom arrow is the diagonal map and the right vertical arrow is defined by functoriality.

We make the construction explicit in the case where the log structures of X and Y are defined by finite families $\mathcal{D} = (D_i)_{i \in I}$ and $\mathcal{E} = (E_i)_{i \in I}$ of Cartier divisors $D_i \subset X$ and of $E_i \subset Y$ with the same index set and the log structure of S is trivial. We define the log product

$$(1.3.1.2) \quad (X \times_S Y)_{\mathcal{D}, \mathcal{E}}^{\sim}$$

to be $X \times_{S, \mathbf{N}^1}^{\log} Y$ defined by the canonical morphisms $\mathbf{N}^1 \rightarrow \Gamma(X, \bar{\mathcal{M}}_X)$, $\mathbf{N}^1 \rightarrow \Gamma(Y, \bar{\mathcal{M}}_Y)$. The canonical morphism $(X \times_S Y)_{\mathcal{D}, \mathcal{E}}^{\sim} \rightarrow X \times_S Y$ is log étale. If $X = Y$ and $\mathcal{D} = \mathcal{E}$, we let $(X \times_S Y)_{\mathcal{D}, \mathcal{E}}^{\sim}$ denoted by $(X \times_S X)_{\mathcal{D}}^{\sim}$. Further if \mathcal{D} is clear from the context, we drop the subscript \mathcal{D} .

Locally, the log product $(X \times_S Y)_{\mathcal{D}, \mathcal{E}}^{\sim}$ is described as follows. Assume that D_i and E_i are defined by $f_i \in \Gamma(X, \mathcal{O}_X)$ and $g_i \in \Gamma(Y, \mathcal{O}_Y)$ respectively. Then, $(f_i)_{i \in I}$ and $(g_i)_{i \in I}$ define maps of monoids $\mathbf{N}^1 \rightarrow \Gamma(X, \mathcal{O}_X)$ and $\mathbf{N}^1 \rightarrow \Gamma(Y, \mathcal{O}_Y)$ and they further induce a map $P = \mathbf{N}^1 \times \mathbf{N}^1 \rightarrow \Gamma(X \times_S Y, \mathcal{O}_{X \times_S Y})$ from the direct sum. We identify the dual monoid $\mathbf{N} = P^*$ with $\mathbf{N}^1 \times \mathbf{N}^1$ and let $N_\sigma = \mathbf{N}^1 \subset \mathbf{N}^1 \times \mathbf{N}^1$ be the diagonal submonoid. Then, the corresponding submonoid $P_\sigma = \{p \in P^{\text{gp}} \mid f(p) \in \mathbf{N} \text{ for } f \in N_\sigma\} \subset P^{\text{gp}} = \mathbf{Z}^1 \times \mathbf{Z}^1$ is equal to $\{(a, b) \in \mathbf{Z}^1 \times \mathbf{Z}^1 \mid a + b \in \mathbf{N}^1\}$. The log product $(X \times_S Y)_{\mathcal{D}, \mathcal{E}}^{\sim}$ is then equal to

$$\begin{aligned}
(1.3.1.3) \quad (\mathbf{X} \times_S \mathbf{Y})_\sigma &= (\mathbf{X} \times_S \mathbf{Y}) \times_{\mathrm{Spec} \mathbf{Z}[P]} \mathrm{Spec} \mathbf{Z}[P_\sigma] \\
&= (\mathbf{X} \times_S \mathbf{Y}) \times_{\mathrm{Spec} \mathbf{Z}[S_i, T_i; i \in I]} \mathrm{Spec} \mathbf{Z}[S_i, T_i, U_i^{\pm 1}; i \in I] / (S_i - U_i T_i; i \in I).
\end{aligned}$$

We have a global embedding as follows. For each $i \in I$, let $\mathcal{I}_{D_i} \subset \mathcal{O}_X$ and $\mathcal{I}_{E_i} \subset \mathcal{O}_Y$ be the ideal sheaves. We consider the \mathbf{P}^1 -bundle $\mathbf{P}(\mathrm{pr}_1^* \mathcal{I}_{D_i} \oplus \mathrm{pr}_2^* \mathcal{I}_{E_i})$ over $\mathbf{X} \times_S \mathbf{Y}$. The complement $P_i \subset \mathbf{P}(\mathrm{pr}_1^* \mathcal{I}_{D_i} \oplus \mathrm{pr}_2^* \mathcal{I}_{E_i})$ of the two sections defined by the surjections $\mathrm{pr}_1^* \mathcal{I}_{D_i} \oplus \mathrm{pr}_2^* \mathcal{I}_{E_i} \rightarrow \mathrm{pr}_1^* \mathcal{I}_{D_i}$ and $\mathrm{pr}_1^* \mathcal{I}_{D_i} \oplus \mathrm{pr}_2^* \mathcal{I}_{E_i} \rightarrow \mathrm{pr}_2^* \mathcal{I}_{E_i}$ is a \mathbf{G}_m -torsor over $\mathbf{X} \times_S \mathbf{Y}$. The log product $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}}^\sim$ is a closed subscheme of the fiber product $\prod_{i \in I} \mathbf{X} \times_S \mathbf{Y} P_i$ over $\mathbf{X} \times_S \mathbf{Y}$.

We consider the variant of log product. Further, let \mathbf{B} be a Cartier divisor of \mathbf{S} and (n_i) be a family of integers $n_i \geq 0$ satisfying $f^* \mathbf{B} = \sum_{i \in I} n_i D_i$ and $g^* \mathbf{B} = \sum_{i \in I} n_i E_i$ for the same family $(n_i)_{i \in I}$ of integers $n_i \geq 1$. We consider the log structure of \mathbf{S} defined by \mathbf{B} and define the log product

$$(1.3.1.4) \quad (\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}/\mathbf{B}}^\sim$$

to be $\mathbf{X} \times_{S, \mathbf{N}^1/\mathbf{N}}^{\log} \mathbf{Y}$ defined by the canonical morphisms $\mathbf{N} \rightarrow \Gamma(\mathbf{S}, \bar{\mathcal{M}}_S)$, $\mathbf{N}^1 \rightarrow \Gamma(\mathbf{X}, \bar{\mathcal{M}}_X)$, $\mathbf{N}^1 \rightarrow \Gamma(\mathbf{Y}, \bar{\mathcal{M}}_Y)$. It is a closed subscheme of $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}}^\sim$. When \mathbf{B} is clear from the context, we let $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}/\mathbf{B}}^\sim$ denoted by $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}}^\sim$ in order to distinguish it from $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}}^\sim$.

The log product $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}/\mathbf{B}}^\sim$ with respect to \mathcal{D}, \mathcal{E} and \mathbf{B} is locally described as follows. Suppose that D_i, E_i and \mathbf{B} are defined by $f_i \in \Gamma(\mathbf{X}, \mathcal{O}_X)$, $g_i \in \Gamma(\mathbf{Y}, \mathcal{O}_Y)$ and $a \in \Gamma(\mathbf{S}, \mathcal{O}_S)$ respectively. We put $a = v \prod_i f_i^{n_i}$ and $a = w \prod_i g_i^{n_i}$ for $v \in \Gamma(\mathbf{X}, \mathcal{O}_X^\times)$ and $w \in \Gamma(\mathbf{Y}, \mathcal{O}_Y^\times)$. Then, $((f_i)_{i \in I}, v)$ and $((g_i)_{i \in I}, w)$ define maps of monoids $\mathbf{N}^1 \times \mathbf{Z} \rightarrow \Gamma(\mathbf{X}, \mathcal{O}_X)$ and $\mathbf{N}^1 \times \mathbf{Z} \rightarrow \Gamma(\mathbf{Y}, \mathcal{O}_Y)$. Let \mathbf{P} be the amalgamate sum $(\mathbf{N}^1 \times \mathbf{Z}) +_{\mathbf{N}} (\mathbf{N}^1 \times \mathbf{Z})$ with respect to the map $\mathbf{N} \rightarrow \mathbf{N}^1 \times \mathbf{Z}$ sending 1 to $((n_i), 1)$. Then, they further induce a map $\mathbf{P} \rightarrow \Gamma(\mathbf{X} \times_S \mathbf{Y}, \mathcal{O}_{\mathbf{X} \times_S \mathbf{Y}})$. We identify the dual monoid $\mathbf{N} = \mathbf{P}^*$ with $\{((a_i), (b_i)) \in \mathbf{N}^1 \times \mathbf{N}^1 \mid \sum_i n_i a_i = \sum_i n_i b_i\}$ and let $\mathbf{N}_\sigma = \mathbf{N}^1 \subset \mathbf{N} \subset \mathbf{N}^1 \times \mathbf{N}^1$ be the diagonal submonoid. Then, the corresponding submonoid $\mathbf{P}_\sigma \subset \mathbf{P}^{\mathrm{sp}} = (\mathbf{Z}^1 \times \mathbf{Z} \oplus \mathbf{Z}^1 \times \mathbf{Z}) / \langle ((n_i), 1, (-n_i), -1) \rangle$ is equal to $\{(\overline{a}, \overline{a'}, \overline{b}, \overline{b'}) \in \mathbf{P}^{\mathrm{sp}} \mid a + b \in \mathbf{N}^1\}$. The log product $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}/\mathbf{B}}^\sim$ is then equal to

$$\begin{aligned}
(1.3.1.5) \quad (\mathbf{X} \times_S \mathbf{Y})_\sigma &= (\mathbf{X} \times_S \mathbf{Y}) \times_{\mathrm{Spec} \mathbf{Z}[P]} \mathrm{Spec} \mathbf{Z}[P_\sigma] \\
&= (\mathbf{X} \times_S \mathbf{Y}) \times_{\mathrm{Spec} \mathbf{Z}[S_i, T_i; i \in I, V^{\pm 1}, W^{\pm 1}]} \\
&\quad \mathrm{Spec} \mathbf{Z}[S_i, T_i, U_i^{\pm 1}; i \in I, V^{\pm 1}, W^{\pm 1}] / \\
&\quad \left(S_i - U_i T_i; i \in I, W - V \prod_i U_i^{n_i} \right).
\end{aligned}$$

In other words, in the presentation (1.3.1.3), it is the closed subscheme defined by the relation $\mathrm{pr}_2^* w / \mathrm{pr}_1^* v = \prod_i U_i^{n_i}$.

We study the boundary of log products. Let $i \in I$ and put $\check{I}_i = I \setminus \{i\}$ and $\check{\mathcal{D}}_i = (\mathcal{D}_j)_{j \in \check{I}_i}$. We define $(\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$ to be $(\mathcal{D}_i \times_S \mathcal{D}_i) \times_{\mathbf{X} \times_S \mathbf{X}} (\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i}$. If $\mathcal{D}_i \cap \mathcal{D}_j = \mathcal{D}_i \times_X \mathcal{D}_j$ is a Cartier divisor of \mathcal{D}_i for every $j \in \check{I}_i$, the scheme $(\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$ is the log product with respect to the family $(\mathcal{D}_i \times_X \mathcal{D}_j)_{j \in \check{I}_i}$ denoted by \mathcal{D}_i .

Lemma 1.3.2. — *Let $\mathbf{X} \rightarrow \mathbf{S}$ and $\mathcal{D} = (\mathcal{D}_i)_{i \in I}$ be as above. Let $i \in I$ and assume that $\mathcal{D}_i \cap \mathcal{D}_j = \mathcal{D}_i \times_X \mathcal{D}_j$ is a Cartier divisor of \mathcal{D}_i for every $j \in I' = I \setminus \{i\}$.*

1. *The scheme $E_i = (\mathcal{D}_i \times_S \mathcal{D}_i) \times_{(\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i}} (\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i}$ is equal to the inverse images $\mathrm{pr}_1^{-1}(\mathcal{D}_i) = \mathrm{pr}_2^{-1}(\mathcal{D}_i)$ of $\mathcal{D}_i \subset \mathbf{X}$ by the projections $(\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i} \rightarrow \mathbf{X}$. It is a \mathbf{G}_m -torsor over $(\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$. The restriction of the log diagonal map $\mathcal{D}_i \rightarrow E_i$ defines a trivialization of the restriction of the \mathbf{G}_m -torsor $E_i \rightarrow (\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$ to $\mathcal{D}_i \subset (\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$.*

2. *Let \mathbf{B} be a Cartier divisor of \mathbf{S} . Assume that $f^*\mathbf{B} = \sum_j n_j \mathcal{D}_j$ and that the coefficient n_i of \mathcal{D}_i in $f^*\mathbf{B}$ is strictly positive $n_i > 0$. Then, the intersection $E_i \cap (\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i/\mathbf{B}}$ is a subscheme of a μ_{n_i} -torsor over $(\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$. The restriction of the log diagonal map $\mathcal{D}_i \rightarrow E_i$ defines a trivialization of the restriction of the μ_{n_i} -torsor $E_i \cap (\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i/\mathbf{B}} \rightarrow (\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$ to $\mathcal{D}_i \subset (\mathcal{D}_i \times_S \mathcal{D}_i)_{\check{\mathcal{D}}_i}$.*

Proof. — 1. Clear from the inductive construction $(\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i} = (\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i} \times_{\mathbf{X} \times_S \mathbf{X}} (\mathbf{X} \times_S \mathbf{X})_{\check{\mathcal{D}}_i}$ of the log product.

2. Clear from the remark after (1.3.1.5). \square

We define a log blow-up $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}}$ of $\mathbf{X} \times_S \mathbf{Y}$ containing the log product $(\mathbf{X} \times_S \mathbf{Y})_{\check{\mathcal{D}}, \mathcal{E}}$ as an open subscheme. For $i \in I$, let $\mathbf{B}_i \subset \mathbf{N} = \mathbf{N}^I \times \mathbf{N}^I$ be the subset $\{((a_k), (b_k)) \in \mathbf{N}^I \times \mathbf{N}^I \mid a_k = b_k = 0 \text{ for } k \neq i \text{ and } (a_i, b_i) \in \{(1, 0), (0, 1), (1, 1)\}\}$ consisting of three elements and we put $\mathbf{B} = \bigcup_{i \in I} \mathbf{B}_i$. Then the set $\Sigma = \{\sigma \mid \sigma \subset \mathbf{B}, \mathrm{Card}(\sigma \cap \mathbf{B}_i) \leq 2 \text{ for every } i \in I\}$ defines a regular proper subdivision of \mathbf{N} . We let the log blow-up $(\mathbf{X} \times_S \mathbf{Y})_{\Sigma}$ denoted by $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}}$. Since the diagonal submonoid $\mathbf{N}^I \subset \mathbf{N}^I \times \mathbf{N}^I$ is generated by the subset $\sigma = \{((a_i), (b_i)) \in \mathbf{B} \mid a_i = b_i \text{ for every } i \in I\}$, the log product $(\mathbf{X} \times_S \mathbf{Y})_{\check{\mathcal{D}}, \mathcal{E}}$ is an open subscheme of $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}}$.

We define a log blow-up $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}/\mathbf{B}}$ of $\mathbf{X} \times_S \mathbf{Y}$ containing the log product $(\mathbf{X} \times_S \mathbf{Y})_{\check{\mathcal{D}}, \mathcal{E}/\mathbf{B}}$ as an open subscheme, assuming $n_i \in \{0, 1\}$ for every $i \in I$. In order to define the log blow-up, we choose and fix a total order of the subset $I' = \{i \in I \mid n_i = 1\}$ of the index set I .

First, we consider the case where $I' = I$ namely $n_i = 1$ for every $i \in I$. The dual $\mathbf{N} = \mathbf{P}^*$ of $\mathbf{P} = \mathbf{N}^I +_{\mathbf{N}} \mathbf{N}^I$ is identified with $\{(a, b) \in \mathbf{N}^I \times \mathbf{N}^I \mid \sum_i a_i = \sum_i b_i\}$. Let (e_i) be the standard basis of \mathbf{N}^I . We identify an element $(i, j) \in I \times I$ with $(e_i, e_j) \in \mathbf{N}$ and regard $I \times I$ as a subset of \mathbf{N} . We consider the product order on the product $I \times I$. Let Σ be the set of totally ordered subsets $\sigma \subset I \times I$. For $\sigma \in \Sigma$, let $\mathbf{N}_{\sigma} \subset \mathbf{N}$ be the submonoid generated by σ . Then, Σ defines a regular proper subdivision of \mathbf{N} . We let the log blow-up $(\mathbf{X} \times_S \mathbf{Y})_{\Sigma}$ denoted $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}/\mathbf{B}}$. Since the diagonal $\sigma = \Delta_I \subset I \times I$ corresponds to

the diagonal submonoid $\mathbf{N}^1 \subset \mathbf{N} \subset \mathbf{N}^1 \times \mathbf{N}^1$, the log product $(\mathbf{X} \times_S \mathbf{Y})_{\mathcal{D}, \mathcal{E}/B}^{\sim}$ is an open subscheme of $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}/B}$.

In the general case, we put $I'' = I \setminus I'$ and consider the subfamilies $\mathcal{D}' = (D_i)_{i \in I'}$, $\mathcal{E}' = (E_i)_{i \in I'}$, $\mathcal{D}'' = (D_i)_{i \in I''}$, $\mathcal{E}'' = (E_i)_{i \in I''}$. Then, we define the log product $(\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}/B}$ as the fiber product by

$$(1.3.2.1) \quad (\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}, \mathcal{E}/B} = (\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}', \mathcal{E}'/B} \times_{\mathbf{X} \times_S \mathbf{Y}} (\mathbf{X} \times_S \mathbf{Y})'_{\mathcal{D}'', \mathcal{E}''}.$$

1.4. Log Lefschetz trace formula over a discrete valuation ring. — We state and prove a log Lefschetz trace formula over a discrete valuation ring. Let L be a henselian discrete valuation field. We regard $T = \text{Spec } \mathcal{O}_L$ as a log scheme with the log structure defined by the closed point t and also regard t as a log point.

Let X be a weakly semi-stable scheme over $T = \text{Spec } \mathcal{O}_L$ with smooth generic fiber X_L and $D \subset X$ be a Cartier divisor with normal crossings relatively to T . Let $j: X_L \rightarrow X$ be the open immersion. In this section, we regard the scheme X as a log scheme with the log structure $\mathcal{O}_X \cap j_* \mathcal{O}_{X_L}^\times$. It is log smooth over T . We consider the fiber $X_t = X \times_T t$ also as a log scheme over a log point t . We put $U = X \setminus D$ and let $j_U: U \rightarrow X$ be the open immersion.

If X is proper, we define the log étale cohomology with compact support by

$$H_{\log c}^q(U_t, \mathbf{Q}_\ell) = H_{\log}^q(X_t, j_{U!} \mathbf{Q}_\ell)$$

where $j_{U!}$ is defined on the log étale site.

Lemma 1.4.1. — *Let X be a proper weakly semi-stable scheme over $T = \text{Spec } \mathcal{O}_L$ and $D \subset X$ be a Cartier divisor with normal crossings relatively to T . Then the cospecialization map*

$$(1.4.1.1) \quad H_{\log c}^*(U_t, \mathbf{Q}_\ell) \longrightarrow H_c^*(U_L, \mathbf{Q}_\ell)$$

is an isomorphism.

Proof. — In the case $D = \emptyset$, it follows from [33, Theorem (3.2)(ii)]. We reduce the general case to this case. Let \bar{D} be the normalization D and let $\pi: \bar{D} \rightarrow X$ be the canonical map. Then, we have an exact sequence $0 \rightarrow j_{U!} \mathbf{Q}_{\ell, U} \rightarrow \mathbf{Q}_{\ell, X} \rightarrow \pi_* \mathbf{Q}_{\ell, \bar{D}} \rightarrow \Lambda^2 \pi_* \mathbf{Q}_{\ell, \bar{D}} \rightarrow \cdots$. Thus the assertion follows. \square

Let X' be another weakly semi-stable scheme over T with smooth generic fiber X'_L and $D' \subset X'$ be a Cartier divisor with normal crossings relatively to T . We also regard X' and X'_t as log schemes over T and over t . Let $\iota_t: X_t \rightarrow X'_t$ be an isomorphism of log schemes over t inducing an isomorphism $D_t \rightarrow D'_t$. Then, it induces an isomorphism $\iota_{t*}: H_{\log c}^q(U_t, \mathbf{Q}_\ell) \rightarrow H_{\log c}^q(U'_t, \mathbf{Q}_\ell)$.

Let $\Gamma \subset U_L \times U'_L$ be a closed subscheme of dimension $d = \dim U_L$. We assume that the second projection $p_2: \Gamma \rightarrow U'_L$ is proper. Then, in [27, Section 2.3], the map

$$(1.4.2.1) \quad \Gamma^*: H_c^q(U'_L, \mathbf{Q}_\ell) \rightarrow H_c^q(U_L, \mathbf{Q}_\ell)$$

is defined as $p_{1*}p_2^*$. We also let Γ^* denote the composition

$$(1.4.2.2) \quad \begin{array}{ccc} H_c^q(U_{\bar{L}}, \mathbf{Q}_\ell) & & H_c^q(U'_{\bar{L}}, \mathbf{Q}_\ell) \xrightarrow{\Gamma^*} H_c^q(U_{\bar{L}}, \mathbf{Q}_\ell) \\ \text{cosp.} \uparrow \simeq & & \text{cosp.} \uparrow \simeq \\ H_{\log c}^q(U_{\bar{t}}, \mathbf{Q}_\ell) & \xrightarrow{\iota_{t*}} & H_{\log c}^q(U'_{\bar{t}}, \mathbf{Q}_\ell) \end{array}$$

by abuse of notation. In this subsection, we give a Lefschetz trace formula computing the alternating sum

$$(1.4.2.3) \quad \text{Tr}(\Gamma^*: H_c^*(U_{\bar{L}}, \mathbf{Q}_\ell)) = \sum_{q=0}^{2d} (-1)^q \text{Tr}(\Gamma^*: H_c^q(U_{\bar{L}}, \mathbf{Q}_\ell)) \in \mathbf{Q}_\ell,$$

assuming that \mathbf{X} is weakly strictly semi-stable.

Let $\mathbf{X}, \mathbf{X}', \mathbf{D}$ and \mathbf{D}' be as above. We assume further that \mathbf{X} and \mathbf{X}' are weakly strictly semi-stable and that $\mathbf{D} = \mathbf{D}_1 + \cdots + \mathbf{D}_n$ and $\mathbf{D}' = \mathbf{D}'_1 + \cdots + \mathbf{D}'_n$ have simple normal crossings with the same indices. Let $\iota_t: \mathbf{X}_t \rightarrow \mathbf{X}'_t$ be an isomorphism of log schemes over t inducing isomorphisms $\mathbf{D}_{i,t} \rightarrow \mathbf{D}'_{i,t}$ for every $1, \dots, n$.

Let \mathbf{T} denote the log scheme \mathbf{T} endowed with the log structure defined by the Cartier divisor t . We consider an fs-monoid \mathbf{P} , a morphism $\mathbf{N} \rightarrow \mathbf{P}$ of monoids and a commutative diagram

$$(1.4.2.4) \quad \begin{array}{ccccc} \mathbf{P} & \longleftarrow & \mathbf{N} & \longrightarrow & \mathbf{P} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(\mathbf{X}, \bar{\mathcal{M}}_{\mathbf{X}}) & \longleftarrow & \Gamma(\mathbf{T}, \bar{\mathcal{M}}_{\mathbf{T}}) & \longrightarrow & \Gamma(\mathbf{X}', \bar{\mathcal{M}}_{\mathbf{X}'}) \end{array}$$

of monoids satisfying the following condition:

- (P) The vertical arrows are locally lifted to charts and compatible with the isomorphism $\iota_t: \mathbf{X}_t \rightarrow \mathbf{X}'_t$.

To define the log product $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$, we define \mathbf{X} to be the log scheme \mathbf{X} defined by the push-out $\bar{\mathcal{M}}'_{\mathbf{X}}$ of the log structure $\bar{\mathcal{M}}_{\mathbf{X}}$ and that defined by the family $\mathcal{D} = (\mathbf{D}_1, \dots, \mathbf{D}_n)$. Similarly, we define \mathbf{X}' . We consider

$$\begin{array}{ccccc} \mathbf{P} \oplus \mathbf{N}^n & \longleftarrow & \mathbf{N} & \longrightarrow & \mathbf{P} \oplus \mathbf{N}^n \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(\mathbf{X}, \bar{\mathcal{M}}_{\mathbf{X}}) & \longleftarrow & \Gamma(\mathbf{T}, \bar{\mathcal{M}}_{\mathbf{T}}) & \longrightarrow & \Gamma(\mathbf{X}', \bar{\mathcal{M}}_{\mathbf{X}'}) \end{array}$$

and define the log product

$$(1.4.2.5) \quad (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$$

to be $\mathbf{X} \times_{\mathbf{T}, \mathbb{P} \oplus \mathbf{N}^n / \mathbf{N}} \mathbf{X}'$. Since the canonical map $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim \rightarrow \mathbf{X} \times_{\mathbf{T}} \mathbf{X}'$ is log étale, the projections $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim \rightarrow \mathbf{X}$ and $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim \rightarrow \mathbf{X}'$ are log smooth. Similarly as Lemma 1.3.1, the projections are strict and hence smooth.

By the universality of $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$, the immersion $\mathbf{X}_t \rightarrow \mathbf{X}$ and the composition $\iota_t: \mathbf{X}_t \rightarrow \mathbf{X}'_t \rightarrow \mathbf{X}$ defines an immersion $\mathbf{X}_t \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$. By identifying \mathbf{X}'_t with \mathbf{X}_t by the isomorphism ι_t , let $\delta_t: \mathbf{X}_t \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$ denote the immersion. The generic fiber $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim \times_{\mathbf{T}} \text{Spec } \mathbf{L}$ is identified with the log product $(\mathbf{X}_{\mathbf{L}} \times_{\mathbf{L}} \mathbf{X}'_{\mathbf{L}})^\sim$ with respect to the families of Cartier divisors $\mathbf{D}_{1,\mathbf{L}}, \dots, \mathbf{D}_{n,\mathbf{L}}$ and $\mathbf{D}'_{1,\mathbf{L}}, \dots, \mathbf{D}'_{n,\mathbf{L}}$.

Definition 1.4.3. — 1. For a scheme \mathbf{S} , let $\mathbf{K}(\mathbf{S})$ denote the Grothendieck group of the exact category of locally free $\mathcal{O}_{\mathbf{S}}$ -modules of finite rank. For a locally free $\mathcal{O}_{\mathbf{S}}$ -modules \mathcal{E} of finite rank, its class is denoted by $[\mathcal{E}] \in \mathbf{K}(\mathbf{S})$.

2. For a noetherian scheme \mathbf{S} , let $\mathbf{G}(\mathbf{S})$ denote the Grothendieck group of the abelian category of coherent $\mathcal{O}_{\mathbf{S}}$ -modules. For a coherent $\mathcal{O}_{\mathbf{S}}$ -modules \mathcal{F} , its class is denoted by $[\mathcal{F}] \in \mathbf{G}(\mathbf{S})$. For an integral closed subscheme \mathbf{V} , the class $[\mathcal{O}_{\mathbf{V}}]$ is also denoted by $[\mathbf{V}]$.

We define a map $\mathbf{G}((\mathbf{X}_{\mathbf{L}} \times_{\mathbf{L}} \mathbf{X}'_{\mathbf{L}})^\sim) \rightarrow \mathbf{G}(\mathbf{X}_t)$ as follows.

Lemma 1.4.4. — Let \mathbf{X} and \mathbf{X}' be weakly strictly semi-stable schemes over $\mathbf{T} = \text{Spec } \mathcal{O}_{\mathbf{L}}$ with smooth generic fibers and $\mathbf{D} = \mathbf{D}_1 + \dots + \mathbf{D}_n \subset \mathbf{X}$ and $\mathbf{D}' = \mathbf{D}'_1 + \dots + \mathbf{D}'_n \subset \mathbf{X}'$ be divisors with simple normal crossings relatively to \mathbf{T} with the same indices. Let $\iota_t: \mathbf{X}_t \rightarrow \mathbf{X}'_t$ be an isomorphism of log schemes compatible with the numberings of \mathbf{D} and \mathbf{D}' and we consider a commutative diagram (1.4.2.4) of monoids satisfying the condition (P).

1. The pull-back $\mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim) \rightarrow \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim$ by the closed immersion $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim \times_{\mathbf{T}} t \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$ induces a map

$$(1.4.4.1) \quad \mathbf{G}((\mathbf{X}_{\mathbf{L}} \times_{\mathbf{L}} \mathbf{X}'_{\mathbf{L}})^\sim) \rightarrow \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim.$$

2. The map $\delta_t: \mathbf{X}_t \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim$ is a regular immersion and it defines a pull-back

$$(1.4.4.2) \quad \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim \rightarrow \mathbf{G}(\mathbf{X}_t).$$

Proof. — 1. Since the projection $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$ is smooth over \mathbf{X} , the scheme $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$ is flat over \mathbf{T} . Hence the closed immersion $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim$ is a regular immersion and is of finite tor-dimension. Thus, it induces a map $\mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim) \rightarrow \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim$.

Since the sequence

$$\mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim \rightarrow \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim) \rightarrow \mathbf{G}((\mathbf{X}_{\mathbf{L}} \times_{\mathbf{L}} \mathbf{X}'_{\mathbf{L}})^\sim) \rightarrow 0$$

is exact and since the composition

$$\mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim \rightarrow \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^\sim) \rightarrow \mathbf{G}((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')_t)^\sim$$

is the zero-map, the assertion follows.

2. Since the projection $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^{\sim} \rightarrow \mathbf{X}$ is smooth, the immersion $\delta_t: \mathbf{X}_t \rightarrow (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^{\sim}_t$ is a section of a smooth map and is a regular immersion. Hence the pull-back on the Grothendieck groups is defined. \square

For an element $\tilde{\Gamma} \in G((\mathbf{X}_L \times_L \mathbf{X}'_L)^{\sim})$, by Lemma 1.4.4, its reduction $\tilde{\Gamma}_t \in G((\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^{\sim}_t)$ and the intersection product

$$(1.4.4.3) \quad (\tilde{\Gamma}_t, \Delta_{\mathbf{X}_t}) = \delta_t^*(\tilde{\Gamma}_t) \in G(\mathbf{X}_t)$$

are defined.

Recall that for a weakly semi-stable scheme \mathbf{X} over \mathbf{T} , a semi-stable modification \mathbf{X}_{Σ} is constructed in Lemma 1.2.1 by patching log blow-ups. By the construction, the pull-back $\mathbf{D}_{\Sigma} = \mathbf{D} \times_{\mathbf{X}} \mathbf{X}_{\Sigma}$ is a divisor of \mathbf{X}_{Σ} with simple normal crossings relatively to \mathbf{T} . The canonical map $\mathbf{X}_{\Sigma} \rightarrow \mathbf{X}$ induces an isomorphism $\mathbf{X}_{\Sigma, L} \rightarrow \mathbf{X}_L$ on the generic fiber.

Corollary 1.4.5. — *Let weakly strictly semi-stable schemes \mathbf{X}, \mathbf{X}' over \mathbf{T} and an isomorphism $\iota_t: \mathbf{X}_t \rightarrow \mathbf{X}'_t$ and a commutative diagram (1.4.2.4) of monoids be as in Lemma 1.4.4. Let $f: \mathbf{X}_{\Sigma} \rightarrow \mathbf{X}$ and $f': \mathbf{X}'_{\Sigma} \rightarrow \mathbf{X}'$ be the semi-stable modification as above. Then, the diagram*

$$\begin{array}{ccc} G((\mathbf{X}_L \times_L \mathbf{X}'_L)^{\sim}) & \longrightarrow & G((\mathbf{X}_{\Sigma})_t) \\ \parallel & & \downarrow f_* \\ G((\mathbf{X}_L \times_L \mathbf{X}'_L)^{\sim}) & \longrightarrow & G(\mathbf{X}_t) \end{array}$$

is commutative.

Proof. — We show that the diagram

$$(1.4.5.1) \quad \begin{array}{ccc} (\mathbf{X}_{\Sigma} \times_{\mathbf{T}} \mathbf{X}'_{\Sigma})^{\sim} & \xrightarrow{\text{pr}_1} & \mathbf{X}_{\Sigma} \\ (f \times f')^{\sim} \downarrow & & \downarrow f \\ (\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^{\sim} & \xrightarrow{\text{pr}_1} & \mathbf{X} \end{array}$$

is Cartesian. By the definition (1.3.1.1) of $(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}')^{\sim}$, it suffices to show that the diagram with $\times_{\mathbf{T}}$ replaced by $\times_{\mathbf{T}}$ is Cartesian. Since the assertion is local on $\mathbf{X} \times_{\mathbf{T}} \mathbf{X}'$, we may assume that we have charts $\mathbf{P} \rightarrow \Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ and $\mathbf{P} \rightarrow \Gamma(\mathbf{X}', \mathcal{O}_{\mathbf{X}'})$. Let \mathbf{Q} be a sub-fs-monoid of \mathbf{P}^{gp} containing \mathbf{P} as a submonoid. Let $\tilde{\mathbf{P}}$ be the inverse image of \mathbf{P} by the sum $\mathbf{P}^{\text{gp}} \oplus \mathbf{P}^{\text{gp}} \rightarrow \mathbf{P}^{\text{gp}}$ and define $\tilde{\mathbf{Q}}$ similarly. Then, since $\mathbf{P}^{\text{gp}} = \mathbf{Q}^{\text{gp}}$, the inclusions $\mathbf{P} \rightarrow \tilde{\mathbf{P}}$, $\mathbf{Q} \rightarrow \tilde{\mathbf{Q}}$ to the first factors induce an isomorphism $\mathbf{Z}[\tilde{\mathbf{P}}] \otimes_{\mathbf{Z}[\mathbf{P}]} \mathbf{Z}[\mathbf{Q}] \rightarrow \mathbf{Z}[\tilde{\mathbf{Q}}]$. Hence the diagram with \mathbf{X}_{Σ} and \mathbf{X}'_{Σ} replaced by $\mathbf{X} \times_{\text{Spec } \mathbf{Z}[\mathbf{P}]} \text{Spec } \mathbf{Z}[\mathbf{Q}]$ and $\mathbf{X}' \times_{\text{Spec } \mathbf{Z}[\mathbf{P}]} \text{Spec } \mathbf{Z}[\mathbf{Q}]$ is Cartesian. Since \mathbf{X}_{Σ} and \mathbf{X}'_{Σ} are defined by patching them, the diagram (1.4.5.1) is Cartesian by the local construction of log product.

By the Cartesian diagram (1.4.5.1), we obtain a commutative diagram

$$\begin{array}{ccc} G((X_\Sigma \times_{\mathbf{T}} X'_\Sigma)^\sim) & \longrightarrow & G((X_\Sigma)_t) \\ (f \times f')^\sim_* \downarrow & & \downarrow f_* \\ G((X \times_{\mathbf{T}} X')^\sim) & \longrightarrow & G(X_t). \end{array}$$

Since the diagram

$$\begin{array}{ccc} G((X_\Sigma \times_{\mathbf{T}} X'_\Sigma)^\sim) & \longrightarrow & G(X_L \times_L X'_L) \\ (f \times f')^\sim_* \downarrow & & \parallel \\ G((X \times_{\mathbf{T}} X')^\sim) & \longrightarrow & G(X_L \times_L X'_L) \end{array}$$

is commutative, the assertion follows. \square

Let X, X', D, D' and $\iota_t: X_t \rightarrow X'_t$ be as above. Assume X and X' are strictly semi-stable and let E_1, \dots, E_m be the irreducible components of X_t and E'_1, \dots, E'_m be the irreducible components of X'_t such that ι_t maps E_j to E'_j for $j = 1, \dots, m$. Then, the log product $(X \times_{\mathbf{T}} X')^\sim$ is equal to $(X \times_{\mathbf{T}} X')_{\mathcal{D} \cup \mathcal{E}, \mathcal{D}' \cup \mathcal{E}'/t}$ defined by the families of Cartier divisors $\mathcal{D} = (D_1, \dots, D_n), \mathcal{E} = (E_1, \dots, E_m)$ of X , $\mathcal{D}' = (D'_1, \dots, D'_n), \mathcal{E}' = (E'_1, \dots, E'_m)$ of X' and t of \mathbf{T} . Consequently, in this case, the log blow-up $(X \times_{\mathbf{T}} X)'$ is defined as (1.3.2.1) and contains $(X \times_{\mathbf{T}} X')^\sim$ as an open subscheme. It contains the log product $(X \times_{\mathbf{T}} X')^\sim$ as an open subscheme. The generic fiber of the log blow-up $(X \times_{\mathbf{T}} X')' \rightarrow X \times_{\mathbf{T}} X'$ is equal to the log blow-up $(X_L \times_L X'_L)' \rightarrow X_L \times_L X'_L$ used in [27]. If $D = D' = \emptyset$, the log blow-up $(X \times_{\mathbf{T}} X')'$ is equal to the log blow-up $(X \times_{\mathbf{T}} X)'$ used in [26].

Lemma 1.4.6. — *Let $(X \times_{\mathbf{T}} X')' \rightarrow X \times_{\mathbf{T}} X'$ be the log blow-up and let $(D \times_{\mathbf{T}} X')', (X \times_{\mathbf{T}} D)'$ be the proper transforms of $D \times_{\mathbf{T}} X', X \times_{\mathbf{T}} D' \subset X \times_{\mathbf{T}} X'$. We consider the open immersions*

$$\begin{array}{ccc} (X \times_{\mathbf{T}} X')' \setminus ((D \times_{\mathbf{T}} X')' \cup (X \times_{\mathbf{T}} D)') & \xrightarrow{j_!} & (X \times_{\mathbf{T}} X')' \setminus (X \times_{\mathbf{T}} D)' \\ \uparrow & & \uparrow j \\ (X_L \times_L X'_L)^\sim & \xrightarrow{j_{!L}} & (X_L \times_L X'_L)' \setminus (X_L \times_L D'_L)'. \end{array}$$

Then the canonical map

$$(1.4.6.1) \quad j_{!L} \mathbf{Q}_\ell \rightarrow Rj_* j_{!L} \mathbf{Q}_\ell$$

is an isomorphism on the log étale site.

Proof. — For an irreducible component D_i of D , let $(D_i \times_{\mathbf{T}} X')' \subset (X \times_{\mathbf{T}} X')'$ denote the proper transform. For a subset $I \subset \{1, \dots, n\}$, we put $\bigcap_{i \in I} (D_i \times_{\mathbf{T}} X')' = (D_I \times_{\mathbf{T}} X')'$ and let $i_I: (D_I \times_{\mathbf{T}} X')' \setminus ((D_I \times_{\mathbf{T}} X')' \cap (X \times_{\mathbf{T}} D')') \rightarrow (X \times_{\mathbf{T}} X')' \setminus (X \times_{\mathbf{T}} D')'$ be the closed immersion. Then, we have an exact sequence

$$0 \rightarrow j_{1!} \mathbf{Q}_\ell \rightarrow \mathbf{Q}_\ell \rightarrow \bigoplus_{i=1}^n i_{i*} \mathbf{Q}_\ell \rightarrow \bigoplus_{1 \leq i < j \leq n} i_{\{i,j\}*} \mathbf{Q}_\ell \rightarrow \dots$$

Since $(D_I \times_{\mathbf{T}} X')'$ is log smooth over \mathbf{T} , the canonical map $i_{I*} \mathbf{Q}_\ell \rightarrow Rj_* i_{I!} \mathbf{Q}_\ell$ is an isomorphism by [33, Theorem (0.2)]. \square

The following theorem is a key ingredient in the proof of a crucial step Proposition 6.3.2 of the proof of the conductor formula.

Theorem 1.4.7. — *Let X and X' be proper weakly strictly semi-stable schemes of relative dimension d over \mathbf{T} with smooth generic fibers and $D = D_1 + \dots + D_n \subset X$ and $D' = D'_1 + \dots + D'_n \subset X'$ be divisors with simple normal crossings relatively to \mathbf{T} . Let $\iota_i: X_t \rightarrow X'_t$ be an isomorphism of proper log smooth schemes over t inducing isomorphisms $(D_i)_t \rightarrow (D'_i)_t$ for $i = 1, \dots, n$ and we consider a commutative diagram (1.4.2.4) of monoids satisfying the condition (P). We put $U = X \setminus D$ and $U' = X' \setminus D'$.*

Let $\Gamma' \subset (X_L \times_L X'_L)'$ be a closed subscheme of dimension d satisfying

$$(1.4.7.1) \quad \Gamma' \cap (D_L \times_L X'_L)' \subset \Gamma' \cap (X_L \times_L D'_L)'$$

and we put $\tilde{\Gamma} = \Gamma' \cap (X_L \times_L X'_L)^\sim$. Then, for $\Gamma = \Gamma' \cap (U_L \times_L U'_L)$, the second projection $p_2: \Gamma \rightarrow U'_L$ is proper and, for the composition Γ^ (1.4.2.2), we have*

$$(1.4.7.2) \quad \mathrm{Tr}(\Gamma^*: H_c^*(U'_L, \mathbf{Q}_\ell)) = \mathrm{deg}(\tilde{\Gamma}_t, \Delta_{X_t}).$$

Note that $(\tilde{\Gamma}_t, \Delta_{X_t})$ in the right hand side is defined in (1.4.4.3) using $\iota_t: X_t \rightarrow X'_t$. The proof is a combination of that of [27, Theorem 2.3.4] and that of [26, Theorem 6.5.1]. In the proper case where $X = U$, it is proved in [26, Theorem 6.5.1]. The proof consists of verifying that the method in the proof of [27, Theorem 2.3.4] to treat the open case also works in this context.

Proof. — By the argument in the beginning of the proof of [27, Theorem 2.3.4], the inclusion (1.4.7.1) implies that the second projection $\mathrm{pr}_2: \Gamma \rightarrow U'_L$ is proper. Thus the endomorphism Γ^* of $H_c^*(U'_L, \mathbf{Q}_\ell)$ is defined.

We prove the equality (1.4.7.2) first in the case where X and X' are strictly semi-stable. We put

$$H_{!*}^{2d}(X_L \times_L X'_L, \mathbf{Q}_\ell(d)) = H^{2d}(X_L \times_L U'_L, (j_U \times 1)_! \mathbf{Q}_\ell(d)),$$

$$\begin{aligned} H_{\log,!}^{2d}(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}', \mathbf{Q}_\ell(d)) &= H_{\log}^{2d}(\mathbf{X} \times_{\mathbf{T}} \mathbf{U}', (j_{\mathbf{U}} \times 1)_! \mathbf{Q}_\ell(d)), \\ H_{\log,!}^{2d}(\mathbf{X}_t \times_t \mathbf{X}'_t, \mathbf{Q}_\ell(d)) &= H_{\log}^{2d}(\mathbf{X}_t \times_t \mathbf{U}'_t, (j_{\mathbf{U}} \times 1)_! \mathbf{Q}_\ell(d)). \end{aligned}$$

Then, in [27, Lemma 2.3.2], the cycle class $[\Gamma] \in H_{!*}^{2d}(\mathbf{X}_{\mathbf{L}} \times_{\mathbf{L}} \mathbf{X}'_{\mathbf{L}}, \mathbf{Q}_\ell(d))$ is defined and the map Γ^* is described in terms of the cycle class $[\Gamma]$ as in the upper line of the diagram (1.4.7.4) below. We consider the image of $[\Gamma]$ by the composition

$$(1.4.7.3) \quad \begin{array}{c} [\Gamma] \in H_{!*}^{2d}(\mathbf{X}_{\mathbf{L}} \times_{\mathbf{L}} \mathbf{X}'_{\mathbf{L}}, \mathbf{Q}_\ell(d)) \\ \uparrow \text{restriction} \simeq \\ H_{\log,!}^{2d}(\mathbf{X} \times_{\mathbf{T}} \mathbf{X}', \mathbf{Q}_\ell(d)) \\ \downarrow \text{restriction} \\ [\Gamma_t] \in H_{\log,!}^{2d}(\mathbf{X}_t \times_t \mathbf{X}'_t, \mathbf{Q}_\ell(d)) \\ \downarrow \delta_t^* \\ \delta_t^*[\Gamma_t] \in H_{\log,c}^{2d}(\mathbf{U}_t, \mathbf{Q}_\ell(d)) \\ \downarrow \text{Tr} \\ \mathbf{Q}_\ell. \end{array}$$

Similarly as Lemma 1.4.6, the first arrow is an isomorphism by [33, Proposition (4.2)].

Since the cospecialization maps are compatible with the pull-back, cup-product and the trace maps, we have a commutative diagram

$$(1.4.7.4) \quad \begin{array}{ccccccc} H_c^*(\mathbf{U}'_{\mathbf{L}}, \mathbf{Q}_\ell) & \xrightarrow{\text{pr}_2^*} & H_{*!}^*(\mathbf{X}_{\mathbf{L}} \times \mathbf{X}'_{\mathbf{L}}, \mathbf{Q}_\ell) & \xrightarrow{\cup[\Gamma]} & H_c^*(\mathbf{U}_{\mathbf{L}} \times \mathbf{U}'_{\mathbf{L}}, \mathbf{Q}_\ell) & \xrightarrow{\text{pr}_{1*}} & H_c^*(\mathbf{U}_{\mathbf{L}}, \mathbf{Q}_\ell) \\ \text{cosp.} \downarrow & & \text{cosp.} \downarrow & & \downarrow \text{cosp.} & & \downarrow \text{cosp.} \\ H_{\log c}^*(\mathbf{U}'_t, \mathbf{Q}_\ell) & \xrightarrow{\text{pr}_2^*} & H_{\log *!}^*(\mathbf{X}_t \times \mathbf{X}'_t, \mathbf{Q}_\ell) & \xrightarrow{\cup[\Gamma_t]} & H_{\log c}^*(\mathbf{U}_t \times \mathbf{U}'_t, \mathbf{Q}_\ell) & \xrightarrow{\text{pr}_{1*}} & H_{\log c}^*(\mathbf{U}_t, \mathbf{Q}_\ell). \end{array}$$

The composition of the arrows in the upper line is the map Γ^* (1.4.2.1). We define Γ_t^* to be the composition of the arrows in the lower line. Then, we have

$$(1.4.7.5) \quad \text{Tr}(\Gamma^* : H_c^*(\mathbf{U}_{\mathbf{L}}, \mathbf{Q}_\ell)) = \text{Tr}(\Gamma_t^* \circ \iota_{t*} : H_{\log c}^*(\mathbf{U}_t, \mathbf{Q}_\ell)).$$

The standard argument of the proof of Lefschetz trace formula using the Künneth formula [32, Theorem (6.2)] and the Poincaré duality [33, Proposition (4.4)] for log étale

cohomology, shows that the diagram

$$\begin{array}{ccc}
\bigoplus_{q=0}^{2d} \text{End}(\mathbf{H}_{\log,c}^q(U_i, \mathbf{Q}_\ell)) & \xrightarrow{\simeq} & \mathbf{H}_{\log,!}^{2d}(X_t \times X'_t, \mathbf{Q}_\ell(d)) \\
\text{Tr} \downarrow & & \downarrow \delta_i^* \\
\mathbf{Q}_\ell & \xleftarrow{\text{Tr}} & \mathbf{H}_{\log,c}^{2d}(U_i, \mathbf{Q}_\ell(d))
\end{array}$$

is commutative. Hence, the right hand side of (1.4.7.5) is equal to $\text{Tr}(\delta_i^*[\Gamma_i])$. Thus the proof of (1.4.7.2) is reduced to showing

$$(1.4.7.6) \quad \text{Tr}(\delta_i^*[\Gamma_i]) = \deg(\tilde{\Gamma}_i, \Delta_{X_i}).$$

In the definition of the left hand side $\text{Tr}(\delta_i^*[\Gamma_i])$, we modify the diagram (1.4.7.3) using Lemma 1.4.6. We consider the log blow-up $(X \times_{\mathbf{T}} X)' \rightarrow X \times_{\mathbf{T}} X'$ defined by (1.3.2.1). We consider the proper transforms $(D \times_{\mathbf{T}} X)'$, $(X \times_{\mathbf{T}} D)'$ $\subset (X \times_{\mathbf{T}} X)'$ of $D \times_{\mathbf{T}} X'$, $X \times_{\mathbf{T}} D' \subset X \times_{\mathbf{T}} X'$ and let $j_i: (X \times_{\mathbf{T}} X)' \setminus ((D \times_{\mathbf{T}} X)' \cup (X \times_{\mathbf{T}} D)') \rightarrow (X \times_{\mathbf{T}} X)' \setminus (D \times_{\mathbf{T}} X)'$ be the open immersion as in Lemma 1.4.6. In the following equalities, we define the left hand sides by the right hand sides

$$\begin{aligned}
\mathbf{H}_{\log,!}^*((X \times_{\mathbf{T}} X)', \mathbf{Q}_\ell) &= \mathbf{H}_{\log}^*((X \times_{\mathbf{T}} X)' \setminus (X \times_{\mathbf{T}} D)', j_{i!} \mathbf{Q}_\ell) \\
\mathbf{H}_{!,*}^*((X_L \times_L X'_L)', \mathbf{Q}_\ell) &= \mathbf{H}^*((X_L \times_L X'_L)' \setminus (X_L \times_L D'_L)', j_{iL!} \mathbf{Q}_\ell).
\end{aligned}$$

We consider a commutative diagram

(1.4.7.7)

$$\begin{array}{ccccc}
\mathbf{H}_{!,*}^{2d}(X_L \times_L X'_L, \mathbf{Q}_\ell(d)) & \longrightarrow & \mathbf{H}_{!,*}^{2d}((X_L \times_L X'_L)', \mathbf{Q}_\ell(d)) & \longrightarrow & \mathbf{H}^{2d}((X_L \times_L X'_L)^\sim, \mathbf{Q}_\ell(d)) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{H}_{\log,!}^{2d}(X \times_{\mathbf{T}} X', \mathbf{Q}_\ell(d)) & \longrightarrow & \mathbf{H}_{\log,!}^{2d}((X \times_{\mathbf{T}} X)', \mathbf{Q}_\ell(d)) & \longrightarrow & \mathbf{H}_{\log}^{2d}((X \times_{\mathbf{T}} X)^\sim, \mathbf{Q}_\ell(d)) \\
\delta_i^* \downarrow & & \downarrow \delta_i^* & & \downarrow \delta_i^* \\
\mathbf{H}_{\log,c}^{2d}(U_i, \mathbf{Q}_\ell(d)) & \longrightarrow & \mathbf{H}_{\log}^{2d}(X_t, \mathbf{Q}_\ell(d)) & \equiv & \mathbf{H}_{\log}^{2d}(X_t, \mathbf{Q}_\ell(d)).
\end{array}$$

The upper vertical arrows are the restrictions to the generic fiber and are isomorphisms by Lemma 1.4.6 and [33, Proposition (4.2)]. The top left horizontal arrow is an isomorphism by [27, Corollary 2.2.2]. By the proof of [27, Theorem 2.3.4], the cycle class $[\Gamma'] \in \mathbf{H}_{!,*}^{2d}((X_L \times_L X'_L)', \mathbf{Q}_\ell(d))$ is the image of $[\Gamma] \in \mathbf{H}_{!,*}^{2d}(X_L \times_L X'_L, \mathbf{Q}_\ell(d))$. Thus, in the diagram (1.4.7.3), we may replace $X \times_{\mathbf{T}} X'$ etc. by $(X \times_{\mathbf{T}} X)'$ etc., $\mathbf{H}_{\log,c}^{2d}(U_i, \mathbf{Q}_\ell(d))$ by $\mathbf{H}_{\log}^{2d}(X_t, \mathbf{Q}_\ell(d))$ and $[\Gamma]$ by $[\Gamma']$. Further, we may replace $(X \times_{\mathbf{T}} X)'$ etc. by $(X \times_{\mathbf{T}} X)^\sim$

etc. and $[\Gamma']$ by $[\tilde{\Gamma}]$ and drop $!_*$. Therefore the proof of (1.4.7.2) is further reduced to showing

$$(1.4.7.8) \quad \mathrm{Tr}(\delta_t^*[\tilde{\Gamma}_t]) = \mathrm{deg}(\tilde{\Gamma}_t, \Delta_{X_t})$$

where the left hand side is defined by

$$(1.4.7.9) \quad \begin{array}{c} [\tilde{\Gamma}] \in H^{2d}((X_L \times_L X'_L)^\sim, \mathbf{Q}_\ell(d)) \\ \text{restriction} \uparrow \simeq \\ H_{\log}^{2d}((X \times_{\mathbf{T}} X')^\sim, \mathbf{Q}_\ell(d)) \\ \text{restriction} \downarrow \\ [\tilde{\Gamma}_t] \in H_{\log}^{2d}((X \times_{\mathbf{T}} X')_t^\sim, \mathbf{Q}_\ell(d)) \\ \delta_t^* \downarrow \\ \delta_t^*[\tilde{\Gamma}_t] \in H_{\log}^{2d}(X_t, \mathbf{Q}_\ell(d)) \\ \mathrm{Tr} \downarrow \\ \mathbf{Q}_\ell. \end{array}$$

We prove (1.4.7.8). The proof goes similarly as that of [26, Theorem 6.5.1]. We identify $\mathrm{Gr}_{\mathbb{F}}^d \mathbf{K}((X_L \times_L X'_L)^\sim)_{\mathbf{Q}}$ with $\mathrm{Gr}_0^{\mathbb{F}} \mathbf{G}((X_L \times_L X'_L)^\sim)_{\mathbf{Q}}$ by the canonical isomorphism, cf. [26, Lemma 2.1.4]. Since we are assuming that \mathbf{X} is strictly semi-stable, the log product $(X \times_{\mathbf{T}} X')^\sim$ is regular. Hence, the restriction map $\mathrm{Gr}_{\mathbb{F}}^d \mathbf{K}((X \times_{\mathbf{T}} X')^\sim) \rightarrow \mathrm{Gr}_{\mathbb{F}}^d \mathbf{K}((X_L \times_L X'_L)^\sim)$ is a surjection and the class $[\tilde{\Gamma}] \in \mathrm{Gr}_{\mathbb{F}}^d \mathbf{K}((X_L \times_L X'_L)^\sim)$ is lifted to an element $[\tilde{\Gamma}] \in \mathrm{Gr}_{\mathbb{F}}^d \mathbf{K}((X \times_{\mathbf{T}} X')^\sim)$. We define the class $[\tilde{\Gamma}] \in H^{2d}((X \times_{\mathbf{T}} X')^\sim, \mathbf{Q}_\ell(d))$ as the Chern character. Since the Chern character is compatible with the pull-back, the class $[\tilde{\Gamma}] \in H^{2d}((X_L \times_L X'_L)^\sim, \mathbf{Q}_\ell(d))$ on the top is the restriction of the class $[\tilde{\Gamma}] \in H^{2d}((X \times_{\mathbf{T}} X')^\sim, \mathbf{Q}_\ell(d))$. Further the trace map $H^{2d}(X_t, \mathbf{Q}_\ell(d)) \rightarrow \mathbf{Q}_\ell$ is the composition of the canonical map $H^{2d}(X_t, \mathbf{Q}_\ell(d)) \rightarrow H_{\log}^{2d}(X_t, \mathbf{Q}_\ell(d))$ with the trace map $H_{\log}^{2d}(X_t, \mathbf{Q}_\ell(d)) \rightarrow \mathbf{Q}_\ell$. Hence the left hand side of the equality (1.4.7.8) is the image of $[\tilde{\Gamma}] \in H^{2d}((X \times_{\mathbf{T}} X')^\sim, \mathbf{Q}_\ell(d))$ in the second line of the diagram (1.4.7.9) with log removed everywhere. Thus the equality (1.4.7.8) follows from the compatibility of the trace map with the degree map [26, Lemma 6.5.4].

We reduce the proof of (1.4.7.2) to the case where \mathbf{X} and \mathbf{X}' are strictly semi-stable. As in Corollary 1.4.5, we consider the semi-stable modifications $X_\Sigma \rightarrow X$ and $X'_\Sigma \rightarrow X'$ constructed in Lemma 1.2.1. The isomorphism $\iota_t: X_t \rightarrow X'_t$ induces an isomorphism $\iota_{t\Sigma}: (X_\Sigma)_t \rightarrow (X'_\Sigma)_t$. Thus, by Corollary 1.4.5, it is reduced to the strictly semi-stable case. \square

1.5. Log stalks. — In the next subsection, we prove an important complement Proposition 1.6.2 to the log Lefschetz trace formula Theorem 1.4.7. As a preliminary, we briefly recall some elementary terminology on log points and the stalks of a tamely ramified sheaf at a log geometric point. For more systematic account, we refer to [19, Section 4]. A reader familiar with the generalities on log schemes may skip this subsection.

Let t be the spectrum of a field F . If t is endowed with the log structure defined by the chart $\mathbf{N} \rightarrow F$ sending 1 to 0, we call t a log point. Let \bar{t} be the spectrum of a separably closed field F of characteristic $p \geq 0$. If \bar{t} is endowed with the log structure defined by the chart $\mathbf{Z}_{(p)} \cap [0, \infty) \rightarrow F$ sending any element $a > 0$ to 0, we call \bar{t} a log geometric point. For a log scheme Y , we call a morphism $t \rightarrow Y$ from a log point a log point of Y . Similarly, we call a morphism $\bar{t} \rightarrow Y$ from a log geometric point a log geometric point of Y .

A typical example of log points and log geometric points are constructed as follows. Let \mathcal{O}_L be a discrete valuation ring and regard $T = \text{Spec } \mathcal{O}_L$ as a log scheme with the log structure defined by the closed point t . Then, the scheme t endowed with the pull-back log structure is a log point. Assume further \mathcal{O}_L is henselian and let L^{tr} denote the maximal tamely ramified extension of L . Then, the limit of the standard log structures on $\text{Spec } \mathcal{O}_{L'}$ for finite extensions L' of L in L^{tr} defines a structure of log geometric point on the closed point \bar{t} of $\text{Spec } \mathcal{O}_{L^{\text{tr}}}$. A morphism $T \rightarrow Y$ of log schemes define a log point $t \rightarrow Y$ and further a log geometric point $\bar{t} \rightarrow Y$.

For a log geometric point \bar{t} of a log scheme Y , the log strict localization $\tilde{Y}_{\bar{t}}$ is defined in [19, 4.5]. The definition of the log stalk $\mathcal{G}_{\bar{t}}$ of a sheaf \mathcal{G} on the Kummer étale site on a log scheme Y at a log geometric point \bar{t} of Y is given in [19, Definition 4.3]. We will make it explicit in a special case.

Let S be a regular noetherian scheme and D be a divisor with normal crossings. We put $W = S \setminus D$ and $j: W \rightarrow S$ the open immersion. Then, the log scheme S with the log structure $\mathcal{M}_S = \mathcal{O}_S \cap j_* \mathcal{O}_W^\times$ is log regular [25]. We consider a locally constant sheaf \mathcal{F} on W tamely ramified along D . The direct image $j_* \mathcal{F}$ on the Kummer étale site of S is a locally constant sheaf by Abhyankar's lemma. Let $g: Y \rightarrow S$ be a morphism of log schemes and we consider the pull-back $\mathcal{G} = g^* j_* \mathcal{F}$ to the Kummer étale site of Y .

Let $\bar{t} \rightarrow Y$ be a log geometric point and let \bar{s} denote the geometric point of S defined by the composition $\bar{t} \rightarrow Y \rightarrow S$. Let $\tilde{g}: \tilde{Y}_{\bar{t}} \rightarrow \tilde{S}_{\bar{s}}$ denote the map of the log strict localizations induced by g and $\tilde{j}: W \times_S \tilde{S}_{\bar{s}} \rightarrow \tilde{S}_{\bar{s}}$ denote the open immersion. The pull-back of \mathcal{F} on $W \times_S \tilde{S}_{\bar{s}}$ is a constant sheaf and hence the direct image $\tilde{\mathcal{F}} = \tilde{j}_*(\mathcal{F}|_{W \times_S \tilde{S}_{\bar{s}}})$ is a constant sheaf on the usual étale site of $\tilde{S}_{\bar{s}}$. The log stalk $\mathcal{G}_{\bar{t}}$ is canonically identified with the stalk $(\tilde{g}^* \tilde{\mathcal{F}})_{\bar{t}}$ at \bar{t} of the pull-back of the constant sheaf $\tilde{\mathcal{F}}$. The map $\tilde{g}: \tilde{Y}_{\bar{t}} \rightarrow \tilde{S}_{\bar{s}}$ induces an isomorphism $(j_* \mathcal{F})_{\bar{s}} \rightarrow \mathcal{G}_{\bar{t}}$ of log stalks.

We consider the log cospecialization map [32, (2.8) 6]. We will use it only in the following situation. Let $g: Y \rightarrow S$ be a morphism of log schemes and $\mathcal{G} = g^* j_* \mathcal{F}$ be as above. Let $V \subset Y$ be an open subscheme where the log structure is trivial and let $\bar{\eta}$ be a geometric point of V . Assume that the image of the log geometric point \bar{t} in Y lies in the closure of the image of $\bar{\eta}$. Then, by choosing a lifting of $\bar{\eta}$ in $\tilde{Y}_{\bar{t}}$, a log cospecialization

map $\mathcal{G}_{\bar{t}} \rightarrow \mathcal{G}_{\bar{\eta}}$ is defined as the usual cospecialization map $(\tilde{g}^* \tilde{\mathcal{F}})_{\bar{t}} \rightarrow (\tilde{g}^* \tilde{\mathcal{F}})_{\bar{\eta}}$. Let $\bar{\xi}$ be an intermediate geometric point of V such that the image of \bar{t} lies in the closure of the image of $\bar{\xi}$ and that the image of $\bar{\xi}$ lies in the closure of the image of $\bar{\eta}$. Then, by choosing liftings $\bar{\xi} \rightarrow \tilde{Y}_{\bar{\xi}}$ and $\bar{\eta} \rightarrow Y_{\bar{\xi}} \rightarrow \tilde{Y}_{\bar{\eta}}$ successively, we obtain the transitivity of cospecialization maps $\mathcal{G}_{\bar{t}} \rightarrow \mathcal{G}_{\bar{\xi}} \rightarrow \mathcal{G}_{\bar{\eta}}$.

The following compatibility of the cospecialization map with the pull-back will be used in the proof of Proposition 1.6.2.

Lemma 1.5.1. — *Let S be a regular noetherian scheme and $W = S \setminus D \subset S$ be the complement of a divisor D with normal crossings. We consider a commutative diagram*

$$\begin{array}{ccccccc} \bar{t} & \longrightarrow & t & \longrightarrow & Y & \xrightarrow{g} & S. \\ & & & & \uparrow h & \nearrow g' & \\ & & & & Y & & \end{array}$$

of morphisms of log schemes. We assume t is a log point and \bar{t} is a log geometric point.

Let $\tilde{h}: \tilde{Y}_{\bar{t}} \rightarrow \tilde{Y}_{\bar{\eta}}$ be the morphism on the log strict localization induced by h . Let $\bar{\eta}$ be a usual geometric point of an open subscheme $V \subset Y$ where the log structure is trivial. We take a lifting $\bar{\eta} \rightarrow \tilde{Y}_{\bar{\eta}}$ and let $\bar{h}: \bar{\eta} \rightarrow \tilde{Y}_{\bar{\eta}}$ be a morphism such that the diagram

$$(1.5.1.1) \quad \begin{array}{ccc} \bar{\eta} & \longrightarrow & \tilde{Y}_{\bar{\eta}} \\ \bar{h} \downarrow & & \downarrow \tilde{h} \\ \bar{\eta} & \longrightarrow & \tilde{Y}_{\bar{\eta}} \end{array}$$

is commutative.

Let \mathcal{F} be a locally constant sheaf on W tamely ramified along D and we put $\mathcal{G} = g^*(j_* \mathcal{F})$ and $\mathcal{G}' = g'^*(j_* \mathcal{F}) = h^* \mathcal{G}$. Then, for the isomorphism $\bar{h}^*: \mathcal{G}_{\bar{\eta}} \rightarrow \mathcal{G}'_{\bar{\eta}}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_{\bar{t}} & \xrightarrow{\text{cosp.}} & \mathcal{G}_{\bar{\eta}} \\ \parallel & & \downarrow \bar{h}^* \\ \mathcal{G}'_{\bar{t}} & \xrightarrow{\text{cosp.}} & \mathcal{G}'_{\bar{\eta}} \end{array}$$

Proof. — By the commutative diagram (1.5.1.1), we have a commutative diagram

$$\begin{array}{ccc} \tilde{g}^*(\tilde{\mathcal{F}})_{\bar{t}} & \xrightarrow{\text{cosp.}} & \tilde{g}^*(\tilde{\mathcal{F}})_{\bar{\eta}} \\ \parallel & & \downarrow \bar{h}^* \\ \tilde{g}'^*(\tilde{\mathcal{F}})_{\bar{t}} & \xrightarrow{\text{cosp.}} & \tilde{g}'^*(\tilde{\mathcal{F}})_{\bar{\eta}} \end{array}$$

Thus it follows from the descriptions of the log stalks and the cospecialization maps. \square

We consider a geometric situation. Let S be a regular noetherian scheme and $W = S \setminus D \subset S$ be the complement of a divisor D with normal crossings as above. Let $f: X \rightarrow S$ be a proper weakly semi-stable scheme such that the base change $X \times_S W \rightarrow W$ is smooth and let $f_U: U \rightarrow S$ be the restriction to the complement $U \subset X$ of a divisor $E \subset X$ with normal crossings relatively to S as in Lemma 1.1.6. Then, for an integer n invertible on S , by Lemma 1.1.6, the higher direct image $\mathcal{F} = R^q f_{U!} \mathbf{Z}/n\mathbf{Z}$ is locally constant on W and is tamely ramified along D .

Let $T = \text{Spec } \mathcal{O}_L$ be the spectrum of a discrete valuation ring with the log structure defined by the closed point $t \in T$ and $g: T \rightarrow S$ be a morphism of log schemes such that the image of the generic point is in W . Then, the pull-back $\mathcal{G} = g^* j_* \mathcal{F}$ is a locally constant sheaf on the Kummer étale site of T and the stalk $\mathcal{G}_{\bar{L}}$ at the geometric point defined by an algebraic closure \bar{L} is identified with $H_c^q(U_{\bar{L}}, \mathbf{Z}/n\mathbf{Z})$ by the usual proper base change theorem. By the proof of [33, Proposition (4.3)], we obtain a commutative diagram

$$(1.5.1.2) \quad \begin{array}{ccc} \mathcal{G}_{\bar{t}} & \xrightarrow{\text{cosp.}} & \mathcal{G}_{\bar{L}} \\ \downarrow & & \downarrow \\ H_{\log c}^q(U_{\bar{t}}, \mathbf{Z}/n\mathbf{Z}) & \longrightarrow & H_c^q(U_{\bar{L}}, \mathbf{Z}/n\mathbf{Z}) \end{array}$$

of isomorphisms.

1.6. Compatibility with cospecializations. — We prove a compatibility with cospecialization maps, that gives an important complement to the log Lefschetz trace formula. We consider the following data:

- (1.6.1.1a) Let Y be a log scheme and $V \subset Y$ be an open subscheme where the log structure is trivial. Let $h: Y \rightarrow Y$ be a morphism of log schemes satisfying $h(V) \subset V$.
- (1.6.1.1b) Let T be the spectrum $\text{Spec } \mathcal{O}_L$ of a discrete valuation ring \mathcal{O}_L regarded as a log scheme with the log structure defined by the closed point t . Let $T \rightarrow Y$ be a morphism of log schemes such that the image of the generic point $\text{Spec } L \in T$ is in V and that the map $t \rightarrow Y$ of log schemes is the same as the composition of $t \rightarrow Y \xrightarrow{h} Y$. Let \bar{t} be a log geometric point above the log point t .
- (1.6.1.1c) Let $\bar{\eta}$ be a geometric point of the strict henselization $V_{\bar{\xi}}$ of V at a geometric point $\bar{\xi}$ above the image $\xi \in V$ of $\text{Spec } L \rightarrow T$ and $\bar{h}: \bar{\eta} \rightarrow \bar{\eta}$ be an automorphism compatible with h .
- (1.6.1.1d) Let $f: X \rightarrow Y$ be a proper and weakly strictly semi-stable scheme of relative dimension d over Y such that the base change $X_V = X \times_Y V \rightarrow V$ is smooth. Let $D = D_1 + \cdots + D_n$ be a divisor of X with simple normal crossings relatively to Y .

The data above are summarized in the following diagram:

$$(1.6.1.1) \quad \begin{array}{ccccc} & & & & X \supset D \\ & & & & \downarrow \\ & & & & Y \\ & & T & \xrightarrow{\quad} & Y \\ & \nearrow & \uparrow & & \leftarrow h \\ & t & L & \xrightarrow{\quad} & V \\ & \uparrow & \uparrow & & \leftarrow h|_V \\ & \bar{t} & V_{\bar{\xi}} & & V_{\bar{\xi}} \\ & & \uparrow & & \uparrow \\ & & \bar{\eta} & \xleftarrow{\quad \bar{h}} & \bar{\eta} \end{array}$$

Let $U = X \setminus D$ be the complement and $f_U: U \rightarrow Y$ be the restriction of f . Let $U_V \subset X_V$ and $U'_V \subset X'_V$ denote the base changes of $U \subset X$ by the inclusion and the restriction $h|_V$ respectively. We consider the log product and the log blow-up $(X_V \times_V X'_V)^\sim \subset (X_V \times_V X'_V)'$ with respect to the pull-backs $(D_{1,V}, \dots, D_{n,V})$ and $(D'_{1,V}, \dots, D'_{n,V})$ by the inclusion and by $h|_V$ of (D_1, \dots, D_n) . Let $(D_V \times_V X'_V)'$, $(X_V \times_V D'_V)'$ $\subset (X_V \times_V X'_V)'$ denote the proper transforms of $D_V \times_V X'_V$ and of $X_V \times_V D'_V$ respectively.

We consider a closed subscheme $\tilde{\Gamma} \subset (X_V \times_V X'_V)^\sim$ flat of relative dimension d over V . Assume that the second projection $\text{pr}_2: \Gamma = \tilde{\Gamma} \cap (U_V \times_V U'_V) \rightarrow U'_V$ is proper. Then, the geometric fiber $\Gamma_{\bar{\eta}} \subset U_{\bar{\eta}} \times_{\bar{\eta}} U'_{\bar{\eta}}$ defines a linear map $\Gamma^*: H_c^q(U'_{\bar{\eta}}, \mathbf{Q}_\ell) \rightarrow H_c^q(U_{\bar{\eta}}, \mathbf{Q}_\ell)$ and the morphism $\text{id} \times \bar{h}: U_{\bar{\eta}} \rightarrow U'_{\bar{\eta}}$ induces an isomorphism $\bar{h}^*: H_c^q(U_{\bar{\eta}}, \mathbf{Q}_\ell) \rightarrow H_c^q(U'_{\bar{\eta}}, \mathbf{Q}_\ell)$. Consequently, the alternating sum

$$(1.6.1.2) \quad \text{Tr}(\Gamma^* \circ \bar{h}^*: H_c^*(U_{\bar{\eta}}, \mathbf{Q}_\ell)) = \sum_{q=0}^{2d} (-1)^q \text{Tr}(\Gamma^* \circ \bar{h}^*: H_c^q(U_{\bar{\eta}}, \mathbf{Q}_\ell)) \in \mathbf{Q}_\ell$$

is defined.

Let $\iota_t: X_t \rightarrow X'_t$ be the isomorphism defined by the assumption that the map $t \rightarrow Y$ is the same as the composition with $h: Y \rightarrow Y$. It induces an isomorphism $\iota_{t*}: H_{\log c}^q(U_t, \mathbf{Q}_\ell) \rightarrow H_{\log c}^q(U'_t, \mathbf{Q}_\ell)$ as in (1.4.2.2). Since $\Gamma_L \subset U_L \times_L U'_L$ induces $\Gamma_L^*: H_c^*(U'_L, \mathbf{Q}_\ell) \rightarrow H_c^*(U_L, \mathbf{Q}_\ell)$, the alternating sum $\text{Tr}(\Gamma^*: H_c^*(U_L, \mathbf{Q}_\ell))$ is defined by (1.4.2.3).

The following complement to the Lefschetz trace formula Theorem 1.4.7 will be used in the proof of a crucial step Proposition 7.3.4 of the proof of the conductor formula.

Proposition 1.6.2. — *Let the notation be as in the diagram (1.6.1.1). We take a lifting of the geometric point $\bar{\eta}$ to the log strict localization $\tilde{Y}_{\bar{\eta}}$ and assume that $\bar{h}: \bar{\eta} \rightarrow \eta$ is compatible with the morphism $\tilde{h}: \tilde{Y}_{\bar{\eta}} \rightarrow \tilde{Y}_{\eta}$ induced by h . Let Γ be a closed subscheme of $U_{\vee} \times_{\vee} U'_{\vee}$ flat of relative dimension d over \vee . We assume that the second projection $\text{pr}_2: \Gamma \rightarrow U'_{\vee}$ is proper.*

We assume the following condition:

(1.6.2.2) *There exist a regular noetherian scheme S , a proper weakly semi-stable scheme $\bar{f}: X_S \rightarrow S$, a divisor $D_S \subset X_S$ with normal crossings relatively to S and a morphism $g: Y_{\bar{\eta}} \rightarrow S$ from the usual strict localization satisfying the following conditions. The pull-back of $D \subset X$ over Y to $Y_{\bar{\eta}}$ is isomorphic to that of $D_S \subset X_S$. There exists a divisor D_S of S with simple normal crossings such that the pull-back of X_S to the complement $W = S \setminus D_S$ is smooth.*

Then, for the alternating sum (1.6.1.2), we have

$$(1.6.2.3) \quad \text{Tr}(\Gamma^* \circ \bar{h}^*: H_c^*(U_{\bar{\eta}}, \mathbf{Q}_{\ell})) = \text{Tr}(\Gamma^*: H_c^*(U_{\bar{L}}, \mathbf{Q}_{\ell})).$$

Proof. — By the definition of $\Gamma^*: H_c^*(U_{\bar{L}}, \mathbf{Q}_{\ell}) \rightarrow H_c^*(U_{\bar{\eta}}, \mathbf{Q}_{\ell})$, it suffices to show the commutativity of the diagram

$$\begin{array}{ccccc}
 H_c^*(U_{\bar{\eta}}, \mathbf{Q}_{\ell}) & \xrightarrow{\bar{h}^*} & H_c^*(U'_{\bar{\eta}}, \mathbf{Q}_{\ell}) & \xrightarrow{\Gamma^*} & H_c^*(U_{\bar{\eta}}, \mathbf{Q}_{\ell}) \\
 \uparrow & & \uparrow & & \uparrow \\
 H_c^*(U_{\bar{L}}, \mathbf{Q}_{\ell}) & & H_c^*(U'_{\bar{L}}, \mathbf{Q}_{\ell}) & \xrightarrow{\Gamma^*} & H_c^*(U_{\bar{L}}, \mathbf{Q}_{\ell}) \\
 \uparrow & \nearrow & & & \\
 H_{\log, c}^*(U_{\bar{\eta}}, \mathbf{Q}_{\ell}) & & & &
 \end{array}$$

where the non-horizontal arrows are the cospecialization maps. For the right square, it is a consequence of the compatibility of a correspondence with usual cospecializations.

We show the commutativity of the left quadrangle. By replacing Y by the strict localization, we may assume $Y = Y_{\bar{\eta}}$. We put $U_S = X_S \setminus D_S$ in (1.6.2.2) and let $f_W: U_W = U_S \times_S W \rightarrow W$ be the restriction of $f_S: X_S \rightarrow S$. We consider the smooth \mathbf{Q}_{ℓ} -sheaf $\mathcal{F} = R^q f_{W!} \mathbf{Q}_{\ell}$ on W tamely ramified along D_S . Let $j_W: W \rightarrow S$ denote the open immersion and define a smooth sheaf $\mathcal{G} = g^* j_{W*} \mathcal{F}$ on $Y_{\bar{\eta}}$. We consider the diagram

$$\begin{array}{ccccc}
 \mathcal{G}_{\bar{\eta}} & \xrightarrow{\text{cosp.}} & \mathcal{G}_{\bar{L}} & \xrightarrow{\text{cosp.}} & \mathcal{G}_{\bar{\eta}} \\
 \downarrow & & \downarrow & & \downarrow \\
 H_{\log, c}^q(U_{\bar{\eta}}, \mathbf{Q}_{\ell}) & \longrightarrow & H_c^q(U_{\bar{L}}, \mathbf{Q}_{\ell}) & \longrightarrow & H_c^q(U_{\bar{\eta}}, \mathbf{Q}_{\ell}).
 \end{array}$$

The right square is the usual commutative diagram for étale cohomology and the left is (1.5.1.2). We have a similar commutative diagram for U' . By the transitivity of cospecialization maps, the commutativity of the left quadrangle follows from Lemma 1.5.1. \square

2. Tamely ramified coverings

In this section, we give a definition for an étale morphism of schemes to be tamely ramified along the boundary. The purpose of studying tame ramification first is to define the wild ramification locus and to focus on it.

First, we formulate the definition of an unramified morphism to be tamely ramified along the boundary using proper modifications and log products in Section 2.1. We give a tameness criterion, Proposition 2.4.4, in terms of valuation rings in Section 2.4 after recalling tamely ramified extensions of valuation fields and the limit of proper modifications in Sections 2.2 and 2.3 respectively. We study the relation with Kummer coverings in 2.5. Finally, we give a criterion for a Galois covering to be tamely ramified in terms of inertia groups in 2.6.

Although we don't need to assume for schemes to be separated in a large part of this section, we will assume it for simplicity.

2.1. Tame ramification and log products. — Recall that a morphism of schemes $V \rightarrow U$ of finite type is said to be unramified if the diagonal map $\delta_V: V \rightarrow V \times_U V$ is an open immersion. We consider a separated scheme Y containing V as an open subscheme and introduce a notion that an unramified morphism $f: V \rightarrow U$ is tamely ramified with respect to Y .

Lemma 2.1.1. — *Let $f: V \rightarrow U$ be an unramified separated morphism of finite type of schemes and $j: V \rightarrow Y$ be an open immersion of separated schemes. Let $\mathcal{D} = (D_i)_{i \in I}$ be a finite family of Cartier divisors of Y such that $V \cap D_i = \emptyset$ for every $i \in I$.*

For a commutative diagram

$$(2.1.1.1) \quad \begin{array}{ccc} Y & \xleftarrow{j} & V \\ \downarrow & & \downarrow f \\ S & \xleftarrow{\quad} & U \end{array}$$

of separated schemes, let $(Y \times_S Y)_{\mathcal{D}}^{\sim}$ be the log product and define a closed subset $\Sigma_{V/U}^{\mathcal{D}} Y \subset Y$ to be the intersection $\Delta_Y^{\log} \cap \overline{W}$ of the log diagonal with the closure of the open and closed subscheme $W = (V \times_U V) \setminus \Delta_V \subset V \times_U V$ in the log product $(Y \times_S Y)_{\mathcal{D}}^{\sim}$.

Then, the closed subset $\Sigma_{V/U}^{\mathcal{D}} Y \subset Y$ is independent of the choice of a diagram (2.1.1.1).

For any morphisms $Y \leftarrow V \rightarrow U$ of schemes, we can complete them into a diagram (2.1.1.1) by putting $S = \text{Spec } \mathbf{Z}$.

Proof. — We consider a commutative diagram (2.1.1.1) with S replaced by another separated scheme S' and show that $\Sigma_{V/U}^{\mathcal{D}} Y \subset Y$ are the same. It suffices to consider the case where $S' = \text{Spec } \mathbf{Z}$. Hence, we may assume that there exists a morphism $S \rightarrow S'$ that makes the diagram

$$\begin{array}{ccc} Y & \longrightarrow & S \\ \downarrow & \nearrow & \uparrow \\ S' & \longleftarrow & U \end{array}$$

commutative. The canonical map $(Y \times_S Y)_{\mathcal{D}}^{\sim} \rightarrow (Y \times_{S'} Y)_{\mathcal{D}}^{\sim}$ is a closed immersion since it is a base change of the diagonal $S \rightarrow S \times_{S'} S$. Hence, the assertion follows. \square

Definition 2.1.2. — *Let $f: V \rightarrow U$ be an unramified separated morphism of finite type of schemes and $j: V \rightarrow Y$ be an open immersion of separated schemes.*

1. *For a finite family of Cartier divisors $\mathcal{D} = (D_i)_{i \in I}$ of Y such that $V \cap D_i = \emptyset$ for every $i \in I$, define a closed subset $\Sigma_{V/U}^{\mathcal{D}} Y \subset Y$ to be the intersection $\Delta_Y^{\log} \cap \overline{W}$ of the log diagonal with the closure of the open and closed subscheme $W = (V \times_U V) \setminus \Delta_V \subset V \times_U V$ in the log product $(Y \times_S Y)_{\mathcal{D}}^{\sim}$ as in Lemma 2.1.1 by taking a commutative diagram (2.1.1.1).*

Define a closed subset $\Sigma_{V/U}^+ Y \subset Y$ to be the intersection $\bigcap_{\mathcal{D}} \Sigma_{V/U}^{\mathcal{D}} Y \subset Y$ where $\mathcal{D} = (D_i)_{i \in I}$ runs through finite families of Cartier divisors of Y as above.

2. *We say $f: V \rightarrow U$ is tamely ramified with respect to Y , if there exists a proper scheme Y' over Y containing V as an open subscheme such that $\Sigma_{V/U}^+ Y' = \emptyset$.*

We will define the wild ramification locus $\Sigma_{V/U} Y$ as a closed subset of $\Sigma_{V/U}^+ Y$ in Definition 2.4.1. Since $\Sigma_{V/U}^{\mathcal{D}} Y \supset \Sigma_{V/U}^{\mathcal{D}'} Y$ for $\mathcal{D} \subset \mathcal{D}'$, there exists a finite family \mathcal{D} of Cartier divisors of Y such that $\Sigma_{V/U}^+ Y = \Sigma_{V/U}^{\mathcal{D}} Y$ if Y is quasi-compact. In particular, the condition $\Sigma_{V/U}^+ Y = \emptyset$ is equivalent to the existence of a finite family \mathcal{D} of Cartier divisors of Y as in Definition 2.1.2.1 satisfying $\Sigma_{V/U}^{\mathcal{D}} Y = \emptyset$.

Lemma 2.1.3. — *Let S be a scheme and let*

$$\begin{array}{ccccc} Y' & \xleftarrow{j'} & V' & \longrightarrow & U' \\ \downarrow g & & \downarrow & & \downarrow \\ Y & \xleftarrow{j} & V & \longrightarrow & U \end{array}$$

be a commutative diagram of separated schemes over S . Assume that $V \rightarrow U$ and $V' \rightarrow U'$ are unramified and that the canonical map $V' \rightarrow V \times_U U'$ is an immersion. Assume also that $j: V \rightarrow Y$

and $j' : V' \rightarrow Y'$ are open immersions. If $\mathcal{O}_{Y'} \rightarrow j'_* \mathcal{O}_{V'}$ is an injection, then we have $g(\Sigma_{V'/U}^+ Y') \subset \Sigma_{V/U}^+ Y$.

Proof. — Let $\mathcal{D} = (D_i)_{i \in I}$ be a finite family of Cartier divisors of Y such that $D_i \cap V = \emptyset$ for every $i \in I$. By the assumption that $\mathcal{O}_{Y'} \rightarrow j'_* \mathcal{O}_{V'}$ is injective, the pull-backs $g^* D_i$ defines a family $\mathcal{D}' = (D'_i)_{i \in I}$ of Cartier divisors of Y' satisfying $D'_i \cap V' = \emptyset$ for every $i \in I$. Hence, the morphism $(g \times g)^\sim : (Y' \times_S Y')_{\mathcal{D}'} \rightarrow (Y \times_S Y)_{\mathcal{D}}$ is defined. By the assumption that $V' \rightarrow V \times_U U'$ is an immersion, the inverse image of Δ_V by $V' \times_{U'} V' \rightarrow V \times_U V$ is $\Delta_{V'}$. Hence $V' \times_{U'} V' \setminus \Delta_{V'} \subset (Y' \times_S Y')_{\mathcal{D}'}$ is a subset of the inverse image of $V \times_U V \setminus \Delta_V$. Thus, we have $\overline{V' \times_{U'} V' \setminus \Delta_{V'}} \subset (g \times g)^{\sim -1}(\overline{V \times_U V \setminus \Delta_V})$ and $\Sigma_{V'/U}^+ Y' \subset g^{-1}(\Sigma_{V/U}^+ Y)$. By taking the intersection, the assertion follows. \square

Whether $\Sigma_{V/U}^+ Y$ is empty or not may depend on Y as the following example shows.

Example 2.1.4. — Let A be a ring where 2 is invertible and let $V = \text{Spec } A[\mathbb{T}_1^{\pm 1}, \mathbb{T}_2^{\pm 1}] \subset Y = \mathbf{A}_A^2 = \text{Spec } A[\mathbb{T}_1, \mathbb{T}_2]$. We define an action of a cyclic group G of order 4 by $\mathbb{T}_1 \mapsto -\mathbb{T}_2, \mathbb{T}_2 \mapsto \mathbb{T}_1$. Then, V is a G -torsor over $U = V/G$. For the blow-up Y' of Y at the 0-section, an elementary computation shows that $\Sigma_{V'/U}^+ Y' = \emptyset$ while $\Sigma_{V/U}^+ Y$ consists of the 0-section of Y .

2.2. Tame ramified extension of valuation fields. — We will study tame ramification defined in the previous subsection in detail in Section 2.4. As preliminaries, we first study tamely ramified extensions of valuation fields in this subsection and limit of compactifications in the next subsection.

We recall the definition of tamely ramified extensions of valuation fields. For generality on valuation rings, we refer to [5, Chapitre 6] and [44, Chapter VI]. If L is a finite separable extension of a field K and if B is the integral closure in L of a valuation ring A of K , then the map from the finite set of maximal ideals of B to the set of valuation rings of L dominating A sending a maximal ideal \mathfrak{m} to the local ring $B_{\mathfrak{m}}$ is a bijection [5, Chapitre 6, Section 8, n^o 3, Remarque].

Definition 2.2.1. — *Let L be a finite separable extension of a field K and B be a valuation ring of L . We put $A = B \cap K$. Let A^{sh} and B^{sh} be strict henselizations and let K^{sh} and L^{sh} be the fraction fields.*

We say that L is tamely ramified over K with respect to B if the degree $[L^{\text{sh}} : K^{\text{sh}}]$ is invertible in B .

If the residue field of B is of characteristic 0, an arbitrary finite separable extension L over K is tamely ramified with respect to B .

We recall some standard terminologies on inertia subgroups. Let M be a finite Galois extension of a field K of Galois group $G = \text{Gal}(M/K)$ and A be a valuation

ring of K . Let $C \subset M$ be the integral closure of A and \mathfrak{m} be a maximal ideal of C . The subgroup $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$ is called the decomposition group of \mathfrak{m} and $I = \text{Ker}(D \rightarrow \text{Aut}(C/\mathfrak{m}))$ the inertia group. The local ring $C_{\mathfrak{m}}$ is a valuation ring. Let $C_{\mathfrak{m}}^{\text{sh}}$ be a strict henselization and let M^{sh} be the fraction field of $C_{\mathfrak{m}}^{\text{sh}}$. Then, the I -fixed part of $C_{\mathfrak{m}}^{\text{sh}}$ is a strict henselization A^{sh} of A .

We regard the value group $\Gamma_A = K^\times/A^\times$ as a subgroup of $\Gamma_{\mathfrak{m}} = M^\times/C_{\mathfrak{m}}^\times$. Then the map $I \times M^\times \rightarrow (C/\mathfrak{m})^\times$ defined by $(\sigma, c) \mapsto \sigma(c)/c$ induces a pairing $I \times \Gamma_{\mathfrak{m}}/\Gamma_A \rightarrow (C/\mathfrak{m})^\times$.

Lemma 2.2.2. — *Let M be a finite Galois extension of a field K of Galois group $G = \text{Gal}(M/K)$ and A be a valuation ring of K . Let $I \subset G$ be the inertia group of a maximal ideal \mathfrak{m} of the integral closure $C \subset M$ of A . Let p be the characteristic of the residue field A/\mathfrak{m}_A .*

1. ([44, Chapter VI, §12, Corollary of Theorem 24]) *If $p = 0$, the pairing $I \times \Gamma_{\mathfrak{m}}/\Gamma_A \rightarrow (C/\mathfrak{m})^\times$ is a perfect pairing of finite abelian groups.*
2. ([Loc. cit. Theorems 24 and 25]) *Assume $p > 0$ and let $(\Gamma_{\mathfrak{m}}/\Gamma_A)'$ denote the prime-to- p part of $\Gamma_{\mathfrak{m}}/\Gamma_A$. Then, the kernel P of the induced map $I \rightarrow \text{Hom}(\Gamma_{\mathfrak{m}}/\Gamma_A, (C/\mathfrak{m})^\times)$ is the unique p -Sylow subgroup of I and the induced pairing $I/P \times (\Gamma_{\mathfrak{m}}/\Gamma_A)' \rightarrow (C/\mathfrak{m})^\times$ is a perfect pairing of finite abelian groups.*

For the rest of this subsection, in the case $p = 0$, we put $(\Gamma_{\mathfrak{m}}/\Gamma_A)' = \Gamma_{\mathfrak{m}}/\Gamma_A$ and $P = 1$.

Corollary 2.2.3. — *Let K be a field and L be a finite separable extension of K . Let A be a valuation ring of K and B be the integral closure of A in L . Then the following conditions are equivalent:*

- (1) *For every maximal ideal \mathfrak{m} of B , L is tamely ramified over K with respect to $B_{\mathfrak{m}}$.*
- (2) *There exist non-zero elements t_1, \dots, t_n of the maximal ideal \mathfrak{m}_A and integers m_1, \dots, m_n invertible in A such that the normalization B' of A in $L[S_1, \dots, S_n]/(S_1^{m_1} - t_1, \dots, S_n^{m_n} - t_n)$ is finite étale over the normalization A' of A in $K[S_1, \dots, S_n]/(S_1^{m_1} - t_1, \dots, S_n^{m_n} - t_n)$.*

Proof. — (1) \Rightarrow (2): We may assume L is a Galois extension of K . Let \mathfrak{m} be a maximal ideal of B and let L^{sh} be the fraction field of the strict henselization of B at a geometric point above \mathfrak{m} and define K^{sh} similarly. By Lemma 2.2.2, L^{sh} is an abelian extension of K^{sh} and the pairing $\text{Gal}(L^{\text{sh}}/K^{\text{sh}}) \times (\Gamma_B/\Gamma_A)' \rightarrow (B/\mathfrak{m})^\times$ is a perfect pairing of finite abelian groups of order prime to p . We take an isomorphism $\mathbf{Z}/m_1\mathbf{Z} \oplus \dots \oplus \mathbf{Z}/m_n\mathbf{Z} \rightarrow (\Gamma_B/\Gamma_A)'$ and its lifting $\gamma: \mathbf{Z}^n \rightarrow \Gamma_B$. Let $e_1, \dots, e_n \in \mathbf{Z}^n$ be the standard basis and we take elements $t_1, \dots, t_n \in \mathfrak{m}_A$ satisfying $v_A(t_i) = m_i\gamma(e_i)$ for $i = 1, \dots, n$. Then, we have $L^{\text{sh}} = K^{\text{sh}}(t_1^{1/m_1}, \dots, t_n^{1/m_n})$ and the assertion follows.

(2) \Rightarrow (1): Since $[L^{\text{sh}}:K^{\text{sh}}]$ divides $m_1 \cdots m_n$, the extension L is tamely ramified. \square

We give a criterion for a finite separable extension of valuation field to be tamely ramified.

Proposition 2.2.4. — *Let L be a finite separable extension of a field K and $B \subset L$ be a valuation ring of L . We put $A = B \cap K$, $U = \text{Spec } K \subset S = \text{Spec } A$ and $V = \text{Spec } L \subset Y = \text{Spec } B$. Then, the following conditions (1)–(4) are equivalent:*

- (1) L is tamely ramified over K with respect to B .
- (2) There exists a finite family $\mathcal{D} = (D_i)_{i \in I}$ of Cartier divisors of Y such that the intersection $\Sigma_{V/U}^{\mathcal{D}} Y = \overline{W} \cap \Delta_Y^{\log}$ with the closure of $W = V \times_U V \setminus \Delta_V$ in the log product $(Y \times_S Y)_{\mathcal{D}}^{\sim}$ is empty.
- (3) Let M be an arbitrary finite separable extension of L and $\sigma : L \rightarrow M$ be a morphism over K different from the inclusion. For an arbitrary valuation ring C of M dominating B and $\sigma(B)$, there exists a non-zero element $b \in B$ such that $\sigma(b)/b \not\equiv 1 \pmod{\mathfrak{m}_C}$.
- (4) Let M be a finite Galois extension of K of Galois group $G = \text{Gal}(M/K)$ containing L as a subextension and \mathfrak{m} be a maximal ideal of the integral closure $C \subset M$ of B such that $B \cap \mathfrak{m} = \mathfrak{m}_B$. Then, the subgroup $H = \text{Gal}(M/L) \subset G = \text{Gal}(M/K)$ contains the conjugates of the p -Sylow subgroup P of the inertia group I of \mathfrak{m} .

In (2), we did not say that $V \rightarrow U$ is tamely ramified with respect to Y because the canonical map $V \rightarrow Y$ may not be an open immersion.

Proof. — (2) \Rightarrow (3) Let $\mathcal{D} = (D_i)_{i \in I}$ be a finite family of Cartier divisors of $Y = \text{Spec } B$ such that $\Sigma_{U/V}^{\mathcal{D}} Y = \emptyset$. Since C dominates B and $\sigma(B)$, the compositions $L^\times \rightarrow M^\times \rightarrow \Gamma_C$ and $L^\times \xrightarrow{\sigma} M^\times \rightarrow \Gamma_C$ are equal. Hence, the map $\gamma = (i, \sigma) : Z = \text{Spec } C \rightarrow Y \times_S Y$ induces a map $\tilde{\gamma} : Z \rightarrow (Y \times_S Y)_{\mathcal{D}}^{\sim}$ to the log product. Since σ is different from the inclusion, the image $\tilde{\gamma}(Z)$ is in the closure \overline{W} of $W = V \times_U V \setminus \Delta_V$. Hence, we have $\gamma(Z) \cap \Delta_Y^{\log} \subset \Sigma_{U/V}^{\mathcal{D}} Y = \emptyset$.

For $i \in I$, let $b_i \in B$ be an element defining the divisor D_i . Then, the closed subscheme $\Delta_Y^{\log} \subset (Y \times_S Y)_{\mathcal{D}}^{\sim}$ is defined by the ideal $(b \otimes 1 - 1 \otimes b; b \in B, (b_i \otimes 1)/(1 \otimes b_i) - 1; i \in I)$. Hence the closed subscheme of Z defined by $(\sigma(b) - b; b \in B, \sigma(b_i)/b_i - 1; i \in I)$ is empty. Namely the ideal of C generated by $\sigma(b) - b$ for $b \in B$ and $\sigma(b_i)/b_i - 1$ for $i \in I$ contains a unit. Thus the assertion is proved.

(3) \Rightarrow (2) We put $W = V \times_U V \setminus \Delta_V = \coprod_{j \in J} \text{Spec } M_j$ where M_j are fields. We regard M_j as an extension of L by the map defined by the first projection and let $\sigma_j : L \rightarrow M_j$ be the map defined by the second projection. For each $j \in J$, the set $\{C_{ij} \mid i \in I_j\}$ of valuation rings of M_j dominating both B and $\sigma(B)$ is a finite set. For each C_{ij} , take a non-zero element $b_{ij} \in B$ such that $\sigma_j(b_{ij})/b_{ij} \not\equiv 1 \pmod{\mathfrak{m}_{C_{ij}}}$ and let D_{ij} be the Cartier divisor of Y defined by b_{ij} . We put $I = \coprod_{j \in J} I_j$ and let $\mathcal{D} = (D_{ij})_{j \in J, i \in I_j}$ be the family of Cartier divisors.

We show that $\Sigma_{V/U}^{\mathcal{D}} Y$ is empty. Let Z_j be the closure of $\text{Spec } M_j$ in the log product $(Y \times_S Y)_{\mathcal{D}}^{\sim}$. Then, we have $\overline{W} = \bigcup_{j \in J} Z_j$ and $\Sigma_{V/U}^{\mathcal{D}} Y = \bigcup_{j \in J} (Z_j \cap \Delta_Y^{\log})$. Hence, if $\Sigma_{V/U}^{\mathcal{D}} Y$ was not empty, the intersection $Z_j \cap \Delta_Y^{\log}$ would contain the closed point y of Y for some index $j \in J$. Take a valuation ring C of M_j dominating the local ring $\mathcal{O}_{Z_j, y}$. Then, C dominates B and $\sigma_j(B)$. Hence it is equal to C_{ij} for some $i \in I_j$ and $\sigma_j(b_{ij})/b_{ij} - 1$ is a unit of

$C = C_{ij}$. On the other hand, since the image y of the closed point of C is in Δ_Y^{\log} , the ideal $(\sigma_j(b) - b; b \in B, \sigma_j(b_{ik})/b_{ik} - 1; k \in J, i \in I_k)$ is different from C , as in the proof of (2) \Rightarrow (3). Thus, we obtain a contradiction.

(1) \Rightarrow (3) By replacing K by an unramified extension, we may assume that the residue field of L is a purely inseparable extension of the residue field of K . We may assume M is a Galois extension and extend $\sigma : L \rightarrow M$ to an element of the Galois group $G = \text{Gal}(M/K)$. Then, since both C and $\sigma(C)$ dominates $\sigma(B)$, there exists $\tau \in G$ such that $\tau(C) = \sigma(C)$ and that $\tau|_{\sigma(L)} : \sigma(L) \rightarrow M$ is the inclusion. Replacing σ by $\tau^{-1}\sigma$ if necessary, we may assume $\sigma(C) = C$. Namely, by the assumption on the residue field, we may assume that σ is in the inertia group $I \subset G$ of the maximal ideal of \mathfrak{m}_C . By the assumption that $\sigma|_L \neq \text{id}_L$, it is not an element of the subgroup $H = \text{Gal}(M/L) \subset G$ corresponding to L .

We put $\bar{C} = C/\mathfrak{m}_C$ and we consider the perfect pairings $I/P \times (\Gamma_{\mathfrak{m}_C}/\Gamma_A)' \rightarrow \bar{C}^\times$ and $(I \cap H)/(P \cap H) \times (\Gamma_{\mathfrak{m}_C}/\Gamma_B)' \rightarrow \bar{C}^\times$. Since $[L^{\text{sh}} : K^{\text{sh}}] = [I : I \cap H]$ is invertible in B , the induced pairing $I/(I \cap H) \times (\Gamma_B/\Gamma_A)' \rightarrow \bar{C}^\times$ is perfect. Since $\sigma|_L$ is not the identity, there exists an element $b \in L^\times$ such that $\sigma(b)/b \not\equiv 1 \pmod{\mathfrak{m}_C}$.

(3) \Rightarrow (4) Let σ be an element of a conjugate $\tau P \tau^{-1}$ of P . We regard L as a subfield of M by $\tau|_L : L \rightarrow M$. Then, the maximal ideal $\mathfrak{m}' = \tau(\mathfrak{m})$ satisfies $\sigma(\mathfrak{m}') = \mathfrak{m}'$ and $\tau|_L^{-1}(\mathfrak{m}') = (\sigma \circ \tau|_L)^{-1}(\mathfrak{m}')$ is equal to $B \cap \mathfrak{m} = \mathfrak{m}_B$. If σ was not an element of H , the condition (3) would imply the existence of an element $b \in L^\times$ such that $\sigma(\tau(b))/\tau(b) \not\equiv 1 \pmod{\mathfrak{m}'}$. This implies that the order of σ is not a power of p . Thus, we get a contradiction.

(4) \Rightarrow (1) Let M be a finite Galois extension of K containing L as a subfield and we put $G = \text{Gal}(M/K) \supset H = \text{Gal}(M/L)$. We take a maximal ideal \mathfrak{m} of the integral closure C above the maximal ideal of B and let I be the inertia group of \mathfrak{m} . Then, the inertia group I is identified with the Galois group $\text{Gal}(M^{\text{sh}}/K^{\text{sh}})$ and $H \cap I$ is the subgroup of I corresponding to the field L^{sh} . Hence $P \subset H$ implies that $[L^{\text{sh}} : K^{\text{sh}}] = [I : I \cap H]$ is prime to p . \square

2.3. Limit of compactifications and valuation rings. — We study local rings of the limit of compactifications. Let S be a separated noetherian scheme and U be a separated scheme of finite type over S . We consider the category $\mathcal{C}_{U/S}$ of compactifications of U over S . Namely an object of $\mathcal{C}_{U/S}$ is a pair (X, j) consisting of a proper scheme X over S and an open immersion $j : U \rightarrow X$ over S . A morphism $f : (X', j') \rightarrow (X, j)$ is a morphism $f : X' \rightarrow X$ of schemes over S such that $f \circ j' = j$. In the following, we omit j from the notation and write simply X for a compactification (X, j) .

Lemma 2.3.1. — *Let S be a separated noetherian scheme and U be a separated scheme of finite type over S .*

1. *The category $\mathcal{C}_{U/S}$ is cofiltered. In particular, it is non-empty.*
2. *The objects containing U as the complement of a Cartier divisor are cofinal in $\mathcal{C}_{U/S}$.*

Proof. — By Nagata's embedding theorem [31], the category $\mathcal{C}_{U/S}$ is non-empty. Since a blow-up $X' \rightarrow X$ is proper, the objects containing U as the complement of a Cartier divisor are cofinal in $\mathcal{C}_{U/S}$. For objects (X, j) and (X', j') of $\mathcal{C}_{U/S}$, a morphism $X \rightarrow X'$ is unique if $\mathcal{O}_X \rightarrow j_* \mathcal{O}_{U'}$ is injective. If X and X' are objects of $\mathcal{C}_{U/S}$, the schematic closure of the diagonal map $U \rightarrow X \times_S X'$ is an object of $\mathcal{C}_{U/S}$. Hence, the category $\mathcal{C}_{U/S}$ is cofiltered. \square

We consider the projective limit $\tilde{X} = \varprojlim_{X \in \mathcal{C}_{U/S}} X$ in the category of locally ringed spaces. The underlying topological space \tilde{X} is known to be quasi-compact [9, Theorem 5.14]. For a point $\tilde{x} = (x_X) \in \tilde{X}$, we have $\mathcal{O}_{\tilde{X}, \tilde{x}} = \varinjlim_{X \in \mathcal{C}_{U/S}} \mathcal{O}_{X, x_X}$. We will describe the limit \tilde{X} and the local rings $\mathcal{O}_{\tilde{X}, \tilde{x}}$ in terms of valuation rings.

Definition 2.3.2. — Let U be a scheme, $u \in U$ be a point and A be a valuation ring of the residue field $\kappa(u)$.

1. We say A is U -external, if $A \not\subseteq \kappa(u)$ and if there exists no intermediate ring $A \subset A' \subsetneq \kappa(u)$ such that the map $u \rightarrow U$ is extended to $\text{Spec } A' \rightarrow U$.

2. Let $U \rightarrow S$ be a separated morphism of schemes. We say A is S -integral, if the composition $u \rightarrow U \rightarrow S$ is extended to a morphism $\text{Spec } A \rightarrow S$.

Let $u \in U$ and $A \subset \kappa(u)$ be an S -integral valuation ring. Then, for an object X of $\mathcal{C}_{U/S}$, the inclusion $u \rightarrow U$ is uniquely extended to a morphism $\text{Spec } A \rightarrow X$ over S by the valuative criterion of properness. The images $x_A \in X \setminus U$ of the closed point of $\text{Spec } A$ define a point $\tilde{x}_A = (x_A)$ of the projective limit $\tilde{X} = \varprojlim_{X \in \mathcal{C}_{U/S}} X$. Thus, we obtain a natural map

$$(2.3.2.1) \quad \coprod_{u \in U} \{S\text{-integral valuation ring of } \kappa(u)\} \rightarrow \tilde{X}.$$

An S -integral valuation ring A of $\kappa(u)$ is U -external if and only if $\{u\} = U \times_X \text{Spec } A \subsetneq \text{Spec } A$ for an object X of $\mathcal{C}_{U/S}$. Consequently, the map (2.3.2.1) induces

$$(2.3.2.2) \quad \coprod_{u \in U} \{U\text{-external and } S\text{-integral valuation ring of } \kappa(u)\} \rightarrow \tilde{X} \setminus U.$$

We show that the map (2.3.2.2) is a bijection.

Lemma 2.3.3. — (Cf. [9, 5.4]) Let S be a separated noetherian scheme and U be a separated scheme of finite type over S . Let $\tilde{x} = (x_X)_X \in \tilde{X} \setminus U$ be a point in the complement and put $\mathcal{O}_{\tilde{X}, \tilde{x}} = \varinjlim \mathcal{O}_{X, x_X}$.

Then, there exists a unique point $u \in U$ such that $U \times_X \text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}} = \text{Spec } \mathcal{O}_{U, u}$ for every object X of $\mathcal{C}_{U/S}$. The canonical map $\mathcal{O}_{\tilde{X}, \tilde{x}} \rightarrow \mathcal{O}_{U, u}$ is injective and its image is the inverse image of a U -external and S -integral valuation ring A of $\kappa(u)$. For each object X of $\mathcal{C}_{U/S}$, the point x_X is the image of the closed point of $\text{Spec } A$ by the unique map $\text{Spec } A \rightarrow X$ over S extending $u \rightarrow U$.

Proof. — Let $\tilde{x} = (x_X)_X \in \tilde{X} \setminus U$ be a point in the complement. For a morphism $X' \rightarrow X$ of $\mathcal{C}_{U/S}$, we have $U \times_X X' = U$ if U is dense in X' . Hence the inverse image $U \times_X \text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}}$ is independent of X . Thus, to show the existence of $u \in U$ such that $U \times_X \text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}} = \text{Spec } \mathcal{O}_{U, u}$, it suffices to show the existence for one object $X \in \mathcal{C}_{U/S}$. Take an object $X \in \mathcal{C}_{U/S}$. We may assume U is the complement of a Cartier divisor $D \subset X$.

By [9, Proposition 5.12], the local ring $\mathcal{O}_{\tilde{X}, \tilde{x}}$ is $I_{\tilde{x}}$ -valuative for $I_{\tilde{x}} = \mathcal{I}_{D, x_X} \mathcal{O}_{\tilde{X}, \tilde{x}}$ in the terminology loc. cit. Hence, $\Gamma(U \times_X \text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}}, \mathcal{O}_{\tilde{X}}) = \varinjlim_{X'} \Gamma(U \times_{X'} \text{Spec } \mathcal{O}_{X', x_{X'}}, \mathcal{O}_{X'})$ is a local ring further by [9, Proposition 5.11]. Since $\text{Spec } \mathcal{O}_{X', x_{X'}} \rightarrow X' \times_X \text{Spec } \mathcal{O}_{X, x_X}$ is a limit of open immersions, its restriction $U \times_{X'} \text{Spec } \mathcal{O}_{X', x_{X'}} \rightarrow U \times_X \text{Spec } \mathcal{O}_{X, x_X}$ is also a limit of open immersions. Hence the local ring $\Gamma(U \times_X \text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}}, \mathcal{O}_{\tilde{X}})$ is a localization of $\Gamma(U \times_X \text{Spec } \mathcal{O}_{X, x_X}, \mathcal{O}_X)$ and is equal to the local ring $\mathcal{O}_{U, u}$ at a point $u \in U$. Further by [9, Proposition 5.11], the canonical map $\mathcal{O}_{\tilde{X}, \tilde{x}} \rightarrow \mathcal{O}_{U, u}$ is an injection and its image is the inverse image of a valuation ring A of $\kappa(u)$ by the surjection $\mathcal{O}_{U, u} \rightarrow \kappa(u)$.

Since $\Gamma(U \times_X \text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}}, \mathcal{O}_{\tilde{X}}) = \mathcal{O}_{U, u}$, the valuation ring A is U -external. Since A is X -integral, it is S -integral. The image x_X of the closed point of $\text{Spec } \mathcal{O}_{\tilde{X}, \tilde{x}}$ is the same as the image of the closed point of $\text{Spec } A$ by the induced map $\text{Spec } A \rightarrow X$. \square

Corollary 2.3.4. — *The map (2.3.2.2) is a bijection. The inverse is defined by sending \tilde{x} to the valuation ring $\mathcal{O}_{\tilde{X}, \tilde{x}}/\mathfrak{m}_u = \text{Image}(\mathcal{O}_{\tilde{X}, \tilde{x}} \rightarrow \kappa(u)) \subset \kappa(u)$, in the notation of Lemma 2.3.3.*

Proof. — Let $u \in U$ be a point and A be a U -external and S -integral valuation ring of $\kappa(u)$. Let $\tilde{x} \in \tilde{X}$ be the point defined by the images of the closed point of $\text{Spec } A$. We consider $A' = \mathcal{O}_{\tilde{X}, \tilde{x}}/\mathfrak{m}_{u'} \subset \kappa(u')$ as in Lemma 2.3.3. We have a natural local homomorphism $\mathcal{O}_{\tilde{X}, \tilde{x}} \rightarrow A$. By [9, Proposition 5.11], the ideal $\mathfrak{m}_{u'} \subset \mathcal{O}_{\tilde{X}, \tilde{x}}$ is the intersection $\bigcap_n I_{\tilde{x}}^n$. Since A is U -external, we have $\bigcap_n I_{\tilde{x}}^n A = 0$. Hence, it induces a local homomorphism $A' \rightarrow A$. Since A' is also U -external, we obtain $u' = u$. Further, since the valuation ring $A \subset \kappa(u)$ dominates $A' \subset \kappa(u)$, we obtain $A' = A$. \square

2.4. Tame ramification and valuation rings. — We give a criterion for tame ramification in terms of valuation rings, in Proposition 2.4.4. We slightly generalize Definition 2.1.2.2.

Definition 2.4.1. — *Let $f: V \rightarrow U$ be an unramified morphism of finite type of separated schemes.*

1. *Let Y be a separated scheme of finite type containing V as an open subscheme. We define the wild ramification locus $\Sigma_{V/U} Y$ to be the closed subset*

$$\Sigma_{V/U} Y = \bigcap_{Y'} \pi_{YY'}(\Sigma_{V/U}^+ Y')$$

where $\pi_{YY'}: Y' \rightarrow Y$ runs through objects of $\mathcal{C}_{V/Y}$.

2. Let \mathbf{T} be a separated noetherian scheme and assume that \mathbf{V} is a separated scheme of finite type over \mathbf{T} . We say $f: \mathbf{V} \rightarrow \mathbf{U}$ is tamely ramified with respect to \mathbf{T} , if there exists an object \mathbf{Y} of $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$ such that $\Sigma_{\mathbf{V}/\mathbf{U}}\mathbf{Y}$ is empty.

Since the category $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$ is cofiltered, the map $f: \mathbf{V} \rightarrow \mathbf{U}$ is tamely ramified with respect to \mathbf{T} if and only if $\Sigma_{\mathbf{V}/\mathbf{U}}\mathbf{Y}$ is empty for every object of $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$. If \mathbf{U} is a scheme over \mathbf{S} and if finite étale morphisms $\mathbf{V} \rightarrow \mathbf{U}$ and $\mathbf{V}' \rightarrow \mathbf{U}$ are tamely ramified with respect to \mathbf{S} , then the fiber product $\mathbf{V} \times_{\mathbf{U}} \mathbf{V}' \rightarrow \mathbf{U}$ is also tamely ramified with respect to \mathbf{S} . In particular, if a finite étale morphism $\mathbf{V} \rightarrow \mathbf{U}$ of connected normal schemes is tamely ramified with respect to \mathbf{S} , its Galois closure $\mathbf{W} \rightarrow \mathbf{U}$ is also tamely ramified with respect to \mathbf{S} .

Lemma 2.4.2. — *Let \mathbf{T} be a separated noetherian scheme, \mathbf{V} be a separated scheme of finite type over \mathbf{T} and let $f: \mathbf{V} \rightarrow \mathbf{U}$ be an unramified morphism of finite type of separated schemes. Then, the objects \mathbf{Y} of $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$ such that there exists a finite family \mathcal{D} of Cartier divisors satisfying $\Sigma_{\mathbf{V}/\mathbf{U}}\mathbf{Y} = \Sigma_{\mathbf{V}/\mathbf{U}}^{\mathcal{D}}\mathbf{Y}$ are cofinal in $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$.*

Proof. — For an object \mathbf{Y} of $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$, let $\pi_{\mathbf{Y}}: \tilde{\mathbf{Y}} = \varprojlim_{\mathbf{Y}' \in \mathcal{C}_{\mathbf{V}/\mathbf{T}}} \mathbf{Y}' \rightarrow \mathbf{Y}$ denote the projection. Since $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$ is cofiltered and $\tilde{\mathbf{Y}}$ is quasi-compact [9, Theorem 5.14], the objects \mathbf{Y} such that $\pi_{\mathbf{Y}}^{-1}(\Sigma_{\mathbf{V}/\mathbf{U}}^+\mathbf{Y}) = \bigcap_{\mathbf{Y}'} \pi_{\mathbf{Y}'}^{-1}(\Sigma_{\mathbf{V}/\mathbf{U}}^+\mathbf{Y}')$ and $\mathbf{Y} = \pi_{\mathbf{Y}}(\tilde{\mathbf{Y}})$ are cofinal in $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$ by Lemma 2.1.3. For such \mathbf{Y} , we have $\Sigma_{\mathbf{V}/\mathbf{U}}^+\mathbf{Y} = \pi_{\mathbf{Y}}(\pi_{\mathbf{Y}}^{-1}(\Sigma_{\mathbf{V}/\mathbf{U}}^+\mathbf{Y})) = \bigcap_{\mathbf{Y}'} \pi_{\mathbf{Y}}(\pi_{\mathbf{Y}'}^{-1}(\Sigma_{\mathbf{V}/\mathbf{U}}^+\mathbf{Y}'))$. Hence the assertion follows from the quasi-compactness of \mathbf{Y} . \square

Lemma 2.4.3. — *Let \mathbf{T} be a separated noetherian scheme and \mathbf{V} be a separated scheme of finite type over \mathbf{T} . Let $f: \mathbf{V} \rightarrow \mathbf{U}$ be an unramified separated morphism of finite type of schemes. Let $\mathfrak{v} \in \mathbf{V}$ be a point and \mathbf{B} be a \mathbf{T} -integral valuation ring of $\kappa(\mathfrak{v})$. We consider the conditions:*

- (1) *There exists a proper scheme \mathbf{Y} over \mathbf{T} containing \mathbf{V} as an open subscheme such that the closed subset $\Sigma_{\mathbf{V}/\mathbf{U}}^+\mathbf{Y} \subset \mathbf{Y}$ does not meet the image of the map $\text{Spec } \mathbf{B} \rightarrow \mathbf{Y}$.*
- (2) *The finite separable extension $\kappa(\mathfrak{v})$ over $\kappa(f(\mathfrak{v}))$ is tamely ramified with respect to \mathbf{B} .*

The condition (1) implies (2). If \mathbf{B} is \mathbf{V} -external (Definition 2.3.2.1), then the conditions (1) and (2) are equivalent.

Proof. — (1) \Rightarrow (2) The condition (1) implies that there exists a finite family $\mathcal{D} = (D_i)_{i \in I}$ of Cartier divisors of $\text{Spec } \mathbf{B}$ such that $\Sigma_{\mathfrak{v}/f(\mathfrak{v})}^{\mathcal{D}} \text{Spec } \mathbf{B}$ is empty, as the pull-back by $\text{Spec } \mathbf{B} \rightarrow \mathbf{Y}$. Then, the condition (2) is satisfied by Proposition 2.2.4 (2) \Rightarrow (1).

(2) \Rightarrow (1) Assume that \mathbf{B} is \mathbf{V} -external. By Proposition 2.2.4 (1) \Rightarrow (2), there exists a finite family $\mathcal{D} = (D_i)_{i \in I}$ of Cartier divisors of $\text{Spec } \mathbf{B}$ such that $\Sigma_{\mathfrak{v}/f(\mathfrak{v})}^{\mathcal{D}} \text{Spec } \mathbf{B}$ is empty. Let $\tilde{\mathbf{Y}} = \varprojlim_{\mathbf{Y} \in \mathcal{C}_{\mathbf{V}/\mathbf{T}}} \mathbf{Y}$ be the limit of compactifications and $\tilde{\mathfrak{y}} = (\mathfrak{y}_{\mathbf{Y}})_{\mathbf{Y}} \in \tilde{\mathbf{Y}}$ be the point corresponding to $\mathbf{B} \subset \kappa(\mathfrak{v})$. We take a non-zero divisor $f_i \in \mathbf{B}$ defining D_i and a lifting $\tilde{f}_i \in \mathcal{O}_{\tilde{\mathbf{Y}}, \tilde{\mathfrak{y}}}$ for each $i \in I$. Since $\mathcal{O}_{\tilde{\mathbf{Y}}, \tilde{\mathfrak{y}}} = \varinjlim_{\mathbf{Y}} \mathcal{O}_{\mathbf{Y}, \mathfrak{y}_{\mathbf{Y}}}$, there exist an object \mathbf{Y} of $\mathcal{C}_{\mathbf{V}/\mathbf{T}}$, an open

neighborhood W of \mathcal{Y}_Y and non-zero divisors $g_i \in \Gamma(W, \mathcal{O}_Y)$ invertible on $V \cap W$ sent to f_i for $i \in I$. By replacing Y by the blow-up of the closure of the divisors of g_i , we may assume that there exists a finite family $\mathcal{D}_Y = (D_{Y,i})_{i \in I}$ of Cartier divisors of Y such that $D_{Y,i} \cap V = \emptyset$ and $D_{Y,i} \cap W$ is defined by g_i for each $i \in I$. Then the inverse image of $\Sigma_{V/U}^{\mathcal{D}_Y} Y$ by the map $\text{Spec } B \rightarrow Y$ is equal to $\Sigma_{v/f(v)}^{\mathcal{D}} \text{Spec } B$ and hence is empty. Thus the assertion is proved. \square

Proposition 2.4.4. — *Let T be a separated noetherian scheme and V be a separated scheme of finite type over T . For an unramified separated morphism $f: V \rightarrow U$ of finite type of schemes, the following conditions are equivalent:*

- (1) $f: V \rightarrow U$ is tamely ramified with respect to T .
- (2) For every point $v \in V$ and for every T -integral and V -external valuation ring (Definition 2.3.2) B of $\kappa(v)$, the extension $\kappa(v)$ over $\kappa(f(v))$ is tamely ramified with respect to B .
- (3) For every point v of V and for every T -integral valuation ring B of $\kappa(v)$, the extension $\kappa(v)$ over $\kappa(f(v))$ is tamely ramified with respect to B .

Proof. — (3) \Rightarrow (2) Clear.

(1) \Rightarrow (3) It follows from Lemma 2.4.3 (1) \Rightarrow (2).

(2) \Rightarrow (1) Let $\tilde{Y} = \varinjlim_{Y \in \mathcal{C}_{V/T}} Y$ be the limit of compactifications and let $\pi_Y: \tilde{Y} \rightarrow Y$ denote the projection. By Lemma 2.4.3 (2) \Rightarrow (1), for every point $\tilde{y} \in \tilde{Y}$ of the boundary, there exists a proper scheme Y over T containing V as an open subscheme such that the inverse image of $\Sigma_{V/U}^+ Y \subset Y$ does not contain \tilde{y} . In other words, the intersection $\bigcap_{Y \in \mathcal{C}_{V/T}} \pi_Y^{-1}(\Sigma_{V/U}^+ Y) \subset \tilde{Y}$ is empty. Since \tilde{Y} is quasi-compact and since the category $\mathcal{C}_{V/T}$ is cofiltered, there exists an object Y of $\mathcal{C}_{V/T}$ such that $\pi_Y^{-1}(\Sigma_{V/U}^+ Y)$ is empty. Thus the assertion follows. \square

Corollary 2.4.5. — *Let T be a separated noetherian scheme and V be a separated scheme of finite type over T . Let $f: V \rightarrow U$ be a separated unramified morphism of finite type. If one of the following conditions is satisfied, then $f: V \rightarrow U$ is tamely ramified with respect to T .*

- (1) T is a scheme over \mathbf{Q} .
- (2) V is a G -torsor over U for a finite group G of order invertible on T .

Proof. — Let $v \in V$ be a point and B be a T -integral and V -external valuation ring of $\kappa(v)$. Then, either of the conditions (1) and (2) implies that the extension $\kappa(v)$ over $\kappa(f(v))$ is tamely ramified with respect to B . Hence the assertion follows from Proposition 2.4.4 (2) \Rightarrow (1). \square

Corollary 2.4.6. — *Let T be a separated noetherian scheme and V be a separated scheme of finite type over T . Let $f: V \rightarrow U$ be an unramified and separated dominant morphism of schemes of*

finite type. Assume that U and V are integral and let $\xi \in U$ and $\eta \in V$ be the generic point respectively. We consider the following conditions:

- (1) $f: V \rightarrow U$ is tamely ramified with respect to T .
- (2) The extension $\kappa(\eta)$ of $\kappa(\xi)$ is tamely ramified with respect to an arbitrary T -integral valuation ring of $\kappa(\eta)$.

Then, the implication (1) \Rightarrow (2) always holds. The other implication (2) \Rightarrow (1) holds if V is regular.

Proof. — (1) \Rightarrow (2) It follows from Proposition 2.4.4 (1) \Rightarrow (3).

(2) \Rightarrow (1) Assume V is regular. Let $v \in V$ and let $B \subset \kappa(v)$ be a T -integral valuation ring. Since V is regular, there exists a valuation ring $B_0 \subset \kappa(\eta)$ dominating $\mathcal{O}_{V,v}$ such that the residue field of B_0 is $\kappa(v)$. Let B_1 be the inverse image of $B \subset \kappa(v)$ by the surjection $B_0 \rightarrow \kappa(v)$. Then $B_1 \subset \kappa(\eta)$ is a valuation ring and T -integral. Thus, by Proposition 2.4.4, it suffices to show that $\kappa(v)$ is tamely ramified over $\kappa(u)$ with respect to B assuming $\kappa(\eta)$ is tamely ramified over $\kappa(\xi)$ with respect to B_1 .

We put $A = B \cap \kappa(u)$ and $A_1 = B_1 \cap \kappa(\xi)$. Then, the map $A_1 \rightarrow A$ is a surjection and we have $\kappa(v) = \kappa(u) \otimes_{A_1} B_1$. Hence we have $B = A \otimes_{A_1} B_1$. Let A_1^{sh} and B_1^{sh} be the strict henselizations and $\kappa(\xi)_1^{\text{sh}}$ and $\kappa(\eta)_1^{\text{sh}}$ be their fraction fields. By the assumption, the degree $[\kappa(\eta)_1^{\text{sh}} : \kappa(\xi)_1^{\text{sh}}]$ is invertible in B . Let A^{sh} and B^{sh} be the strict henselizations and $\kappa(u)^{\text{sh}}$ and $\kappa(v)^{\text{sh}}$ be their fraction fields. Then, A^{sh} and B^{sh} are quotients of A_1^{sh} and of B_1^{sh} and the canonical map $\kappa(u)^{\text{sh}} \otimes_{A_1^{\text{sh}}} B_1^{\text{sh}} \rightarrow \kappa(v)^{\text{sh}}$ is an isomorphism. Hence, we have $[\kappa(v)^{\text{sh}} : \kappa(u)^{\text{sh}}] = [\kappa(\eta)_1^{\text{sh}} : \kappa(\xi)_1^{\text{sh}}]$ and the assertion follows. \square

The following example shows that the condition (2) need not imply (1) if we replace “regular” by “normal”.

Example 2.4.7. — Let k be an algebraically closed field of characteristic $p > 0$, E be an ordinary elliptic curve over k and \mathcal{L} be a very ample invertible \mathcal{O}_E -module on E e.g. $\mathcal{O}(3 \cdot [0])$. Let $X_0 = \text{Spec} \bigoplus \Gamma(E, \mathcal{L}^{\otimes n})$ be the affine cone. The blow-up X_1 of X_0 at the origin is the line bundle over E associated to \mathcal{L} . Let $Y_1 \rightarrow X_1$ be the base change of the map $V: E^{(p)} \rightarrow E$ and $Y_0 \rightarrow X_0$ be the Stein factorization of the composition $Y_1 \rightarrow X_1 \rightarrow X_0$.

Let $C \rightarrow C'$ be a finite étale cyclic covering of affine curves of degree p . We assume that the map $\overline{C} \rightarrow \overline{C}'$ of the compactifications is wildly ramified. We put $V = Y_0 \times C$, $Y = Y_0 \times \overline{C}$ and we consider the action of $E[p](k) \times \text{Gal}(C/C') \simeq (\mathbf{Z}/p\mathbf{Z})^2$ on Y . Let $G \subset E[p](k) \times \text{Gal}(C/C')$ be a diagonal subgroup and $X = Y/G$ be the quotient. Since the action of G on V is free, the map $f: V \rightarrow U = V/G$ is finite and étale.

We show that $Y \rightarrow X$ satisfies (2). Since $V: E^{(p)} \rightarrow E$ is finite étale, the blow-up $Y_1 \rightarrow X_1$ of $Y_0 \rightarrow X_0$ is finite étale. Hence, the action of G on $Y_1 \times \overline{C}$ is free and the map $Y_1 \times \overline{C} \rightarrow (Y_1 \times \overline{C})/G$ is finite étale.

We show that $Y \rightarrow X$ does not satisfy (1). The inclusion $\overline{C} \rightarrow Y$ at the origin of Y_0 induces $\overline{C}' \rightarrow X$. By the assumption, the covering $\overline{C} \rightarrow \overline{C}'$ is widely ramified at the boundary $c \in \overline{C} \setminus C$. Since the valuation ring $\mathcal{O}_{\overline{C},c}$ is Y -integral, the assertion follows.

2.5. Tame ramification and Kummer coverings. — We consider a finite étale morphism $f: V \rightarrow U$ of separated schemes of finite type over a separated noetherian scheme S . We study the condition for $f: V \rightarrow U$ to be tamely ramified with respect to S in terms of Kummer coverings.

Definition 2.5.1. — Let X be a scheme, $U \subset X$ be an open subscheme and $f: V \rightarrow U$ be a finite étale morphism.

1. Let $x \in X \setminus U$ be a point of boundary. We say $f: V \rightarrow U$ is of Kummer type at x if there exist an open neighborhood W of x , functions $t_1, \dots, t_n \in \Gamma(W, \mathcal{O}_X)$ invertible on $U_W = U \cap W$, integers $m_1, \dots, m_n \geq 1$ invertible on W and an étale surjective morphism $W' \rightarrow W$, such that the base change of $f: V \rightarrow U$ by the étale map $U \times_X (W[S_1, \dots, S_n]/(S_1^{m_1} - t_1, \dots, S_n^{m_n} - t_n) \times_W W') \rightarrow U$ is a constant étale covering.

2. We say $f: V \rightarrow U$ is of Kummer type with respect to X if it is of Kummer type at every point of $x \in X \setminus U$.

Lemma 2.5.2. — Let S be a separated noetherian scheme and $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S . Let $u \in U$ be a point let A be an S -integral valuation ring of $\kappa(u)$. We consider the conditions:

- (1) For every point $v \in f^{-1}(u)$ and for every valuation ring B of $\kappa(v)$ dominating A , the extension $\kappa(v)$ over $\kappa(u)$ is tamely ramified with respect to B .
- (2) There exists a proper scheme X over S containing U as an open subscheme such that $V \rightarrow U$ is of Kummer type at the image of the closed point of $\text{Spec } A$ by $\text{Spec } A \rightarrow X$.

The condition (1) implies (2). If A is U -external, then the conditions (1) and (2) are equivalent.

Proof. — (2) \Rightarrow (1) Since the assertion is étale local on $\text{Spec } A$, it follows from Corollary 2.2.3 (2) \Rightarrow (1).

(1) \Rightarrow (2) Assume that A is U -external. By Corollary 2.2.3 (1) \Rightarrow (2), there exists non-zero elements $t_1, \dots, t_n \in \mathfrak{m}_A$ and integers $m_1, \dots, m_n \geq 1$ invertible in A such that, for every $v \in f^{-1}(u)$, the normalization of A in $\kappa(v)[S_1, \dots, S_n]/(S_1^{m_1} - t_1, \dots, S_n^{m_n} - t_n)$ is finite étale over the normalization of A in $\kappa(u)[S_1, \dots, S_n]/(S_1^{m_1} - t_1, \dots, S_n^{m_n} - t_n)$. We put $A_1 = A[S_1, \dots, S_n]/(S_1^{m_1} - t_1, \dots, S_n^{m_n} - t_n)$ and let A^{sh} denote the strict henselization of A . Then, the étale covering $V_u = V \times_U u$ of u is trivialized by the base change $A \rightarrow A^{\text{sh}} \otimes_A A_1$. Consequently, there exist an étale A -algebra A' and a maximal ideal \mathfrak{m}' of A' above the maximal ideal \mathfrak{m}_A such that the étale covering $V_u = V \times_U u$ of u is trivialized by the base change $A \rightarrow A' \otimes_A A_1$.

Let $\tilde{X} = \varprojlim_{X \in \mathcal{C}_{U/S}} X$ be the limit of compactifications and $\tilde{x} \in \tilde{X}$ be the point corresponding to $A \subset \kappa(u)$. We write $\tilde{A} = \mathcal{O}_{\tilde{X}, \tilde{x}}$ and let $\mathfrak{p} = \mathfrak{m}_u \subset \tilde{A}$ denote the kernel of the surjection $\tilde{A} \rightarrow A$. We take liftings $\tilde{t}_1, \dots, \tilde{t}_n \in \tilde{A}$ of $t_1, \dots, t_n \in A$ and put $\tilde{A}_1 = \tilde{A}[S_1, \dots, S_n]/(S_1^{m_1} - \tilde{t}_1, \dots, S_n^{m_n} - \tilde{t}_n)$. We also take an étale \tilde{A} -algebra \tilde{A}' such that $\tilde{A}' \otimes_{\tilde{A}} A = A'$ and put $\tilde{A}'_1 = \tilde{A}' \otimes_{\tilde{A}} \tilde{A}_1$. Then, we have an isomorphism $\tilde{A}'_1 \otimes_{\tilde{A}} A \rightarrow A' \otimes_A A_1$. Let $\tilde{\mathfrak{m}}'$ be the maximal ideal of \tilde{A}' above \mathfrak{m}' and $\tilde{\mathfrak{m}}'_1$ be a maximal ideal \tilde{A}'_1 above $\tilde{\mathfrak{m}}'$. We will apply the following lemma to the localization $\tilde{A}'_{1, \tilde{\mathfrak{m}}'_1}$.

Lemma 2.5.3. — *Let A be a local ring and \mathfrak{p} be a prime ideal of A such that A is canonically isomorphic to the inverse image of A/\mathfrak{p} by the surjection $A_{\mathfrak{p}} \rightarrow \kappa(\mathfrak{p})$. Let B be a finite A -algebra such that $B \otimes_A A_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ and that $B \otimes_A \kappa(\mathfrak{p})$ is isomorphic to the product $\kappa(\mathfrak{p})^n$. We define an A -subalgebra B' of $B \otimes_A A_{\mathfrak{p}}$ to be the inverse image of $(A/\mathfrak{p})^n$ by the surjection $B \otimes_A A_{\mathfrak{p}} \rightarrow B \otimes_A \kappa(\mathfrak{p})$. Then, B' is finite étale over A and the canonical maps $B \otimes_A A_{\mathfrak{p}} \rightarrow B' \otimes_A A_{\mathfrak{p}}$ and $B'/\mathfrak{p}B' \rightarrow (A/\mathfrak{p})^n$ are isomorphisms.*

Proof. — We may assume B is \mathfrak{p} -torsion free and identify $B \subset B \otimes_A A_{\mathfrak{p}}$. By the assumption on A , the prime ideal $\mathfrak{p} = \text{Ker}(A \rightarrow A/\mathfrak{p})$ is equal to the maximal ideal $\mathfrak{p}A_{\mathfrak{p}} = \text{Ker}(A_{\mathfrak{p}} \rightarrow \kappa(\mathfrak{p}))$. Hence, we have $\mathfrak{p}B' \subset \mathfrak{p}A_{\mathfrak{p}}(B \otimes_A A_{\mathfrak{p}}) = \mathfrak{p}A_{\mathfrak{p}} \cdot B = \mathfrak{p}B \subset \mathfrak{p}B'$. Thus, we have an equality $\mathfrak{p}B' = \mathfrak{p}A_{\mathfrak{p}}(B \otimes_A A_{\mathfrak{p}})$ and an isomorphism $B'/\mathfrak{p}B' \rightarrow (A/\mathfrak{p})^n$. Since the A -module B'/B is isomorphic to $(A/\mathfrak{p})^n/(B/\mathfrak{p}B)$, the A -module B' is of finite type. Hence, by Nakayama's lemma, B' is finite flat over A and hence is étale over A . The canonical map $B \otimes_A \kappa(\mathfrak{p}) \rightarrow B' \otimes_A \kappa(\mathfrak{p})$ is an isomorphism and hence $B \otimes_A A_{\mathfrak{p}} \rightarrow B' \otimes_A A_{\mathfrak{p}}$ is also an isomorphism. \square

We go back to the proof of Lemma 2.5.2. By [9, Proposition 5.11], the local ring \tilde{A} is canonically isomorphic to the inverse image of A/\mathfrak{p} by the surjection $\tilde{A}_{\mathfrak{p}} \rightarrow \kappa(\mathfrak{p})$. We show that the local ring $\tilde{A}'_{1, \tilde{\mathfrak{m}}'_1}$ also satisfies the condition of Lemma 2.5.3. Since \tilde{A}'_1 is flat over \tilde{A} , it follows that \tilde{A}'_1 is canonically isomorphic to the inverse image of $\tilde{A}'_1/\mathfrak{p}\tilde{A}'_1 = A' \otimes_A A_1$ by the surjection $\tilde{A}'_1 \otimes_{\tilde{A}} \tilde{A}_{\mathfrak{p}} \rightarrow \tilde{A}'_1 \otimes_{\tilde{A}} \kappa(\mathfrak{p})$. Thus the claim follows by localization.

By Zariski's main theorem, there exists a finite \tilde{A} -algebra \tilde{B} such that $\tilde{B} \otimes_{\tilde{A}} \mathcal{O}_{U, u}$ is isomorphic to $\Gamma(V \times_U \text{Spec } \mathcal{O}_{U, u}, \mathcal{O}_V)$. Hence, by Lemma 2.5.3, the base change of $V \times_U \text{Spec } \mathcal{O}_{U, u} \rightarrow \text{Spec } \mathcal{O}_{U, u}$ by $\tilde{A} \rightarrow \tilde{A}'_1$ is extended to a finite étale covering on a neighborhood of $\tilde{\mathfrak{m}}'_1$. By replacing A' by an étale algebra contained in A^{sh} , we may assume that the base change of $V \times_U \text{Spec } \mathcal{O}_{U, u} \rightarrow \text{Spec } \mathcal{O}_{U, u}$ by $\tilde{A} \rightarrow \tilde{A}'_1$ is a constant finite étale covering.

Since $\tilde{A} = \varinjlim_X \mathcal{O}_{X, x_X}$, there exist an object X of $\mathcal{C}_{U/S}$, an open neighborhood W of x_X and non-zero divisors $f_1, \dots, f_n \in \Gamma(W, \mathcal{O}_X)$ invertible on $U \cap W$ sent to \tilde{t}_i for $i = 1, \dots, n$ and an étale morphism $W' \rightarrow W$ such that \tilde{A}'_1 is the pull-back of $W' \rightarrow W$ by $\text{Spec } \tilde{A} \rightarrow W$. We put $W_1 = W[S_1, \dots, S_n]/(S_1^{m_1} - f_1, \dots, S_n^{m_n} - f_n)$. Then further by $\tilde{A} =$

$\varinjlim_{\mathbf{X}} \mathcal{O}_{\mathbf{X}, \tilde{\mathbf{X}}}$, replacing \mathbf{X} if necessary, the base change of $V \rightarrow U$ by $W_1 \times_W W' \rightarrow \mathbf{X}$ is a constant finite étale covering. \square

Proposition 2.5.4. — *Let S be a separated noetherian scheme and $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S . Then, the following conditions are equivalent:*

- (1) $f: V \rightarrow U$ is tamely ramified with respect to S .
- (2) There exists a proper scheme \mathbf{X} over S containing U as an open subscheme such that $V \rightarrow U$ is of Kummer type with respect to \mathbf{X} .

Proof. — (2) \Rightarrow (1) It follows from Lemma 2.5.2 (2) \Rightarrow (1) and Proposition 2.4.4 (2) \Rightarrow (1).

(1) \Rightarrow (2) By Proposition 2.4.4 (1) \Rightarrow (2) and Lemma 2.5.2 (1) \Rightarrow (2), for every point $\tilde{x} \in \tilde{\mathbf{X}} = \varprojlim_{\mathbf{X} \in \mathcal{C}_{U/S}} \mathbf{X}$ of the boundary, there exists a proper scheme \mathbf{X} over S containing U such that the maximum open subscheme $W_{\mathbf{X}} \subset \mathbf{X}$ where $V \rightarrow U$ is of Kummer type contains the image $\pi_{\mathbf{X}}(\tilde{x}) \in \mathbf{X}$ of $\tilde{x} \in \tilde{\mathbf{X}}$ by the projection $\pi_{\mathbf{X}}: \tilde{\mathbf{X}} \rightarrow \mathbf{X}$. In other words, the family $(\pi_{\mathbf{X}}^{-1}(W_{\mathbf{X}}))_{\mathbf{X} \in \mathcal{C}_{U/S}}$ is an open covering of $\tilde{\mathbf{X}}$. Since $\tilde{\mathbf{X}}$ is quasi-compact and since the category $\mathcal{C}_{U/S}$ is cofiltered, there exists an object \mathbf{X} of $\mathcal{C}_{U/S}$ such that $\pi_{\mathbf{X}}^{-1}(W_{\mathbf{X}}) = \tilde{\mathbf{X}}$. Thus the assertion follows. \square

2.6. Tame ramification of Galois coverings. — The following proposition shows that the definition of tame ramification here is equivalent to that given by Gabber [43, Section 2.1] for Galois coverings. Let \mathbf{X} be a normal scheme, $U \subset \mathbf{X}$, be a dense open subscheme and $V \rightarrow U$ be a finite G -torsor. Then, for a geometric point \bar{x} of \mathbf{X} , the inertia subgroup $I_{\bar{x}}$ of G is defined up to conjugacy.

Proposition 2.6.1. — *Let S be a separated noetherian scheme and U be a separated normal integral scheme of finite type over S . For a G -torsor $f: V \rightarrow U$ for a finite group G , the following conditions are equivalent:*

- (1) $f: V \rightarrow U$ is tamely ramified with respect to S .
- (2) There exists a proper normal scheme \mathbf{X} over S and an open immersion $U \rightarrow \mathbf{X}$ over S such that for every geometric point \bar{x} of \mathbf{X} , the order of the inertia subgroup $I_{\bar{x}} \subset G$ is invertible at \bar{x} .

Proof. — (1) \Rightarrow (2) By Proposition 2.5.4 (1) \Rightarrow (2), there exists a proper normal scheme \mathbf{X} over S and an open immersion $U \rightarrow \mathbf{X}$ over S such that $V \rightarrow U$ is of Kummer type with respect to \mathbf{X} . Hence the assertion follows.

(2) \Rightarrow (1) For every point $u \in U$, for every S -integral and U -external valuation ring A of $\kappa(u)$, for every $v \in f^{-1}(u)$ and for every valuation ring B of $\kappa(v)$ dominating A , the order of the inertia group $I_{B/A}$ is invertible in A . Hence, by Proposition 2.4.4 (2) \Rightarrow (1), the map $V \rightarrow U$ is tamely ramified with respect to S . \square

Corollary 2.6.2. — *Let $f: V \rightarrow U$ be a finite étale morphism and let X be a normal scheme containing U as a dense open subscheme. We assume $f: V \rightarrow U$ is a G -torsor for a finite group G . We consider the following conditions:*

- (1) $f: V \rightarrow U$ is tamely ramified with respect to X .
- (2) For every geometric point \bar{x} of X , the order of the inertia subgroup $I_{\bar{x}} \subset G$ is invertible at \bar{x} .
- (3) Let x be an arbitrary point of X such that the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring. Then, $f: V \rightarrow U$ is tamely ramified with respect to $\mathcal{O}_{X,x}$.

Then, we have implications (2) \Rightarrow (1) \Rightarrow (3). If X is a regular separated noetherian scheme and if U is the complement of a divisor with normal crossings, then (3) implies (2).

Proof. — (2) \Rightarrow (1) It follows from Proposition 2.6.1 (2) \Rightarrow (1).

(1) \Rightarrow (3) It follows from Proposition 2.4.4 (1) \Rightarrow (2).

(3) \Rightarrow (2) By [37, Proposition 5.2] (Lemme d'Abhyankhar absolu), the condition (3) implies that $V \rightarrow U$ is of Kummer type with respect to X . Hence (3) implies (2). \square

If we drop the assumption that U is the complement of a divisor with normal crossings, the implication (1) \Rightarrow (2) nor (3) \Rightarrow (1) need not hold even if X is regular, as the following examples show. The authors thank M. Raynaud for the help to find Example 2.6.3.2.

Example 2.6.3. — 1. Let k be an algebraically closed field of characteristic $p > 0$ and $V_0 \rightarrow U_0 = \text{Spec } k[t, t^{-1}, (t-1)^{-1}] = \mathbf{P}_k^1 \setminus \{0, 1, \infty\}$ be a finite étale connected Galois covering of degree divisible by p tamely ramified at $0, 1, \infty$. We put $X = \mathbf{A}_k^2 = \text{Spec } k[x, y] \supset U = \text{Spec } k[x, y, (xy(x-y))^{-1}]$ and define a map $U \rightarrow U_0$ by sending t to x/y . Let $V = V_0 \times_{U_0} U$ be the pull-back by the map $U \rightarrow U_0$. Then, since $V_0 \rightarrow U_0$ is assumed tamely ramified, the covering $V \rightarrow U$ is tamely ramified with respect to X and satisfies the condition (1). Since the inertia group at the origin $0 \in X$ is equal to the Galois group $\text{Gal}(U_0/V_0)$, the condition (2) is not satisfied.

2. Let k be a field of characteristic $p \geq 3$ and $m \geq 1$ be an integer. We consider the cyclic covering

$$\begin{array}{c} Z = \text{Spec } k[x, y, z] / (z^{p-1} - (x^{2m(p-1)-1} + x^{m(p-1)}y^{m(p-1)} + y^{2m(p-1)})) \\ \pi \downarrow \\ X = \mathbf{A}_k^2 = \text{Spec } k[x, y] \end{array}$$

of degree $p-1$ ramified at the divisor $D = (x^{2m(p-1)-1} + x^{m(p-1)}y^{m(p-1)} + y^{2m(p-1)})$. We put $U = X \setminus D$ and $W = Z \times_X U$. We consider the Artin-Schreier covering of $Z \times_X (\mathbf{G}_m \times \mathbf{A}^1)$ defined by $T^p - T = y^{m(p-2)}z/x^{mp}$. Since

$$\begin{aligned}
& \frac{y^{m(p-2)}z}{x^{mp}} - \left(\left(\frac{z}{x^m y^m} \right)^p - \frac{z}{x^m y^m} \right) \\
&= \frac{(y^{2m(p-1)} - (x^{2m(p-1)-1} + x^{m(p-1)}y^{m(p-1)} + y^{2m(p-1)} + x^{m(p-1)}y^{m(p-1)})z}{x^{mp}y^{mp}} \\
&= -\frac{x^{m(p-2)-1}z}{y^{mp}},
\end{aligned}$$

it is extended to a finite étale covering on $Z \setminus \pi^{-1}(0)$. Hence it defines a finite étale Galois covering $V \rightarrow U$ of Galois group $\mathbf{F}_p^\times \ltimes \mathbf{F}_p$, tamely ramified with respect to $X \setminus \{0\}$. Thus the Galois covering $V \rightarrow U \subset X$ satisfies the condition (3) in Corollary 2.6.2.

Let $X' \rightarrow X$ be the blow-up at the origin and Z' be the normalization of X' in W . We put $t = x/y$. Then, since $x^{2m(p-1)-1} + x^{m(p-1)}y^{m(p-1)} + y^{2m(p-1)} = y^{2m(p-1)-1}(t^{2m(p-1)-1} + t^{m(p-1)}y + y)$, the cyclic covering $Z' \rightarrow X'$ is totally ramified along the exceptional divisor $E \subset X'$ and the valuation of $y^{m(p-2)}z/x^{mp} = z/(x^{2m}t^{m(p-2)})$ at the generic point of the inverse image $E' = E \times_{X'} Z'$ is $(2m(p-1) - 1) - 2m(p-1) = -1$. Hence the Artin-Schreier covering $V \rightarrow W$ is totally ramified along E' and $V \rightarrow U$ is not tamely ramified with respect to X . Thus the Galois covering $V \rightarrow U \subset X$ does not satisfy the condition (1) in Corollary 2.6.2.

3. Complements on localized intersection products

We compute certain tor-sheaves in Section 3.1. This computation plays a crucial role in the proof of the excision formula and of the blow-up formula in Section 6.2. We recall the definition of the localized intersection product and some useful formulas in Section 3.3.

The results in Sections 3.2 and 3.4 are used only in an explicit computation of the logarithmic different in Section 5.1. In Section 3.4, we prove a refinement Proposition 3.4.3 of the excess intersection formula [26, Proposition 3.4.2], which relates the classes of certain tor-sheaves defining the localized intersection product with the mapping cone of exterior derived power complexes. In Section 3.2, we show that the class of the mapping cone of exterior derived power complexes is given by the localized Chern class.

3.1. A computation of Tor sheaves. — We compute certain Tor sheaves related to blow-up. We recall some terminology on tor-sheaves. For a Cartesian diagram

$$\begin{array}{ccc}
Y & \longleftarrow & Z \\
\text{(3.1.0.1)} \quad \downarrow g & & \downarrow \\
S & \xleftarrow{f} & X
\end{array}$$

of schemes, a quasi-coherent \mathcal{O}_X -module \mathcal{F} , a quasi-coherent \mathcal{O}_Y -module \mathcal{G} and an integer $q \geq 0$, a quasi-coherent \mathcal{O}_Z -module $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ is defined in [11, (6.5.3)]. If $S = \text{Spec } A$, $X = \text{Spec } B$, $Y = \text{Spec } C$ are affine and if \mathcal{F} and \mathcal{G} are the quasi-coherent sheaves associated to an B -module M and an C -module N respectively, then $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ is associated to the $B \otimes_A C$ -module $\mathcal{T}or_q^A(M, N)$. If $\mathcal{F} = \mathcal{O}_X$, we put $L_q f^* \mathcal{G} = \mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{G})$.

Definition 3.1.1. — 1. ([16, Definition 1.5]) *Let X and Y be schemes over a scheme S . We say that X and Y are tor-independent over S if $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for every $q > 0$.*

2. ([16, Definition 3.1]) *Let $f: X \rightarrow S$ be a morphism of schemes. We say that f is of finite tor-dimension, if there exists an integer $n \geq 0$ such that, for every quasi-coherent \mathcal{O}_S -module \mathcal{F} and every integer $q > n$, we have $L_q f^* \mathcal{F} = 0$.*

If X or Y is flat over S , then X and Y are tor-independent over S .

Lemma 3.1.2. — *We consider morphisms*

$$Y \xrightarrow{g} S \xleftarrow{f} X \xleftarrow{f'} X'$$

of schemes. Assume that X and Y are tor-independent over S . Then, X' and Y are tor-independent over S if and only if X' and $X \times_S Y$ are tor-independent over X .

Proof. — By the assumption that $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for every $q > 0$, we obtain an isomorphism $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{O}_{X'}, \mathcal{O}_Y) \rightarrow \mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_{X \times_S Y})$ for every $q > 0$. \square

Lemma 3.1.3. — *Assume that the schemes S and X are noetherian and Y is of finite type over S in the diagram (3.1.0.1). Then, for a coherent \mathcal{O}_X -module \mathcal{F} and for a coherent \mathcal{O}_Y -module \mathcal{G} , the \mathcal{O}_Z -modules $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. — Since the question is local on Z , we may assume that schemes S, X, Y and Z are affine. We take a closed immersion $Y \rightarrow P = \mathbf{A}_S^n$ to an affine space. Since an \mathcal{O}_Z -module is coherent if it is coherent as an $\mathcal{O}_{X \times_S P}$ -module, we may replace Y by P . Hence, we may assume further that Y is flat over S . Then, a resolution \mathcal{L} of \mathcal{G} by free \mathcal{O}_Y -modules of finite rank is a resolution by flat \mathcal{O}_S -modules. Since $\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ is a cohomology sheaf of the complex $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{L}$, it is a coherent \mathcal{O}_Z -module. \square

Let S be a regular noetherian scheme of finite dimension. Then, for a scheme $f: X \rightarrow S$ of finite type over S , the dimension function $X \rightarrow \mathbf{N}$ is defined as in [26, Section 2.1]. Namely, for a point $x \in X$ and $s = f(x)$, we put $\dim x = \text{tr.deg}(\kappa(x)/\kappa(s)) + \dim S - \dim \mathcal{O}_{S,s}$. Using this dimension function, the topological filtration $F_\bullet G(X)$ and the lower numbering Chow groups $\text{CH}_\bullet(X)$ are defined. We have a canonical map $\text{CH}_\bullet(X) \rightarrow \text{Gr}_\bullet^F G(X)$ sending the class $[V]$ of an integral closed subscheme V of X to the class $[\mathcal{O}_V]$ also denoted by $[V]$.

By Lemma 3.1.3, for a morphism $f: X \rightarrow Y$ of noetherian schemes of finite tor-dimension and for a scheme A of finite type over Y , the pull-back map

$$f^*: G(A) \rightarrow G(A \times_Y X)$$

is defined by $f^*([\mathcal{F}]) = \sum_{q \geq 0} (-1)^q [\mathcal{T}or_q^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)]$ for a coherent \mathcal{O}_A -module \mathcal{F} .

Lemma 3.1.4. — *Let S be a regular noetherian scheme of finite dimension and $f: X \rightarrow Y$ be a quasi-projective morphism locally of complete intersection of relative virtual dimension r of schemes of finite type over S .*

Then, for a scheme A of finite type over Y , the pull-back $f^: G(A) \rightarrow G(A \times_Y X)$ preserves topological filtration in the sense that f^* maps $F_\bullet G(A)$ to $F_{\bullet+r} G(A \times_Y X)$. Further, for an integer $q \geq 0$, we have a commutative diagram*

$$(3.1.4.1) \quad \begin{array}{ccc} \mathrm{CH}_q(A) & \xrightarrow{f^!} & \mathrm{CH}_{q+r}(A \times_Y X) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_q^F G(A) & \xrightarrow{f^*} & \mathrm{Gr}_{q+r}^F G(A \times_Y X). \end{array}$$

Proof. — By the assumption on f , it is the composition $X \rightarrow P \rightarrow Y$ of a regular immersion $X \rightarrow P$ and a smooth morphism $P \rightarrow Y$. Since, it is clear for a smooth map, it is reduced to the case where f is a regular immersion. Then, it follows from [26, Proposition 2.2.2]. \square

We compute $\mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)$ for morphisms $Y \rightarrow X' \rightarrow X$ under certain conditions. Corollaries 3.1.6 and 3.1.7 of the following proposition are crucial in the proof of Proposition 6.2.1 and Theorem 6.2.2 respectively.

Proposition 3.1.5. — *Let X be a scheme and \mathcal{N} be a locally free \mathcal{O}_X -module of finite rank c . Let $\alpha: \mathcal{N} \rightarrow \mathcal{O}_X$ be an \mathcal{O}_X -linear map and $C \subset X$ be the closed subscheme defined by $\mathcal{I}_C = \mathrm{Im}(\alpha: \mathcal{N} \rightarrow \mathcal{O}_X)$. Let $P \subset \mathbf{P}(\mathcal{N})$ be an open subscheme of the associated \mathbf{P}^{c-1} -bundle $\mathbf{P}(\mathcal{N}) = \mathcal{P}roj(S^\bullet \mathcal{N})$ and $p: P \rightarrow X$ be the canonical map.*

Let $\mathcal{E} = \mathrm{Ker}(p^ \mathcal{N} \rightarrow \mathcal{O}_P(1)) = \Omega_{P/X}^1(1)$ be the kernel of the canonical surjection. Let $X' \subset P$ be the closed subscheme defined by the image $\mathcal{I}_{X'} = \mathrm{Im}(\alpha': \mathcal{E} \rightarrow \mathcal{O}_P)$ of the restriction $\alpha' = p^* \alpha|_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{O}_P$ to $\mathcal{E} \subset p^* \mathcal{N}$ and $q: X' \rightarrow X$ denote the composition.*

We consider a Cartesian diagram

$$(3.1.5.1) \quad \begin{array}{ccccccc} E_Y & \longrightarrow & E & \longrightarrow & P_C & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{g} & X' & \xrightarrow{c} & P & \xrightarrow{p} & X \end{array}$$

of schemes. Let $f: Y \rightarrow X$ denote the composition of the bottom arrows. We assume that $E_Y = E \times_{X'} Y$ is a Cartier divisor of Y .

1. The upper middle arrow $E \rightarrow P_C = P \times_X C$ is an isomorphism. The restriction $X' \setminus E \rightarrow X \setminus C$ of $q: X' \rightarrow X$ is an isomorphism. The composition of the immersions $X' \setminus E \rightarrow P \setminus P_C \rightarrow \mathbf{P}(\mathcal{N})$ is the composition of the isomorphism $X' \setminus E \rightarrow X \setminus C$ and the section $X \setminus C \rightarrow \mathbf{P}(\mathcal{N})$ defined by the surjection $\alpha|_{X \setminus C}: \mathcal{N}|_{X \setminus C} \rightarrow \mathcal{O}_{X \setminus C}$.

The map $\alpha: \mathcal{N} \rightarrow \mathcal{O}_X$ induces a surjection $f^*\mathcal{N} \rightarrow \mathcal{I}_{E_Y} \subset \mathcal{O}_Y$. The composition $Y \rightarrow X' \rightarrow \mathbf{P}(\mathcal{N})$ is the section defined by the surjection $f^*\mathcal{N} \rightarrow \mathcal{I}_{E_Y}$.

2. We assume that the immersion $X' \rightarrow P$ is a regular immersion of codimension $c - 1$ [4, Definition 1.4]. Let $\gamma: Y \rightarrow P_Y = P \times_X Y$ be the section defined by g and $\Gamma \subset P_Y$ be the image of γ regarded as a closed subscheme of P_Y . Let $\text{pr}_1: P_Y \rightarrow P$ and $\text{pr}_2: P_Y \rightarrow Y$ denote the projections.

Then, the composition $q: X' \rightarrow X$ is of finite tor-dimension and there exists a spectral sequence $E_{p,q}^1 \Rightarrow E_r$ of $\mathcal{O}_{X' \times_X Y}$ -modules such that

$$(3.1.5.2) \quad E_r = \begin{cases} \text{Ker}(\mathcal{O}_{X' \times_X Y} \rightarrow \mathcal{O}_\Gamma) & \text{if } r = 0, \\ \text{Tor}_r^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) & \text{if } r \neq 0, \end{cases}$$

$$E_{p,q}^1 = \text{pr}_1^* \Omega_{P/X}^{\beta+q+1}(p+q+1) \otimes \text{pr}_2^* \mathcal{N}_{E_Y/Y}^{-(q+1)} \quad \text{if } p \geq 0, q \geq 0, p+q \leq c-2$$

and $E_{p,q}^1 = 0$ otherwise.

3. Assume that Y is noetherian. The surjection $\mathcal{O}_{X' \times_X Y} \rightarrow \mathcal{O}_\Gamma$ is an isomorphism outside $E \times_C E_Y$ and we have equalities

$$(3.1.5.3) \quad \begin{aligned} & [\text{Ker}(\mathcal{O}_{X' \times_X Y} \rightarrow \mathcal{O}_\Gamma)] + \sum_{r>0} (-1)^r [\text{Tor}_r^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)] \\ &= \sum_{\beta=1}^{c-1} (-1)^{\beta-1} \sum_{q=1}^{\beta} [\text{pr}_1^* \Omega_{E/C}^{\beta}(q) \otimes \text{pr}_2^* \mathcal{N}_{E_Y/Y}^{-q}] \end{aligned}$$

$$(3.1.5.4) \quad = \sum_{s=2}^c (-1)^s [\Lambda^s \pi^* \mathcal{N}] \cdot \sum_{q \geq 1, r \geq 1, q+r \leq s} [\text{pr}_1^* \mathcal{O}(-r)] \cdot [\text{pr}_2^* \mathcal{N}_{E_Y/Y}^{-q}]$$

in $G(E \times_C E_Y)$ where $\pi: E \times_C E_Y \rightarrow X$ denotes the canonical map.

Proof. — 1. The first paragraph is clear from the definition of X' and of C . The map $\alpha: \mathcal{N} \rightarrow \mathcal{O}_X$ defines a surjection $\mathcal{N} \rightarrow \mathcal{I}_C \subset \mathcal{O}_X$ and hence induces a surjection $f^*\mathcal{N} \rightarrow \mathcal{I}_{E_Y} \subset \mathcal{O}_Y$. By the first paragraph, the kernel of the surjection $f^*\mathcal{N} \rightarrow g^*(\mathcal{O}_{X'}(1))$ is equal to the kernel of $f^*\mathcal{N} \rightarrow \mathcal{I}_{E_Y}$ on the complement $Y \setminus E_Y$. Since a section $Y \rightarrow \mathbf{P}(\mathcal{N})$ is uniquely determined by its restriction to the complement of a divisor, the assertion follows.

2. By the definition of regular immersion [4, Definition 1.4], the Koszul complex

$$\mathcal{K} = \text{Kos}(\alpha') = [\Lambda^{c-1} \mathcal{E} \rightarrow \cdots \rightarrow \mathcal{E} \xrightarrow{\alpha'} \mathcal{O}_P]$$

is a resolution of an \mathcal{O}_P -module $\mathcal{O}_{X'}$. Since P is flat over X and $X' \rightarrow P$ is assumed to be a regular immersion, the composition $q: X' \rightarrow X$ is of finite tor-dimension. Further, we obtain an isomorphism

$$\mathcal{H}_r(\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \rightarrow \text{Tor}_r^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)$$

from the resolution $\mathcal{K} \rightarrow \mathcal{O}_{X'}$.

We construct a resolution of the \mathcal{O}_{P_Y} -module \mathcal{O}_Γ . Let $\beta: \text{pr}_1^* \mathcal{E} \rightarrow \text{pr}_2^* \mathcal{I}_{E_Y}$ be the restriction of the map $\text{pr}_1^* p^* \mathcal{N} = \text{pr}_2^* f^* \mathcal{N} \rightarrow \text{pr}_2^* \mathcal{I}_{E_Y}$ to the kernel $\text{pr}_1^* \mathcal{E} = \text{Ker}(\text{pr}_1^* p^* \mathcal{N} \rightarrow \text{pr}_1^* \mathcal{O}_P(1))$. The map β induces the pull-back $\text{pr}_1^* \mathcal{E} \rightarrow \mathcal{O}_{P_Y}$ of α' . Since the section $\gamma: Y \rightarrow P_Y$ is defined by the surjection $f^* \mathcal{N} \rightarrow \mathcal{I}_{E_Y}$ by 1, the closed subscheme $\Gamma \subset P_Y$ is characterized by the condition that the cokernel $\text{Coker}(\beta: \text{pr}_1^* \mathcal{E} \rightarrow \text{pr}_2^* \mathcal{I}_{E_Y})$ is an invertible \mathcal{O}_Γ -module. Hence the Koszul complex $\mathcal{K}' = \text{Kos}(\beta')$ defined by the twist $\beta': \text{pr}_1^* \mathcal{E} \otimes \text{pr}_2^* \mathcal{I}_{E_Y}^{-1} \rightarrow \mathcal{O}_{P_Y}$ of β is a resolution of the \mathcal{O}_{P_Y} -module \mathcal{O}_Γ .

We consider the morphism of complexes $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \rightarrow \mathcal{K}'$ induced by the inclusion $\text{pr}_1^* \mathcal{E} \rightarrow \text{pr}_1^* \mathcal{E} \otimes \text{pr}_2^* \mathcal{I}_{E_Y}^{-1}$. Then, it induces the canonical surjection $\mathcal{O}_{X' \times_X Y} \rightarrow \mathcal{O}_\Gamma$. Hence, for the complex $\mathcal{M} = (\mathcal{K}' / (\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{O}_Y))[-1]$, we have an isomorphism $\mathcal{H}_r \mathcal{M} \rightarrow E_r$.

The p -th component \mathcal{M}_p of the complex \mathcal{M} is given by

$$\begin{aligned} \mathcal{M}_p &= (\Lambda^{\beta+1} \text{pr}_1^* \mathcal{E} \otimes \text{pr}_2^* \mathcal{I}_{E_Y}^{-(\beta+1)}) / \Lambda^{\beta+1} \text{pr}_1^* \mathcal{E} \\ &= \Lambda^{\beta+1} \text{pr}_1^* \mathcal{E} \otimes (\text{pr}_2^* \mathcal{I}_{E_Y}^{-(\beta+1)} / \mathcal{O}_{P_Y}). \end{aligned}$$

We define an increasing filtration F_\bullet on \mathcal{M} by $F_q \mathcal{M}_p = \Lambda^{\beta+1} \text{pr}_1^* \mathcal{E} \otimes (\text{pr}_2^* \mathcal{I}_{E_Y}^{-(q+1)} / \mathcal{O}_{P_Y})$. Then, we obtain a spectral sequence

$$E_{p,q}^1 = \text{Gr}_{-q}^F \mathcal{M}_{p+q} \Rightarrow \mathcal{H}_r \mathcal{M}.$$

Since $\text{Gr}_{-q}^F \mathcal{M}_p = \Lambda^{\beta+1} \text{pr}_1^* \mathcal{E} \otimes \text{pr}_2^* \mathcal{N}_{E_Y/Y}^{-(q+1)}$ and $\mathcal{E} = \Omega_{P/X}^1(1)$, the assertion follows.

3. By 2, we have the equality (3.1.5.3). By the exact sequence $0 \rightarrow \mathcal{E} \rightarrow p^* \mathcal{N} \rightarrow \mathcal{O}(1) \rightarrow 0$, we have an exact sequence $0 \rightarrow \Lambda^c p^* \mathcal{N}(-c-\beta) \rightarrow \cdots \rightarrow \Lambda^{\beta+1} p^* \mathcal{N}(-1) \rightarrow \Lambda^\beta \mathcal{E} \rightarrow 0$ and an equality $[\Lambda^\beta \mathcal{E}] = \sum_{\beta+r=\beta+1}^c (-1)^{r-1} [\Lambda^{\beta+r} p^* \mathcal{N}(-r)]$. Substituting this and putting $\beta+r=s$, we obtain the second equality (3.1.5.4). \square

Corollary 3.1.6. — *Let X be a noetherian scheme and $C \subset X$ be a closed subscheme such that the immersion $C \rightarrow X$ is a regular immersion of codimension c . Let $q: X' \rightarrow X$ be the blow-up at C and let $E = X' \times_X C$ denote the exceptional divisor.*

Let Y be a noetherian scheme over X' such that $E_Y = E \times_{X'} Y \subset Y$ is a Cartier divisor and let Γ denote the image of the section $Y \rightarrow X' \times_X Y$. Let $\text{pr}_1: E \times_C E_Y \rightarrow E$ and $\text{pr}_2: E \times_C E_Y \rightarrow E_Y$ denote the projections.

Assume that $q: X' \rightarrow X$ is locally of complete intersection. Then, we have an equality

$$(3.1.6.1) \quad \begin{aligned} & [\mathrm{Ker}(\mathcal{O}_{X' \times_X Y} \rightarrow \mathcal{O}_\Gamma)] + \sum_{r>0} (-1)^r [\mathrm{Tor}_r^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)] \\ &= \sum_{\beta=1}^{c-1} (-1)^{\beta-1} \sum_{q=1}^{\beta} [\mathrm{pr}_1^* \Omega_{E/C}^\beta(\beta) \otimes \mathrm{pr}_2^* \mathcal{N}_{E_Y/Y}^{-q}] \end{aligned}$$

in $G(E \times_C E_Y)$.

Proof. — Locally on X , there exists a surjection $\mathcal{N} = \mathcal{O}_X^{\oplus c} \rightarrow \mathcal{I}_C$ of \mathcal{O}_X -modules. Hence, applying Proposition 3.1.5.3 to $\mathbf{P} = \mathbf{P}(\mathcal{N})$ and the closed immersion $Y = X' \rightarrow \mathbf{P}$, we obtain a spectral sequence (3.1.5.2), locally on X . By the proof of Proposition 3.1.5.3, it suffices to construct the spectral sequence (3.1.5.2) globally.

If we have another locally free \mathcal{O}_X -module \mathcal{N}' of rank c and a surjection $\mathcal{N}' \rightarrow \mathcal{I}_C$, then locally we have an isomorphism $\mathcal{N} \rightarrow \mathcal{N}'$ compatible with the surjections to \mathcal{I}_C . It induces an isomorphism of spectral sequences and the assertion follows. \square

The authors do not know how to construct globally the spectral sequence (3.1.5.2) under the assumption of Corollary 3.1.6 without using patching.

Corollary 3.1.7. — *Let U be a scheme of finite type over a noetherian scheme S and $D \subset U$ be a Cartier divisor. Let $q: (U \times_S U)^\sim \rightarrow U \times_S U$ be the log product with respect to D and assume that $q: (U \times_S U)^\sim \rightarrow U \times_S U$ is locally of complete intersection of relative dimension 0. Let $\Delta_U \subset U \times_S U$ and $\Delta_U^{\mathrm{log}} \subset (U \times_S U)^\sim$ denote the diagonal and the log diagonal respectively and identify the inverse image of $\Delta_D \subset U \times_S U$ by $(U \times_S U)^\sim \rightarrow U \times_S U$ with $\mathbf{G}_{m,D}$.*

Then, the kernel of the surjection $q^ \mathcal{O}_{\Delta_U} \rightarrow \mathcal{O}_{\Delta_U^{\mathrm{log}}}$ and $L_r q^* \mathcal{O}_{\Delta_U}$ for $r > 0$ are coherent $\mathcal{O}_{\mathbf{G}_{m,D}}$ -modules and we have*

$$[\mathrm{Ker}(q^* \mathcal{O}_{\Delta_U} \rightarrow \mathcal{O}_{\Delta_U^{\mathrm{log}}})] + \sum_{r>0} (-1)^r [L_r q^* \mathcal{O}_{\Delta_U}] = [\mathbf{G}_{m,D}]$$

in $G(\mathbf{G}_{m,D})$.

Proof. — We define a locally free $\mathcal{O}_{U \times_S U}$ -module of rank 2 by $\mathcal{N} = \mathrm{pr}_1^* \mathcal{I}_D \oplus \mathrm{pr}_2^* \mathcal{I}_D$ and define a \mathbf{P}^1 -bundle $\mathbf{P}(\mathrm{pr}_1^* \mathcal{I}_D \oplus \mathrm{pr}_2^* \mathcal{I}_D)$ over $U \times_S U$. The complement $\mathbf{P} \subset \mathbf{P}(\mathrm{pr}_1^* \mathcal{I}_D \oplus \mathrm{pr}_2^* \mathcal{I}_D)$ of the sections defined by the surjections $\mathcal{N} = \mathrm{pr}_1^* \mathcal{I}_D \oplus \mathrm{pr}_2^* \mathcal{I}_D \rightarrow \mathrm{pr}_i^* \mathcal{I}_D$ for $i = 1, 2$ is a \mathbf{G}_m -bundle on $U \times_S U$. We regard the log product $(U \times_S U)^\sim$ as a subscheme of \mathbf{P} . By the assumption that $(U \times_S U)^\sim \rightarrow U \times_S U$ is locally of complete intersection of relative dimension 0, the immersion $(U \times_S U)^\sim \rightarrow \mathbf{P}$ is a regular immersion of codimension 1 by Lemma below.

We apply Proposition 3.1.5 to the diagram

$$(3.1.7.1) \quad \begin{array}{ccccccc} D & \longrightarrow & P_{D \times_S D} & \xlongequal{\quad} & P_{D \times_S D} & \longrightarrow & D \times_S D \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & (U \times_S U)^\sim & \xrightarrow{\quad c \quad} & P & \longrightarrow & U \times_S U \end{array}$$

where the image of the first arrow $U \rightarrow (U \times_S U)^\sim$ in the bottom is Δ_U^{\log} and that of the composition $U \rightarrow U \times_S U$ is Δ_U . In the notation there, we have $C = D \times_S D$, $Y = U$, $E_Y = D$ and E is the \mathbf{G}_m -bundle $P_{D \times_S D}$. Hence $E \times_C E_Y$ is $P_D = \mathbf{G}_{m,D}$.

In the right hand side of (3.1.5.4), the pull back of $\mathcal{N} = \mathrm{pr}_1^* \mathcal{I}_D \oplus \mathrm{pr}_2^* \mathcal{I}_D$ to D is $\mathcal{N}_{D/U}^{\oplus 2}$ and the second exterior power $\Lambda^2 \pi^* \mathcal{N}$ is $\mathcal{N}_{D/U}^{\otimes 2}$. Since $\mathcal{N}_{E_Y/Y} = \mathcal{N}_{D/U}$ and the pull-back of $\mathcal{O}(1)$ to D is also $\mathcal{N}_{D/U}$ by Proposition 3.1.5.1, the assertion follows. \square

Lemma 3.1.8. — *Let S be a noetherian scheme, $X \rightarrow S$ be a scheme locally of complete intersection of relative dimension d and $P \rightarrow S$ be a smooth scheme of relative dimension n . Then, an immersion $X \rightarrow P$ over S is a regular immersion of codimension $n - d$.*

Proof. — Since the assertion is local on X , we may take a regular immersion $X \rightarrow Q$ of codimension c over S to a smooth scheme Q of relative dimension $d + c$ over S . The immersion $X \rightarrow P \times_S Q$ is the composition of the section $X \rightarrow P \times_S X$ of a smooth morphism of relative dimension n and the smooth base change $P \times_S X \rightarrow P \times_S Q$ of the regular immersion $X \rightarrow Q$ of codimension c and is a regular immersion of codimension $n + c$. It is also the composition of the section $X \rightarrow X \times_S Q$ of a smooth morphism of relative dimension $d + c$ and an immersion $X \times_S Q \rightarrow P \times_S Q$. Hence, the immersion $X \times_S Q \rightarrow P \times_S Q$ is a regular immersion of codimension $(n + c) - (d + c) = n - d$ on the image of X by [13, Proposition 19.1.5]. Since $X \times_S Q \rightarrow P \times_S Q$ is a smooth base change of $X \rightarrow P$, the assertion follows. \square

3.2. Derived exterior power and localized Chern classes. — We study the relation between derived exterior power complexes and localized Chern classes. Let X be a noetherian scheme, \mathcal{E} and \mathcal{E}' be locally free \mathcal{O}_X -modules of the same finite rank n and $e: \mathcal{E} \rightarrow \mathcal{E}'$ be a morphism. For an integer $k \geq 0$, we consider the complex $[\Lambda^k \mathcal{E} \rightarrow \Lambda^k \mathcal{E}']$ where $\Lambda^k \mathcal{E}'$ is put on degree 0. Let D be a closed subset of X such that $\mathcal{K} = [\mathcal{E} \rightarrow \mathcal{E}']$ is acyclic on the complement $X \setminus D$. We define a morphism

$$(3.2.0.1) \quad [\Lambda^k \mathcal{E} \rightarrow \Lambda^k \mathcal{E}']_D: G(X) \longrightarrow G(D)$$

by sending the class $[\mathcal{F}]$ of a coherent \mathcal{O}_X -module \mathcal{F} to $[\mathrm{Coker}(\mathcal{F} \otimes \Lambda^k \mathcal{E} \rightarrow \mathcal{F} \otimes \Lambda^k \mathcal{E}')] - [\mathrm{Ker}(\mathcal{F} \otimes \Lambda^k \mathcal{E} \rightarrow \mathcal{F} \otimes \Lambda^k \mathcal{E}')]_D$.

Recall that homomorphisms $\lambda_t, \gamma_t: K(X) \rightarrow 1 + tK(X)[[t]] \subset K(X)[[t]]^\times$ are defined by $\lambda_t([\mathcal{E}]) = \sum_{q=0}^n [\Lambda^q \mathcal{E}] t^q$ and $\gamma_t([\mathcal{E}]) = \lambda_{\frac{t}{1-t}}([\mathcal{E}])$. We define operators $\gamma_k(\mathcal{K})_D$:

$G(\mathbf{X}) \rightarrow G(\mathbf{D})$ for $k \geq 1$ by requiring that $\gamma_i(\mathcal{K})_{\mathbf{D}} = \sum_{k=1}^n \gamma_k(\mathcal{K})_{\mathbf{D}} \cdot t^k$ is given by

$$(3.2.0.2) \quad \gamma_i(\mathcal{K})_{\mathbf{D}} = \left(\sum_{k=1}^n [\Lambda^k \mathcal{E} \rightarrow \Lambda^k \mathcal{E}']_{\mathbf{D}} \cdot \left(\frac{t}{1-t} \right)^k \right) \cdot \gamma_i([\mathcal{E}])^{-1}.$$

If $\gamma_k(\mathcal{K})$ without the suffix \mathbf{D} denotes the composition with $G(\mathbf{D}) \rightarrow G(\mathbf{X})$, we have $1 + \gamma_i(\mathcal{K}) = 1 + (\gamma_i([\mathcal{E}']) - \gamma_i([\mathcal{E}])) \cdot \gamma_i([\mathcal{E}])^{-1} = \gamma_i([\mathcal{E}'] - [\mathcal{E}])$.

Let $c_k(\mathcal{K})_{\mathbf{D}}: \mathrm{CH}_{\bullet}(\mathbf{X}) \rightarrow \mathrm{CH}_{\bullet-k}(\mathbf{D})$ be the localized Chern class map defined by using the graph construction in [10, Section 18.1].

Proposition 3.2.1. — *Let S be a regular noetherian scheme of finite dimension. Let \mathbf{X} be a scheme of finite type over S , \mathcal{E} and \mathcal{E}' locally free $\mathcal{O}_{\mathbf{X}}$ -modules of the same finite rank and $e: \mathcal{E} \rightarrow \mathcal{E}'$ be a morphism such that the restriction on the complement of \mathbf{D} is an isomorphism. For the complex $\mathcal{K} = [e: \mathcal{E} \rightarrow \mathcal{E}']$ and an integer $k > 0$, we have the following.*

1. *The map $\gamma_k(\mathcal{K})_{\mathbf{D}}: G(\mathbf{X}) \rightarrow G(\mathbf{D})$ sends the topological filtration $F_j G(\mathbf{X})$ to $F_{j-k} G(\mathbf{D})$.*
2. *The diagram*

$$(3.2.1.1) \quad \begin{array}{ccc} \mathrm{CH}_j(\mathbf{X}) & \xrightarrow{c_k(\mathcal{K})_{\mathbf{D}}} & \mathrm{CH}_{j-k}(\mathbf{D}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_j^F G(\mathbf{X}) & \xrightarrow{\gamma_k(\mathcal{K})_{\mathbf{D}}} & \mathrm{Gr}_{j-k}^F G(\mathbf{D}) \end{array}$$

is commutative.

Proof. — 1. It suffices to show $\gamma_k(\mathcal{K})_{\mathbf{D}}([\mathbf{X}]) \in F_{d-k} G(\mathbf{D})$ assuming that \mathbf{X} is integral of dimension d and $\mathbf{D} \neq \mathbf{X}$, by a standard argument. We show this by using the most elementary case of MacPherson's graph construction cf. [10, Section 18.1].

Let n be the rank of \mathcal{E} and $p: G \rightarrow \mathbf{X}$ be the Grassmann scheme $\mathrm{Grass}_n(\mathcal{E} \oplus \mathcal{E}')$ classifying subbundles of rank n . The second factor $\mathcal{E}' \subset \mathcal{E} \oplus \mathcal{E}'$ defines a section $s_0: \mathbf{X} \rightarrow G$. Let t denote the coordinate of $\mathbf{G}_{m,\mathbf{X}}$. Then, the graph of $t^{-1} \cdot e: \mathcal{E}_{\mathbf{G}_{m,\mathbf{X}}} \rightarrow \mathcal{E}'_{\mathbf{G}_{m,\mathbf{X}}}$ defines a section $\tilde{s}: \mathbf{G}_{m,\mathbf{X}} \rightarrow \mathbf{G}_{m,G}$. At $t = 1$, the restriction $\tilde{s}|_1: \mathbf{X} \rightarrow G$ is the section defined by the graph of e .

On the complement $U = \mathbf{X} \setminus \mathbf{D}$, the restriction $e|_U: \mathcal{E}_U \rightarrow \mathcal{E}'_U$ is an isomorphism. The transpose of the graph of $t \cdot e|_U^{-1}: \mathcal{E}'_{\mathbf{A}_U^1} \rightarrow \mathcal{E}_{\mathbf{A}_U^1}$ defines a section $\bar{s}: \mathbf{A}_U^1 \rightarrow \mathbf{A}_{G_U}^1$. At $t = 0$, the restriction $\bar{s}|_0: U \rightarrow G_U$ is the restriction $s_0|_U$. The restrictions of \tilde{s} and \bar{s} on $\mathbf{G}_{m,U} = \mathbf{G}_{m,\mathbf{X}} \cap \mathbf{A}_U^1$ are the same. Let $\tilde{\mathbf{X}} \subset \mathbf{A}_G^1$ denote the schematic closure of $\tilde{s}(\mathbf{G}_{m,\mathbf{X}}) \cup \bar{s}(\mathbf{A}_U^1)$ and let $\pi: \tilde{\mathbf{X}} \rightarrow \mathbf{A}_X^1$ be the projection. Since $\bar{s}|_0: U \rightarrow G_U$ is the restriction of s_0 , the fiber $\tilde{\mathbf{X}}_0 = \tilde{\mathbf{X}} \times_{\mathbf{A}_X^1} X$ at $t = 0$ contains $s_0(\mathbf{X})$ as a closed subscheme.

Let $\tilde{\mathcal{E}} \subset \mathcal{E}_{\tilde{\mathbf{X}}} \oplus \mathcal{E}'_{\tilde{\mathbf{X}}}$ be the restriction to $\tilde{\mathbf{X}} \subset \mathbf{A}_G^1$ of the tautological subbundle and $\tilde{e}: \tilde{\mathcal{E}} \rightarrow \mathcal{E}'_{\tilde{\mathbf{X}}}$ be the restriction of the second projection. We consider the complex $\tilde{\mathcal{K}} =$

$[\tilde{\mathcal{E}} \rightarrow \mathcal{E}'_{\tilde{X}}]$. The restriction of $\tilde{\mathcal{K}}$ to $\tilde{U} = \bar{s}(\mathbf{A}_U^1)$ is acyclic. We put $\tilde{D} = \mathbf{A}_D^1 \times_{\mathbf{A}_X^1} \tilde{X}$ and let $\pi_D: \tilde{D} \rightarrow \mathbf{A}_D^1$ be the projection. We consider the composition

$$G(\tilde{X}) \xrightarrow{\gamma_k(\tilde{\mathcal{K}})_{\tilde{D}}} G(\tilde{D}) \xrightarrow{\pi_{D*}} G(\mathbf{A}_D^1).$$

The pull-backs $i_1^*, i_0^*: G(\mathbf{A}_D^1) \rightarrow G(D)$ by the sections $D \rightarrow \mathbf{A}_D^1$ at $t = 1, 0$ are the same isomorphisms. Since the fiber $\tilde{\mathcal{K}}_1$ at $t = 1$ recovers the original complex \mathcal{K} on X , we have $\gamma_k(\mathcal{K})_D([X]) = i_1^* \pi_{D*}(\gamma_k(\tilde{\mathcal{K}})_{\tilde{D}}([\tilde{X}]))$. Let $\tilde{\mathcal{K}}_0$ denote the pull-back of $\tilde{\mathcal{K}}$ to \tilde{X}_0 and let $\pi_0: \tilde{X}_0 \rightarrow X_0$ be the projection. We put $\tilde{D}_0 = \tilde{D} \cap \tilde{X}_0$. We have further $\gamma_k(\mathcal{K})_D([X]) = i_0^* \pi_{D*}(\gamma_k(\tilde{\mathcal{K}})_{\tilde{D}}([\tilde{X}])) = \pi_{0*}(\gamma_k(\tilde{\mathcal{K}}_0)_{\tilde{D}_0}([\tilde{X}_0]))$.

Let $[\bar{X}_0]$ denote the class of the kernel $\text{Ker}(\mathcal{O}_{\tilde{X}_0} \rightarrow \mathcal{O}_{s_0(X)})$ of the surjection. Since the restriction $\tilde{\mathcal{K}}|_{s_0(X)}$ is acyclic and $\bar{X}_0 \subset \tilde{D}_0$, we have $\gamma_k(\tilde{\mathcal{K}}_0)_{\tilde{D}_0}([\tilde{X}_0]) = \gamma_k(\tilde{\mathcal{K}}_0)_{\tilde{D}_0}([\bar{X}_0]) = \gamma_k([\mathcal{E}'] - [\tilde{\mathcal{E}}])([\bar{X}_0])$. Hence,

$$(3.2.1.2) \quad \gamma_k(\mathcal{K})_D([X]) = \pi_{0*}(\gamma_k([\mathcal{E}'] - [\tilde{\mathcal{E}}])([\bar{X}_0]))$$

is an element of $F_{d-k}G(D)$ as required.

2. In the notation above, the localized Chern class $c_k(\mathcal{K})_D \cap [X] \in \text{CH}_{d-k}(D)$ is defined as $\pi_{0*}(c(\mathcal{E}')c(\mathcal{E})^{-1}([\bar{X}_0]))_{\dim=d-k}$ [10, Section 18.1]. Hence, the assertion follows. \square

The proof of Proposition 3.2.1 shows that the system of maps $\gamma_k(\mathcal{K})_D$ is characterized by the compatibility with the Gysin maps for regular immersions and the normalization property that the composition with the natural map $G(D) \rightarrow G(X)$ is equal to the map $\gamma_k([\mathcal{E}'] - [\mathcal{E}])$. Similarly as (3.2.1.2), we have

$$(3.2.1.3) \quad [\Lambda^k(\mathcal{E}) \rightarrow \Lambda^k(\mathcal{E}')]_D([X]) = \pi_{0*}([\Lambda^k(\mathcal{E}')] - [\Lambda^k(\tilde{\mathcal{E}})])([\bar{X}_0]).$$

Similarly as [10, Section 18.1], we have the following properties.

Corollary 3.2.2. — *Let X be a scheme of finite type over a regular noetherian scheme S of finite dimension and $D \subset X$ be a closed subscheme.*

1. *Let $\mathcal{E} \rightarrow \mathcal{E}'$ be a morphism of locally free $\mathcal{O}_{\mathbf{A}_X^1}$ -modules of finite rank such that the complex of $\mathcal{K} = [\mathcal{E} \rightarrow \mathcal{E}']$ is acyclic outside \mathbf{A}_D^1 . Let \mathcal{K}_0 and \mathcal{K}_1 be the pull-back of \mathcal{K} by the 0-section and the 1-section respectively. Then, we have*

$$\gamma_k(\mathcal{K}_0)_D = \gamma_k(\mathcal{K}_1)_D.$$

2. *Let $\mathcal{E}_1 \rightarrow \mathcal{E}'_1$ and $\mathcal{E}_2 \rightarrow \mathcal{E}'_2$ be morphisms of locally free \mathcal{O}_X -modules of finite rank such that the complexes $\mathcal{K}_1 = [\mathcal{E}_1 \rightarrow \mathcal{E}'_1]$ and $\mathcal{K}_2 = [\mathcal{E}_2 \rightarrow \mathcal{E}'_2]$ are acyclic outside D . If $\mathcal{K}_1 \rightarrow \mathcal{K}_2$ is a quasi-isomorphism, we have*

$$\gamma_k(\mathcal{K}_1)_D = \gamma_k(\mathcal{K}_2)_D.$$

Let \mathcal{K} be a complex of \mathcal{O}_X -modules such that there exist a morphism $\mathcal{E} \rightarrow \mathcal{E}'$ of locally free \mathcal{O}_X -modules of finite rank and a quasi-isomorphism $[\mathcal{E} \rightarrow \mathcal{E}'] \rightarrow \mathcal{K}$. Then, we define the map $\gamma_k(\mathcal{K})_D: G(X) \rightarrow G(D)$ to be $\gamma_k([\mathcal{E} \rightarrow \mathcal{E}'])_D$. This is well-defined by Corollary 3.2.2.2. The localized Chern class $c_k(\mathcal{K})_D: CH_j(X) \rightarrow CH_{j-k}(D)$ is defined similarly as $c_k([\mathcal{E} \rightarrow \mathcal{E}'])_D$ if X is of finite type over a regular noetherian scheme S of finite dimension.

We consider coherent \mathcal{O}_X -modules $\mathcal{F}, \mathcal{F}'$ and a morphism $f: \mathcal{F} \rightarrow \mathcal{F}'$ of \mathcal{O}_X -modules satisfying the following condition:

- (3.2.3.1) There exists a locally free \mathcal{O}_X -module \mathcal{E}' of finite rank and a surjection $\mathcal{E}' \rightarrow \mathcal{F}'$. The kernel $\mathcal{E} = \text{Ker}(\mathcal{F} \oplus \mathcal{E}' \rightarrow \mathcal{F}')$ is a locally free \mathcal{O}_X -module of the same finite rank as \mathcal{E}' .

Let $D \subset X$ be a closed subscheme such that f is an isomorphism outside D . Then, since the map $[\mathcal{E} \rightarrow \mathcal{E}'] \rightarrow \mathcal{K} = [\mathcal{F} \rightarrow \mathcal{F}']$ is a quasi-isomorphism, the map $\gamma_k(\mathcal{F} \rightarrow \mathcal{F}')_D: G(X) \rightarrow G(D)$ and the localized Chern class $c_k(\mathcal{F} \rightarrow \mathcal{F}')_D: CH_j(X) \rightarrow CH_{j-k}(D)$ are defined.

We further assume that \mathcal{F} is of tor-dimension ≤ 1 and let r be the virtual rank. In other words, for a surjection $\mathcal{E} \rightarrow \mathcal{F}$ as in (3.2.3.1), the kernel $\mathcal{L} = \text{Ker}(\mathcal{E} \rightarrow \mathcal{F})$ is locally free of rank $\text{rank } \mathcal{E} - r$. Since the canonical map $\text{Ker}(\mathcal{E} \rightarrow \mathcal{F}) \rightarrow \text{Ker}(\mathcal{E}' \rightarrow \mathcal{F}')$ is an isomorphism, the sheaf \mathcal{F}' is also of tor-dimension ≤ 1 and of virtual rank r .

We also define a map $\delta_k(\mathcal{F} \rightarrow \mathcal{F}')_D: G(X) \rightarrow G(D)$ for $k > 0$ by requiring that $\delta_t(\mathcal{F} \rightarrow \mathcal{F}')_D = \sum_{k=1}^{\infty} \delta_k(\mathcal{F} \rightarrow \mathcal{F}')_D \cdot t^k$ is given by

$$\delta_t(\mathcal{F} \rightarrow \mathcal{F}')_D = \gamma_t(\mathcal{F} \rightarrow \mathcal{F}')_D \cdot \gamma_t([\mathcal{F}] - r).$$

We recall that the localized Chern class $c_k(\mathcal{F}' - \mathcal{F})_D: CH_j(X) \rightarrow CH_{j-k}(D)$ is defined by requiring

$$(3.2.3.2) \quad \sum_{k>0} c_k(\mathcal{F}' - \mathcal{F})_D \cdot t^k = \left(\sum_{k>0} c_k(\mathcal{F} \rightarrow \mathcal{F}')_D \cdot t^k \right) \cdot c_t(\mathcal{F})$$

in [27, (3.24)].

For the definition and properties of the derived exterior power $L\Lambda^k \mathcal{F}$, we refer to [26, Section 1.2]. We recall that for a locally free resolution $[\mathcal{L} \rightarrow \mathcal{E}] \rightarrow \mathcal{F}$ as above, we have a quasi-isomorphism $[\Gamma^k \mathcal{L} \rightarrow \Gamma^{k-1} \mathcal{L} \otimes \mathcal{E} \rightarrow \dots \rightarrow \mathcal{L} \otimes \Lambda^{k-1} \mathcal{E} \rightarrow \Lambda^k \mathcal{E}] \rightarrow L\Lambda^k \mathcal{F}$, where Γ^\bullet denotes the divided power. For an integer $k > 0$, the mapping cone $[L\Lambda^k \mathcal{F} \rightarrow L\Lambda^k \mathcal{F}']$ of the derived exterior powers is defined. We define a map

$$[L\Lambda^k \mathcal{F} \rightarrow L\Lambda^k \mathcal{F}']_D: G(X) \rightarrow G(D)$$

sending $[\mathcal{G}]$ to $\sum_q (-1)^q \text{Tor}_q^{\mathcal{O}_X}([L\Lambda^k \mathcal{F} \rightarrow L\Lambda^k \mathcal{F}'], \mathcal{G})$. We describe the map $[L\Lambda^k \mathcal{F} \rightarrow L\Lambda^k \mathcal{F}']_D$ using the operators $\gamma_k(\mathcal{F} \rightarrow \mathcal{F}')_D: G(X) \rightarrow G(D)$.

Proposition 3.2.4. — *Let S be a regular noetherian scheme of finite dimension and X be a scheme of finite type over S . Let $\mathcal{F}, \mathcal{F}'$ be coherent \mathcal{O}_X -modules of tor-dimension ≤ 1 and of virtual rank r and $f: \mathcal{F} \rightarrow \mathcal{F}'$ be a morphism of \mathcal{O}_X -modules satisfying the condition (3.2.3.1). Let $D \subset X$ be a closed subscheme such that f is an isomorphism outside D .*

1. *We put formally $\beta_t = \sum_{k=1}^{\infty} [\mathrm{L}\Lambda^k \mathcal{F} \rightarrow \mathrm{L}\Lambda^k \mathcal{F}']_{\mathrm{D}} (\frac{t}{1-t})^k: \mathrm{G}(X) \rightarrow \mathrm{G}(\mathrm{D})[[t]]$. Then, we have $\delta_t(\mathcal{F} \rightarrow \mathcal{F}')_{\mathrm{D}} = (1-t)^r \beta_t$.*

2. *For $n = r + 1$, we have*

$$(3.2.4.1) \quad [\mathrm{L}\Lambda^n \mathcal{F} \rightarrow \mathrm{L}\Lambda^n \mathcal{F}']_{\mathrm{D}} = \delta_n(\mathcal{F} \rightarrow \mathcal{F}')_{\mathrm{D}}$$

and it sends the topological filtration $F_j \mathrm{G}(X)$ to $F_{j-n} \mathrm{G}(\mathrm{D})$. Further the diagram

$$(3.2.4.2) \quad \begin{array}{ccc} \mathrm{CH}_j(X) & \xrightarrow{c_n(\mathcal{F}' - \mathcal{F})_{\mathrm{D}}} & \mathrm{CH}_{j-n}(D) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_j^{\mathrm{F}} \mathrm{G}(X) & \xrightarrow{[\mathrm{L}\Lambda^n \mathcal{F} \rightarrow \mathrm{L}\Lambda^n \mathcal{F}']_{\mathrm{D}}} & \mathrm{Gr}_{j-n}^{\mathrm{F}} \mathrm{G}(D) \end{array}$$

is commutative.

In the case where $\mathcal{F} = 0$ and $D = X$, Proposition 3.2.4.2 is proved in [26, Proposition 2.4.4].

Proof. — 1. The equality $\delta_t(\mathcal{F} \rightarrow \mathcal{F}')_{\mathrm{D}} = (1-t)^r \beta_t$ is reduced to the equality

$$(3.2.4.3) \quad \delta_t(\mathcal{F} \rightarrow \mathcal{F}')_{\mathrm{D}}([\mathrm{X}]) = (1-t)^r \beta_t([\mathrm{X}])$$

in $\mathrm{G}(\mathrm{D})[[t]]$, by a standard argument. In fact, since $\mathrm{G}(X)$ is generated by the classes of integral closed subschemes V , it suffices to prove the formula for $[V]$ and to take the push-forward. By the assumption that \mathcal{F} and \mathcal{F}' are of tor-dimension ≤ 1 , the kernel $\mathcal{L} = \mathrm{Ker}(\mathcal{E} \rightarrow \mathcal{F}) = \mathrm{Ker}(\mathcal{E}' \rightarrow \mathcal{F}')$ is locally free and we have quasi-isomorphisms $[\mathcal{L} \rightarrow \mathcal{E}] \rightarrow \mathcal{F}$ and $[\mathcal{L} \rightarrow \mathcal{E}'] \rightarrow \mathcal{F}'$. Hence, we obtain $[\mathrm{L}\Lambda^k \mathcal{F} \rightarrow \mathrm{L}\Lambda^k \mathcal{F}']_{\mathrm{D}}([\mathrm{X}]) = \sum_{q=0}^k (-1)^q [\Lambda^{k-q} \mathcal{E} \rightarrow \Lambda^{k-q} \mathcal{E}']_{\mathrm{D}}[\Gamma^q \mathcal{L}]([\mathrm{X}])$. We apply the graph construction to $\mathcal{E} \rightarrow \mathcal{E}'$ and use the notation in the proof of Proposition 3.2.1. Then by (3.2.1.3), we have

$$\begin{aligned} & [\mathrm{L}\Lambda^k \mathcal{F} \rightarrow \mathrm{L}\Lambda^k \mathcal{F}']_{\mathrm{D}}([\mathrm{X}]) \\ &= \pi_{0*} \left(\sum_{q=0}^k (-1)^q ([\Lambda^{k-q} \mathcal{E}'] - [\Lambda^{k-q} \tilde{\mathcal{E}}]) [\Gamma^q \mathcal{L}]([\bar{X}_0]) \right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\beta_i([\mathbf{X}]) &= \pi_{0*}((\gamma_i([\mathcal{E}']) - \gamma_i([\tilde{\mathcal{E}}]))\gamma_i([\mathcal{L}])^{-1}([\bar{\mathbf{X}}_0])) \\
&= \gamma_i([\mathcal{F}'])\pi_{0*}\left(\left(1 - \frac{1}{1 + \gamma_i([\mathcal{E} \rightarrow \tilde{\mathcal{E}}])}\right)([\bar{\mathbf{X}}_0])\right) \\
&= \gamma_i([\mathcal{F}'])\left(1 - \frac{1}{1 + \gamma_i([\mathcal{E} \rightarrow \mathcal{E}'])_{\mathbb{D}}}\right)([\mathbf{X}]) \\
&= \gamma_i(\mathcal{F})\gamma_i([\mathcal{E} \rightarrow \mathcal{E}'])_{\mathbb{D}}([\mathbf{X}]).
\end{aligned}$$

Since $\gamma_i(\mathcal{F}) = \gamma_i(\mathcal{F} - r)\gamma_i(1)^r = \gamma_i(\mathcal{F} - r)(1 - t)^{-r}$, we obtain the equality (3.2.4.3).

2. For $n = r + 1$, we have

$$\begin{aligned}
\delta_i(\mathcal{F} \rightarrow \mathcal{F}')_{\mathbb{D}} &= (1 - t)^r \beta_i \\
&\equiv \sum_{k=1}^r [\mathrm{L}\Lambda^k \mathcal{F} \rightarrow \mathrm{L}\Lambda^k \mathcal{F}']_{\mathbb{D}} \cdot t^k (1 - t)^{r-k} \\
&\quad + [\mathrm{L}\Lambda^n \mathcal{F} \rightarrow \mathrm{L}\Lambda^n \mathcal{F}']_{\mathbb{D}} \cdot t^n (1 - t)^{-1} \bmod t^{n+1}.
\end{aligned}$$

Comparing the coefficients of t^n , we obtain $\delta_n(\mathcal{F} \rightarrow \mathcal{F}')_{\mathbb{D}} = [\mathrm{L}\Lambda^n \mathcal{F} \rightarrow \mathrm{L}\Lambda^n \mathcal{F}']_{\mathbb{D}}$. The remaining assertion follows from this and Proposition 3.2.1. \square

Lemma 3.2.5. — *Let \mathcal{E} and \mathcal{E}' be locally free \mathcal{O}_X -modules of the same rank n and let $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X$ be morphisms of \mathcal{O}_X -modules. We consider the Koszul complexes $\mathcal{K} = \mathrm{Kos}(\mathcal{E} \rightarrow \mathcal{O}_X)$ and $\mathcal{K}' = \mathrm{Kos}(\mathcal{E}' \rightarrow \mathcal{O}_X)$ and the induced morphism $\mathcal{K}' \rightarrow \mathcal{K}$ of complexes.*

Let $\mathcal{M} = [\mathcal{K}^ \rightarrow \mathcal{K}'^*]$ be the mapping cone of the morphism of the dual complexes. Then, the homology sheaves $\mathcal{H}_q(\mathcal{M})$ are $\mathcal{H}_0(\mathcal{K}') = \mathcal{O}_X/(\mathrm{Image} \mathcal{E}')$ -modules.*

Proof. — The product $\mathcal{K} \otimes \mathcal{K} \rightarrow \mathcal{K}$ of Koszul complex induces a canonical map $\mathcal{K} \otimes \mathcal{K}^* \rightarrow \mathcal{K}^*$. The canonical maps $\mathcal{K} \otimes \mathcal{K}^* \rightarrow \mathcal{K}^*$ and $\mathcal{K}' \otimes \mathcal{K}'^* \rightarrow \mathcal{K}'^*$ induces $\mathcal{K}' \otimes \mathcal{M} \rightarrow \mathcal{M}$. This defines a multiplication $\mathcal{H}_0(\mathcal{K}') \otimes \mathcal{H}_q(\mathcal{M}) \rightarrow \mathcal{H}_q(\mathcal{M})$ compatible with the \mathcal{O}_X -module structure. Thus the assertion follows. \square

Let X be a scheme and $\mathcal{K} \rightarrow \mathcal{K}'$ be a morphism of chain complexes of flat \mathcal{O}_X -modules. For an integer $q \geq 0$, we consider the mapping cone $[\mathrm{L}\Lambda^q \mathcal{K} \rightarrow \mathrm{L}\Lambda^q \mathcal{K}']$. The canonical maps $\mathrm{L}\Lambda^{q+1} \mathcal{K} \rightarrow \mathrm{L}\Lambda^q \mathcal{K} \otimes \mathcal{K}$ and $\mathrm{L}\Lambda^{q+1} \mathcal{K}' \rightarrow \mathrm{L}\Lambda^q \mathcal{K}' \otimes \mathcal{K}'$ [26, (1.2.1.4)] induce a map

$$(3.2.5.1) \quad [\mathrm{L}\Lambda^{q+1} \mathcal{K} \rightarrow \mathrm{L}\Lambda^{q+1} \mathcal{K}'] \rightarrow [\mathrm{L}\Lambda^q \mathcal{K} \rightarrow \mathrm{L}\Lambda^q \mathcal{K}'] \otimes \mathcal{K}'.$$

We consider the following condition on a complex \mathcal{K} of \mathcal{O}_X -modules.

(L(n)) For every $x \in X$, there exist an open neighborhood U of x , locally free \mathcal{O}_U -modules \mathcal{E} and \mathcal{L} of rank n and 1 respectively and a quasi-isomorphism $[\mathcal{L} \rightarrow \mathcal{E}] \rightarrow \mathcal{K}|_U$.

If a complex \mathcal{K} of \mathcal{O}_X -modules satisfies the condition (L(n)), it is a perfect complex of tor-dimension ≤ 1 . For a perfect complex \mathcal{K} of \mathcal{O}_X -modules satisfying the condition (L(n)), let Z denote the closed subscheme of X defined by the annihilator ideal $\mathcal{I}_Z = \text{Ann} \Lambda^n \mathcal{H}_0(\mathcal{K})$ and $i: Z \rightarrow X$ denote the immersion. Then, for a quasi-isomorphism $[\mathcal{L} \rightarrow \mathcal{E}] \rightarrow \mathcal{K}|_U$ on an open subscheme U of X as in (L(n)), the intersection $Z \cap U \subset U$ is the largest closed subscheme where the map $\mathcal{L} \rightarrow \mathcal{E}$ is the zero-map. Further, a quasi-isomorphism $[\mathcal{L} \rightarrow \mathcal{E}] \rightarrow \mathcal{K}|_U$ induces an isomorphism $i^* \mathcal{L} \rightarrow \mathcal{L}_Z|_{Z \cap U} = L_1 i^* \mathcal{K}|_{Z \cap U}$. Hence, $\mathcal{L}_Z = L_1 i^* \mathcal{K}$ is an invertible \mathcal{O}_Z -module [26, Lemma 2.4.1.1].

Corollary 3.2.6. — *Let \mathcal{K} and \mathcal{K}' be complexes of \mathcal{O}_X -modules satisfying the condition (L(n)) and $\mathcal{K} \rightarrow \mathcal{K}'$ be a morphism such that the mapping cone $[\mathcal{K} \rightarrow \mathcal{K}']$ is of tor-dimension ≤ 1 . For $q > 0$, let $[\mathbf{L}\Lambda^q \mathcal{K} \rightarrow \mathbf{L}\Lambda^q \mathcal{K}']$ denote the mapping cone.*

Let Z and Z' denote the closed subschemes of X defined by the annihilator ideals $\mathcal{I}_Z = \text{Ann} \Lambda^n \mathcal{H}_0(\mathcal{K})$ and $\mathcal{I}_{Z'} = \text{Ann} \Lambda^n \mathcal{H}_0(\mathcal{K}')$. Let $i: Z \rightarrow X$ and $i': Z' \rightarrow X$ denote the immersions and let $\mathcal{L}_Z = L_1 i^ \mathcal{K}$ and $\mathcal{L}'_{Z'} = L_1 i'^* \mathcal{K}'$ be the invertible \mathcal{O}_Z -module and $\mathcal{O}_{Z'}$ -module respectively.*

1. *The scheme Z is a closed subscheme of Z' and the canonical map $\mathcal{L}'_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_Z \rightarrow \mathcal{L}_Z$ is an isomorphism.*

2. *The homology sheaf $\mathcal{H}_p([\mathbf{L}\Lambda^q \mathcal{K} \rightarrow \mathbf{L}\Lambda^q \mathcal{K}'])$ is an $\mathcal{O}_{Z'}$ -module if $p > 0$ or $q \geq n$.*

3. *The map (3.2.5.1) induces an isomorphism*

$$(3.2.6.1) \quad \mathcal{H}_{p+1}([\mathbf{L}\Lambda^{q+1} \mathcal{K} \rightarrow \mathbf{L}\Lambda^{q+1} \mathcal{K}']) \rightarrow \mathcal{H}_p([\mathbf{L}\Lambda^q \mathcal{K} \rightarrow \mathbf{L}\Lambda^q \mathcal{K}']) \otimes \mathcal{L}'_{Z'}$$

for $p > 0$ or $q \geq n$.

In the case where $\mathcal{K} = 0$, Corollary 3.2.6.2 (resp. 3.2.6.3) is proved in [26, Lemma 2.4.2.1 (resp. 2.4.2.2)].

Proof. — 1. Since the question is local, we may assume that there is a quasi-isomorphism $[\mathcal{L} \rightarrow \mathcal{E}'] \rightarrow \mathcal{K}'$ for locally free sheaves \mathcal{E}' and \mathcal{L} of rank n and 1 . Then, the mapping cone $[\mathcal{E}' \oplus \mathcal{K} \rightarrow \mathcal{K}']$ is a perfect complex of tor-amplitude $[1, 1]$ and hence is quasi-isomorphic to $\mathcal{E}[1]$ for a locally free sheaf \mathcal{E} of rank n . Thus, we obtain a quasi-isomorphism $[\mathcal{L} \rightarrow \mathcal{E}] \rightarrow \mathcal{K}$. Since the pull-back of $\mathcal{L} \rightarrow \mathcal{E}$ to Z is the zero-map, the pull-back of $\mathcal{L} \rightarrow \mathcal{E}'$ to Z is also the zero-map. This shows that Z is a subscheme of Z' . Further, the canonical map $\mathcal{L}'_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_Z \rightarrow \mathcal{L}_Z$ is induced by the identity of \mathcal{L} and is an isomorphism.

2. Since the assertion is local, we may assume, as in the proof of 1, that $\mathcal{K} = [\mathcal{O}_X \xrightarrow{e} \mathcal{E}]$ and $\mathcal{K}' = [\mathcal{O} \xrightarrow{e'} \mathcal{E}']$ where \mathcal{E} and \mathcal{E}' are locally free of rank n , and $\mathcal{K} \rightarrow \mathcal{K}'$ is induced by $\mathcal{E} \rightarrow \mathcal{E}'$ compatible with e and e' . Then for $q \geq 0$, the derived exterior

power complex $L\Lambda^q\mathcal{K}$ is identified with the complex $[\mathcal{O}_X \xrightarrow{e} \mathcal{E} \xrightarrow{\wedge^e} \Lambda^2\mathcal{E} \xrightarrow{\wedge^e} \cdots \xrightarrow{\wedge^e} \Lambda^q\mathcal{E}]$. Consequently, if $q \geq n$, its shift $(L\Lambda^q\mathcal{K})[-q]$ is isomorphic to the dual of the Koszul complex $\text{Kos}(e^*: \mathcal{E}^* \rightarrow \mathcal{O}_X)$ associated to the dual of $e: \mathcal{O}_X \rightarrow \mathcal{E}$ and similarly for the complex $(L\Lambda^q\mathcal{K}')[-q]$. Since the ideal sheaf $\mathcal{I}_{Z'} \subset \mathcal{O}_X$ of Z' is the image of $e^*: \mathcal{E}^* \rightarrow \mathcal{O}_X$, the assertion follows from Lemma 3.2.5.

3. We show that (3.2.5.1) induces (3.2.6.1). The map (3.2.5.1) induces a homomorphism

$$\begin{aligned} \mathcal{H}_{p+1}([\Lambda^{q+1}\mathcal{K} \rightarrow \Lambda^{q+1}\mathcal{K}']) &\rightarrow \mathcal{H}_{p+1}([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}'] \otimes \mathcal{K}') \\ &= \mathcal{T}or_{p+1}^{\mathcal{O}_X}([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}'], \mathcal{K}'). \end{aligned}$$

Since \mathcal{K}' is of tor-dimension ≤ 1 , the spectral sequence

$$\begin{aligned} E_{s,t}^2 &= \mathcal{T}or_s^{\mathcal{O}_X}(\mathcal{H}_t([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}']), \mathcal{K}') \\ &\Rightarrow \mathcal{T}or_{s+t}^{\mathcal{O}_X}([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}'], \mathcal{K}') \end{aligned}$$

induces a homomorphism

$$\mathcal{T}or_{p+1}^{\mathcal{O}_X}([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}'], \mathcal{K}') \rightarrow \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{H}_p([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}']), \mathcal{K}').$$

Since $\mathcal{H}_p([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}'])$ is an $\mathcal{O}_{Z'}$ -module, we have an isomorphism

$$\begin{aligned} \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{H}_p([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}']), \mathcal{K}') &\rightarrow \mathcal{H}_p([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}']) \otimes L_1 i'^*\mathcal{K}' \\ &= \mathcal{H}_p([\Lambda^q\mathcal{K} \rightarrow \Lambda^q\mathcal{K}']) \otimes_{\mathcal{O}_{Z'}} \mathcal{L}'_{Z'}. \end{aligned}$$

(cf. [26, Lemma 2.4.1.3]). They define a canonical map (3.2.6.1).

By [26, (2.4.2.2)], the canonical maps

$$L_{p+1}\Lambda^{q+1}\mathcal{K} \rightarrow L_p\Lambda^q\mathcal{K} \otimes_{\mathcal{O}_Z} \mathcal{L}_Z, \quad L_{p+1}\Lambda^{q+1}\mathcal{K}' \rightarrow L_p\Lambda^q\mathcal{K}' \otimes_{\mathcal{O}_{Z'}} \mathcal{L}'_{Z'}$$

are isomorphisms. Further by 1, the canonical map $L_p\Lambda^q\mathcal{K} \otimes_{\mathcal{O}_Z} \mathcal{L}_Z \rightarrow L_p\Lambda^q\mathcal{K} \otimes_{\mathcal{O}_{Z'}} \mathcal{L}_{Z'}$ is an isomorphism. Thus the map (3.2.6.1) is an isomorphism. \square

3.3. Localized intersection product. — We briefly recall terminologies and properties on cotangent complexes [14, Chapitre II] and excess conormal complexes [26, Definition 1.6.3] which will play the central roles in this and the next subsections.

For a morphism of schemes $X \rightarrow S$, the cotangent complex $L_{X/S}$ is defined in [14, Chapitre II] as an object of the derived category $D_{qcoh}^-(\mathcal{O}_X)$ of the category of quasi-coherent \mathcal{O}_X -modules. We have $\mathcal{H}_0(L_{X/S}) = \Omega_{X/S}^1$ and $\mathcal{H}_i(L_{X/S}) = 0$ for $i < 0$. For a morphism $f: X \rightarrow Y$ of schemes over S , we have a distinguished triangle

$$(3.3.0.1) \quad Lf^*L_{Y/S} \rightarrow L_{X/S} \rightarrow L_{X/Y} \rightarrow .$$

If $X \rightarrow P$ is a regular closed immersion over S and if P is smooth over S , then there exists a canonical isomorphism

$$(3.3.0.2) \quad L_{X/S} \rightarrow [N_{X/P} \xrightarrow{d} \Omega_{P/S}^1 \otimes_{\mathcal{O}_P} \mathcal{O}_X]$$

to the complex where the conormal sheaf $N_{X/P} = \mathcal{I}_X/\mathcal{I}_X^2$ is put on degree 1 and $\Omega_{P/S}^1 \otimes_{\mathcal{O}_P} \mathcal{O}_X$ is put on degree 0. If $T \rightarrow S$ is flat, then the canonical map $Lp_1^*L_{X/S} \rightarrow L_{X \times_S T/T}$ is an isomorphism where $p_1: X \times_S T \rightarrow X$ denotes the first projection.

For an immersion $V \rightarrow X$ of schemes, the conormal complex $M_{V/X}$ is defined in [26, Definition 1.6.3] to be the shift $L_{V/X}[-1]$ of the cotangent complex $L_{V/X}$. We have a canonical isomorphism $\mathcal{H}_0(M_{V/X}) \rightarrow N_{V/X}$ to the conormal sheaf.

Lemma 3.3.1. — *If $X \rightarrow S$ is flat, then there exists a canonical isomorphism $L_{X/S} \rightarrow M_{X/(X \times_S X)}$.*

Proof. — We consider the distinguished triangle (3.3.0.1) for $X \rightarrow X \times_S X \rightarrow X$, where the first arrow is the diagonal $\delta: X \rightarrow X \times_S X$ and the second arrow is the second projection. Since $L_{X/X} = 0$, we obtain an isomorphism $M_{X/(X \times_S X)} = L_{X/(X \times_S X)}[-1] \rightarrow L\delta^*L_{(X \times_S X)/X}$. Since $X \rightarrow S$ is assumed to be flat, the canonical map $p_1^*L_{X/S} \rightarrow L_{(X \times_S X)/X}$ is an isomorphism where p_1 denotes the first projection. It induces an isomorphism $L_{X/S} \rightarrow L\delta^*Lp_1^*L_{X/S} \rightarrow L\delta^*L_{(X \times_S X)/X}$. \square

For an immersion $V \rightarrow X$ and for a morphism $W \rightarrow X$ such that $T = V \times_X W \rightarrow W$ is a regular immersion, the excess conormal complex $M'_{V/X,W}$ is defined in [26, Definition 1.6.3] as an object of the derived category $D_{qcoh}^-(\mathcal{O}_V)$ of the category of quasi-coherent \mathcal{O}_V -modules. We have a distinguished triangle

$$M'_{V/X,W} \rightarrow Lg^*M_{V/X} \rightarrow N_{T/W} \rightarrow$$

where g is the morphism $T \rightarrow V$.

We recall the definition of locally a hypersurface [26, Definition 3.1.1].

Definition 3.3.2. — *Let S be a scheme. A scheme X of finite presentation over S is called locally a hypersurface of relative virtual dimension $n - 1$ if, locally on X , it is a Cartier divisor of a smooth scheme of relative dimension n over S .*

For such X , if $i: X \rightarrow P$ is a regular immersion to a smooth scheme over S , the cotangent complex $L_{X/S}$ is canonically quasi-isomorphic to the complex $[N_{X/P} \rightarrow i^*\Omega_{P/S}^1]$ inducing a canonical isomorphism $\mathcal{H}_0(L_{X/S}) \rightarrow \Omega_{X/S}^1$. Consequently, the complex $L_{X/S}$ satisfies the condition $(L(n))$ in Section 3.2.

We recall the definition of the localized intersection product [26, Definition 3.2.2]. Let S be a regular noetherian scheme of finite dimension and let X be a scheme of finite

type over S that is locally a hypersurface of relative virtual dimension $n - 1$. Let Z be the closed subscheme of X defined by the annihilator of $\Omega_{X/S}^n$, $i: Z \rightarrow X$ be the closed immersion and let $\mathcal{L}_Z = L_1 i^* L_{X/S}$ be the invertible \mathcal{O}_Z -module. The underlying set of Z is the complement of the largest open subscheme of X smooth over S .

Let V be a closed subscheme of X and W be a noetherian scheme over X . We put $T = V \times_X W$. Then, for a coherent \mathcal{O}_V -module \mathcal{F} , a complex $\mathcal{G} \in D_{\text{coh}}^b(\mathcal{O}_W)$ of \mathcal{O}_W -modules with bounded coherent cohomology sheaves and for a sufficiently large integer q , the \mathcal{O}_T -module $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is supported on the inverse image $Z_T = Z \times_X T$ and the class $[\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]$ in $G(Z_T)_{/\mathcal{L}_Z} = \text{Coker}(\mathcal{L}_Z - 1: G(Z_T) \rightarrow G(Z_T))$ depends only on the parity of q by [26, Theorem 3.2.1].

Definition 3.3.3. — *Let S be a regular noetherian scheme of finite dimension and let X be a scheme of finite type over S that is locally a hypersurface of relative virtual dimension $n - 1$. Let Z be the closed subscheme of X defined by the annihilator of $\Omega_{X/S}^n$, $i: Z \rightarrow X$ be the closed immersion and let $\mathcal{L}_Z = L_1 i^* L_{X/S}$ be the invertible \mathcal{O}_Z -module. Let V be a closed subscheme of X and W be a noetherian scheme over X . We put $T = V \times_X W$, $Z_T = Z \times_X T$ and $G(Z_T)_{/\mathcal{L}_Z} = \text{Coker}(\mathcal{L}_Z - 1: G(Z_T) \rightarrow G(Z_T))$.*

Then, the localized intersection product

$$(3.3.3.1) \quad ((,))_X: G(V) \times G(W) \rightarrow G(Z_T)_{/\mathcal{L}_Z}$$

is a biadditive pairing defined by

$$(3.3.3.2) \quad (([\mathcal{F}], [\mathcal{G}]])_X = (-1)^q ([\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})] - [\mathcal{T}or_{q+1}^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})])$$

for a coherent \mathcal{O}_V -module \mathcal{F} , a coherent \mathcal{O}_W -module \mathcal{G} and for a sufficiently large integer q .

In [26], it is denoted $[[,]]$. We have changed the notation to emphasize the similarity with the usual intersection pairing $(,)$ of algebraic cycles. It is proved in [26, Theorem 3.2.1], that the right hand side of (3.3.3.2) is independent of q sufficiently large and defines a pairing.

For $\mathcal{G} = \mathcal{O}_W$, we put

$$(3.3.3.3) \quad (([\mathcal{F}], [\mathcal{O}_W]))_X = (([\mathcal{F}], W))_X.$$

The localized intersection product with W defines a map

$$(3.3.3.4) \quad ((, W))_X: G(V) \rightarrow G(Z_T)_{/\mathcal{L}_Z}.$$

Similarly, we define a map $((V,))_X: G(W) \rightarrow G(Z_T)_{/\mathcal{L}_Z}$. If $\Sigma \subset W$ is a closed subset such that the restriction $W \setminus \Sigma \rightarrow X$ is of finite tor-dimension, the map (3.3.3.4) is lifted to a map

$$(3.3.3.5) \quad ((, W))_X: G(V) \rightarrow G(Z_T \times_W \Sigma)_{/\mathcal{L}_Z}.$$

We recall some formulas on localized intersection product.

Lemma 3.3.4. — *Let X and X' be locally hypersurfaces of relative virtual dimension $n - 1$ and $n' - 1$ over a regular noetherian scheme S of finite dimension and*

$$\begin{array}{ccc} V & \longleftarrow & T \\ \cap \downarrow & & \downarrow \cap \\ X' & \longleftarrow & W' \\ f \downarrow & & \downarrow \\ X & \longleftarrow & W \end{array}$$

be a Cartesian diagram of noetherian schemes over S where the vertical arrows are closed immersions.

We assume that the immersion $f: X' \rightarrow X$ is of finite tor-dimension. We also assume that $Z \times_X X'$ is a closed subscheme of the closed subscheme Z' of X' defined by the annihilator of $\Omega_{X'/S}^{n'}$ and that the map $G(Z_T) \rightarrow G(Z'_T)$ induced by the inclusion $Z_T \rightarrow Z'_T$ induces a map $G(Z_T)_{/\mathcal{L}_Z} \rightarrow G(Z'_T)_{/\mathcal{L}_{Z'}}$.

Then, the composition

$$G(V) \xrightarrow{((, W))_X} G(Z_T)_{/\mathcal{L}_Z} \rightarrow G(Z'_T)_{/\mathcal{L}_{Z'}}$$

is equal to the localized intersection product

$$((, [L_f^* \mathcal{O}_W]))_{X'}: G(V) \rightarrow G(Z'_T)_{/\mathcal{L}_{Z'}}$$

with $[L_f^* \mathcal{O}_W] \in G(W')$.

Proof. — It follows from the canonical isomorphism $\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_W) \rightarrow \mathrm{Tor}_q^{\mathcal{O}_{X'}}(\mathcal{F}, L_f^* \mathcal{O}_W)$ for a coherent \mathcal{O}_V -module \mathcal{F} [26, Lemma 1.5.1]. \square

Lemma 3.3.5. — *Let X be locally a hypersurface of relative dimension $n - 1$ over a regular noetherian scheme S of finite dimension and*

$$\begin{array}{ccc} V & \longleftarrow & T \\ \cap \downarrow & & \downarrow \cap \\ X & \longleftarrow & W \end{array}$$

be a Cartesian diagram of noetherian schemes over S where the vertical arrows are closed immersions. We assume V is regular.

Then, the map $((, W))_X: G(V) \rightarrow G(Z_T)_{/\mathcal{L}_Z}$ is equal to the usual intersection product

$$((, (V, W))_X)_V: G(V) \rightarrow G(Z_T)_{/\mathcal{L}_Z}$$

with the localized intersection product $((V, W))_X \in G(Z_T)_{/\mathcal{L}_Z}$.

Assume in addition that V is locally of complete intersection of relative dimension $n - c$ over S , that W is of dimension p and that the immersion $T \rightarrow W$ is a regular immersion of codimension c' . Let $M'_{V/X,W}$ denote the excess conormal complex. Then, we have

$$(3.3.5.1) \quad ((V, W))_X = (-1)^{c-c'} c_{c-c'}^T(M'_{V/X,W}) \cap [T]$$

in $\mathrm{Gr}_{p-c}^F(G(Z_T)/\mathcal{L}_Z)$.

Proof. — We apply [26, Lemma 3.3.1] to the spectral sequence $E_{p,q}^2 = \mathcal{T}or_p^{\mathcal{O}_V}(\mathcal{F}, \mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)) \Rightarrow E_{p+q} = \mathcal{T}or_{p+q}^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_W)$. Then, similarly as in the proof of [26, Proposition 3.3.2.1], we obtain an equality $((\mathcal{F}, W))_X = (\mathcal{F}, ((V, W))_X)_V$ for a coherent \mathcal{O}_V -module \mathcal{F} .

The equality (3.3.5.1) is [26, (3.4.4.1)]. \square

Lemma 3.3.6. — *Let X be locally a hypersurface over a regular noetherian scheme S of finite dimension and*

$$\begin{array}{ccccc} V & \longleftarrow & T & \longleftarrow & T' \\ \cap \downarrow & & \downarrow \cap & & \downarrow \cap \\ X & \longleftarrow & W & \xleftarrow{g} & W' \end{array}$$

be a Cartesian diagram of noetherian schemes over S where the vertical arrows are closed immersions. We assume that $g: W' \rightarrow W$ is of finite tor-dimension.

Then, the map $((, W'))_X: G(V) \rightarrow G(Z_T)/\mathcal{L}_Z$ is equal to the composition of

$$G(V) \xrightarrow{((, W))_X} G(Z_T)/\mathcal{L}_Z \xrightarrow{g^*} G(Z_{T'})/\mathcal{L}_Z.$$

Proof. — It suffices to put $\mathcal{G} = \mathcal{O}_W$ and $\mathcal{H} = \mathcal{O}_{W'}$ in [26, Proposition 3.3.2.1]. \square

Lemma 3.3.7. — *Let X be locally a hypersurface over a regular noetherian scheme S of finite dimension and let X' be locally a hypersurface over a regular noetherian scheme S' of finite dimension. We consider a Cartesian diagram*

$$\begin{array}{ccccc} V & \longleftarrow & V' & \longleftarrow & T \\ \cap \downarrow & & \cap \downarrow & & \downarrow \cap \\ X & \xleftarrow{f} & X' & \longleftarrow & W \end{array}$$

of noetherian schemes over S where the vertical arrows are closed immersions. We assume that $f: X' \rightarrow X$ is of finite tor-dimension. We also assume that $Z_T = Z \times_X T$ is a subset of $Z'_T = Z' \times_{X'} T$ set-theoretically and that the canonical morphism $G(Z_T) \rightarrow G(Z'_T)$ induces a morphism $G(Z_T)/\mathcal{L}_Z \rightarrow G(Z'_T)/\mathcal{L}_{Z'}$.

Then, the composition $((\cdot, W))_X: G(V) \rightarrow G(Z_T)_{/\mathcal{L}_Z} \rightarrow G(Z'_T)_{/\mathcal{L}_{Z'}}$ is equal to the composition of

$$G(V) \xrightarrow{f^*} G(V') \xrightarrow{((\cdot, W))_{X'}} G(Z'_T)_{/\mathcal{L}_{Z'}}.$$

Proof. — Similarly as [26, Corollary 3.3.4.3] applied by taking $W = X'$ and V' to be W , it follows from [26, Proposition 3.3.3]. \square

3.4. Relative excess intersection formula. — We establish a refinement of the excess intersection formula [26, Proposition 3.4.2]. The result in this subsection will be used in the proof of an explicit computation of the logarithmic different in Proposition 5.1.2. Most of the proofs in this subsection are immediate variations of those in [26, Sections 1.6, 1.7]. The reader is recommended to read them before following the proof here. The authors apologize for possible inconvenience.

We consider a commutative diagram

$$(3.4.0.1) \quad \begin{array}{ccc} T & \xrightarrow{c} & W \\ \downarrow & & \downarrow g' \\ V' & \xrightarrow{c} & X' \\ \downarrow & & \downarrow f \\ V & \xrightarrow{c} & X \end{array}$$

of schemes satisfying the following condition:

(3.4.0.2) The horizontal arrows are closed immersions and the immersion $T \rightarrow W$ is a regular immersion. The upper square and the tall rectangle are Cartesian.

Let $g: W \rightarrow X$ denote the composition of the right vertical arrows. Recall that a simplicial algebra $A_{V/X,W}$ on T such that the normal complex $N(A_{V/X,W})$ computes $Lg^*\mathcal{O}_V$ is defined in [26, 1.6.3, 1.6.4]. Further, an ideal $I_{V/X,W} \subset A_{V/X,W}$ is defined as the kernel of the surjection $A_{V/X,W} \rightarrow \mathcal{O}_T$ and the excess conormal complex $M'_{V/X,W} = N(I_{V/X,W}/I_{V/X,W}^2)[-1]$ is defined as a complex of \mathcal{O}_T -modules loc. cit.

We construct a variant of the spectral sequence $E_{p,q}^1 = \mathcal{H}_{2p+q} L\Lambda^{-p} M'_{V/X,W} \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_{X'}}(\mathcal{O}_V, \mathcal{O}_W)$ [26, (1.6.4.3)]. We further assume that the map $f: X' \rightarrow X$ is of finite tor-dimension. We define an object \mathcal{C} of $D_{\text{coh}}^-(\mathcal{O}_{X'})$ fitting in the distinguished triangle

$$(3.4.0.3) \quad \rightarrow \mathcal{C} \rightarrow Lf^*\mathcal{O}_V \rightarrow \mathcal{O}_{V'} \rightarrow$$

as follows. Since $Lf^*\mathcal{O}_V$ is acyclic in degree > 0 , the 0-th subcomplex $\tau_{\leq 0} Lf^*\mathcal{O}_V$ in the canonical filtration is quasi-isomorphic to $Lf^*\mathcal{O}_V$. Namely, for a complex \mathcal{C}' defining $Lf^*\mathcal{O}_V$ in the derived category and the subcomplex $\tau_{\leq 0}\mathcal{C}'$ defined by replacing the

components \mathcal{C}'_q by 0 for $q > 0$ and \mathcal{C}'_0 by $\text{Ker}(\mathcal{C}'_0 \rightarrow \mathcal{C}'_{-1})$, the inclusion $\tau_{\leq 0}\mathcal{C}' \rightarrow \mathcal{C}'$ is a quasi-isomorphism. Let \mathcal{C} be the subcomplex of \mathcal{C}' obtained by further replacing $\text{Ker}(\mathcal{C}'_0 \rightarrow \mathcal{C}'_{-1})$ on degree 0 by the kernel of the surjection $\text{Ker}(\mathcal{C}'_0 \rightarrow \mathcal{C}'_{-1}) \rightarrow \mathcal{O}_{V'}$. Then, the complex \mathcal{C} fits in the distinguished triangle (3.4.0.3) and is independent of the choice of \mathcal{C}' up to a canonical quasi-isomorphism.

We use the notation in [26, Proposition 1.6.4]. A simplicial algebra $A_{V/X,W}$ and an increasing filtration $F^\bullet A_{V/X,W} = I^\bullet_{V/X,W}$ are defined in [26, Definition 1.6.3]. Similarly $A_{V'/X',W}$ and its filtration F^\bullet are defined. They define a filtered chain complex $\tilde{\mathcal{C}} = [A_{V/X,W} \rightarrow A_{V'/X',W}]$ of simplicial modules. Here $A_{V'/X',W}$ is put on degree 0 and $A_{V/X,W}$ is on degree 1. Then, since $P_X(\mathcal{O}_V)$ and $P_{X'}(\mathcal{O}_{V'})$ are resolutions of \mathcal{O}_V and of $\mathcal{O}_{V'}$ by free simplicial algebras [14, 1.5.5.6], we have a quasi-isomorphism $\int N\tilde{\mathcal{C}} \rightarrow \text{Lg}^{g^*}\mathcal{C}$ from the associated simple complex. Thus, we obtain a spectral sequence

$$(3.4.0.4) \quad E_{p,q}^1 = \mathcal{H}_{p+q} \text{Gr}_F^{-p} \int N\tilde{\mathcal{C}} \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_{X'}}(\mathcal{C}, \mathcal{O}_W).$$

Lemma 3.4.1. — *Let the notation be as above. Then, for the E^1 -term of (3.4.0.4), there exists a canonical isomorphism*

$$(3.4.1.1) \quad E_{p,q}^1 \rightarrow \mathcal{H}_{2p+q}[\text{L}\Lambda^{-p}M'_{V/X,W} \rightarrow \text{L}\Lambda^{-p}M'_{V'/X',W}]$$

of \mathcal{O}_T -modules.

By Lemma 3.4.1, we obtain a spectral sequence

$$(3.4.1.2) \quad E_{p,q}^1 = \mathcal{H}_{2p+q}[\text{L}\Lambda^{-p}M'_{V/X,W} \rightarrow \text{L}\Lambda^{-p}M'_{V'/X',W}] \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_{X'}}(\mathcal{C}, \mathcal{O}_W)$$

of \mathcal{O}_T -modules.

Proof. — The proof is similar to [26, Proposition 1.6.4]. Similarly as loc. cit., the canonical map $S^p(\text{Gr}_F^1\tilde{\mathcal{C}}) \rightarrow \text{Gr}_F^p\tilde{\mathcal{C}}$ is an isomorphism. The normal complex of the graded piece $\text{Gr}_F^1\tilde{\mathcal{C}} = [\text{Gr}_F^1 A_{V/X,W} \rightarrow \text{Gr}_F^1 A_{V'/X',W}]$ is defined by the canonical map $M'_{V/X,W}[1] \rightarrow M'_{V'/X',W}[1]$ of conormal complexes. Hence the normal complex $N\text{Gr}_F^p\tilde{\mathcal{C}} = N[\text{Gr}_F^p A_{V/X,W} \rightarrow \text{Gr}_F^p A_{V'/X',W}]$ is canonically quasi-isomorphic to the mapping cone of $\text{NS}^p(M'_{V/X,W}[1]) \rightarrow \text{NS}^p(M'_{V'/X',W}[1])$. Thus by [26, Proposition 1.2.8], the E^1 -term $E_{p,q}^1 = \mathcal{H}_{p+q} \text{Gr}_F^{-p} \int N\tilde{\mathcal{C}}$ is given by $\mathcal{H}_{2p+q}[\text{L}\Lambda^{-p}M'_{V/X,W} \rightarrow \text{L}\Lambda^{-p}M'_{V'/X',W}]$ as required. \square

We consider a commutative diagram

$$(3.4.2.1) \quad \begin{array}{ccccc} T & \xrightarrow{c} & W & & \\ \downarrow & & \downarrow & & \\ V' & \xrightarrow{c} & X' & \xrightarrow{c} & P' \\ \downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow{c} & X & \xrightarrow{c} & P \end{array}$$

of schemes satisfying the following conditions:

- (3.4.2.2) The horizontal arrows are closed immersions and the immersions $X \rightarrow P$ and $X' \rightarrow P'$ are regular immersions of the same codimension. The immersion $T \rightarrow W$ is also a regular immersion. The upper square, the left tall rectangle and the right square are Cartesian.

In [26, (1.7.2.1)], a map $\lambda_{V/X,P,W}: L\Lambda^\beta M'_{V/X,W} \rightarrow N_{X/P} \otimes L\Lambda^{\beta-1} M'_{V/X,W}[1]$ is defined as the composition of the maps

$$(3.4.2.3) \quad L\Lambda^\beta M'_{V/X,W} \rightarrow M'_{V/X,W} \otimes L\Lambda^{\beta-1} M'_{V/X,W} \rightarrow N_{X/P} \otimes L\Lambda^{\beta-1} M'_{V/X,W}[1]$$

induced by the canonical map $M'_{V/X,W} \rightarrow Lg_T^* M_{V/X} \rightarrow Lg_T^* M_{X/P}[1] = N_{X/P} \otimes \mathcal{O}_T[1]$. We identify $N_{X'/P'} = f^* N_{X/P}$ and define $\lambda_{V'/X',P',W}: L\Lambda^\beta M'_{V'/X',W} \rightarrow N_{X/P} \otimes L\Lambda^{\beta-1} M'_{V'/X',W}[1]$ similarly. We will construct a map (3.4.2.5) below compatible with $\lambda_{V/X,P,W}$ and $\lambda_{V'/X',P',W}$.

We use the notation in the proof of [26, Proposition 1.7.2]. Then, we have a commutative diagram of exact sequences

$$(3.4.2.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J_{B'}/J_{B'}^2 & \longrightarrow & B'/J_{B'}^2 & \longrightarrow & A' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & J_B/J_B^2 & \longrightarrow & B/J_B^2 & \longrightarrow & A \longrightarrow 0 \end{array}$$

of filtered simplicial modules. The lower exact sequence is constructed for $V \rightarrow X \leftarrow W$ and the upper one is for $V' \rightarrow X' \leftarrow W'$. Further we have a quasi-isomorphism $[A \rightarrow A'] \rightarrow [A_{V/X,W} \rightarrow A_{V'/X',W}]$. By the assumption that the right square in (3.4.2.1) is Cartesian, we have an isomorphism $N_{X/P} \otimes [A \rightarrow A'] \rightarrow [J_B/J_B^2 \rightarrow J_{B'}/J_{B'}^2]$ by iii in the step 1 of the proof of [26, Proposition 1.7.2]. They induce a canonical map

$$(3.4.2.5) \quad \lambda: [L\Lambda^\beta M'_{V/X,W} \rightarrow L\Lambda^\beta M'_{V'/X',W}] \rightarrow N_{X/P} \otimes [L\Lambda^{\beta-1} M'_{V/X,W} \rightarrow L\Lambda^{\beta-1} M'_{V'/X',W}][1].$$

By the step 3 in the proof of [26, Proposition 1.7.2], we see that it fits in a commutative diagram of distinguished triangles

(3.4.2.6)

$$\begin{array}{ccc}
\mathrm{L}\Lambda^{\beta}\mathrm{M}'_{\mathrm{V}/\mathrm{X},\mathrm{W}} & \xrightarrow{\lambda_{\mathrm{V}/\mathrm{X}/\mathrm{P},\mathrm{W}}} & \mathrm{N}_{\mathrm{X}/\mathrm{P}} \otimes \mathrm{L}\Lambda^{\beta-1}\mathrm{M}'_{\mathrm{V}/\mathrm{X},\mathrm{W}}[1] \\
\downarrow & & \downarrow \\
\mathrm{L}\Lambda^{\beta}\mathrm{M}'_{\mathrm{V}'/\mathrm{X}',\mathrm{W}} & \xrightarrow{\lambda_{\mathrm{V}'/\mathrm{X}'/\mathrm{P}',\mathrm{W}}} & \mathrm{N}_{\mathrm{X}/\mathrm{P}} \otimes \mathrm{L}\Lambda^{\beta-1}\mathrm{M}'_{\mathrm{V}'/\mathrm{X}',\mathrm{W}}[1] \\
\downarrow & & \downarrow \\
[\mathrm{L}\Lambda^{\beta}\mathrm{M}'_{\mathrm{V}/\mathrm{X},\mathrm{W}} \rightarrow \mathrm{L}\Lambda^{\beta}\mathrm{M}'_{\mathrm{V}'/\mathrm{X}',\mathrm{W}}] & \xrightarrow{\lambda} & \mathrm{N}_{\mathrm{X}/\mathrm{P}} \otimes [\mathrm{L}\Lambda^{\beta-1}\mathrm{M}'_{\mathrm{V}/\mathrm{X},\mathrm{W}} \rightarrow \mathrm{L}\Lambda^{\beta-1}\mathrm{M}'_{\mathrm{V}'/\mathrm{X}',\mathrm{W}}][1] \\
\downarrow & & \downarrow
\end{array}$$

Lemma 3.4.2. — *Let $E_{\mathcal{T}}$ denote the spectral sequence $E_{p,q}^1 \Rightarrow E_{p+q}$ (3.4.1.2) and let $E_{\mathcal{T}}[-1, 3]$ denote the spectral sequence $E_{p+1,q-3}^1 \Rightarrow E_{p+q-2}$.*

Then, there exists a map

$$(3.4.2.7) \quad \alpha : E_{\mathcal{T}} \rightarrow \mathrm{N}_{\mathrm{X}/\mathrm{P}} \otimes E_{\mathcal{T}}[-1, 3]$$

of spectral sequences where the maps of E^1 -terms are induced by the map λ in (3.4.2.5).

Proof. — This is a variant of the construction in [26, Proposition 1.7.2] where the corresponding map for the spectral sequence $E_{p,q}^1 = \mathcal{H}_{2p+q}\mathrm{L}\Lambda^{-\beta}\mathrm{M}'_{\mathrm{V}/\mathrm{X},\mathrm{W}} \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_{\mathrm{X}}}(\mathcal{O}_{\mathrm{V}}, \mathcal{O}_{\mathrm{W}})$ is defined. Similarly as the step 1 in the proof loc. cit., we obtain a map of spectral sequences. The assertion on the map of E^1 -terms is clear from the definition of λ . \square

We prove a relative version of the excess intersection formula [26, Proposition 3.4.2].

Proposition 3.4.3. — *Let S be a regular noetherian scheme of finite dimension and X, X' be locally hypersurfaces over S . We consider a commutative diagram*

$$(3.4.3.1) \quad \begin{array}{ccc}
\mathrm{T} & \xrightarrow{\mathrm{c}} & \mathrm{W} \\
\downarrow & & \downarrow \\
\mathrm{V}' & \xrightarrow{\mathrm{c}} & \mathrm{X}' \\
\downarrow & & \downarrow f \\
\mathrm{V} & \xrightarrow{\mathrm{c}} & \mathrm{X}
\end{array}$$

of schemes over S satisfying the condition (3.4.0.2). Let c' be the codimension of the regular immersion $T \rightarrow W$. Let $M'_{V/X,W}$ and $M'_{V'/X',W}$ be the excess conormal complexes.

We assume that, locally on X , there exists a commutative diagram

$$(3.4.3.2) \quad \begin{array}{ccccc} V' & \xrightarrow{c} & X' & \xrightarrow{c} & P' \\ \downarrow & & \downarrow f & & \downarrow \tilde{f} \\ V & \xrightarrow{c} & X & \xrightarrow{c} & P \end{array}$$

of schemes over S satisfying the following condition:

- (3.4.3.3) *The maps $X \rightarrow P$ and $X' \rightarrow P'$ are regular immersion of codimension 1 and that $V \rightarrow P$ and $V' \rightarrow P'$ are regular immersion of codimension n . The right square is Cartesian. The schemes P and P' are smooth of relative dimension n over S .*

Let $U' \subset X'$ be an open subscheme such that $V' \times_{X'} U' \rightarrow V \times_X U'$ is an open immersion and put $\Sigma = X' \setminus U'$. Let $Z' \subset X'$ be the closed subscheme defined by the annihilator of $\Omega_{X'/S}^n$ and we put $\mathcal{L}_{Z'} = L_1 i^* \mathcal{L}_{X'/S}$ where $i: Z' \rightarrow X'$ is the closed immersion. Then, we have the following.

1. The map $f: X' \rightarrow X$ is of finite tor-dimension.
2. The canonical map $L\Lambda^{n-c'} M'_{V/X,W} \rightarrow L\Lambda^{n-c'} M'_{V'/X',W}$ is a quasi-isomorphism on the complement $T \setminus (Z'_T \times_{X'} \Sigma)$.
3. We define $f^1[V] - [V'] \in G(\Sigma \cap (V \times_X X'))$ by

$$f^1[V] - [V'] = [\mathrm{Ker}(f^* \mathcal{O}_V \rightarrow \mathcal{O}_{V'})] + \sum_{q>0} (-1)^q [L_q f^* \mathcal{O}_V]$$

and put $d = \dim W$. Then, we have

$$(3.4.3.4) \quad (((f^1[V] - [V']), W))_{X'} = [L\Lambda^{n-c'}(M'_{V/X,W} \rightarrow M'_{V'/X',W})]$$

in $F_{d-n+c'} G(Z'_T \times_{X'} \Sigma)_{/\mathcal{L}_{Z'}}$.

Proof. — 1. In the diagram (3.4.3.2), the schemes X and P' are tor-independent over P . Since \tilde{f} is of finite tor-dimension, the map f is also of finite tor-dimension.

2. The assertion is local on X . Recall that the diagram (3.4.3.2) defines a quasi-isomorphism $M'_{V/X,W} \rightarrow [N_{X/P} \otimes \mathcal{O}_T \rightarrow \mathrm{Ker}(N_{V/P} \otimes \mathcal{O}_T \rightarrow N_{T/W})]$ [26, Lemma 1.7.1] and similarly for $M'_{V'/X',W}$. Hence the excess conormal complexes $M'_{V/X,W}$ and $M'_{V'/X',W}$ satisfy the condition $(L(n-c'))$ in Section 3.2.

The map $V' \rightarrow V \times_X X'$ defined by the diagram (3.4.3.1) is an open immersion outside Σ . Hence the canonical map $M'_{V/X,W} \rightarrow M'_{V'/X',W}$ is a quasi-isomorphism on the complement $T \setminus T \times_{X'} \Sigma$ by the description of the excess conormal complex recalled above.

Let $Z'_1 \subset T$ be the closed subscheme defined by the annihilator ideal of $\Lambda^{n-c'} \mathcal{H}_0(M'_{V'/X',W})$. By the assumption that the right square in (3.4.3.2) is Cartesian,

the canonical map $N_{X/P} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \rightarrow N_{X'/P'}$ is an isomorphism. Hence the assumption that the mapping cone $[M'_{V/X,W} \rightarrow M'_{V'/X',W}]$ is of tor-dimension ≤ 1 is satisfied. Thus by Lemma 3.2.5, the homology sheaves $\mathcal{H}_q([L\Lambda^{n-c'}M'_{V/X,W} \rightarrow L\Lambda^{n-c'}M'_{V'/X',W}])$ are $\mathcal{O}_{Z'_1}$ -module for $q \geq 0$.

Since the diagram (3.4.3.2) defines an isomorphism $[N_{X'/P'} \rightarrow \Omega_{P'/S}^1 \otimes_{\mathcal{O}_{P'}} \mathcal{O}_{X'}] \rightarrow L_{X'/S}$, we have an inclusion $Z'_1 \subset Z'_T$ of closed subschemes of T . Hence the assertion follows.

3. The proof is similar to [26, Proposition 3.4.2] using Lemmas 3.4.1 and 3.4.2.

By 2, the right hand side of (3.4.3.4) is defined as an element of $F_{d-n+c'}G(Z'_T \times_{X'} \Sigma)$. Since X and P' are tor-independent over P , we see that the canonical map $L_{f^*}\mathcal{O}_V \rightarrow L_{f'^*}\mathcal{O}_V$ is a quasi-isomorphism. Hence, the complex \mathcal{C} of $\mathcal{O}_{X'}$ -modules in Lemma 3.4.1 is acyclic outside Σ . We consider the spectral sequence (3.4.1.2) and will show the equality

$$(3.4.3.5) \quad \sum_{p+q=r,r+1} (-1)^{p+q} [E_{p,q}^1] = (-1)^r [E_r] + (-1)^{r+1} [E_{r+1}]$$

for sufficiently large r . The right hand side of (3.4.3.5) is independent of sufficiently large r by [26, Theorem 3.2.1] and defines the left hand side of (3.4.3.4).

We show that the left hand side of (3.4.3.5) equals the right hand side of (3.4.3.4). By [26, Lemma 3.4.1], the map $\lambda_{V/X,P,W}$ induces an isomorphism $L_{p+1}\Lambda^{q+1}M'_{V/X,W} \rightarrow N_{X/P} \otimes L_p\Lambda^q M'_{V/X,W}$ for $q \geq n - c'$ and similarly for $\lambda_{V'/X'/P',W}$. By the commutative diagram (3.4.2.6), it induces an isomorphism $E_{p,q}^1 \rightarrow \mathcal{L}_{Z'} \otimes E_{p+1,q-3}^1$ if $-(p+1) \geq n - c'$. Hence, the left hand side of (3.4.3.5) is equal to $\sum_q (-1)^{-(n-c')+q} [E_{-(n-c'),q}^1]$ and to the right hand side of (3.4.3.4).

We show the equality (3.4.3.5) applying [26, Lemma 3.3.1]. Note that, in [26, Lemma 3.3.1], the map α_r is used only to show that the right hand side is independent of $r \geq r_0$ and that we can drop the compatibility assumption with the restriction $\alpha_r|_{T \cap U}$ since it is not used in the proof. We apply [26, Lemma 3.3.1], to the maps (3.4.2.5) and [26, (3.1.3.1)] and the map (3.4.2.7). Then, by Lemma 3.4.2, the maps of E^1 -terms of (3.4.2.7) are isomorphisms for sufficiently large $p+q$ and the assumption of [26, Lemma 3.3.1] is satisfied. Thus, we obtain the equality (3.4.3.5). \square

4. Intersection product with the log diagonal

From this section on, we fix a complete discrete valuation field K with perfect residue field F of characteristic $p > 0$. We put $S = \text{Spec } \mathcal{O}_K$ and $s = \text{Spec } F$. Both 0 and p are allowed as the characteristic of K . A morphism of schemes over S is always a morphism over S .

We introduce in Section 4.3 the localized intersection product with the log diagonal by applying Definition 3.3.3. We establish an important property that the localized

intersection product with the log diagonal is independent of the boundary in Proposition 4.3.5. In preliminary subsections 4.1 and 4.2, we study local structure of log products and the logarithmic cotangent complex respectively.

4.1. Log products. — In this subsection, we study the log self-product of a regular flat scheme X of finite type over $S = \text{Spec } \mathcal{O}_K$ with respect to a divisor $D \subset X$ with simple normal crossings. We do not assume inclusion between D and the closed fiber X_F . The case where $D = X_F$ and X_K is smooth over K is treated in [26, Sections 5.1, 5.2]. First, we study the local structure of X .

Lemma 4.1.1. — *Let X be a regular flat scheme of finite type over S and D be a divisor with simple normal crossings. Then, for every point x of X , there exist an open neighborhood U of x , a smooth scheme P over S , a divisor \tilde{D} of P with simple normal crossings relatively to S and a regular immersion $U \rightarrow P$ of codimension 1 such that $D \cap U = \tilde{D} \times_P U$.*

Proof. — It suffices to prove the cases where x is a closed point of X_F and of X_K respectively. First, we show the case where x is a closed point of X_F . Let D_1, \dots, D_m be the irreducible components of D containing x and take $t_1, \dots, t_m \in \mathfrak{m}_x$ defining D_1, \dots, D_m on a neighborhood of x . We extend it to a minimal system $t_1, \dots, t_n \in \mathfrak{m}_x$ of generators. Then, the map $U \rightarrow \mathbf{A}_S^n$ defined by t_1, \dots, t_n on an open neighborhood U of x is unramified. Hence, after shrinking U if necessary, there is an étale scheme $P \rightarrow \mathbf{A}_S^n$ and a regular closed immersion $U \rightarrow P$ of codimension 1 such that $D \cap U$ is the sum of the pull-back of the first m coordinate hyperplanes by [13, Corollaire (18.4.7)].

Next, we show the case where x is a closed point of X_K . We take a minimal system $t_1, \dots, t_n \in \mathfrak{m}_x$ of generators as above. There exists an element $t_0 \in \mathcal{O}_{X,x}$ such that the residue field $\kappa(x)$ is a finite separable extension of $K(\bar{t}_0)$ and that $K(\bar{t}_0)$ is purely inseparable over K by Lemma 4.1.2 below. Then, the map $U \rightarrow \mathbf{A}_S^{n+1}$ defined by t_0, t_1, \dots, t_n on an open neighborhood U of x is unramified. Thus, we conclude similarly as above. \square

Lemma 4.1.2. — *Let k be a field of characteristic $p > 0$ such that $[k : k^p] = p$. Then, for a finite extension L of k , there exists an integer $e \geq 0$ such that L is a separable extension of k^{1/p^e} .*

Proof. — Let L_1 be the separable closure of k in L and put $[L : L_1] = p^e$. Then since $[L_1 : L_1^p] = [k : k^p] = p$, it follows that L is a unique purely inseparable extension L_1^{1/p^e} of L_1 of degree p^e and is a separable extension of k^{1/p^e} . \square

We show a relative version of Lemma 4.1.1.

Lemma 4.1.3. — *Let X and Y be regular flat schemes of finite type over S and $f : Y \rightarrow X$ be a morphism over S . Let $D \subset X$ and $E \subset Y$ be divisors with simple normal crossings such that $Y \setminus E \subset f^{-1}(X \setminus D)$. Let y be a point of Y and we put $x = f(y) \in X$.*

Then, there exists open neighborhoods U and V of x and y respectively satisfying $f(V) \subset U$ and a Cartesian diagram

$$(4.1.3.1) \quad \begin{array}{ccc} V & \longrightarrow & Q \supset \tilde{E} \\ f|_V \downarrow & & \downarrow \tilde{f} \\ U & \longrightarrow & P \supset \tilde{D} \end{array}$$

of schemes over S satisfying the following conditions:

- The schemes P and Q are smooth over S and $\tilde{D} = \sum_i \tilde{D}_i \subset P$ and $\tilde{E} = \sum_j \tilde{E}_j \subset Q$ are divisors with relative simple normal crossings relatively to S respectively. For each i , we have $\tilde{f}^{-1}(\tilde{D}_i) = \sum_j e_{ij} \tilde{E}_j$ for some integers $e_{ij} \geq 0$. The horizontal arrows are regular immersions of codimension 1 and $D \cap U = \tilde{D} \times_P U$ and $E \cap V = \tilde{E} \times_Q V$ as in Lemma 4.1.1.

Proof. — It suffices to prove the cases where y is a closed point of Y_F and of Y_K respectively. First we prove the case where y is a closed point of Y_F and hence x is a closed point of X_F . By the proof of Lemma 4.1.1, we obtain a diagram (4.1.3.1) without \tilde{f} together with étale morphisms $P \rightarrow \mathbf{A}_S^n = \text{Spec } \mathcal{O}_K[T_1, \dots, T_n]$ and $Q \rightarrow \mathbf{A}_S^{n'} = \text{Spec } \mathcal{O}_K[S_1, \dots, S_{n'}]$ such that \tilde{D} and \tilde{E} are the pull-back of the first m and m' coordinate hyperplanes respectively and that the maximal ideals $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ and $\mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ are generated by $t_i = T_i|_U$ for $i = 1, \dots, n$ and $s_j = S_j|_V$ for $j = 1, \dots, n'$ respectively.

For $i = 1, \dots, m$, we put $f^*t_i = v_i \prod_j s_j^{e_{ij}}$ for some units v_i on V . For $i = m+1, \dots, n$, we also put $f^*t_i = \sum_j a_{ij} s_j$ for some functions a_{ij} on V . After shrinking Q , we take units \tilde{v}_i on Q lifting v_i and functions \tilde{a}_{ij} on Q lifting a_{ij} and define a map $g: Q \rightarrow \mathbf{A}_S^n$ by sending T_i to $\tilde{v}_i \prod_j s_j^{e_{ij}}$ for $i = 1, \dots, m$ and to $\sum_j \tilde{a}_{ij} s_j$ for $i = m+1, \dots, n$. By replacing Q by an étale neighborhood $Q \times_{\mathbf{A}_S^n} P$, we obtain a map $\tilde{f}: Q \rightarrow P$ that makes (4.1.3.1) a commutative diagram.

The equalities $\tilde{f}^{-1}(\tilde{D}_i) = \sum_j e_{ij} \tilde{E}_j$ follow from the definition of g . We show that the diagram (4.1.3.1) thus obtained is Cartesian on a neighborhood of y . Let $\tilde{\mathfrak{m}}_x \subset \mathcal{O}_{P,x}$ and $\tilde{\mathfrak{m}}_y \subset \mathcal{O}_{Q,y}$ be the maximal ideals. Since the horizontal arrows are regular immersions of codimension 1, it suffices to show that the canonical map $\tilde{f}^*: \tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2$ induces an isomorphism $\text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2) \rightarrow \text{Ker}(\tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2 \rightarrow \mathfrak{m}_y/\mathfrak{m}_y^2)$ on the subspaces of dimension 1. Since $\mathfrak{m}_x/\mathfrak{m}_x^2 = \langle t_1, \dots, t_n \rangle$ and $\mathfrak{m}_y/\mathfrak{m}_y^2 = \langle s_1, \dots, s_{n'} \rangle$, we have $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2) \oplus \langle T_1, \dots, T_n \rangle$ and $\tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2 = \text{Ker}(\tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2 \rightarrow \mathfrak{m}_y/\mathfrak{m}_y^2) \oplus \langle S_1, \dots, S_{n'} \rangle$. By the definition of g , the map \tilde{f}^* sends the subspace $\langle T_1, \dots, T_n \rangle \subset \tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2$ into $\langle S_1, \dots, S_{n'} \rangle \subset \tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2$. Since π, T_1, \dots, T_n and $\pi, S_1, \dots, S_{n'}$ are bases of $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2) \oplus \langle T_1, \dots, T_n \rangle$ and of $\tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2$ respectively, the image of $\text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2)$ is not in $\langle S_1, \dots, S_{n'} \rangle$. Hence the map $\text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2) \rightarrow \text{Ker}(\tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2 \rightarrow \mathfrak{m}_y/\mathfrak{m}_y^2)$ induced by \tilde{f}^* is an isomorphism as required.

Next, we prove the case where y is a closed point of Y_K and hence x is a closed point of X_K . Let π be a prime element of \mathcal{O}_K . We may take $t_0 \in \mathcal{O}_{X,x}$ and $s_0 \in \mathcal{O}_{Y,y}$ such

that $t_0 \equiv \pi \pmod{\mathfrak{m}_x}$ and $s_0 \equiv \pi \pmod{\mathfrak{m}_y}$ if \mathbf{K} is of characteristic 0 and $t_0^{p^a} \equiv \pi \pmod{\mathfrak{m}_x}$ and $s_0^{p^{a+b}} \equiv \pi \pmod{\mathfrak{m}_y}$ for some integers $a \geq 0, b \geq 0$ and $\kappa(x)$ and $\kappa(y)$ are separable extensions over $\mathbf{K}(\bar{t}_0)$ and over $\mathbf{K}(\bar{s}_0)$ respectively if \mathbf{K} is of characteristic $p > 0$ by Lemma 4.1.2. Then, we obtain a diagram (4.1.3.1) without \tilde{f} together with étale morphisms $\mathbf{P} \rightarrow \mathbf{A}_S^{n+1} = \text{Spec } \mathcal{O}_K[T_0, T_1, \dots, T_n]$ and $\mathbf{Q} \rightarrow \mathbf{A}_S^{n'+1} = \text{Spec } \mathcal{O}_K[S_0, S_1, \dots, S_{n'}]$ similarly as above. We put $f^*t_0 = s_0^{p^b} + \sum_{j>0} a_{0j}s_j$ and take liftings \tilde{a}_{0j} as above. By the same procedure for $i > 0$ as above, we define a map $\tilde{f}: \mathbf{Q} \rightarrow \mathbf{P}$ that makes (4.1.3.1) a commutative diagram satisfying the equalities $\tilde{f}^{-1}(\tilde{\mathbf{D}}_i) = \sum_j e_{ij}\tilde{\mathbf{E}}_j$.

We show that the diagram (4.1.3.1) thus obtained is Cartesian on a neighborhood of y . By the definition of \tilde{f} , the map \tilde{f}^* sends the subspace $\langle T_1, \dots, T_n \rangle \subset \tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2$ into $\langle S_1, \dots, S_{n'} \rangle \subset \tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2$. Since $T_0^{p^a} - \pi, T_1, \dots, T_n$ and $S_0^{p^{a+b}} - \pi, S_1, \dots, S_{n'}$ are bases of $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2) \oplus \langle T_1, \dots, T_n \rangle$ and of $\tilde{\mathfrak{m}}_y/\tilde{\mathfrak{m}}_y^2$ respectively, the image of $\text{Ker}(\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2)$ is not in $\langle S_1, \dots, S_{n'} \rangle$. This implies the required assertion as above. \square

Let \mathbf{X} be a regular flat separated scheme of finite type over \mathbf{S} and $\mathbf{D} \subset \mathbf{X}$ be a divisor with simple normal crossings. We consider the log product $(\mathbf{X} \times_{\mathbf{S}} \mathbf{X})^\sim$ defined as $(\mathbf{X} \times_{\mathbf{S}} \mathbf{X})_{\tilde{\mathcal{D}}}^\sim$ with respect to the family $\mathcal{D} = (\mathbf{D}_i)_{i \in I}$ of irreducible components of \mathbf{D} .

Lemma 4.1.4. — *Let \mathbf{X} be a regular flat separated scheme of finite type over \mathbf{S} and $\mathbf{D} \subset \mathbf{X}$ be a divisor with simple normal crossings.*

Then, the log product $(\mathbf{X} \times_{\mathbf{S}} \mathbf{X})^\sim$ is locally a hypersurface (Definition 3.3.2) over \mathbf{X} with respect to either of the projections.

Proof. — Let $x \in \mathbf{X}$ be a point, $\mathbf{U} \subset \mathbf{X}$ be an open neighborhood of x and $\mathbf{U} \rightarrow \mathbf{P}$ be a regular immersion of codimension 1 to a smooth scheme \mathbf{P} over \mathbf{S} satisfying the condition in Lemma 4.1.1. We define the log product $(\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim$ similarly. Since the second projection $(\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim \rightarrow \mathbf{X}$ is log smooth and strict by Lemma 1.3.1, it is smooth. Hence the log product $(\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim$ is regular.

By the universality of log product, we have a Cartesian diagram

$$(4.1.4.1) \quad \begin{array}{ccc} (\mathbf{U} \times_{\mathbf{S}} \mathbf{X})^\sim & \xrightarrow{\text{pr}_1} & \mathbf{U} \\ \downarrow & & \downarrow \\ (\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim & \xrightarrow{\text{pr}_1} & \mathbf{P}. \end{array}$$

Hence the ideal defining the immersion $(\mathbf{U} \times_{\mathbf{S}} \mathbf{X})^\sim \rightarrow (\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim$ is locally monogenic. Thus, to conclude that $(\mathbf{U} \times_{\mathbf{S}} \mathbf{X})^\sim$ is a divisor of $(\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim$, it is sufficient to show that the immersion $(\mathbf{U} \times_{\mathbf{S}} \mathbf{X})^\sim \rightarrow (\mathbf{P} \times_{\mathbf{S}} \mathbf{X})^\sim$ is nowhere dominant. Thus, it is reduced to the case where \mathbf{D} is empty. In this case, $\mathbf{U} \times_{\mathbf{S}} \mathbf{X}$ is a divisor of $\mathbf{P} \times_{\mathbf{S}} \mathbf{X}$ since \mathbf{X} is flat over \mathbf{S} . \square

Corollary 4.1.5. — *Let X and Y be regular flat separated schemes of finite type and $D \subset X$ and $E \subset Y$ be divisors with simple normal crossings. Let $f: Y \rightarrow X$ be a morphism over S satisfying $f^{-1}(D) \subset E$ set-theoretically.*

Then, the map $(f \times f)^\sim: (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ is locally of complete intersection and hence is of finite tor-dimension.

Proof. — Since the assertion is local, we may take a Cartesian diagram (4.1.3.1). By the Cartesian diagram (4.1.4.1) and the corresponding one for Y , the diagram

$$(4.1.5.1) \quad \begin{array}{ccc} (V \times_S Y)^\sim & \xrightarrow{(f_V \times f)^\sim} & (U \times_S X)^\sim \\ \downarrow & & \downarrow \\ (Q \times_S Y)^\sim & \xrightarrow{(\tilde{f} \times f)^\sim} & (P \times_S X)^\sim \end{array}$$

is Cartesian. Since $(P \times_S X)^\sim$ is smooth over X and $(Q \times_S Y)^\sim$ is smooth over Y , they are regular. Hence the bottom horizontal arrow $(\tilde{f} \times f)^\sim$ is locally of complete intersection. Since the vertical arrows are regular immersion of codimension 1 and the diagram is Cartesian, $(U \times_S X)^\sim$ and $(Q \times_S Y)^\sim$ are tor-independent over $(P \times_S X)^\sim$. Hence the top arrow $(f_V \times f)^\sim$ is locally of complete intersection of the same virtual relative dimension. \square

We study the local structure of the log product $(X \times_S X)^\sim$ inductively on the number of irreducible components of D . Let X , D and the log product $(X \times_S X)^\sim$ be as in the beginning of this subsection. Let X_1 be a regular divisor of X such that $D_1 = D \cap X_1$ is a divisor of X_1 with simple normal crossings. Let $\mathcal{D} = (D_i)_{i \in I}$ be the family of irreducible components of D and we consider the family $\mathcal{D}_1 = (D_i \cap X_1)_{i \in I}$ of smooth divisors of X_1 . Then, the log product $(X_1 \times_S X_1)^\sim$ with respect to \mathcal{D}_1 is identified with the inverse image of $X_1 \times_S X_1$ by the canonical map $(X \times_S X)^\sim \rightarrow X \times_S X$.

The sum $D' = D \cup X_1$ is a divisor of X with simple normal crossings. We consider the log product $(X \times_S X)^\approx$ with respect to D' . By the inductive construction of the log product, we have a canonical isomorphism $(X \times_S X)^\approx \rightarrow (X \times_S X)^\sim \times_{X \times_S X} (X \times_S X)^\approx_{X_1}$. The inverse image E of $(X_1 \times_S X_1)^\sim$ by the canonical map $(X \times_S X)^\approx \rightarrow (X \times_S X)^\sim$ is a \mathbf{G}_m -torsor over $(X_1 \times_S X_1)^\sim$ by Lemma 1.3.2. They are summarized in the Cartesian diagram

$$(4.1.5.2) \quad \begin{array}{ccccc} E & \longrightarrow & (X_1 \times_S X_1)^\sim & \longrightarrow & X_1 \times_S X_1 \\ \downarrow & & \downarrow & & \downarrow \\ (X \times_S X)^\approx & \longrightarrow & (X \times_S X)^\sim & \longrightarrow & X \times_S X \end{array}$$

where the vertical arrows are closed immersions. The morphism $(X \times_S X)^\approx \rightarrow (X \times_S X)^\sim$ is an isomorphism on the complements of E and $(X_1 \times_S X_1)^\sim$. The subscheme

$E \subset (\mathbf{X} \times_S \mathbf{X})^\sim$ is the inverse image of $X_1 \subset X$ by the composition of the canonical map $(\mathbf{X} \times_S \mathbf{X})^\sim \rightarrow X \times_S X$ with either of the projections $X \times_S X \rightarrow X$.

To understand the local structure of the log product, it suffices to study it on a neighborhood of E by the inductive construction of the log product.

Lemma 4.1.6. — *Let the notations \mathbf{X}, D, X_1, E etc. be as above.*

1. *Assume X_1 is flat over S . Then the immersion $(X_1 \times_S X_1)^\sim \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim$ is a regular immersion of codimension 2 and E is a Cartier divisor of $(\mathbf{X} \times_S \mathbf{X})^\sim$.*

2. *Assume X_1 is a subscheme of the closed fiber X_F and put $d = \dim X_F$. Then the scheme $(X_1 \times_F X_1)^\sim$ is smooth of dimension $2d$ over F .*

Proof. — 1. The immersion $(X_1 \times_S X_1)^\sim \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim$ is locally of complete intersection by Corollary 4.1.5. Hence, the assertion follows from Lemma 3.1.8.

2. Since the projections $(X_1 \times_F X_1)^\sim \rightarrow X_1$ are smooth of relative dimension d , the assertion follows. \square

We show some tor-independences (Definition 3.1.1.1). Its consequences Corollary 4.1.8.1 and 4.1.8.2 will be used in the proof of Propositions 6.1.1 and 6.1.2 respectively.

Lemma 4.1.7. — *Let X and Y be regular flat separated schemes over S of finite type and $f: Y \rightarrow X$ be a morphism over S . Let $D \subset X$ be a regular divisor such that $D_Y = D \times_X Y$ is a divisor of Y and let D' be a divisor of Y with simple normal crossings. We assume that either both D and D_Y are flat over S or they are schemes over F .*

1. *The fiber products $D \times_S D$ and $Y \times_S Y$ are tor-independent over $X \times_S X$.*

2. *Let $(\mathbf{X} \times_S \mathbf{X})^\sim$ and $(\mathbf{Y} \times_S \mathbf{Y})^\sim$ be the log product with respect to D and D' respectively. Assume that $D_Y = D \times_X Y$ is a subset of D' set-theoretically. Further assume that either D and D' are flat over S or D and D' are schemes over F .*

Let $E \subset (\mathbf{X} \times_S \mathbf{X})^\sim$ be the pull-back of $D \subset X$ by either of the two projections. Then E and $(\mathbf{Y} \times_S \mathbf{Y})^\sim$ are tor-independent over $(\mathbf{X} \times_S \mathbf{X})^\sim$.

Proof. — 1. By Lemma 3.1.2, it suffices to show that $D \times_S D$ and $X \times_S Y$ are tor-independent over $X \times_S X$ and that $D \times_S D_Y$ and $Y \times_S Y$ are tor-independent over $X \times_S Y$.

By the assumption that $D_Y = D \times_X Y$ is a divisor of Y , it follows that D and Y are tor-independent over X . Either if D is flat over S or if D is a scheme over F , the fiber product $D \times_S D$ is flat over D . Hence $D \times_S D$ and Y are tor-independent over X with respect to the second projection $D \times_S D \rightarrow X$ by Lemma 3.1.2. Thus, by applying Lemma 3.1.2 to $Y \rightarrow X \leftarrow X \times_S X \leftarrow D \times_S D$, we conclude that $D \times_S D$ and $X \times_S Y$ are tor-independent over $X \times_S X$.

Similarly, either if D_Y is flat over S or if D and D_Y are both schemes over F , the fiber product $D \times_S D_Y$ is flat over D . Hence $D \times_S D_Y$ and Y are tor-independent over

X with respect to the first projection $D \times_S D_Y \rightarrow X$ by Lemma 3.1.2. Thus, by applying Lemma 3.1.2 to $Y \rightarrow X \leftarrow X \times_S Y \leftarrow D \times_S D_Y$, we conclude that $D \times_S D_Y$ and $Y \times_S Y$ are tor-independent over $X \times_S Y$ as required.

2. First, we show the case where D and D' are flat over S . By Lemma 4.1.6.1, E is a divisor of $(X \times_S X)^\sim$. Since every D'_i is flat over S in this case, the pull-back $E'_i \subset (Y \times_S Y)^\sim$ of an irreducible component D'_i of D' by either of the two projections $(Y \times_S Y)^\sim \rightarrow Y$ is also a divisor. Hence, if $D_Y = \sum_i e_i D'_i$, the pull-back $(f \times f)^\sim * E = \sum_i e_i E'_i$ is also a divisor and the assertion follows in this case.

Assume $D = D_F$. Since the assertion is local, we may take a Cartesian diagram (4.1.3.1) satisfying the condition in Lemma 4.1.3 and we consider the diagram (4.1.5.1). By applying Lemma 3.1.2 to $U \rightarrow P \leftarrow (X \times_S P)^\sim \leftarrow (Y \times_S Q)^\sim$, we conclude that the schemes $(X \times_S U)^\sim$ and $(Y \times_S Q)^\sim$ are tor-independent over $(X \times_S P)^\sim$. Since $E \cap (X \times_S U)^\sim$ is a divisor of $(X \times_S P)^\sim$ and $(f \times f)^\sim * E \cap (Y \times_S V)^\sim$ is a divisor of $(Y \times_S Q)^\sim$, the schemes $E \cap (X \times_S U)^\sim$ and $(Y \times_S Q)^\sim$ are tor-independent over $(X \times_S P)^\sim$. Then, applying Lemma 3.1.2 to $(Y \times_S Q)^\sim \rightarrow (X \times_S P)^\sim \leftarrow (X \times_S U)^\sim \leftarrow E \cap (X \times_S U)^\sim$, we conclude that $E \cap (X \times_S U)^\sim$ and $(Y \times_S V)^\sim$ are tor-independent over $(X \times_S U)^\sim$. \square

Corollary 4.1.8. — *Let the notation be as in Lemma 4.1.7 and we put $D_Y = D \times_X Y = \sum_i e_i D'_i$ and $c = \dim Y_K - \dim X_K$.*

1. *Let $f_i: D'_i \rightarrow D$ be the restriction of $f: Y \rightarrow X$ and $(f_i \times f_j)^*: G(D \times_S D) \rightarrow G(D'_i \times_S D'_j)$ be the pull-back. Then, the map $(f \times f)^*: \mathrm{Gr}_{\bullet}^F G(D \times_S D) \rightarrow \mathrm{Gr}_{\bullet+2c}^F G(D_Y \times_S D_Y)$ defined by $f \times f: Y \times_S Y \rightarrow X \times_S X$ is the composition of*

$$\mathrm{Gr}_{\bullet}^F G(D \times_S D) \xrightarrow{(f_i \times f_j)^*_{i,j}} \bigoplus_{i,j} \mathrm{Gr}_{\bullet+2c}^F G(D'_i \times_S D'_j) \xrightarrow{\sum_{i,j} e_i \cdot e_j \cdot \mathrm{can}} \mathrm{Gr}_{\bullet+2c}^F G(D_Y \times_S D_Y)$$

2. *We put $E' = E \times_{(X \times_S X)^\sim} (Y \times_S Y)^\sim$ and let $E'_i \subset (Y \times_S Y)^\sim$ be the pull-back of $D'_i \subset Y$ by either of the projections. Then, the restriction $g_i: E'_i \rightarrow E$ of $(f \times f)^\sim: (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ is of finite tor-dimension. Further, the map $(f \times f)^\sim *: \mathrm{Gr}_{\bullet}^F G(E) \rightarrow \mathrm{Gr}_{\bullet+2c}^F G(E')$ defined by $(f \times f)^\sim: (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ is the composition of*

$$\mathrm{Gr}_{\bullet}^F G(E) \xrightarrow{(g_i^*)_{i,j}} \bigoplus_i \mathrm{Gr}_{\bullet+2c}^F G(E'_i) \xrightarrow{\sum_i e_i \cdot \mathrm{can}} \mathrm{Gr}_{\bullet+2c}^F G(E').$$

Proof. — 1. By Corollary 4.1.5 applied to $(\coprod_i D'_i) \times_S (\coprod_i D'_i) \rightarrow D \times_S D$ with the trivial divisor, the map $f_i \times f_j: D'_i \times_S D'_j \rightarrow D \times_S D$ is of finite tor-dimension. Hence the map $(f_i \times f_j)^*: G(D \times_S D) \rightarrow G(D'_i \times_S D'_j)$ is defined. Let $f_D: D_Y \rightarrow D$ be the base change of f . Then, by Lemma 4.1.7.1, the map $(f \times f)^*: G(D \times_S D) \rightarrow G(D_Y \times_S D_Y)$ is the same as the pull-back by $f_D \times f_D: D_Y \times_S D_Y \rightarrow D \times_S D$. By the assumption, either

every D'_i is flat over S or is a scheme over F . Hence, there exists a filtration on $\mathcal{O}_{D_Y \times_S D_Y}$ such that graded pieces are invertible $\mathcal{O}_{D'_i \times_S D'_j}$ -modules with multiplicities $e_i e_j$. Thus the assertion follows.

2. Both E'_i and $E \times_{D \times_S D} (D'_i \times_S D'_i)^\sim$ are \mathbf{G}_m -torsors over $(D'_i \times_S D'_i)^\sim$. The map $E'_i \rightarrow E \times_{D \times_S D} (D'_i \times_S D'_i)^\sim$ induced by $g_i: E'_i \rightarrow E$ is compatible with the e_i -th power map of \mathbf{G}_m and is flat. Hence, the map g_i is of finite tor-dimension by Corollary 4.1.5. By Lemma 4.1.7.2, the map $(f \times f)^{\sim*}: G(E) \rightarrow G(E')$ is the same as the pull-back by the restriction $g_E: E' \rightarrow E$. Since there exists a filtration on $\mathcal{O}_{E'}$ such that graded pieces are invertible $\mathcal{O}_{E'_i}$ -modules with multiplicities e_i , the assertion follows. \square

4.2. Logarithmic cotangent complex. — We define a logarithmic version of the cotangent complex.

Definition 4.2.1. — Let X be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and D be a divisor of X with simple normal crossings. We put $n = \dim X_K + 1$. Let $(X \times_S X)^\sim$ denote the log product with respect to the family (D_i) of Cartier divisors consisting of the irreducible components of the divisor D . We regard X as a closed subscheme of $(X \times_S X)^\sim$ by the log diagonal map $\delta: X \rightarrow (X \times_S X)^\sim$.

Define the logarithmic cotangent complex $L_{X/S}(\log D)$ to be the conormal complex $M_{X/(X \times_S X)^\sim} = L_{X/(X \times_S X)^\sim}[1]$ and a coherent \mathcal{O}_X -module $\Omega_{X/S}^1(\log D)$ to be the conormal sheaf $N_{X/(X \times_S X)^\sim} = \mathcal{H}_0(L_{X/S}(\log D))$. Define a closed subscheme $\Sigma_{X/S}$ of X to be that defined by the annihilator of the n -th exterior power $\Omega_{X/S}^n(\log D) = \Lambda^n \Omega_{X/S}^1(\log D)$.

If the characteristic of K is 0, the coherent sheaf $\Omega_{X/S}^1(\log D)$ is locally free of rank $n - 1$ on the generic fiber and hence $\Sigma_{X/S}$ is supported on the closed fiber set-theoretically.

Since the log diagonal map $\delta: X \rightarrow (X \times_S X)^\sim$ is a section of the projection $(X \times_S X)^\sim \rightarrow X$, the pull-back $L\delta^* L_{(X \times_S X)^\sim/X}$ of the cotangent complex [14] is canonically identified with the conormal complex $M_{X/(X \times_S X)^\sim}$ [26, Definition 1.6.3.1] and hence with the logarithmic cotangent complex $L_{X/S}(\log D)$.

Lemma 4.2.2. — Let X be a regular flat separated scheme of finite type over S and D be a divisor with simple normal crossings.

1. Let U be an open subscheme of X and $U \rightarrow P$ be a regular immersion of codimension 1 into a smooth scheme P over S satisfying $D \cap U = \tilde{D} \times_P U$ as in Lemma 4.1.1. Then, we have a quasi-isomorphism

$$(4.2.2.1) \quad [N_{U/P} \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U] \rightarrow L_{X/S}(\log D)|_U.$$

Consequently, the logarithmic cotangent complex $L_{X/S}(\log D)$ satisfies the condition $(L(n))$ in Section 3.2 and we have an exact sequence

$$(4.2.2.2) \quad N_{U/P} \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U \rightarrow \Omega_{U/S}^1(\log D) \rightarrow 0.$$

2. Let D_1, \dots, D_m be the irreducible components of D . Then, we have a distinguished triangle

$$(4.2.2.3) \quad \rightarrow L_{X/S} \rightarrow L_{X/S}(\log D) \rightarrow \bigoplus_{i=1}^m \mathcal{O}_{D_i} \rightarrow$$

and consequently an exact sequence

$$(4.2.2.4) \quad 0 \rightarrow \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^1(\log D) \rightarrow \bigoplus_{i=1}^m \mathcal{O}_{D_i} \rightarrow 0.$$

Proof. — 1. The distinguished triangle (3.3.0.1) for the immersions $U \rightarrow (U \times_S X)^\sim \rightarrow (U \times_S P)^\sim$ defines a distinguished triangle

$$\delta|_{U^*} N_{(U \times_S X)^\sim / (U \times_S P)^\sim} \rightarrow N_{U / (U \times_S P)^\sim} \rightarrow M_{U / (U \times_S X)^\sim} \rightarrow .$$

Since $U \rightarrow (U \times_S P)^\sim$ is a section of the smooth morphism $(U \times_S P)^\sim \rightarrow U$, the isomorphism $\mathrm{pr}_2^* \Omega_{P/S}^1(\log \tilde{D}) \rightarrow \Omega_{(U \times_S P)^\sim / U}^1$ induces a canonical isomorphism $N_{U / (U \times_S P)^\sim} \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U$. By the Cartesian diagram (4.1.4.1), the canonical map $N_{U/P} \rightarrow \delta|_{U^*} N_{(U \times_S X)^\sim / (U \times_S P)^\sim}$ is an isomorphism. Thus the assertion follows.

2. The distinguished triangle (3.3.0.1) for $X \rightarrow (X \times_S X)^\sim \rightarrow X \times_S X$ defines a distinguished triangle

$$(4.2.2.5) \quad L_{X/S} \rightarrow L_{X/S}(\log D) \rightarrow L\delta^* L_{(X \times_S X)^\sim / X \times_S X} \rightarrow .$$

Let E_1, \dots, E_m be the inverse images by either of the two projections $(X \times_S X)^\sim \rightarrow X$. Then, we have a canonical isomorphism $\Omega_{(X \times_S X)^\sim / X \times_S X}^1 \rightarrow \bigoplus_{i=1}^m \mathcal{O}_{E_i}$. It suffices to show that this induces an isomorphism $L\delta^* L_{(X \times_S X)^\sim / X \times_S X} \rightarrow \bigoplus_{i=1}^m \mathcal{O}_{E_i}$. The assertion is local on X . By comparing the isomorphism (4.2.2.1) with the isomorphism $[N_{U/P} \rightarrow \Omega_{P/S}^1 \otimes_{\mathcal{O}_P} \mathcal{O}_U] \rightarrow L_{U/S}$, the distinguished triangle (4.2.2.5) implies that $L\delta^* L_{(X \times_S X)^\sim / X \times_S X} \rightarrow \bigoplus_{i=1}^m \mathcal{O}_{E_i}$ is an isomorphism. \square

If there exists a dense open subscheme of X smooth over S , the coherent \mathcal{O}_X -module $\Omega_{X/S}^1(\log D)$ is locally free of rank $n - 1$ on it and the first map $N_{U/P} \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U$ in (4.2.2.2) is an injection.

On an open subscheme $U \subset X$ with a regular immersion $U \rightarrow P$ as in Lemma 4.1.1, if e_1, \dots, e_n is a basis of $\Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U$ and if $a_1 e_1 + \dots + a_n e_n$ is the image of a basis of $N_{U/P}$, then the restriction $\Omega_{X/S}^1(\log D)|_U$ is isomorphic to $\mathcal{O}_U / (a_1, \dots, a_n)$ and hence the annihilator ideal $\mathrm{Ann} \Omega_{X/S}^1(\log D)|_U \subset \mathcal{O}_U$ is generated by a_1, \dots, a_n .

We study the logarithmic cotangent complex $L_{X/S}(\log D)$ more in detail. First, we consider the case where there exists a dense open subscheme of X smooth over S .

Lemma 4.2.3. — We put $X_F = \sum_i l_i D_i$ as a divisor of X and define a Cartier divisor D' of X by $D' = \sum_{D_i \subset X_F, D_i \subset D} l_i D_i$. We put $Z = \Sigma_{X/S}$ and let $i : Z \rightarrow X$ be the closed immersion. We

also put $n = \dim X_K + 1$. Assume that there exists a dense open subscheme of X smooth over S . Then, we have the following

1. There exists a unique \mathcal{O}_X -linear map

$$\cdot d \log \pi : \mathcal{O}_{D'}(D' - X_F) = \mathcal{O}_{D'} \otimes_{\mathcal{O}_X} \mathcal{I}_{X_F - D'} \rightarrow \Omega_{X/S}^1(\log D)$$

sending a local generator g of the ideal $\mathcal{I}_{X_F - D'} \subset \mathcal{O}_X$ to $dg + g \cdot d \log(\pi/g)$ for a prime element π of K . The map $\cdot d \log \pi$ is independent of the choice of a uniformizer π .

2. The map $\cdot d \log \pi : \mathcal{O}_{D'}(D' - X_F) \rightarrow \Omega_{X/S}^1(\log D)$ is an injection and the cokernel $\Omega_{X/S}^1(\log D)/\mathcal{O}_{D'}(D' - X_F)$ is an \mathcal{O}_X -module of tor-dimension ≤ 1 .

3. The \mathcal{O}_Z -module $L_1 i^* \Omega_{X/S}^1(\log D)$ is invertible. For a normal scheme W over F and a morphism $\varphi : W \rightarrow Z$ over S and for the pull-back $\varphi^* L_1 i^* \Omega_{X/S}^1(\log D)$, there exists a canonical isomorphism

$$N_{s/S} \otimes \mathcal{O}_W \rightarrow \varphi^* L_1 i^* \Omega_{X/S}^1(\log D)$$

of trivial invertible \mathcal{O}_W -modules, where $N_{s/S}$ denotes the conormal sheaf $\mathfrak{m}_K/\mathfrak{m}_K^2$ of the closed point s of S .

The complement $X \setminus Z$ is the largest open subscheme of X smooth over S , which is assumed to be dense in X . In the case $X \setminus D \subset X_K$, we have $D' = X_F$ and the cokernel $\Omega_{X/S}^1(\log D)/\mathcal{O}_{X_F}$ will be denoted by $\Omega_{X/S}^1(\log D/\log F)$.

Proof. — 1. The local section $d \log(\pi/g)$ of $\Omega_{X/S}^1(\log D)$ is independent of the choice of a prime element π . We have $d(ug) + (ug) \cdot d \log(\pi/ug) = u(dg + g \cdot d \log(\pi/g)) + g(du - ud \log u)$ for a unit u and the last term is 0. Hence the \mathcal{O}_X -linear map $d \log \pi : \mathcal{O}_X(D' - X_F) \rightarrow \Omega_{X/S}^1(\log D)$ is well-defined.

Since $(\pi/g)(dg + g \cdot d \log(\pi/g)) = (\pi/g)dg + gd(\pi/g) = d\pi = 0$, it induces $\mathcal{O}_{D'}(D' - X_F) \rightarrow \Omega_{X/S}^1(\log D)$.

2. For the injectivity, it suffices to show it at the generic point of each irreducible component D_i of D' . Hence, we may assume $X_K = X \setminus D$. Then, it follows from [26, Lemma 5.3.4.2].

We show that $\Omega_{X/S}^1(\log D)/\mathcal{O}_{D'}(D' - X_F)$ is of tor-dimension ≤ 1 . Since the question is local, we take an immersion $U \rightarrow P$ as in Lemma 4.1.1. Let \tilde{g} be a function on P lifting $g = \pi / \prod_i t_i^{l_i}$. Then, on a neighborhood of U , the divisor $U \subset P$ is defined by an equation $\pi = \tilde{g} \cdot \prod_i T_i^{l_i}$. Hence, the image of $N_{U/P} \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U$ is generated by $d(\tilde{g} \cdot \prod_i T_i^{l_i})$. Since the image of the section $(\prod_i t_i^{l_i})^{-1} d(\tilde{g} \cdot \prod_i T_i^{l_i}) = d\tilde{g} + \tilde{g} \cdot \sum_i l_i d \log T_i$ of $\Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U$ in $\Omega_{X/S}^1(\log D)|_U$ is $dg + g \cdot d \log(\pi/g)$, we have a locally free resolution $0 \rightarrow N_{U/P}(D') \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U \rightarrow (\Omega_{X/S}^1(\log D)/\mathcal{O}_{D'}(D' - X_F))|_U \rightarrow 0$.

3. Since $\Omega_{X/S}^1(\log D)$ satisfies the condition $L(n)$ in Section 3.2, the \mathcal{O}_Z -module $L_1 i^* \Omega_{X/S}^1(\log D)$ is invertible.

We may assume W is integral. Let $i_W = i \circ \varphi: W \rightarrow X$ denote the composition. Then, since $\Omega_{X/S}^1(\log D)$ is of tor-dimension ≤ 1 and the \mathcal{O}_Z -modules $i^*\Omega_{X/S}^1(\log D)$ and $L_1 i^*\Omega_{X/S}^1(\log D)$ are locally free, the canonical map $\varphi^*L_1 i^*\Omega_{X/S}^1(\log D) \rightarrow L_1 i_W^*\Omega_{X/S}^1(\log D)$ is an isomorphism.

First we consider the case $i_W(W) \not\subset D$. By $i_W(W) \not\subset D$, we have $L_1 i_W^*\mathcal{O}_{D_i} = 0$. Hence the exact sequence (4.2.2.4) induces an isomorphism $L_1 i_W^*\Omega_{X/S}^1 \rightarrow L_1 i_W^*\Omega_{X/S}^1(\log D)$ of invertible \mathcal{O}_W -modules. Let Z' be the closed subscheme of X defined by the annihilator ideal of $\Omega_{X/S}^n = \Lambda^n \Omega_{X/S}^1$. For a morphism $g: T \rightarrow X$ of schemes, the \mathcal{O}_T -module $L_1 g^*\Omega_{X/S}^1$ is invertible if and only if g factors through Z' , since $\Omega_{X/S}^1$ satisfies the condition $L(n)$ in Section 3.2. Since $L_1 i_W^*\Omega_{X/S}^1$ is invertible, the map $i_W: W \rightarrow X$ factors through the closed subscheme Z' . Hence, the assertion follows from [26, Lemma 5.1.3.1].

Next, we consider the case $i_W(W) \subset D$. Then the exact sequence $0 \rightarrow \mathcal{O}_X(-X_F) \rightarrow \mathcal{O}_X(D' - X_F) \rightarrow \mathcal{O}_{D'}(D' - X_F) \rightarrow 0$ defines an isomorphism $i_W^*\mathcal{O}_X(-X_F) = N_{S/S} \otimes \mathcal{O}_W \leftarrow L_1 i_W^*\mathcal{O}_{D'}(D' - X_F)$. Further the map $d \log \pi \cdot: \mathcal{O}_{D'}(D' - X_F) \rightarrow \Omega_{X/S}^1(\log D)$ induces a map $L_1 i_W^*\mathcal{O}_{D'}(D' - X_F) \rightarrow L_1 i_W^*\Omega_{X/S}^1(\log D)$. We show that it is an isomorphism.

Since the question is local, we take an immersion $U \rightarrow P$ as in Lemma 4.1.1. Then, by the proof of 2, we have a commutative diagram of exact sequences

(4.2.3.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{U/P} & \longrightarrow & N_{U/P}(D') & \longrightarrow & \mathcal{O}_{D'}(D' - X_F)|_U & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow d \log \pi \cdot|_U & & \\ 0 & \longrightarrow & N_{U/P} & \longrightarrow & \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U & \longrightarrow & \Omega_{X/S}^1(\log D)|_U & \longrightarrow & 0. \end{array}$$

Then both $L_1 i_W^*\mathcal{O}_{D'}(D' - X_F)$ and $L_1 i_W^*\Omega_{X/S}^1(\log D)$ are identified with $i_W^*N_{U/P}$ and the assertion follows. \square

Next, we consider the case where $\Sigma_{X/S} = X$. This occurs only if the characteristic of K is $p > 0$.

Lemma 4.2.4. — *Assume $\Sigma_{X/S} = X$. Then, there exists a canonical isomorphism $\Omega_S^1 \otimes \mathcal{O}_X \rightarrow \mathcal{H}_1(L_{X/S}(\log D))$ of invertible \mathcal{O}_X -modules.*

Proof. — Let $K_0 = K^p \subset K$ and put $S_0 = \text{Spec } \mathcal{O}_{K_0}$. The composition of closed immersions $X \rightarrow (X \times_S X)^\sim \rightarrow (X \times_{S_0} X)^\sim$ defines a distinguished triangle

$$(4.2.4.1) \quad \rightarrow L\delta^*L_{(X \times_S X)^\sim / (X \times_{S_0} X)^\sim}[1] \rightarrow L_{X/S_0}(\log D) \rightarrow L_{X/S}(\log D) \rightarrow$$

of cotangent complexes. By the Cartesian diagram

$$\begin{array}{ccc} (X \times_S X)^\sim & \longrightarrow & (X \times_{S_0} X)^\sim \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \times_{S_0} S, \end{array}$$

we obtain a surjection $\Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_X \rightarrow N_{(X \times_S X)^\sim / (X \times_{S_0} X)^\sim}$. We claim that this map and the map $\mathcal{H}_1 L_{X/S}(\log D) \rightarrow N_{(X \times_S X)^\sim / (X \times_{S_0} X)^\sim}$ defined by (4.2.4.1) are isomorphisms. This will complete the proof since $\Omega_{S/S_0}^1 = \Omega_S^1$ is an invertible \mathcal{O}_S -module.

To show the claim, it suffices to show that the canonical map $\mathcal{H}_1 L_{X/S_0}(\log D) \rightarrow \mathcal{H}_1 L_{X/S}(\log D)$ is the 0-map since $\mathcal{H}_0 L_{X/S_0}(\log D) \rightarrow \mathcal{H}_0 L_{X/S}(\log D)$ is an isomorphism and $\mathcal{H}_1 L_{X/S}(\log D)$ is invertible. We show that $\mathcal{H}_1 L_{X/S_0}(\log D) \rightarrow \mathcal{H}_1 L_{X/S}(\log D)$ is the 0-map. Since the assertion is local on X , we may assume that there is a regular immersion $X \rightarrow P_0$ of codimension 1 to a smooth scheme P_0 over S_0 as in Lemma 4.1.1. It induces an immersion $X \rightarrow P = P_0 \times_{S_0} S$ to a smooth scheme P over S . Then, the map $\mathcal{H}_1 L_{X/S_0}(\log D) \rightarrow \mathcal{H}_1 L_{X/S}(\log D)$ is identified with the canonical map $N_{X/P_0} \rightarrow N_{X/P}$ of the conormal sheaves induced by the projection $P = P_0 \times_{S_0} S \rightarrow P_0$. Since $X \times_{S_0} S = X \times_{P_0} P$ regarded as a Cartier divisor of P is p -times the Cartier divisor X of P , the assertion follows. \square

We study a consequence of Lemma 4.1.3.

Lemma 4.2.5. — *Let $f: Y \rightarrow X$ be as in Lemma 4.1.3 and put $n = \dim X_K + 1$ and $n' = \dim Y_K + 1$ respectively. Then, for the closed subschemes $\Sigma_{X/S} \subset X$ and $\Sigma_{Y/S} \subset Y$ defined by the annihilators $\Lambda^n \Omega_{X/S}^1(\log D)$ and $\Lambda^{n'} \Omega_{Y/S}^1(\log E)$, the pull-back $\Sigma_{X/S} \times_X Y$ is a subscheme of $\Sigma_{Y/S}$.*

Proof. — The assertion is local on Y . We consider a Cartesian diagram (4.1.3.1) satisfying the condition in Lemma 4.1.3. Then, we have a commutative diagram of exact sequences

(4.2.5.1)

$$\begin{array}{ccccccc} 0 & \longrightarrow & f|_V^* N_{U/P} & \longrightarrow & f|_V^*(\Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U) & \longrightarrow & f|_V^* \Omega_{X/S}^1(\log D)|_U \longrightarrow 0 \\ & & \simeq \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_{V/Q} & \longrightarrow & \Omega_{Q/S}^1(\log \tilde{E}) \otimes_{\mathcal{O}_Q} \mathcal{O}_V & \longrightarrow & \Omega_{Y/S}^1(\log E)|_V \longrightarrow 0 \end{array}$$

The closed subscheme $(\Sigma_{X/S} \times_X Y) \cap V$ is the largest closed subscheme where the pull-back $f|_V^* N_{U/P} \rightarrow f|_V^*(\Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U)$ is the zero-map. Similarly $\Sigma_{Y/S} \cap V$ is the largest closed subscheme where the pull-back $N_{V/Q} \rightarrow \Omega_{Q/S}^1(\log \tilde{E}) \otimes_{\mathcal{O}_Q} \mathcal{O}_V$ is the zero-map. Hence the assertion follows from (4.2.5.1). \square

Next, we study consequences of Lemma 4.1.6. As loc. cit., we assume that X_1 is a regular divisor of X and that $D_1 = D \cap X_1$ is a divisor with simple normal crossings. Let $(X \times_S X)^\sim$ and $(X \times_S X)^\approx$ be the log product with respect to D and $D' = D \cup X_1$ respectively and E be the inverse image of X_1 by either of the two projections

$(X \times_S X)^\sim \rightarrow X$. We have Cartesian diagrams

$$(4.2.6.1) \quad \begin{array}{ccccc} X_1 & \xrightarrow{c} & E & & X_1 & \xrightarrow{c} & (X_1 \times_S X_1)^\sim \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{c} & (X \times_S X)^\sim, & & X & \xrightarrow{c} & (X \times_S X)^\sim \end{array}$$

by the definition of E and by the universality of log product, respectively.

First, we consider the case where X_1 is flat over S .

Lemma 4.2.6. — *Let the notation be as above and assume that X_1 is flat over S . Let $i : X_1 \rightarrow X$ denote the closed immersion. Then, the left Cartesian diagram (4.2.6.1) defines a distinguished triangle*

$$L_{X_1/S}(\log D_1) \rightarrow Li^*L_{X/S}(\log D') \rightarrow \mathcal{O}_{X_1} \rightarrow$$

on X_1 .

Proof. — By Lemma 4.1.6.1, the vertical arrows in the left Cartesian diagram (4.2.6.1) are regular immersion of codimension 1. Hence the canonical map $Li^*L_{X/S}(\log D') \rightarrow M_{X_1/E}$ to the conormal complex is a quasi-isomorphism. Since $E \rightarrow (X_1 \times_S X_1)^\sim$ is smooth, the morphisms $X_1 \rightarrow E \rightarrow (X_1 \times_S X_1)^\sim$ define a distinguished triangle $L_{X_1/S}(\log D_1) \rightarrow M_{X_1/E} \rightarrow \Omega_{E/(X_1 \times_S X_1)^\sim}^1 \otimes \mathcal{O}_{X_1}$. Since E is a \mathbf{G}_m -torsor over $(X_1 \times_S X_1)^\sim$ splitting on X_1 , we have a canonical trivialization $\Omega_{E/(X_1 \times_S X_1)^\sim}^1 \otimes \mathcal{O}_{X_1} \rightarrow \mathcal{O}_{X_1}$. Thus, the assertion follows. \square

Next, we assume that X_1 is contained in the closed fiber X_F . Since the immersions $X_1 \rightarrow E$ and $X_1 \rightarrow (X_1 \times_F X_1)^\sim$ are regular immersions of smooth schemes over F , the Cartesian diagram (4.2.6.1) define the excess conormal complexes $M'_{X/(X \times_S X)^\sim, E}$ and $M'_{X/(X \times_S X)^\sim, (X_1 \times_F X_1)^\sim}$ as complexes of \mathcal{O}_{X_1} -modules as in [26, Definition 1.6.3.2]. By the definitions $L_{X/S}(\log D) = M_{X/(X \times_S X)^\sim}$ and $L_{X/S}(\log D') = M_{X/(X \times_S X)^\sim}$, they fit in the distinguished triangles

$$(4.2.7.1) \quad \rightarrow M'_{X/(X \times_S X)^\sim, E} \rightarrow Li_1^*L_{X/S}(\log D') \rightarrow N_{X_1/E} \rightarrow,$$

$$(4.2.7.2) \quad \rightarrow M'_{X/(X \times_S X)^\sim, (X_1 \times_F X_1)^\sim} \rightarrow Li_1^*L_{X/S}(\log D) \rightarrow \Omega_{X_1/F}^1(\log D_1) \rightarrow$$

by [26, Proposition 1.6.4.2]. More concretely, they are described as follows.

Lemma 4.2.7. — *Let the notation be as above and assume that X_1 is a smooth scheme of dimension d over F . Let $N_{s/S}$ be the conormal sheaf $\mathfrak{m}_K/\mathfrak{m}_K^2$ of the closed point s of S .*

1. *There exists a canonical isomorphism $M'_{X/(X \times_S X)^\sim, E} \rightarrow N_{s/S} \otimes_F \mathcal{O}_{X_1}[1]$.*
2. *There exists a canonical isomorphism $M'_{X/(X \times_S X)^\sim, (X_1 \times_F X_1)^\sim} \rightarrow [N_{s/S} \otimes_F \mathcal{O}_{X_1} \rightarrow N_{X_1/X}]$. The complex $M'_{X/(X \times_S X)^\sim, (X_1 \times_F X_1)^\sim}$ on X_1 is acyclic outside the intersection $X_1 \cap \Sigma_{X/S}$ where $\Sigma_{X/S}$ is defined in Definition 4.2.1.*

Proof. — 1. Let $i_1 : X_1 \rightarrow X$ be the closed immersion. Since the immersion $X_1 \rightarrow E$ is a regular immersion of codimension $d + 1$, the canonical surjection $i_1^* \Omega_{X/S}^1(\log D') \rightarrow N_{X_1/E}$ is an isomorphism. Hence, by the distinguished triangle (4.2.7.1), we obtain an isomorphism $M'_{X/(X \times_S X)^\sim, E} \rightarrow L_1 i_1^* L_{X/S}(\log D')$ [1]. Thus the assertion follows from Lemma 4.2.3.3 and Lemma 4.2.4.

2. Similarly as Lemma 4.2.2.2, we have a distinguished triangle $\rightarrow L_{X/S}(\log D) \rightarrow L_{X/S}(\log D') \rightarrow \mathcal{O}_{X_1} \rightarrow$ and hence a distinguished triangle $\rightarrow L_1^* L_{X/S}(\log D) \rightarrow L_1^* L_{X/S}(\log D') \rightarrow L_1^* \mathcal{O}_{X_1} \rightarrow$. By (4.2.7.1), (4.2.7.2) and by the exact sequence $0 \rightarrow \Omega_{X_1/F}^1(\log D_1) \rightarrow N_{X_1/E} \rightarrow \mathcal{O}_{X_1} \rightarrow 0$, we obtain a distinguished triangle

$$\rightarrow M'_{X/(X \times_S X)^\sim, (X_1 \times_F X_1)^\sim} \rightarrow M'_{X/(X \times_S X)^\sim, E_1} \rightarrow L_1 i_1^* \mathcal{O}_{X_1}[1] \rightarrow .$$

Hence the assertion follows from 1 and the isomorphism $L_1 i_1^* \mathcal{O}_{X_1} \rightarrow N_{X_1/X}$.

On the complement of $\Sigma_{X/S}$, the immersion $X_1 \rightarrow X_F$ is an open immersion and the complex $M'_{X/(X \times_S X)^\sim, (X_1 \times_F X_1)^\sim}$ is acyclic. \square

Assume that the underlying set of the closed fiber X_F is a subset of D and put $n = \dim X_K + 1$. We give a variant of Lemma 4.1.4 and Lemma 4.2.3. We define a variant $(X \times_{\mathbf{s}} X)^\sim$ of the log product by the Cartesian diagram

$$\begin{array}{ccc} (X \times_{\mathbf{s}} X)^\sim & \longrightarrow & (X \times_S X)^\sim \\ \downarrow & & \downarrow \\ S & \longrightarrow & (S \times_S S)^\sim \end{array}$$

where $(S \times_S S)^\sim$ is the log product defined with respect to the Cartier divisor s of S .

Lemma 4.2.8. — *Let X be a regular flat scheme of finite type over S and $D \subset X$ be a divisor with simple normal crossings. Assume that the underlying set of the closed fiber X_F is a subset of D .*

Then the scheme $(X \times_{\mathbf{s}} X)^\sim$ is flat and locally a hypersurface over X with respect to either of the projections.

Proof. — The log scheme X with the log structure defined by D is log flat ([26, Section 4.3]) and log locally of complete intersection ([26, Definition 4.4.2]) over S with the log structure defined by the closed point, similarly as [26, Lemma 5.2.1]. Since the projection $(X \times_{\mathbf{s}} X)^\sim \rightarrow X$ is strict, it is flat. Since the assertion is local, we take a regular immersion $U \rightarrow P$ of codimension 1 as in Lemma 4.1.1. Then, we have a closed immersion $(U \times_{\mathbf{s}} X)^\sim \rightarrow (P \times_S X)^\sim$. Then, since $(P \times_S X)^\sim$ is smooth over X and $(U \times_{\mathbf{s}} X)^\sim$ is locally of complete intersection, the immersion $(U \times_{\mathbf{s}} X)^\sim \rightarrow (P \times_S X)^\sim$ is a regular immersion by Lemma 3.1.8. We verify that it is a regular immersion of codimension 1 by reducing to the case where D is empty. \square

We also define a variant $L_{X/S}(\log D/\log F)$ of the logarithmic cotangent complex to be the conormal complex $M_{X/(X \times_S X)^\sim}$. The coherent \mathcal{O}_X -module $\Omega_{X/S}^1(\log D/\log F)$

defined as $\mathcal{H}_0(\mathbf{L}_{X/S}(\log D/\log F))$ is the conormal sheaf $N_{X/(X \times_S X)^\sim}$. For a regular immersion $U \rightarrow P$ as in Lemma 4.1.1, we have shown in the proof of Lemma 4.2.8 that the immersion $(U \times_S X)^\sim \rightarrow (P \times_S X)^\sim$ is a regular immersion of codimension 1. The immersions $U \rightarrow (U \times_S X)^\sim \rightarrow (P \times_S X)^\sim$ define a distinguished triangle $\delta|_U^* N_{(U \times_S X)^\sim/(P \times_S X)^\sim} \rightarrow N_{U/(P \times_S X)^\sim} \rightarrow M_{U/(U \times_S X)^\sim} \rightarrow$. Similarly as for $L_{X/S}(\log D)$, it defines a quasi-isomorphism

$$(4.2.9.1) \quad [N_{(U \times_S X)^\sim/(P \times_S X)^\sim} \otimes \mathcal{O}_U \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U] \rightarrow L_{X/S}(\log D/\log F)|_U.$$

This shows that the logarithmic cotangent complex $L_{X/S}(\log D/\log F)$ satisfies the condition $(L(n))$ in Section 3.2. If there exists a dense open subscheme of X smooth over S , the coherent \mathcal{O}_X -module $\Omega_{X/S}^1(\log D/\log F)$ is locally free of rank $n - 1$ on a dense open subscheme and the map $N_{(U \times_S X)^\sim/(P \times_S X)^\sim} \otimes \mathcal{O}_U \rightarrow \Omega_{P/S}^1(\log \tilde{D}) \otimes_{\mathcal{O}_P} \mathcal{O}_U$ in (4.2.9.1) is an injection.

The immersions $X \rightarrow (X \times_S X)^\sim \rightarrow (X \times_S X)^\sim$ define a distinguished triangle

$$(4.2.9.2) \quad L\delta^* L_{(X \times_S X)^\sim/(X \times_S X)^\sim} \rightarrow L_{X/S}(\log D) \rightarrow L_{X/S}(\log D/\log F) \rightarrow .$$

Since $(S \times_S S)^\sim = \text{Spec } \mathcal{O}_K[U^{\pm 1}]/((U - 1)\pi)$ for a prime element π , the conormal sheaf $N_{S/(S \times_S S)^\sim}$ is isomorphic to F and is generated by $d \log \pi$. Hence, the distinguished triangle (4.2.9.2) gives an exact sequence

$$(4.2.9.3) \quad \mathcal{O}_{X_F} \rightarrow \Omega_{X/S}^1(\log D) \rightarrow \Omega_{X/S}^1(\log D/\log F) \rightarrow 0.$$

If there exists a dense open subscheme of X smooth over S , the first arrow $\mathcal{O}_{X_F} \rightarrow \Omega_{X/S}^1(\log D)$ is an injection by Lemma 4.2.3.2. If X is nowhere smooth over S , the second arrow $\Omega_{X/S}^1(\log D) \rightarrow \Omega_{X/S}^1(\log D/\log F)$ is an isomorphism of locally free \mathcal{O}_X -modules of rank $n + 1$. We put $\Omega_{X/S}^n(\log D/\log F) = \Lambda^n \Omega_{X/S}^1(\log D/\log F)$.

Lemma 4.2.9. — *Let Z be the closed subscheme of X defined by the ideal $\text{Ann } \Omega_{X/S}^n(\log D/\log F)$ and $i : Z \rightarrow X$ be the closed immersion. Then the restriction of the invertible \mathcal{O}_Z -module $L_1 i^* \Omega_{X/S}^1(\log D/\log F)$ to the reduced closed fiber $Z_{F,\text{red}}$ is trivial.*

Proof. — It suffices to consider the cases where the complement $X \setminus Z$ is dense and $Z = X$ respectively. First, we prove the case where the complement $X \setminus Z$ is dense. In this case, the proof is similar to [26, Lemma 5.3.5.1]. Let $i' : Z_{F,\text{red}} \rightarrow X$ be the immersion. Similarly as in Lemma 4.2.3.3, the restriction of the invertible \mathcal{O}_Z -module $L_1 i^* \Omega_{X/S}^1(\log D/\log F)$ to $Z_{F,\text{red}}$ is isomorphic to $L_1 i'^* \Omega_{X/S}^1(\log D/\log F)$.

By the exact sequence (4.2.9.3) together with the injectivity of $\mathcal{O}_{X_F} \rightarrow \Omega_{X/S}^1(\log D)$, we obtain an exact sequence

$$\begin{aligned} 0 &\rightarrow L_1 i'^* \mathcal{O}_{X_F} \rightarrow L_1 i'^* \Omega_{X/S}^1(\log D) \rightarrow L_1 i'^* \Omega_{X/S}^1(\log D/\log F) \\ &\rightarrow i'^* \mathcal{O}_{X_F} \rightarrow i'^* \Omega_{X/S}^1(\log D) \rightarrow i'^* \Omega_{X/S}^1(\log D/\log F) \rightarrow 0. \end{aligned}$$

By a local description of Z similar to that given before Lemma 4.2.3, it follows that the last map $i^* \Omega_{X/S}^1(\log D) \rightarrow i^* \Omega_{X/S}^1(\log D/\log F)$ is an isomorphism of locally free $\mathcal{O}_{Z_{F,\text{red}}}$ -modules. Hence the map $L_1 i^* \Omega_{X/S}^1(\log D/\log F) \rightarrow i^* \mathcal{O}_{X_F}$ is a surjection of invertible $\mathcal{O}_{Z_{F,\text{red}}}$ -modules and is an isomorphism. Therefore, the map $L_1 i^* \mathcal{O}_{X_F} \rightarrow L_1 i^* \Omega_{X/S}^1(\log D)$ is also an isomorphism of invertible $\mathcal{O}_{Z_{F,\text{red}}}$ -modules. Since $L_1 i^* \mathcal{O}_{X_F}$ is isomorphic to $\mathcal{O}_{Z_{F,\text{red}}}$, the assertion follows.

We show the case where $Z = X$. The proof in this case is similar to that of Lemma 4.2.4. We show that there exists a canonical isomorphism $\Omega_S^1 \otimes \mathcal{O}_X \rightarrow \mathcal{H}_1 L_{X/S}(\log D/\log F)$ of invertible \mathcal{O}_X -modules. Let $K_0 = K^b \subset K$ and put $S_0 = \text{Spec } \mathcal{O}_{K_0}$ as in the proof of Lemma 4.2.4. The composition of closed immersions $X \rightarrow (X \times_{\mathbf{S}} X)^\sim \rightarrow (X \times_{S_0} X)^\sim$ defines a distinguished triangle

$$(4.2.9.4) \quad \rightarrow L\delta^* L_{(X \times_{\mathbf{S}} X)^\sim / (X \times_{S_0} X)^\sim} [1] \rightarrow L_{X/S_0}(\log D) \rightarrow L_{X/S}(\log D/\log F) \rightarrow$$

of cotangent complexes. By the Cartesian diagram

$$\begin{array}{ccc} (X \times_{\mathbf{S}} X)^\sim & \longrightarrow & (X \times_{S_0} X)^\sim \\ \downarrow & & \downarrow \\ S & \longrightarrow & (S \times_{S_0} S)^\sim, \end{array}$$

we obtain a surjection $\Omega_{S/S_0}^1(\log F) \otimes_{\mathcal{O}_S} \mathcal{O}_X \rightarrow N_{(X \times_{\mathbf{S}} X)^\sim / (X \times_{S_0} X)^\sim}$. Similarly as in the proof of Lemma 4.2.4, this map and the map $\mathcal{H}_1 L_{X/S}(\log D/\log F) \rightarrow N_{(X \times_{\mathbf{S}} X)^\sim / (X \times_{S_0} X)^\sim}$ defined by (4.2.9.4) are isomorphisms. This complete the proof since $\Omega_{S/S_0}^1(\log F)$ is isomorphic to \mathcal{O}_S . \square

4.3. Intersection product with the log diagonal. — Let X be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and $D \subset X$ be a divisor with simple normal crossings. We put $n = \dim X_K + 1$. We define the localized intersection product with the log diagonal as follows.

We recall the notation from the previous subsection. The log product $(X \times_S X)^\sim$ is defined with respect to the family $\mathcal{D} = (D_i)_{i \in I}$ of irreducible components of D . The logarithmic cotangent complex $L_{X/S}(\log D)$ is defined as the conormal complex $M_{X/(X \times_S X)^\sim}$ of the log diagonal $\delta: X = \Delta_X^{\log} \rightarrow (X \times_S X)^\sim$ and we have a canonical isomorphism $\mathcal{H}_0(L_{X/S}(\log D)) \rightarrow \Omega_{X/S}^1(\log D)$. We define a closed subscheme $\Sigma_{X/S}$ of X to be that defined by the annihilator ideal of $\Omega_{X/S}^n(\log D) = \Lambda^n(\Omega_{X/S}^1(\log D))$. Let $\mathcal{L}_{\Sigma_{X/S}}$ denote the invertible $\mathcal{O}_{\Sigma_{X/S}}$ -module $L_1 i^* L_{X/S}(\log D)$ where $i: \Sigma_{X/S} \rightarrow X$ denote closed immersion.

We consider the following special case of Definition 3.3.3. We consider X and the log product $(X \times_S X)^\sim$ as S and X in Definition 3.3.3. The log product $(X \times_S X)^\sim$ is locally a hypersurface of relative dimension $n - 1$ over X by either of the two projections by Lemma 4.1.4. As V in Definition 3.3.3, we take X regarded as a closed subscheme of $(X \times_S X)^\sim$ by the log diagonal. Since the canonical map $\delta^* \Omega_{(X \times_S X)^\sim / X}^1 \rightarrow \Omega_{X/S}^1(\log D)$

is an isomorphism, the intersection $Z \times_X V \subset V$ in Definition 3.3.3 is $\Sigma_{X/S} \subset X$ in our setting.

We consider a scheme W of finite type over S and a morphism $g: W \rightarrow (X \times_S X)^\sim$ over S as $W \rightarrow X$ in Definition 3.3.3. Then, $Z_T \subset T \subset W$ in Definition 3.3.3 are the inverse images $g^{-1}(\Sigma_{X/S}) \subset g^{-1}(\Delta_X^{\log}) \subset W$ where Δ_X^{\log} denotes X regarded as a closed subscheme of $(X \times_S X)^\sim$ by the log diagonal. Since the canonical map $L\delta^* L_{(X \times_S X)^\sim/X} \rightarrow L_{X/S}(\log D)$ is an isomorphism, the pull-back of \mathcal{L}_Z to $Z \times_X V$ in Definition 3.3.3 is the invertible sheaf $\mathcal{L}_{\Sigma_{X/S}} = L_1 i^* L_{X/S}(\log D)$ on $\Sigma_{X/S}$ in our setting. By Lemma below, under the assumption (A) there, the group $G(Z_T)_{/\mathcal{L}_Z}$ in Definition 3.3.3 is $G(g^{-1}(\Sigma_{X/S}))$ in our setting.

Lemma 4.3.1. — *Let the notation be as above. In particular, let W be a scheme of finite type over S and $g: W \rightarrow (X \times_S X)^\sim$ a morphism over S . Assume that the morphism $g: W \rightarrow (X \times_S X)^\sim$ satisfies the condition:*

(A) *The inverse image $g^{-1}(\Sigma_{X/S})$ is supported on the closed fiber W_F set-theoretically.*

Then the multiplication of the pull-back of the invertible $\mathcal{O}_{\Sigma_{X/S}}$ -module $\mathcal{L}_{\Sigma_{X/S}}$ on the Grothendieck group $G(g^{-1}(\Sigma_{X/S}))$ is the identity.

Proof. — Since the Grothendieck group $G(g^{-1}(\Sigma_{X/S}))$ is generated by the push forward of the classes of the normalizations of integral closed subschemes, it follows from Lemma 4.2.3.3 and Lemma 4.2.4. \square

Thus, we make the following definition as in [26, Definition 5.1.5].

Definition 4.3.2. — *Let X be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and $D \subset X$ be a divisor with simple normal crossings. Let W be a scheme of finite type over S and $g: W \rightarrow (X \times_S X)^\sim$ be a morphism over S satisfying the condition (A) in Lemma 4.3.1. Then, we define the localized intersection product with the log diagonal*

$$(4.3.2.1) \quad ((\Delta_X^{\log},))_{(X \times_S X)^\sim} : G(W) \rightarrow G(g^{-1}(\Sigma_{X/S}))$$

as the product (3.3.3.1) with the class of $\mathcal{F} = \mathcal{O}_X$.

In [26, Definition 5.1.5], we defined the localized intersection product with the log diagonal under the assumption that the generic fiber is smooth and $D = X_F$. Here, we replace the assumption by the condition (A) in Lemma 4.3.1.

The logarithmic localized intersection product $((\Delta_X^{\log},))_{(X \times_S X)^\sim} : G(W) \rightarrow G(g^{-1}(\Sigma_{X/S}))$ preserves the topological filtration in the sense that it induces a map

$$F_q G(W) \rightarrow F_{q-n} G(g^{-1}(\Sigma_{X/S}))$$

[26, Theorem 3.4.3.1].

If we take a closed immersion $A \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim$ as $W \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim$, the condition (A) in Lemma 4.3.1 can be written as

(A') The intersection $\delta^{-1}(A) \cap \Sigma_{\mathbf{X}/S}$ is supported on the closed fiber X_F set-theoretically.

Under this assumption, the localized intersection product with the log diagonal

$$(4.3.2.2) \quad ((\cdot, \Delta_{\mathbf{X}}^{\log}))_{(\mathbf{X} \times_S \mathbf{X})^\sim} : G(A) \rightarrow G(\delta^{-1}(A) \cap \Sigma_{\mathbf{X}/S})$$

is defined as the product (3.3.3.1) with the class of $\mathcal{G} = \mathcal{O}_{\mathbf{X}}$ by taking $\mathbf{X} \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim$ as $W \rightarrow \mathbf{X}$ in Definition 3.3.3. By the symmetry of $\mathcal{T}or$, we have

$$((\cdot, \Delta_{\mathbf{X}}^{\log}))_{(\mathbf{X} \times_S \mathbf{X})^\sim} = ((\Delta_{\mathbf{X}}^{\log}, \cdot))_{(\mathbf{X} \times_S \mathbf{X})^\sim}.$$

The localized intersection product with log diagonal has the following functoriality.

Lemma 4.3.3. — *Let Y be another regular flat separated scheme of finite type over S and $E \subset Y$ be a divisor with simple normal crossings. Let $(Y \times_S Y)^\sim$ be the log product with respect to E and we put $V = Y \setminus E$. Let $f : Y \rightarrow X$ be a morphism over S such that $f(V) \subset U = X \setminus D$ and we consider the map $(f \times f)^\sim : (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ of log products.*

Let A be a closed subscheme of $(X \times_S X)^\sim$ satisfying the condition (A') after Definition 4.3.2 and assume that $A_Y = (f \times f)^{\sim -1}(A) \subset (Y \times_S Y)^\sim$ also satisfies the corresponding condition that $\delta_Y^{-1}(A_Y) \cap \Sigma_{Y/S}$ is supported on the closed fiber Y_F set-theoretically. Let $f^ : G(\delta_X^{-1}(A) \cap \Sigma_{X/S}) \rightarrow G(\delta_Y^{-1}(A_Y) \cap \Sigma_{Y/S})$ be the pull-back by $f : Y \rightarrow X$.*

*Then the pull-back $(f \times f)^{\sim *} : G(A) \rightarrow G(A_Y)$ by $(f \times f)^\sim : (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ is defined. Further, the diagram*

$$\begin{array}{ccc} G(A) & \xrightarrow{((\cdot, \Delta_{\mathbf{X}}^{\log}))} & G(\delta_X^{-1}(A) \cap \Sigma_{X/S}) \\ (f \times f)^{\sim *} \downarrow & & \downarrow f^* \\ G(A_Y) & \xrightarrow{((\cdot, \Delta_{\mathbf{Y}}^{\log}))} & G(\delta_Y^{-1}(A_Y) \cap \Sigma_{Y/S}) \end{array}$$

is commutative.

Proof. — By Lemma 4.2.5, we have $f^{-1}(\Sigma_{X/S}) \subset \Sigma_{Y/S}$. Since $f^{-1}(\delta_X^{-1}(A)) = \delta_Y^{-1}(A_Y)$, the map $G(\delta_X^{-1}(A) \cap \Sigma_{X/S}) \rightarrow G(\delta_Y^{-1}(A_Y) \cap \Sigma_{Y/S})$ is defined by the assumption (A').

By Corollary 4.1.5, the map $(f \times f)^\sim : (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ is of finite tor-dimension. Hence, the pull-back $(f \times f)^{\sim *} : G(A) \rightarrow G(A_Y)$ by $(f \times f)^\sim : (Y \times_S Y)^\sim \rightarrow (X \times_S X)^\sim$ is defined.

We apply the associativity, Lemma 3.3.6, by taking $A \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim \leftarrow \mathbf{X} \leftarrow Y$ as $V \rightarrow \mathbf{X} \leftarrow W \leftarrow W'$. Then the composition via upper right is equal to the

map $((, \Delta_Y^{\log}))_{(X \times_S X)^\sim}$. We also apply the associativity, Lemma 3.3.7, by taking $A \rightarrow (X \times_S X)^\sim \leftarrow (Y \times_S Y)^\sim \leftarrow Y$ as $V \rightarrow X \leftarrow X' \leftarrow W'$. Then, the composition via lower left is also equal to the same map. \square

We establish an important property that the localized intersection product with the log diagonal is independent of the boundary, in Proposition 4.3.5 below. We begin with preliminary computations. Let X_1 be a regular divisor of X such that the intersection $D_1 = X_1 \cap D$ is a divisor of X_1 with simple normal crossings. Let $\mathcal{D} = (D_i)_{i \in I}$ be the family of irreducible components of D and we consider the family $\mathcal{D}_1 = (D_i \cap X_1)_{i \in I}$ of smooth divisors of X_1 . We identify the log product $(X_1 \times_S X_1)^\sim$ with respect to \mathcal{D}_1 with the inverse image of $X_1 \times_S X_1$ by the canonical map $(X \times_S X)^\sim \rightarrow X \times_S X$. The sum $D' = D \cup X_1$ is a divisor of X with simple normal crossings.

We consider the log product $(X \times_S X)^\sim$ with respect to D' , the log diagonal map $\delta: X \rightarrow (X \times_S X)^\sim$ and the canonical map $(X \times_S X)^\sim \rightarrow (X \times_S X)^\sim$. The inverse image E of $(X_1 \times_S X_1)^\sim \subset (X \times_S X)^\sim$ by $(X \times_S X)^\sim \rightarrow (X \times_S X)^\sim$ is a \mathbf{G}_m -torsor over $(X_1 \times_S X_1)^\sim$ by Lemma 1.3.2. The pull-back of the \mathbf{G}_m -torsor E by the log diagonal $X_1 \rightarrow (X_1 \times_S X_1)^\sim$ is trivialized by the restriction to X_1 of the log diagonal $X \rightarrow (X \times_S X)^\sim$. We identify $E \times_{(X_1 \times_S X_1)^\sim} X_1$ with \mathbf{G}_{m, X_1} and the restriction of the log diagonal $X_1 \rightarrow E \times_{(X_1 \times_S X_1)^\sim} X_1$ with the 1-section 1_{X_1} . They are summarized in the Cartesian diagram

$$(4.3.4.1) \quad \begin{array}{ccccc} \mathbf{G}_{m, X_1} & \longrightarrow & E & \longrightarrow & (X \times_S X)^\sim \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & (X_1 \times_S X_1)^\sim & \longrightarrow & (X \times_S X)^\sim. \end{array}$$

Lemma 4.3.4. — *Let X_1 be a regular irreducible divisor of X such that $D_1 = X_1 \cap D$ is a divisor of X_1 with simple normal crossings. Let $(X \times_S X)^\sim$ denote the log product with respect to $D' = D \cup X_1$ and let $\Sigma'_{X/S}$ be the closed subscheme of X defined by the annihilator ideal $\text{Ann } \Omega_{X/S}^n(\log D')$.*

Let A be a closed subscheme of $E \subset (X \times_S X)^\sim$ satisfying the condition (A') after Definition 4.3.2. We identify $E \times_{(X_1 \times_S X_1)^\sim} X_1$ with \mathbf{G}_{m, X_1} and the section $X_1 \rightarrow E \times_{(X_1 \times_S X_1)^\sim} X_1$ defined by the restriction of the log diagonal with $1_X: X_1 \rightarrow \mathbf{G}_{m, X_1}$. Then, the intersection product

$$(4.3.4.2) \quad ((, \Delta_X^{\log}))_{(X \times_S X)^\sim} : G(A) \rightarrow G(\delta^{-1}(A) \cap \Sigma'_{X/S})$$

with the log diagonal satisfies the following

1. *Assume that X_1 is flat over S . Then the composition $A \rightarrow E \rightarrow (X_1 \times_S X_1)^\sim$ satisfies the condition (A) in Lemma 4.3.1 and the localized intersection product*

$$(4.3.4.3) \quad ((\Delta_{X_1}^{\log},))_{(X_1 \times_S X_1)^\sim} : G(A) \rightarrow G(A \cap \mathbf{G}_{m, \Sigma_{X_1/S}})$$

with the log diagonal $X_1 \rightarrow (X_1 \times_S X_1)^\sim$ is defined. Further, we have $\Sigma_{X_1/S} = \Sigma'_{X/S} \cap X_1$ and the map (4.3.4.2) is induced by the composition

$$(4.3.4.4) \quad G(A) \xrightarrow{((\Delta_{X_1}^{\log}, \cdot)_{(X_1 \times_S X_1)^\sim})} G(A \cap \mathbf{G}_{m, \Sigma_{X_1/S}}) \xrightarrow{(\cdot, 1_{X_1})_{\mathbf{G}_{m, X_1}}} G(\delta^{-1}(A) \cap \Sigma_{X_1/S}).$$

2. Assume that X_1 is a subscheme of the closed fiber X_F . Then, we have $X_1 \subset \Sigma'_{X/S}$ and the map on the graded quotients $((\cdot, \Delta_X^{\log}))_{(X \times_S X)^\sim} : \mathrm{Gr}_{\bullet}^F G(A) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma'_{X/S})$ induced by (4.3.4.2) is induced by the usual intersection product

$$(4.3.4.5) \quad G(A) \xrightarrow{(\cdot, 1_{X_1})_E} G(\delta^{-1}(A) \cap X_1).$$

Proof. — 1. By Lemma 4.2.6, we have an exact sequence $0 \rightarrow \Omega_{X_1/S}^1(\log D_1) \rightarrow \Omega_{X/S}^1(\log D') \otimes \mathcal{O}_{X_1} \rightarrow \mathcal{O}_{X_1} \rightarrow 0$. This defines an isomorphism $\Omega_{X_1/S}^{n-1}(\log D_1) \rightarrow \Omega_{X/S}^n(\log D') \otimes \mathcal{O}_{X_1}$. Hence, we have $\Sigma_{X_1/S} = \Sigma'_{X/S} \cap X_1$. Thus the condition (A') for $A \rightarrow (X \times_S X)^\sim$ implies the condition (A) for $A \rightarrow (X_1 \times_S X_1)^\sim$.

By Lemma 4.1.6.1, E is a Cartier divisor of $(X \times_S X)^\sim$ and we have $(E, \Delta_X^{\log})_{(X \times_S X)^\sim} = [\Delta_{X_1}^{\log}]$. We apply Lemma 3.3.4 to the diagram

$$\begin{array}{ccc} A & \longleftarrow & A \cap X_1 \\ \downarrow & & \downarrow \\ E & \longleftarrow & X_1 \\ \downarrow & & \downarrow \\ (X \times_S X)^\sim & \longleftarrow & X \end{array}$$

Then, the map (4.3.4.2) is the same as $((\cdot, 1_{X_1}))_E : G(A) \rightarrow G(\delta^{-1}(A) \cap \Sigma_{X_1/S})$. Further we apply Lemma 3.3.6 by taking the upper line in the Cartesian diagram

$$\begin{array}{ccccc} E & \longleftarrow & \mathbf{G}_{m, X_1} & \longleftarrow & X_1 \\ \downarrow & & \downarrow & & \\ (X_1 \times_S X_1)^\sim & \longleftarrow & X_1 & & \end{array}$$

as $X \leftarrow W \leftarrow W'$ on the lower line in the diagram of Lemma 3.3.6. Then the map $((\cdot, 1_{X_1}))_E : G(A) \rightarrow G(\delta^{-1}(A) \cap \Sigma_{X_1/S})$ is equal to the composition of $((\cdot, [\mathbf{G}_{m, X_1}]))_E : G(A) \rightarrow G(A \cap \mathbf{G}_{m, \Sigma_{X_1/S}})$ with the usual intersection product $(\cdot, X_1)_{\mathbf{G}_{m, X_1}} : G(A \cap \mathbf{G}_{m, \Sigma_{X_1/S}}) \rightarrow G(\delta^{-1}(A) \cap \Sigma_{X_1/S})$. Since E is flat over $(X_1 \times_S X_1)^\sim$, the first map $((\cdot, [\mathbf{G}_{m, X_1}]))_E : G(A) \rightarrow G(A \cap \mathbf{G}_{m, \Sigma_{X_1/S}})$ is the same as (4.3.4.3)

2. We show $X_1 \subset \Sigma'_{X/S}$. Since X_1 is assumed irreducible, it suffices to show that the generic point ξ_1 of X_1 is contained in $\Sigma'_{X/S}$. It suffices to consider the case where the complement $X \setminus \Sigma'_{X/S}$ is dense. Then, we have an exact sequence $0 \rightarrow \mathcal{O}_{X_F} \rightarrow$

$\Omega_{X/S}^1(\log D') \rightarrow \Omega_{X/S}^1(\log D'/\log F) \rightarrow 0$ on a neighborhood ξ_1 . By the assumption that the complement $X \setminus \Sigma'_{X/S}$ is dense, the free part of the module $\Omega_{X/S}^1(\log D')_{\xi_1}$ over the discrete valuation ring \mathcal{O}_{X,ξ_1} has rank $n - 1$. Since it is not torsion free, the smallest number of generators is n . Hence ξ_1 is contained in $\Sigma'_{X/S}$ as required.

We apply Lemma 3.3.5 by taking $E \rightarrow (X \times_S X)^\approx \leftarrow \Delta_X^{\log}$ as $V \rightarrow X \leftarrow W$. Then, the localized intersection product $((, \Delta_X^{\log}))_{(X \times_S X)^\approx}$ is equal to the usual intersection product $(, ((E, \Delta_X^{\log}))_{(X \times_S X)^\approx})_E$. By the excess intersection formula (3.3.5.1), we have $((E, \Delta_X^{\log}))_{(X \times_S X)^\approx} = c_0(M'_{X/(X \times_S X)^\approx, E}) \cap [X_1]$. By Lemma 4.2.7.1, the right hand side is equal to 1_{X_1} . Hence, the assertion follows. \square

Proposition 4.3.5. — *Let X be a regular flat separated scheme of finite type over S and $D \subset X$ be a divisor with simple normal crossings. We put $U = X \setminus D$. Let A be a closed subscheme of $(X \times_S X)^\sim$ satisfying the condition (A') after Definition 4.3.2 and the following condition:*

- (B) *For each irreducible component D_i of D , we regard \mathbf{G}_{m, D_i} as a closed subscheme of $(X \times_S X)^\sim$ as in (4.3.4.1). Then, there exists an integer $l_i \geq 1$ such that the intersection $A \cap \mathbf{G}_{m, D_i}$ is supported on the subscheme $\mu_{l_i, D_i} \subset \mathbf{G}_{m, D_i}$.*

We put $A^\circ = A \cap (U \times_S U)$. Then, there exists a unique map $\mathrm{Gr}_{\bullet}^F G(A^\circ) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S})$, also denoted by $((, \Delta_X^{\log}))$, that makes the diagram

$$(4.3.5.1) \quad \begin{array}{ccc} \mathrm{Gr}_{\bullet}^F G(A) & \xrightarrow{((, \Delta_X^{\log}))} & \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S}) \\ \text{restriction} \downarrow & \nearrow & \\ \mathrm{Gr}_{\bullet}^F G(A^\circ) & & \end{array}$$

commutative.

Proof. — For each irreducible component D_i of D , let E_i be its inverse image by either of the two projections $(X \times_S X)^\sim \rightarrow X$ and we put $A_i = A \cap E_i$. By the exact sequence $\bigoplus_i \mathrm{Gr}_{\bullet}^F G(A_i) \rightarrow \mathrm{Gr}_{\bullet}^F G(A) \rightarrow \mathrm{Gr}_{\bullet}^F G(A^\circ) \rightarrow 0$, it suffices to show that the composition of

$$(4.3.5.2) \quad \mathrm{Gr}_{\bullet}^F G(A_i) \longrightarrow \mathrm{Gr}_{\bullet}^F G(A) \xrightarrow{((, \Delta_X^{\log}))} \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S})$$

is the zero map for each i .

First, we consider the case where D_i is flat over S . By Lemma 4.3.4.1, the composition of (4.3.5.2) is induced by the composition of

$$G(A_i) \xrightarrow{((, \Delta_{D_i}^{\log}))_{(D_i \times_S D_i)^\sim}} G(A_i \cap \mathbf{G}_{m, \Sigma_{D_i/S}}) \xrightarrow{((, 1_{D_i})_{\mathbf{G}_{m, D_i}})} G(\delta^{-1}(A_i) \cap \Sigma_{D_i/S}).$$

Hence, it suffices to show the second map is the zero-map. By the assumption (B), the intersection $A_i \cap \mathbf{G}_{m, D_i}$ is a closed subscheme of μ_{l_i, D_i} . Hence, if the characteristic of \mathbf{K} is $p > 0$, the assertion follows from Lemma 4.3.6 below. If the characteristic of \mathbf{K} is 0, the generic fiber $\Sigma_{D_i/S} \times_S \text{Spec } \mathbf{K}$ is empty. Hence it also follows from Lemma 4.3.6 below.

Next, we consider the case where D_i is a subscheme of the closed fiber \mathbf{X}_F . In this case, by Lemma 4.3.4.2, the composition of (4.3.5.2) is induced by the composition of

$$G(A_i) \xrightarrow{(\cdot, \Delta_{D_i}^{\log})_{(D_i \times_F D_i)^\sim}} G(A_i \cap \mathbf{G}_{m, D_i}) \xrightarrow{(\cdot, 1_{D_i})_{\mathbf{G}_{m, D_i}}} G(A_i \cap \Delta_{D_i}^{\log}).$$

Hence, it suffices to show the second map is the zero-map. By the assumption (B), the assertion in this case is also reduced to the following Lemma 4.3.6.

Lemma 4.3.6. — *Let D be a noetherian scheme over \mathbf{F}_p and $l \geq 1$ be an integer. Let A be a closed subscheme of $\mu_{l, D} \subset \mathbf{G}_{m, D}$. Then, the intersection product*

$$(\cdot, 1_D)_{\mathbf{G}_{m, D}} : G(A) \rightarrow G(A \cap 1_D)$$

with the unit section $1_D \subset \mu_{l, D}$ is the zero map.

Proof. — By replacing l by its p -part l' , we may assume that l is a power of the characteristic $p > 0$ of \mathbf{F} since $\mu_{l', D}$ is a closed and open subscheme of $\mu_{l, D}$ and has the same intersection with the 1-section. Further, since the closed immersion $A \cap 1_D \rightarrow A$ defined by a nilpotent ideal induces an isomorphism $G(A \cap 1_D) \rightarrow G(A)$, we may assume A is a closed subscheme of 1_D . For a coherent \mathcal{O}_{1_D} -module \mathcal{F} , we have

$$\begin{aligned} ([\mathcal{F}], 1_D)_{\mathbf{G}_{m, D}} &= [\mathcal{T}or_0^{\mathcal{O}_{\mathbf{G}_{m, D}}}(\mathcal{F}, \mathcal{O}_{1_D})] - [\mathcal{T}or_1^{\mathcal{O}_{\mathbf{G}_{m, D}}}(\mathcal{F}, \mathcal{O}_{1_D})] \\ &= [\mathcal{F}] - [\mathcal{F}] = 0. \end{aligned}$$

Hence the assertion follows. □

The following Lemma, analogous to Lemma 4.3.4, will be used in the proof of Proposition 6.1.1, which in turn will be used in the proof of a blow-up formula Proposition 6.2.1.

Lemma 4.3.7. — *Let X_1 be a regular divisor of \mathbf{X} such that $D_1 = X_1 \cap D$ is a divisor of X_1 with simple normal crossings. Let $(\mathbf{X} \times_S \mathbf{X})^\sim$ denote the log product with respect to D and let $\Sigma_{X/S}$ be the closed subscheme of \mathbf{X} defined by the annihilator ideal $\text{Ann } \Omega_{X/S}^n(\log D)$. We regard $(X_1 \times_S X_1)^\sim$ as a closed subscheme of $(\mathbf{X} \times_S \mathbf{X})^\sim$ as in (4.3.4.1).*

Let A be a closed subscheme of $(X_1 \times_S X_1)^\sim$ satisfying the condition (A') after Definition 4.3.2 and let $i: (X_1 \times_S X_1)^\sim \rightarrow (\mathbf{X} \times_S \mathbf{X})^\sim$ be the closed immersion. Then, the map on the graded pieces

$$(4.3.7.1) \quad \left((\cdot, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} : \mathrm{Gr}_{\bullet}^F G(A) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S})$$

induced by the intersection product with the log diagonal is computed as follows.

1. Assume X_1 is flat over S . Then, we have $\Sigma_{X_1/S} = \Sigma_{X/S} \cap X_1$ and the map (4.3.7.1) is the composition of

$$(4.3.7.2) \quad \begin{array}{ccc} \mathrm{Gr}_{\bullet}^F G(A) & \xrightarrow{((\cdot, \Delta_{X_1}^{\log}))_{(X_1 \times_S X_1)^\sim}} & \mathrm{Gr}_{\bullet-(n-1)}^F G(\delta^{-1}(A) \cap \Sigma_{X_1/S}) \\ & \xrightarrow{-i_* \circ c_1(N_{X_1/X})} & \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X_1/S}). \end{array}$$

2. Assume that X_1 is a subscheme of the closed fiber X_F . Then, the map on the graded pieces induced by (4.3.7.1) is the composition of

$$(4.3.7.3) \quad \begin{array}{ccc} \mathrm{Gr}_{\bullet}^F G(A) & \xrightarrow{((\cdot, \Delta_{X_1}^{\log}))_{(X_1 \times_F X_1)^\sim}} & \mathrm{Gr}_{\bullet-(n-1)}^F G(\delta^{-1}(A)) \\ & \xrightarrow{-c_1([N_{S/S} \otimes \mathcal{O}_{X_1} \rightarrow N_{X_1/X}])_{X_1 \cap \Sigma_{X/S}}} & \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S}). \end{array}$$

Proof. — 1. We have shown the equality $\Sigma_{X_1/S} = \Sigma_{X/S} \cap X_1$ at the beginning of the proof of Lemma 4.3.4.1. By Lemma 4.1.6.1, the immersion $(X_1 \times_S X_1)^\sim \rightarrow (X \times_S X)^\sim$ is a regular immersion of codimension 2. Since the excess conormal sheaf $\mathrm{Ker}((\mathrm{pr}_1^* N_{X_1/X} \oplus \mathrm{pr}_2^* N_{X_1/X})|_{X_1} \rightarrow N_{X_1/X})$ is isomorphic to $N_{X_1/X}$, we have $((X_1 \times_S X_1)^\sim, \Delta_X^{\log})_{(X \times_S X)^\sim} = -c_1(N_{X_1/X}) \cap \Delta_{X_1}^{\log}$. We apply Lemma 3.3.4 by taking $(X \times_S X)^\sim \supset (X_1 \times_S X_1)^\sim, X \supset X_1$ as $X \supset X', W \supset W'$. Then, the map (4.3.7.1) is induced by $((\cdot, -c_1(N_{X_1/X}) \cap \Delta_{X_1}^{\log}))_{(X_1 \times_S X_1)^\sim} : G(A) \rightarrow G(\delta^{-1}(A) \cap \Sigma_{X_1/S})$. Further, it is equal to the composition of (4.3.7.2).

2. We apply Lemma 3.3.5 by taking $(X_1 \times_S X_1)^\sim \rightarrow (X \times_S X)^\sim \leftarrow \Delta_X^{\log}$ as $V \rightarrow X \leftarrow W$. Then, the map (4.3.7.1) is equal to the usual intersection product

$$\left((\cdot, ((X_1 \times_S X_1)^\sim, \Delta_X^{\log}))_{(X \times_S X)^\sim} \right)_{(X_1 \times_S X_1)^\sim}$$

with $((X_1 \times_S X_1)^\sim, \Delta_X^{\log})_{(X \times_S X)^\sim} = -c_1(M'_{X/(X \times_S X)^\sim, (X_1 \times_S X_1)^\sim}) \cap [X_1]$. By Lemma 4.2.7.2, the right hand side is equal to $-c_1([N_{S/S} \otimes \mathcal{O}_{X_1} \rightarrow N_{X_1/X}])_{X_1 \cap \Sigma_{X/S}} \cap \Delta_{X_1}^{\log}$. Hence, the assertion follows. \square

If $(X \setminus D)_F = \emptyset$, we have an alternative construction. Let $A \subset (X \times_S X)^\sim \subset (X \times_S X)^\sim$ be a closed subscheme satisfying the condition (A). Then, a localized intersection product

$$(4.3.7.4) \quad \left((\cdot, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} : G(A) \rightarrow G(\delta^{-1}(A) \cap \Sigma_{X/S}).$$

is defined similarly as (4.3.2.1), by Lemma 4.2.9. We show that it gives the same invariants.

Proposition 4.3.8. — *Let X be a regular flat separated scheme of finite type over S and $D \subset X$ be a divisor with simple normal crossings. We assume $(X \setminus D)_F = \emptyset$.*

Then, for a closed subscheme A of $(X \times_S X)^\sim$ satisfying the condition (A), we have an equality

$$\left((\cdot, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} = \left((\cdot, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim}$$

of maps $\mathrm{Gr}_{\bullet}^F G(A) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S})$.

Proof. — Let $W \subset A$ be an integral closed subscheme. If W is a subscheme of Δ_X^{\log} , let $\pi : W' \rightarrow W$ denote the identity of W . If not, let $\pi : W' \rightarrow W$ be the blow-up at $W' \cap \Delta_X^{\log}$. Since $G(A)$ is generated by the classes $\pi_*[W']$ for integral closed subschemes W of A , it suffices to show the equality $\left((W', \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} = \left((W', \Delta_X^{\log}) \right)_{(X \times_S X)^\sim}$.

We put $T' = W' \times_{(X \times_S X)^\sim} \Delta_X^{\log}$, $Z = \Sigma_{X/S}$, $d = \dim W_K + 1$ and let $\varphi : T' \rightarrow X$ denote the canonical map. Then, by the excess intersection formula [26, Theorem 3.4.3], we have

$$\begin{aligned} \left((W', \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} &= \pi_* \left((-1)^d c_{dZ_{T'}}^{T'} \left(M'_{X/(X \times_S X)^\sim, W'} \right) \cap [T'] \right), \\ \left((W', \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} &= \pi_* \left((-1)^d c_{dZ_{T'}}^{T'} \left(M'_{X/(X \times_S X)^\sim, W'} \right) \cap [T'] \right). \end{aligned}$$

By the distinguished triangles

$$\begin{aligned} \rightarrow \mathcal{O}_{X_F} &\rightarrow L_{X/S}(\log D) \rightarrow L_{X/S}(\log D / \log F) \rightarrow 0, \\ \rightarrow M'_{X/(X \times_S X)^\sim, W'} &\rightarrow L\varphi^* L_{X/S}(\log D) \rightarrow N_{T'/W'} \rightarrow, \\ \rightarrow M'_{X/(X \times_S X)^\sim, W'} &\rightarrow L\varphi^* L_{X/S}(\log D / \log F) \rightarrow N_{T'/W'} \rightarrow, \end{aligned}$$

and by $c_1(\mathcal{O}_{X_F}) \cap [T'] = 0$, we obtain an equality $c_{dZ_{T'}}^{T'} \left(M'_{X/(X \times_S X)^\sim, W'} \right) \cap [T'] = c_{dZ_{T'}}^{T'} \left(M'_{X/(X \times_S X)^\sim, W'} \right) \cap [T']$. Thus the assertion follows. \square

The following Proposition shows that the localized intersection product does not depend on the choice of the base S .

Proposition 4.3.9. — *Let X be a regular flat separated scheme of finite type over S and $D \subset X$ be a divisor with normal crossings. Assume that K is a finite extension of a complete discrete valuation field K' and put $\mathcal{O}_{K'} = K' \cap \mathcal{O}_K$. Let A be a closed subscheme of $(X \times_S X)^\sim$ satisfying the condition (A') after Definition 4.3.2 with respect to $S' = \mathrm{Spec} \mathcal{O}_{K'}$. Namely, we assume that $\delta^{-1}(A) \cap \Sigma_{X/S'}$ is contained in X_F .*

Then, we have an inclusion $\Sigma_{X/S} \subset \Sigma_{X/S'}$ and an equality

$$\left((\cdot, \Delta_X^{\log}) \right)_{(X \times_{S'} X)^\sim} = \left((\cdot, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim}$$

of maps $\mathrm{Gr}_{\bullet}^F G(A) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\delta^{-1}(A) \cap \Sigma_{X/S'})$.

Proof. — The proof is similar to that of Proposition 4.3.8. Let $W \subset A$ be an integral closed subscheme and let $\pi : W' \rightarrow W$ be as in the proof of Proposition 4.3.8. We put $T' = W' \times_{(X \times_S X)^\sim} \Delta_X^{\log}$, $Z' = \Sigma_{X/S'}$, and $d = \dim W_K + 1$. Then, by the excess intersection formula [26, Theorem 3.4.3], we have

$$\begin{aligned} ((W', \Delta_X^{\log}))_{(X \times_S X)^\sim} &= \pi_*((-1)^d c_{d, Z_{T'}}^{T'}(M'_{X/(X \times_S X)^\sim, W'}) \cap [T']), \\ ((W', \Delta_X^{\log}))_{(X \times_{S'} X)^\sim} &= \pi_*((-1)^d c_{d, Z_{T'}}^{T'}(M'_{X/(X \times_{S'} X)^\sim, W'}) \cap [T']). \end{aligned}$$

Using the distinguished triangle $\Omega_{S/S'}^1 \otimes^L \mathcal{O}_X \rightarrow L_{X/S'}(\log D) \rightarrow L_{X/S}(\log D) \rightarrow$, we complete the proof similarly as in the proof of Proposition 4.3.8. \square

5. Invariants of wild ramification

We keep fixing a complete discrete valuation field K with perfect residue field F of characteristic > 0 and $S = \text{Spec } \mathcal{O}_K$.

In Section 5.3, we define invariants of wild ramification for a finite étale morphism $f : V \rightarrow U$ of regular flat separated schemes of finite type over S , such that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K (Definition 2.4.1). The definition uses the localized intersection product with the log diagonal constructed in Section 4.3. The definition is extended to cover the case where U and V are not assumed regular at the end of Section 6.2 as a consequence of the excision formula, Theorem 6.2.2. On the counterpart for a finite étale morphism $f : V \rightarrow U$ of smooth separated schemes of finite type over F defined in [27], we also state some complements. In Section 5.4, we establish elementary properties of the invariants of wild ramification defined in Section 5.3. We define the logarithmic different and the Lefschetz classes and derive their basic properties analogous to the classical ones.

Before defining the invariants in the general case, we define and compute the logarithmic different and the Lefschetz class using regular schemes containing U and V as the complements of divisors with simple normal crossings in Section 5.1. We introduce the target groups where the invariants of wild ramification take values as certain projective limits with respect to the system of compactifications in Section 5.2. We also introduce in Theorem 5.3.9 a variant that will be used in the case where K is of characteristic 0, in Section 7.5.

5.1. Logarithmic different and the Lefschetz class. — Let Y be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and $V \subset Y$ be the complement of a divisor E with simple normal crossings. Let $f : V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S . We consider the family $\mathcal{E} = (E_i)_{i \in I}$ of irreducible components of E and we assume that the closed subscheme $\Sigma_{V/U}^{\mathcal{E}} Y \subset Y$ (Definition 2.1.2.1) is supported on the closed fiber.

By the assumption that $V \rightarrow U$ is finite étale, the diagonal Δ_V is an open and closed subscheme of $V \times_U V$. The closure A of $(V \times_U V) \setminus \Delta_V$ in the log product $(Y \times_S Y)^\sim = (Y \times_S Y)^\sim_{\mathcal{E}}$ satisfies the condition (A') after Definition 4.3.2 since $\Sigma_{V/U}^{\mathcal{E}}$ is the inverse image $\delta^{-1}(A)$ by the log diagonal $\delta: Y \rightarrow (Y \times_S Y)^\sim$.

We also assume that there exists a separated scheme X of finite type over S containing U as the complement of a Cartier divisor and that $f: V \rightarrow U$ is extended to a morphism $\bar{f}: Y \rightarrow X$ over S satisfying $\bar{f}^{-1}(U) = V$. Then, by Lemma 1.3.2.2, the same A satisfies the condition (B) in Proposition 4.3.5. Thus, by applying the map (4.3.5.1), we obtain

$$\left((V \times_U V) \setminus \Delta_V, \Delta_V^{\log} \right)_{(Y \times_S Y)^\sim} \in F_0G(\Sigma_{V/U}^{\mathcal{E}} Y).$$

Definition 5.1.1. — *Let Y be a regular flat separated scheme of finite type over S and $V \subset Y$ be the complement of a divisor E of Y with simple normal crossings. Let $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S .*

We assume that the closed subset $\Sigma_{V/U}^{\mathcal{E}}$ (Definition 2.1.2.1) defined for the family $\mathcal{E} = (E_i)_{i \in I}$ of irreducible component of E is supported on the closed fiber. We also assume that there exists a separated scheme X of finite type over S containing U as the complement of a Cartier divisor and that $f: V \rightarrow U$ is extended to a morphism $\bar{f}: Y \rightarrow X$ over S satisfying $\bar{f}^{-1}(U) = V$.

Then, we define the logarithmic different $D_{V/U, Y}^{\log} \in F_0G(\Sigma_{V/U}^{\mathcal{E}} Y)$ by

$$D_{V/U, Y}^{\log} = \left((V \times_U V) \setminus \Delta_V, \Delta_V^{\log} \right)_{(Y \times_S Y)^\sim}.$$

We compute the logarithmic different explicitly using regular models. It will imply in particular (Corollary 5.1.3) that if $U = \text{Spec } L$ and $V = \text{Spec } M$ for finite separable extensions $L \subset M$ of K , we have

$$(5.1.1.1) \quad D_{V/U, Y}^{\log} = \text{length}_{\mathcal{O}_M} \Omega_{\mathcal{O}_M/\mathcal{O}_L}^1(\log/\log) = \text{length}_{\mathcal{O}_M} \Omega_{\mathcal{O}_M/\mathcal{O}_L}^1 - (e_{M/L} - 1)$$

in $\mathbf{Z} = F_0G(\text{Spec } \mathcal{O}_M/\mathfrak{m}_M)$. Recall that $\text{length}_{\mathcal{O}_M} \Omega_{\mathcal{O}_M/\mathcal{O}_L}^1$ is the classical different.

We consider a Cartesian diagram

$$(5.1.1.2) \quad \begin{array}{ccc} V & \longrightarrow & Y \\ f \downarrow & & \downarrow \bar{f} \\ U & \longrightarrow & X \end{array}$$

of regular flat separated schemes of finite type over S . Suppose that $f: V \rightarrow U$ is finite étale and that $U = X \setminus D$ and $V = Y \setminus E$ are the complements of divisors D and E with simple normal crossings respectively. Using the diagram (5.1.1.2), the logarithmic different $D_{V/U, Y}^{\log}$ can be computed as follows.

We put $n = \dim X_K + 1 = \dim Y_K + 1$. We consider the map $\bar{f}^* \Omega_{X/S}^1(\log D) \rightarrow \Omega_{Y/S}^1(\log E)$ of coherent \mathcal{O}_Y -modules. Let $\Sigma = \Sigma_{Y/X} \subset Y$ be the closed subscheme

defined by the annihilator $\mathcal{I}_\Sigma = \text{Ann}(\text{Coker}(\bar{f}^* \Omega_{X/S}^1(\log D) \rightarrow \Omega_{Y/S}^1(\log E))) \subset \mathcal{O}_Y$. Since Y is regular, there exist a locally free \mathcal{O}_Y -module \mathcal{V} and a surjection $\mathcal{V} \rightarrow \Omega_{Y/S}^1(\log E)$ by [15, Corollaire 2.2.7.1]. Hence, the localized Chern class $c_n(\Omega_{Y/S}^1(\log E) - \bar{f}^* \Omega_{X/S}^1(\log D))_\Sigma \cap [Y] \in F_0 G(\Sigma_{Y/X})$ is defined in [27, (3.24)] (cf. (3.2.3.2)).

The image of the logarithmic different $D_{V/U,Y}^{\log} \in F_0 G(\Sigma_{V/U}^\mathcal{E} Y)$ in $F_0 G(\Sigma_{Y/X})$ is computed using the localized Chern class $c_n(\Omega_{Y/S}^1(\log E) - \bar{f}^* \Omega_{X/S}^1(\log D))_{\Sigma_{Y/X}} \cap [Y]$ as follows.

Proposition 5.1.2. — *Let X and Y be regular flat separated schemes of finite type over S and let $U = X \setminus D$ and $V = Y \setminus E$ be the complements of divisors $D \subset X$ and $E \subset Y$ with simple normal crossings. Let $\bar{f}: Y \rightarrow X$ be a morphism over S such that $\bar{f}^{-1}(U) = V$ and the restriction $f = \bar{f}|_V: V \rightarrow U$ is finite étale.*

We assume that the support $\Sigma = \Sigma_{Y/X} \subset Y$ of the cokernel $\Omega_Y^1(\log E)/\bar{f}^ \Omega_X^1(\log D)$ is supported on the closed fiber Y_F . We also assume that there exists a dense open subscheme of X smooth over S .*

Then, we have $\Sigma_{V/U}^\mathcal{E} Y \subset \Sigma_{Y/X}$ and, for $n = \dim X_K + 1$,

$$(5.1.2.1) \quad D_{V/U,Y}^{\log} = (-1)^n c_n(\Omega_{Y/S}^1(\log E) - \bar{f}^* \Omega_{X/S}^1(\log D))_\Sigma \cap ([Y])$$

in $F_0 G(\Sigma_{Y/X})$.

Proof. — We consider the log products $(X \times_S X)^\sim$ and $(Y \times_S Y)^\sim$ with respect to D and E respectively and will apply Proposition 3.4.3 to the commutative diagram

$$(5.1.2.2) \quad \begin{array}{ccc} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & (Y \times_S Y)^\sim \\ \bar{f} \downarrow & & \downarrow (\bar{f} \times \bar{f})^\sim \\ X & \xrightarrow{\delta_X} & (X \times_S X)^\sim \end{array}$$

where the upper square and the tall rectangle are Cartesian. We put $(Y \times_X Y)^\sim = (Y \times_S Y)^\sim \times_{(X \times_S X)^\sim} \Delta_X^{\log}$. Since the cokernel $\text{Coker}(\bar{f}^* \Omega_{X/S}^1(\log D) \rightarrow \Omega_{Y/S}^1(\log E))$ is the conormal sheaf $N_{Y/(Y \times_X Y)^\sim}$, the restriction of the log diagonal map $\delta_Y: Y \rightarrow (Y \times_X Y)^\sim$ to the complement $\tilde{V} = Y \setminus \Sigma$ is an open immersion. Hence the complement $A = (Y \times_X Y)^\sim \setminus \Delta_{\tilde{V}}^{\log}$ is a closed subset of $(Y \times_S Y)^\sim$ such that $\delta^{-1}(A) = \Sigma_{Y/X}$. Since A contains $(V \times_U V) \setminus \Delta_V$ as a subset, we have an inclusion $\Sigma_{V/U}^\mathcal{E} Y \subset \Sigma_{Y/X}$.

We define a bounded complex \mathcal{C} of $\mathcal{O}_{(Y \times_S Y)^\sim}$ -modules fitting in the distinguished triangle $\rightarrow \mathcal{C} \rightarrow L(\bar{f} \times \bar{f})^* \mathcal{O}_{\Delta_X^{\log}} \rightarrow \mathcal{O}_{\Delta_Y^{\log}} \rightarrow$ as in (3.4.0.3). We have $A \cap (V \times_S V) = (V \times_U V) \setminus \Delta_V$ and the restriction map $F_n G(A) \rightarrow F_n G((V \times_U V) \setminus \Delta_V)$ sends $[\mathcal{C}]$

to $[(V \times_U V) \setminus \Delta_V]$. Hence, by Proposition 4.3.5, the image of the logarithmic different $D_{V/U, Y}^{\log}$ by the map $F_0G(\Sigma_{V/U}^{\tilde{E}} Y) \rightarrow F_0G(\Sigma_{Y/X})$ is the localized intersection product $(([C], \Delta_Y^{\log}))_{(Y \times_S Y)^\sim}$.

In order to apply Proposition 3.4.3 to the diagram (5.1.2.2), we check that its assumption is satisfied. For a point y of the closed fiber of Y , we have an open neighborhood V' of y , an open neighborhood U' of $\tilde{f}(y)$ and a Cartesian diagram

$$\begin{array}{ccc} V' & \longrightarrow & Q \\ \downarrow & & \downarrow \\ U' & \longrightarrow & P \end{array}$$

as in Lemma 4.1.3. Then, in the diagram

$$\begin{array}{ccccc} V' & \longrightarrow & (V' \times_S Y)^\sim & \longrightarrow & (Q \times_S Y)^\sim \\ \downarrow & & \downarrow & & \downarrow \\ U' & \longrightarrow & (U' \times_S X)^\sim & \longrightarrow & (P \times_S X)^\sim, \end{array}$$

the right square is Cartesian and the horizontal arrows in the right square are regular immersions of codimension 1. The compositions of the horizontal arrows are both sections of smooth morphisms of relative dimension n and hence are regular immersions of codimension n . Thus the condition (3.4.3.3) is satisfied.

By the assumption that there exists a dense open subscheme of X smooth over S , the excess conormal complexes $M'_{Y/(Y \times_S Y)^\sim, Y} = M_{Y/(Y \times_S Y)^\sim} = L_{Y/S}(\log E)$ and $M'_{X/(X \times_S X)^\sim, Y} = Lf^*M_{X/(X \times_S X)^\sim} = Lf^*L_{X/S}(\log D)$ are quasi-isomorphic to $\Omega_{Y/S}^1(\log E)$ and $f^*\Omega_{X/S}^1(\log D)$ respectively. Applying Proposition 3.4.3.3 to the diagram

$$\begin{array}{ccc} Y & \longrightarrow & (Y \times_S Y)^\sim \\ \downarrow & & \downarrow \\ X & \longrightarrow & (X \times_S X)^\sim \end{array}$$

and to $T = W = Y$, we obtain

$$\begin{aligned} & ((f \times f)^*[\Delta_U] - [\Delta_V], \Delta_Y^{\log})_{(Y \times_S Y)^\sim} \\ &= [L\Lambda^n f^* \Omega_{X/S}^1(\log D) \rightarrow L\Lambda^n \Omega_{Y/S}^1(\log E)] \end{aligned}$$

in $F_0G(\Sigma_{Y/X})$. The right hand side is equal to the image of the localized Chern class $c_n(\Omega_{Y/S}^1(\log E) - f^*\Omega_{X/S}^1(\log D))_\Sigma \cap ([Y])$ in $F_0G(\Sigma_{Y/X})$ by Proposition 3.2.4. \square

Corollary 5.1.3. — *Let $L \subset M$ be finite separable extensions of K and put $U = \text{Spec } L$ and $V = \text{Spec } M$. Let X and Y be the normalizations of S in U and V respectively. Then, we have $D_{V/U, Y}^{\log} = \text{length}_{\mathcal{O}_M} \Omega_{\mathcal{O}_M/\mathcal{O}_L}^1(\log / \log) \in \mathbf{Z}$ (5.1.1.1).*

For an automorphism of a scheme over S , we define the Lefschetz class as the intersection product of the graph with the log diagonal as follows.

Definition 5.1.4. — *Let X be a regular flat separated scheme of finite type over S and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. Let σ be an automorphism of X over S such that $\sigma(U) = U$ and $U^\sigma = \emptyset$.*

1. *Let $\tilde{\Gamma}_\sigma \subset (X \times_S X)^\sim$ be the schematic closure of the graph $\Gamma_\sigma \subset U \times_S U$ of the restriction of σ . We define the logarithmic fixed part $X_{\log}^\sigma \subset X$ by*

$$X_{\log}^\sigma = \Delta_X^{\log} \times_{(X \times_S X)^\sim} \tilde{\Gamma}_\sigma.$$

We assume that the intersection $X_{\log}^\sigma \cap \Sigma_{X/S}$ with the support $\Sigma_{X/S}$ of $\Omega_{X/S}^n(\log D)$ is supported in the closed fiber set-theoretically. We call the logarithmic intersection product

$$\left((\tilde{\Gamma}_\sigma, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} \in F_0G(X_{\log}^\sigma \cap \Sigma_{X/S})$$

the logarithmic Lefschetz class.

2. *We say σ is admissible if the following condition is satisfied: For each irreducible component D_i of D , we have either $\sigma(D_i) = D_i$ or $\sigma(D_i) \cap D_i = \emptyset$.*

We compute the logarithmic Lefschetz class using the Segre classes [10, 4.2], under a slightly weaker assumption than in [26, Lemma 5.4.8].

Lemma 5.1.5. — *Let X be a regular flat separated scheme of finite type over S and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. Let σ be an automorphism of X over S such that $\sigma(U) = U$ and $U^\sigma = \emptyset$. We assume σ is admissible. Let D_1, \dots, D_m be the irreducible components of D and we put $\tilde{U} = X \setminus \bigcup_{i:\sigma(D_i) \cap D_i = \emptyset} D_i$. Assume further that there exists a dense open scheme of X smooth over S . Then,*

1. *Let $\gamma_{\tilde{U}} : \tilde{U} \rightarrow X \times_S X$ be the restriction of $\gamma = (1, \sigma) : X \rightarrow X \times_S X$. Then it induces a closed immersion $\tilde{\gamma} : \tilde{U} \rightarrow (X \times_S X)^\sim$. The image $\tilde{\Gamma}_\sigma = \tilde{\gamma}(\tilde{U}) \subset (X \times_S X)^\sim$ is the schematic closure of $\Gamma_\sigma \subset U \times_S U$.*

2. *Assume that the generic fiber $X_{\log, K}^\sigma$ is empty and let $s(X_{\log}^\sigma, X)$ denote the Segre class of $X_{\log}^\sigma \subset X$. We put $n = \dim X_K + 1$. Then the log Lefschetz class $\left((\tilde{\Gamma}_\sigma, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} \in F_0G(X_{\log}^\sigma)$ is equal to the image of*

$$\begin{aligned} & \left\{ c(\Omega_{X/S}^1(\log D))^* \cap s(X_{\log}^\sigma, X) \right\}_{\dim 0} \\ &= \sum_{i=0}^n (-1)^i c_i(\Omega_{X/S}^1(\log D))_{s_{n-i}}(X_{\log}^\sigma, X). \end{aligned}$$

In particular, if the logarithmic fixed part X_{\log}^σ is a Cartier divisor of X , we have

$$\left((\tilde{\Gamma}_\sigma, \Delta_X^{\log}) \right)_{(X \times_S X)^\sim} = \left\{ c(\Omega_{X/S}^1(\log D))^* \cap (1 + X_{\log}^\sigma)^{-1} \cap [X_{\log}^\sigma] \right\}_{\dim 0}.$$

Proof. — 1. We set $(X \times_S X)^0 = X \times_S X - \bigcup_{(i,j): D_i \cap D_j = \emptyset} D_i \times D_j$. By the definition of $(X \times_S X)^\sim$, we have $\text{pr}_1^{-1}(D_i) = \text{pr}_2^{-1}(D_i)$ in $(X \times_S X)^\sim$ for every irreducible component D_i of D . Hence $(X \times_S X)^\sim$ is a scheme over $(X \times_S X)^0$. By the definition of \tilde{U} , it is the inverse image of $(X \times_S X)^0 \subset X \times_S X$ by the map $\gamma : X \rightarrow X \times_S X$. Hence its restriction $\gamma_{\tilde{U}} : \tilde{U} \rightarrow (X \times_S X)^0$ is a closed immersion.

By the assumption that σ is admissible, the map $\gamma_{\tilde{U}} : \tilde{U} \rightarrow (X \times_S X)^0$ induces a map $\tilde{\gamma} : \tilde{U} \rightarrow (X \times_S X)^\sim$. Since $\gamma_{\tilde{U}} : \tilde{U} \rightarrow (X \times_S X)^0$ is a closed immersion, the induced map $\tilde{\gamma} : \tilde{U} \rightarrow (X \times_S X)^\sim$ is also a closed immersion.

2. By the assumption that $X_{\log, K}^\sigma$ is empty, the underlying set of X_{\log}^σ is a subset of the closed fiber X_F . We apply [26, Corollary 3.4.6], by taking $X \rightarrow (X \times_S X)^\sim \rightarrow X$ to be $V \rightarrow X \rightarrow S$ and $X_{\log}^\sigma \rightarrow \tilde{\Gamma}_\sigma \rightarrow (X \times_S X)^\sim$ to be $T \rightarrow W \rightarrow X$ in [26, Corollary 3.4.6]. Since $M_{X/(X \times_S X)^\sim} = \Omega_{X/S}^1(\log D)$, we obtain $((X, \tilde{\Gamma}_\sigma))_{(X \times_S X)^\sim} = \{c(\Omega_{X/S}^1(\log D))^* \cap s(X_{\log}^\sigma, \tilde{\Gamma}_\sigma)\}_{\dim 0}$. Since the open immersion $\tilde{\Gamma}_\sigma \rightarrow X$ induces the identity on X_σ^{\log} , we have $s(X_{\log}^\sigma, \Gamma_\sigma) = s(X_{\log}^\sigma, \Delta_{\tilde{U}}) = s(X_{\log}^\sigma, X)$. Thus the assertion is proved. \square

Corollary 5.1.6. — *Assume further that σ is of finite order and let i be an integer prime to the order of σ . Then, we have $((\Gamma_\sigma, \Delta_V^{\log})) = ((\Gamma_{\sigma^i}, \Delta_V^{\log}))$.*

Proof. — Since $X_{\log}^\sigma = X_{\log}^{\sigma^i}$, the assertion follows from Lemma 5.1.5.2. \square

For isolated singular points, we have the following formula similarly as [27, Lemma 3.4.14].

Proposition 5.1.7. — *Let X be a regular flat separated scheme of finite type over S and σ be an automorphism of X over S . Assume that there exists a dense open subscheme of X smooth over S . Let $x \in X$ be a closed point in the closed fiber and assume that the fixed part X^σ is set-theoretically equal to the set $\{x\}$.*

Let $f : X' \rightarrow X$ be the blow-up at x and D be the exceptional divisor. Let $(X' \times_S X')^\sim$ denote the log product with respect to D and $\tilde{\Gamma}_\sigma \subset (X' \times_S X')^\sim$ be the proper transform of the graph $\Gamma_\sigma \subset X \times_S X$ of σ . Then, we have

$$(5.1.7.1) \quad f_*((\tilde{\Gamma}_\sigma, \Delta_{X'})_{(X' \times_S X')^\sim}) = [\mathcal{O}_{X^\sigma}] - [x]$$

where $[\mathcal{O}_{X^\sigma}] = \text{length } \mathcal{O}_{X^\sigma} \cdot [x]$.

Proof. — We apply Lemma 5.1.5 to the automorphism σ of X' admissible with respect to the exceptional divisor D . By the exact sequence $0 \rightarrow \Omega_{D/F}^1 \rightarrow \Omega_{X'/S}^1(\log D) \otimes \mathcal{O}_D \xrightarrow{\text{res}} \mathcal{O}_D \rightarrow 0$, the total Chern class satisfies $c(\Omega_{X'/S}^1(\log D)) = c(\Omega_{D/F}^1) = c(\mathcal{O}(-1))^n = (1 - H)^n$ on D where H denote the class of the hyperplane of the projective space D . Hence, by Lemma 5.1.5, we obtain

$$((\tilde{\Gamma}_\sigma, \Delta_{X'})_{(X' \times_S X')^\sim}) = \{(1 + H)^n s(X_{\log}^\sigma, X')\}_{\dim 0}.$$

Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J} \subset \mathcal{O}_{X'}$ denote the ideal sheaf of X^σ and the ideal sheaf of X'_{\log} respectively. Then, since \mathcal{I} is generated by $\sigma(t_i) - t_i$ for a minimal system (t_i) of generators of the maximal ideal \mathfrak{m}_x , we compute $f^*\mathcal{I} = \mathcal{J} \cdot \mathcal{I}_D$. This means that X'_{\log} is the residual scheme [10, Definition 9.2.1] to the Cartier divisor D in the inverse image $f^*(X^\sigma)$. Hence by [10, Proposition 9.2], it implies that the Segre class satisfies

$$s(f^*(X^\sigma), X')_{\dim 0} = H^{n-1} + \{(1+H)^n s(X'_{\log}, X')\}_{\dim 0}$$

since the self intersection $D \cdot D$ is $-H$. Thus, we obtain

$$f_*((\tilde{\Gamma}_\sigma, \Delta_{X'}))_{(X' \times_S X')^\sim} = f_* s(f^*(X^\sigma), X')_{\dim 0} - f_* H^{n-1}.$$

By $f_* s(f^*(X^\sigma), X')_{\dim 0} = s(X^\sigma, X)_{\dim 0} = [X^\sigma]$ [10, Proposition 4.2(2)] and $f_* H^{n-1} = [x]$, the assertion follows. \square

In the case $X = \text{Spec } \mathcal{O}_L$ for a finite separable extension L of K , we obtain the following.

Corollary 5.1.8. — *Let L be a finite separable extension of K and σ be a non-trivial automorphism of L over K . We put $X = \text{Spec } \mathcal{O}_L$ and let $J_\sigma \subset \mathcal{O}_L$ be the ideal generated by $\sigma(a) - a$ for $a \in \mathcal{O}_L$ and $\sigma(b)/b - 1$ for $b \in \mathcal{O}_L$ and $b \neq 0$. Then, we have*

$$((\Gamma_\sigma, \Delta_X^{\log}))_{(X \times_S X)^\sim} = \text{length}_{\mathcal{O}_L} \mathcal{O}_L/J_\sigma.$$

5.2. The target groups. — Let $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over $S = \text{Spec } \mathcal{O}_K$. In this subsection, we define an abelian group $F_0G(\partial_{V/U}W)$ and a \mathbf{Q} -vector space $F_0G(\partial_{V/U}W)_{\mathbf{Q}}$ for a separated scheme W of finite type over V . Assuming U and V are regular, for a finite étale scheme V' over V , invariants of wild ramification of $V' \rightarrow U$ will be defined as elements of the group $F_0G(\partial_{V/U}V')_{\mathbf{Q}}$ in Section 5.4. Without assuming the regularity of U and V , the definition is extended at the end of Section 6.2 as a consequence of the excision formula, Theorem 6.2.2.

Let $f: V \rightarrow U$ be a morphism of separated schemes of finite type over S . Recall that an open immersion $j: V \rightarrow Y$ is schematically dense if the canonical map $\mathcal{O}_Y \rightarrow j_*\mathcal{O}_V$ is an injection. We define a category $\mathcal{C}_{V \rightarrow U}$ of compactifications of $f: V \rightarrow U$ as follows:

- An object is a morphism $\bar{f}: Y \rightarrow X$ of proper schemes over S such that X and Y contain U and V respectively as schematically dense open subschemes and that the diagram

$$(5.2.0.1) \quad \begin{array}{ccc} V & \longrightarrow & Y \\ f \downarrow & & \downarrow \bar{f} \\ U & \longrightarrow & X \end{array}$$

is commutative.

- A morphism $(g, h): (\bar{f}': Y' \rightarrow X') \rightarrow (\bar{f}: Y \rightarrow X)$ is a pair of morphisms $g: X' \rightarrow X$ and $h: Y' \rightarrow Y$ of schemes over S extending the identities of U and of V such that the diagram

$$(5.2.0.2) \quad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ h \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

is commutative.

Lemma 5.2.1. — *Let $f: V \rightarrow U$ be a morphism of separated schemes of finite type over S .*

1. *The category $\mathcal{C}_{V \rightarrow U}$ is cofiltered and in particular non-empty.*
2. *If $f: V \rightarrow U$ is finite flat, then the objects $\bar{f}: Y \rightarrow X$ such that the diagram (5.2.0.1) is Cartesian and that \bar{f} are finite flat are cofinal in the category $\mathcal{C}_{V \rightarrow U}$.*
3. *If V is a G -torsor over U for a finite group G , the objects $\bar{f}: Y \rightarrow X$ such that the diagram (5.2.0.1) is Cartesian, that \bar{f} are finite flat and that the action of G is extended to an action on Y over X are cofinal in the category $\mathcal{C}_{V \rightarrow U}$.*

Proof. — 1. By Nagata's embedding theorem [31], there exists a proper scheme X over S containing U as an open subscheme. After replacing X by the schematic closure of U , the open subscheme U is schematically dense in X . Further by Nagata's embedding theorem [31], there exists a proper scheme Y over X containing V as an open subscheme. After replacing Y by the schematic closure of V as above, we obtain an object $Y \rightarrow X$ of $\mathcal{C}_{V \rightarrow U}$.

Let $Y \rightarrow X$ and $Y' \rightarrow X'$ be objects of $\mathcal{C}_{V \rightarrow U}$. If there exists a map $(Y \rightarrow X) \rightarrow (Y' \rightarrow X')$ of $\mathcal{C}_{V \rightarrow U}$, it is unique since V is assumed schematically dense. Let X'' be the schematic closure of U in $X \times_S X'$ and Y'' be the schematic closure of V in $X'' \times_{X \times_S X'} (Y \times_S Y')$. Then $Y'' \rightarrow X''$ is an object of $\mathcal{C}_{V \rightarrow U}$ and there exist unique maps $(Y'' \rightarrow X'') \rightarrow (Y \rightarrow X)$ and $(Y'' \rightarrow X'') \rightarrow (Y' \rightarrow X')$.

2. Let $Y \rightarrow X$ be an object of $\mathcal{C}_{V \rightarrow U}$. Since V is schematically dense in Y , the diagram (5.2.0.1) is Cartesian. Then it follows from [36, 5.7.10] that there exists a blow-up $X' \rightarrow X$ inducing an isomorphism $U \times_X X' \rightarrow U$ such that the proper transform Y' of Y is finite flat over X' . After replacing X' by the schematic closure of U , the immersion $U \rightarrow X$ is schematically dense. Since Y is flat over X , the immersion $V \rightarrow Y$ is also schematically dense.

3. Let $Y \rightarrow X$ be an object of $\mathcal{C}_{V \rightarrow U}$. By replacing Y by the schematic closure of the diagonal image of V in the fibered product $\prod_{\sigma \in G, X} Y$ over X , we may assume that Y carries an action of G . Then Y' constructed in the proof of 2 also carries an action of G . \square

Recall that for an object Y of the category $\mathcal{C}_{V/S}$ of compactifications of V over S , the wild ramification locus $\Sigma_{V/U} Y \subset Y$ is defined as a closed subscheme in Definition

2.4.1. Further, if V is schematically dense in Y , the closed subsets $\Sigma_{V/U}Y$ form a projective system by Lemma 2.1.3.

Definition 5.2.2. — *Let $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S .*

1. *We define an abelian group $F_0G(\partial_{V/U}U)$ and a \mathbf{Q} -vector space $F_0G(\partial_{V/U}U)_{\mathbf{Q}}$ as the inverse limits:*

$$\begin{aligned} F_0G(\partial_{V/U}U) &= \varprojlim_{(\tilde{f}: Y \rightarrow X) \in \mathcal{C}_{V \rightarrow U}} F_0G(\tilde{f}(\Sigma_{V/U}Y)) \\ F_0G(\partial_{V/U}U)_{\mathbf{Q}} &= \varprojlim_{(\tilde{f}: Y \rightarrow X) \in \mathcal{C}_{V \rightarrow U}} (F_0G(\tilde{f}(\Sigma_{V/U}Y)) \otimes_{\mathbf{Z}} \mathbf{Q}) \end{aligned}$$

with respect to the proper push-forward maps.

2. *Let W be a separated scheme of finite type over V . We define an abelian group $F_0G(\partial_{V/U}W)$ and a \mathbf{Q} -vector space $F_0G(\partial_{V/U}W)_{\mathbf{Q}}$ as the inverse limits:*

$$\begin{aligned} F_0G(\partial_{V/U}W) &= \varprojlim_{(Z \rightarrow Y) \in \mathcal{C}_{W \rightarrow V}} F_0G(\Sigma_{V/U}Y \times_Y Z) \\ F_0G(\partial_{V/U}W)_{\mathbf{Q}} &= \varprojlim_{(Z \rightarrow Y) \in \mathcal{C}_{W \rightarrow V}} (F_0G(\Sigma_{V/U}Y \times_Y Z) \otimes_{\mathbf{Z}} \mathbf{Q}) \end{aligned}$$

with respect to the proper push-forward maps.

In the rest of this subsection, we will establish properties for $F_0G(\partial_{V/U}W)$. The same proof also works for $F_0G(\partial_{V/U}W)_{\mathbf{Q}}$.

For an object $Z \rightarrow Y$ of $\mathcal{C}_{W \rightarrow V}$, we have a canonical map

$$\mathrm{pr}_Z: F_0G(\partial_{V/U}W) \rightarrow F_0G(\Sigma_{V/U}Y \times_Y Z).$$

Since we will assume that the covering of the generic fibers $V_K \rightarrow U_K$ is tamely ramified with respect to K in the definition of the invariants in the next subsection, the group $F_0G(\partial_{V/U}W)$ is generated by the classes supported on the closed fibers, in practice. The assumption is always satisfied if K is of characteristic 0, by Corollary 2.4.5.

Lemma 5.2.3. — *Let*

$$(5.2.3.1) \quad \begin{array}{ccccc} U' & \longleftarrow & V' & \longleftarrow & W' \\ \downarrow & & \downarrow & & \downarrow^g \\ U & \longleftarrow & V & \longleftarrow & W \end{array}$$

be a commutative diagram of separated schemes of finite type over S such that the left square is Cartesian and that the map $V \rightarrow U$ is finite étale. Then, the push-forward maps induce

$$(5.2.3.2) \quad \begin{aligned} g_! &: F_0G(\partial_{V'/U'}W') \rightarrow F_0G(\partial_{V/U}W), \\ g_! &: F_0G(\partial_{V'/U'}W')_{\mathbf{Q}} \rightarrow F_0G(\partial_{V/U}W)_{\mathbf{Q}}. \end{aligned}$$

If g is proper, we write $g_* = g_!$ in (5.2.3.2).

Proof. — We define a category $\mathcal{C}_{W' \rightarrow V'/W \rightarrow V}$ consisting of commutative diagrams

$$\begin{array}{ccc} Y' & \longleftarrow & Z' \\ \bar{h} \downarrow & & \downarrow \bar{g} \\ Y & \longleftarrow & Z \end{array}$$

of schemes over S compatible with the right square of (5.2.3.1) such that $Z \rightarrow Y$ and $Z' \rightarrow Y'$ are objects of $\mathcal{C}_{W \rightarrow V}$ and of $\mathcal{C}_{W' \rightarrow V'}$ respectively. By Lemma 2.1.3, for an object of $\mathcal{C}_{W' \rightarrow V'/W \rightarrow V}$ we have $\bar{h}(\Sigma_{V'/U'}Y') \subset \Sigma_{V/U}Y$ and the push-forward map $\bar{g}_*: F_0G(\Sigma_{V'/U'}Y' \times_{Y'} Z') \rightarrow F_0G(\Sigma_{V/U}Y \times_Y Z)$ is defined. Similarly as in Lemma 5.2.1, the image of $\mathcal{C}_{W' \rightarrow V'/W \rightarrow V}$ in $\mathcal{C}_{W \rightarrow V}$ is cofinal. Hence the map $g_!: F_0G(\partial_{V'/U'}W') \rightarrow F_0G(\partial_{V/U}W)$ is defined as the limit. The map $g_!: F_0G(\partial_{V'/U'}W')_{\mathbf{Q}} \rightarrow F_0G(\partial_{V/U}W)_{\mathbf{Q}}$ is defined similarly. \square

Lemma 5.2.4. — *Let $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S and let $g: W' \rightarrow W$ be a finite flat morphism of separated schemes of finite type over V .*

1. *The pull-back maps induce a map*

$$g^*: F_0G(\partial_{V/U}W) \rightarrow F_0G(\partial_{V/U}W').$$

2. *Assume that $g: W' \rightarrow W$ is of degree d . Then, the composition $g_* \circ g^*: F_0G(\partial_{V/U}W) \rightarrow F_0G(\partial_{V/U}W)$ is multiplication by d .*

3. *Assume that W' is a G -torsor over W for a finite group G . Then, the composition $g^* \circ g_*: F_0G(\partial_{V/U}W') \rightarrow F_0G(\partial_{V/U}W')$ is equal to $\sum_{\sigma \in G} \sigma^*$. Consequently, $g^*: F_0G(\partial_{V/U}W)_{\mathbf{Q}} \rightarrow F_0G(\partial_{V/U}W')_{\mathbf{Q}}$ is an isomorphism to the G -fixed part.*

Proof. — 1. By Lemma 5.2.1.2, it suffices to show the following: Let $(h', h): ((\bar{g}_1, \text{id}_Y): (Z'_1 \rightarrow Y) \rightarrow (Z_1 \rightarrow Y)) \rightarrow ((\bar{g}, \text{id}_Y): (Z' \rightarrow Y) \rightarrow (Z \rightarrow Y))$ be a morphism in the category $\mathcal{C}_{W' \rightarrow V'/W \rightarrow V}$ defined in the proof of Lemma 5.2.3 such that $\bar{g}_1: Z'_1 \rightarrow Z_1$ and $\bar{g}: Z' \rightarrow Z$ are finite flat and that the maps $W' \rightarrow W \times_{Z_1} Z'_1$ and $W' \rightarrow W \times_Z Z'$ are

isomorphisms. Then, the diagram

$$\begin{array}{ccc} F_0G(\Sigma_{V/U}Y \times_Y Z_1) & \xrightarrow{\bar{g}_1^*} & F_0G(\Sigma_{V/U}Y \times_Y Z'_1) \\ h_* \downarrow & & \downarrow h'_* \\ F_0G(\Sigma_{V/U}Y \times_Y Z) & \xrightarrow{\bar{g}^*} & F_0G(\Sigma_{V/U}Y \times_Y Z') \end{array}$$

is commutative.

Since the diagram is commutative if $Z'_1 = Z' \times_Z Z_1$, we may assume h is the identity. Let $z \in \Sigma_{V/U}Y \times_Y Z$ be a closed point. Then since the base changes of the finite flat morphisms $Z' \rightarrow Z$ and $Z'_1 \rightarrow Z_1 = Z$ to the henselization are decomposed into the disjoint unions of the spectra of local rings, the class $\bar{g}^*[z] = [\bar{g}^{-1}(z)]$ is equal to $h'_*(\bar{g}_1^*[z]) = h'_*([\bar{g}_1^{-1}(z)])$. Hence the assertion follows.

2. By Lemma 5.2.1.2, it suffices to consider objects $\bar{g}: Z' \rightarrow Z$ of $\mathcal{C}_{W' \rightarrow W}$ such that \bar{g} is finite flat of degree d . Then, for a closed point $z \in \Sigma_{V/U}Y \times_Y Z$, we have $\bar{g}_*\bar{g}^*[z] = d[z]$ and the assertion follows.

3. Similarly as in the proof of 2, by Lemma 5.2.1.3, it suffices to consider finite flat objects $Z' \rightarrow Z$ of $\mathcal{C}_{W' \rightarrow W}$ such that the G -action is extended. Then, for a closed point $z \in \Sigma_{V/U}Y \times_Y Z'$, we have $\bar{g}^*\bar{g}_*[z] = \sum_{\sigma \in G} [\sigma z]$ and the assertion follows. \square

Similarly, the push-forward map

$$f_*: F_0G(\partial_{V/U}V) \rightarrow F_0G(\partial_{V/U}U)$$

and, if V is a G -torsor over U , the pull-back map

$$f^*: F_0G(\partial_{V/U}U) \rightarrow F_0G(\partial_{V/U}V)$$

are defined. The following Lemma is proved in the same way as Lemma 5.2.4.

Lemma 5.2.5. — *Let $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S and assume that V is a G -torsor over U for a finite group G of order d .*

The composition $f_ \circ f^*: F_0G(\partial_{V/U}U) \rightarrow F_0G(\partial_{V/U}U)$ is multiplication by d and the composition $f^* \circ f_*: F_0G(\partial_{V/U}V) \rightarrow F_0G(\partial_{V/U}V)$ is equal to $\sum_{\sigma \in G} \sigma^*$. Consequently, $f^*: F_0G(\partial_{V/U}U)_{\mathbf{Q}} \rightarrow F_0G(\partial_{V/U}V)_{\mathbf{Q}}$ is an isomorphism to the G -fixed part.*

Let $V' \rightarrow V \rightarrow U$ be finite étale morphisms of separated schemes of finite type over S . Then, for an object $g: Y' \rightarrow Y$ of $\mathcal{C}_{V' \rightarrow V}$, we have an inclusion $\Sigma_{V'/U}Y' \subset g^{-1}(\Sigma_{V/U}Y)$ by Lemma 2.1.3. Hence, for a separated scheme W of finite type over V' , a canonical map

$$(5.2.5.1) \quad F_0G(\partial_{V/U}W) \rightarrow F_0G(\partial_{V'/U}W)$$

is defined.

Similarly, let $V \rightarrow U \rightarrow U'$ be finite étale morphisms of separated schemes of finite type over S . Then, for an object Y of $\mathcal{C}_{V/S}$, we have an inclusion $\Sigma_{V/U}Y \subset \Sigma_{V/U'}Y$ by Lemma 2.1.3. Hence, for a separated scheme W of finite type over V , a canonical map

$$(5.2.5.2) \quad F_0G(\partial_{V/U}W) \rightarrow F_0G(\partial_{V/U'}W)$$

is defined.

We introduce a variant.

Definition 5.2.6. — Let U be a separated scheme of finite type over S and $\mathcal{C}_{U/S}$ be the category of compactifications defined in the beginning of Section 2.3.

1. We define an abelian group $F_0G(\partial_F U)$ and a \mathbf{Q} -vector space $F_0G(\partial_F U)_{\mathbf{Q}}$ as the inverse limits:

$$F_0G(\partial_F U) = \varprojlim_{X \in \mathcal{C}_{U/S}} F_0G(X \times_S F),$$

$$F_0G(\partial_F U)_{\mathbf{Q}} = \varprojlim_{X \in \mathcal{C}_{U/S}} (F_0G(X \times_S F) \otimes_{\mathbf{Z}} \mathbf{Q})$$

with respect to the proper push-forward maps.

2. For a morphism $f: V \rightarrow U$ of separated schemes of finite type over S , we define a map

$$f_!: F_0G(\partial_F V) \rightarrow F_0G(\partial_F U)$$

to be the limit of the push-forward maps.

If f is proper, we write $f_* = f_!$.

Let $f: V \rightarrow U$ be a finite étale morphism of separated schemes of finite type over S such that the generic fiber $f_K: V_K \rightarrow U_K$ is tamely ramified with respect to K . Then, the objects Y of the category $\mathcal{C}_{V/S}$ of compactifications of V satisfying set-theoretical inclusions $\Sigma_{V/U}Y \subset Y_F$ are cofinal in $\mathcal{C}_{V/S}$. Hence, we have a canonical map

$$(5.2.6.1) \quad F_0G(\partial_{V/U}V) \rightarrow F_0G(\partial_F V).$$

5.3. Definition of invariants of wild ramification. — In this subsection, we define invariants of wild ramification without assuming the regularity of compactification.

First, we recall the existence of an alteration.

Lemma 5.3.1 ([6, Theorem 6.5]). — Let X be a flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and $U \subset X$ be a dense open subscheme. Then, there exist a scheme Z over S and a morphism $\bar{h}: Z \rightarrow X$ over S satisfying the following conditions:

(5.3.1.1) The scheme Z is regular flat separated of finite type over S and $W = \bar{h}^{-1}(U)$ is the complement of a divisor D with simple normal crossings.

(5.3.1.2) *The morphism $\bar{h}: Z \rightarrow X$ is proper, surjective and generically finite.*

We give some sufficient conditions for simultaneous good alterations for a scheme Y and for a weakly semi-stable scheme Y' over Y . This will be used in Proposition 6.3.2.

Corollary 5.3.2. — *Let Y be a flat separated scheme of finite type over S and $V \subset Y$ be a dense open subscheme. Let $Y' \rightarrow Y$ be a weakly semi-stable scheme such that the base change $Y'_V = Y' \times_Y V$ is smooth over V . We assume that either of the following conditions is satisfied:*

(5.3.2.1a) $Y' \rightarrow Y$ is a curve.

(5.3.2.1b) *There exist a morphism $\bar{g}_0: Z_0 \rightarrow Y$ of schemes over S satisfying the conditions (5.3.1.1) and (5.3.1.2) with X and \bar{h} replaced by Y and \bar{g}_0 and a log blow-up $Z'_0 \rightarrow Y' \times_Y Z_0$ inducing an isomorphism $Z'_0 \times_{Z_0} W_0 \rightarrow Y' \times_Y W_0$ where $W_0 = \bar{g}_0^{-1}(V)$. Further, $Z'_0 \rightarrow Z_0$ is weakly strictly semi-stable and satisfies the condition (1.2.3.1) with $X \rightarrow S$ replaced by $Z'_0 \rightarrow Z_0$.*

Then, there exist a regular flat scheme Z over S , a proper surjective and generically finite morphism $\bar{g}: Z \rightarrow Y$ and a log blow-up $Z' \rightarrow Y' \times_Y Z$ satisfying the following conditions: The inverse image $W = \bar{g}^{-1}(V)$ is the complement of a divisor D_Z with simple normal crossings, the induced map $Z' \times_Z W \rightarrow Y' \times_Y W$ is an isomorphism, the map $Z' \rightarrow Z$ is weakly strictly semi-stable, the scheme Z' is regular and the divisor $Z' \times_Z D_Z$ has simple normal crossings.

Only the case (5.3.2.1a) will be used in the proof of the conductor formula.

Proof. — First, we consider the case where Y' is a curve over Y . By replacing Y by a finite covering obtained by adjoining some square roots, we may assume that Y satisfies the condition (1.1.4.1) in Lemma 1.1.4. By Lemma 5.3.1, there exist a regular flat scheme Z over S , a proper surjective and generically finite morphism $\bar{g}: Z \rightarrow Y$ such that inverse image $W = \bar{g}^{-1}(V)$ is the complement of a divisor D_Z with simple normal crossings. Then, it suffices to apply Lemmas 1.1.4 and 1.2.2.

If (5.3.2.1b) is satisfied, it suffices to apply Lemma 1.2.3. \square

Proposition 5.3.3. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S . Let Y be a flat separated scheme of finite type over S containing V as an open subscheme and let $\mathcal{D} = (D_i)_{i \in I}$ be a finite family of Cartier divisors of Y satisfying $V = Y \setminus \bigcup_{i \in I} D_i$ and $\sum_{V/U}^{\mathcal{D}} Y_K = \emptyset$. Let $A_{\mathcal{D}} \subset (Y \times_S Y)_{\tilde{\mathcal{D}}}$ denote the closure $\overline{(V \times_U V \setminus \Delta_V)}$.*

Let

$$(5.3.3.1) \quad \begin{array}{ccccc} Y & \xleftarrow{\bar{g}} & Z & \xrightarrow{\bar{h}} & X \\ \uparrow & & \uparrow & & \uparrow \\ U & & U & & U \\ V & \xleftarrow{g} & W & \xrightarrow{h} & U \end{array}$$

be a Cartesian diagram of separated scheme of finite type over S satisfying the following conditions:

(5.3.3.1a) The scheme Z is regular and flat over S and W is the complement of a divisor D with simple normal crossings.

(5.3.3.1b) The scheme X contains U as the complement of a Cartier divisor B and we have $h = f \circ g$.

Let $(Z \times_X Z)^\sim \subset (Z \times_S Z)^\sim$ be the log products defined by the family (D'_j) of irreducible components of the complement $Z \setminus W$ and by $B \subset X$ (5.3.3.1b). Let $A \subset (Z \times_S Z)^\sim$ denote the intersection $(\bar{g} \times \bar{g})^{\sim -1}(A_{\mathcal{D}}) \cap (Z \times_X Z)^\sim$.

We put $n = \dim Z$. Then, there exists a unique map

$$((, \Delta_Z^{\log})) : \mathrm{Gr}_{\bullet}^F G(W \times_U W \setminus W \times_V W) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z)$$

that makes the diagram

$$(5.3.3.2) \quad \begin{array}{ccc} \mathrm{Gr}_{\bullet}^F G(W \times_U W \setminus W \times_V W) & \xrightarrow{((, \Delta_Z^{\log}))} & \mathrm{Gr}_{\bullet-n}^F G(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z) \\ \uparrow \text{restriction} & \nearrow & \\ \mathrm{Gr}_{\bullet}^F G(A) & & \end{array} \quad ((, \Delta_Z^{\log}))_{(Z \times_S Z)^\sim}$$

commutative.

In the characterization of the map $((, \Delta_Z^{\log})) : \mathrm{Gr}_{\bullet}^F G(W \times_U W \setminus W \times_V W) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z)$ in (5.3.3.2), we may replace A by the closure of $W \times_U W \setminus W \times_V W$.

Since the definition of the log product $(Z \times_X Z)^\sim$ involves also the Cartier divisor $B \subset X$, it could be better denoted by $(Z \times_{\mathbf{X}} Z)^\sim$. However, since we always consider the log product with B as long as the base is X , we will use the notation $(Z \times_X Z)^\sim$.

Proof. — We show that the condition (A') after Definition 4.3.2 and (B) in Proposition 4.3.5 are satisfied. By the assumption that the generic fiber of $\Sigma_{V/U}^{\mathcal{D}} Y = A_{\mathcal{D}} \cap \Delta_Y^{\log}$ is empty, the intersection $A \cap \Delta_Z^{\log} \subset (\bar{g} \times \bar{g})^{\sim -1}(A_{\mathcal{D}} \cap \Delta_Y^{\log})$ is supported on the closed fiber. Hence, the condition (A') after Definition 4.3.2 is satisfied and the map $((, \Delta_Z^{\log}))_{(Z \times_S Z)^\sim} : G(A) \rightarrow G(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z)$ is defined.

Let $(D'_j)_j$ be the irreducible components of $Z \setminus W$ and we put $\bar{h}^* B = \sum_j l_j D'_j$. Since $W = \bar{h}^{-1}(U)$, we have $l_j > 0$ for each irreducible component D'_j . By Lemma 1.3.2.2, the intersection $A \cap \mathbf{G}_{m, D'_j} \subset (Z \times_X Z)^\sim \cap \mathbf{G}_{m, D'_j}$ is a closed subscheme of μ_{l_j, D'_j} for each irreducible component D'_j . Hence the condition (B) in Proposition 4.3.5 is also satisfied. Since the intersection $A^\circ = A \cap (W \times_U W)$ is equal to $W \times_U W \setminus W \times_V W$, there exists a unique map $((, \Delta_Z^{\log})) : \mathrm{Gr}_{\bullet}^F G(W \times_U W \setminus W \times_V W) \rightarrow \mathrm{Gr}_{\bullet-n}^F G(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z)$ making the diagram (5.3.3.2) commutative by Proposition 4.3.5. \square

The map $((, \Delta_Z^{\log}))$ is compatible with the pull-back as follows.

Corollary 5.3.4. — We keep the notation in Proposition 5.3.3. Further, let Z' be a regular separated scheme of finite type over S and $\pi : Z' \rightarrow Z$ be a morphism over S such that $W' = \pi^{-1}(W)$ is the complement of a divisor with simple normal crossings.

We assume $\dim Z'_K + 1 = \dim Z_K + 1 = n$. Then, we have a commutative diagram

$$(5.3.4.1) \quad \begin{array}{ccc} \mathrm{Gr}_{\bullet}^F \mathrm{G}(W \times_U W \setminus W \times_V W) & \xrightarrow{((\cdot, \Delta_Z^{\log}))} & \mathrm{Gr}_{\bullet-n}^F \mathrm{G}(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z) \\ (\pi \times \pi)^* \downarrow & & \downarrow \pi^* \\ \mathrm{Gr}_{\bullet}^F \mathrm{G}(W' \times_U W' \setminus W' \times_V W') & \xrightarrow{((\cdot, \Delta_{Z'}^{\log}))} & \mathrm{Gr}_{\bullet-n}^F \mathrm{G}(\Sigma_{V/U}^{\mathcal{D}} Y \times_Y Z'). \end{array}$$

Proof. — It suffices to apply Lemma 4.3.3.1. \square

We introduce a category of alterations. Let $f : V \rightarrow U$ be a proper morphism of separated reduced schemes of finite type over S . We define a category $\mathcal{A}_{V \rightarrow U}$ of alterations as follows:

- An object is a proper, surjective and generically finite morphism $\bar{g} : Z \rightarrow Y$ of proper schemes over S such that Y contains V as a schematically dense open subscheme satisfying the following conditions:

(5.3.5.1a) The scheme Z is regular and flat over S and $W = \bar{g}^{-1}(V)$ is the complement of a divisor D with simple normal crossings. There exists a dense open subscheme V_0 of V such that $\bar{g}^{-1}(V_0) \rightarrow V_0$ is finite flat of constant rank.

(5.3.5.1b) There exists a proper scheme X over S containing U as the complement of a Cartier divisor B and a Cartesian diagram

$$(5.3.5.1) \quad \begin{array}{ccc} W & \xrightarrow{c} & Z \\ f \circ g \downarrow & & \downarrow \bar{h} \\ U & \xrightarrow{c} & X \end{array}$$

of schemes over S where $g : W \rightarrow V$ is the restriction of $\bar{g} : Z \rightarrow Y$.

- A morphism $(\bar{\pi}, \varphi) : (\bar{g}' : Z' \rightarrow Y') \rightarrow (\bar{g} : Z \rightarrow Y)$ is a pair of a proper, surjective and generically finite morphism $\bar{\pi} : Z' \rightarrow Z$ over S and a morphism $\varphi : Y' \rightarrow Y$ of schemes over S such that the diagram

$$(5.3.5.2) \quad \begin{array}{ccccc} Z' & \xrightarrow{\bar{g}'} & Y' & \xleftarrow{\supset} & V \\ \bar{\pi} \downarrow & & \downarrow \varphi & & \parallel \\ Z & \xrightarrow{\bar{g}} & Y & \xleftarrow{\supset} & V \end{array}$$

is commutative and that there exists a dense open subscheme Z_0 of Z such that $\bar{\pi}^{-1}(Z_0) \rightarrow Z_0$ is finite flat of constant rank.

Lemma 5.3.5. — *Let $f: V \rightarrow U$ be a proper, surjective and generically finite morphism of separated schemes of finite type over S .*

1. *For an object Y of the category $\mathcal{C}_{V/S}$ of compactifications containing V as a schematically dense open subscheme, there exists an object $\bar{g}: Z \rightarrow Y$ of the category $\mathcal{A}_{V \rightarrow U}$ of alterations.*
2. *The category $\mathcal{A}_{V \rightarrow U}$ is cofiltered.*

Proof. — 1. By Nagata's embedding theorem [31], there exists a proper scheme X over S containing U as an open subscheme. By replacing X by a blow-up at the complement $X \setminus U$ if necessary, we may assume $U \subset X$ is the complement of a Cartier divisor B . By replacing Y by the closure of the graph $\Gamma_f \subset X \times_S Y$, we may assume there exists a morphism $\bar{f}: Y \rightarrow X$ such that $\bar{f}^{-1}(U) = V$ and $\bar{f}|_V = f$. Then, it suffices to apply Lemma 5.3.1 to the open immersion $V \rightarrow Y$ and to take some disjoint union of connected component.

2. It suffices to apply Lemma 5.3.1 to the open immersion $W \times_V W' \rightarrow \overline{W \times_V W'} \subset Z \times_S Z'$ and to take some disjoint union of connected component. \square

Note that the condition (5.3.5.1b) is satisfied if we have an object $\bar{f}: Y \rightarrow X$ of $\mathcal{C}_{V \rightarrow U}$ such that X contains U as the complement of a Cartier divisor B . If $V \rightarrow U$ is a Galois covering, such an object may be constructed as follows.

Lemma 5.3.6. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular separated scheme of finite type over S and $V \rightarrow Y$ be an open immersion of separated schemes of finite type over S . Let $\mathcal{D} = (D_i)_{i \in I}$ be a finite family of Cartier divisors of Y such that V is the complement of the union $\bigcup_{i \in I} D_i$. Assume that V is a G -torsor over U and that the action of G is extended to Y and on \mathcal{D} . Assume further that the action of G on Y is admissible in the sense that the quotient $X = Y/G$ is defined as a scheme.*

Then, the canonical map $\bar{f}: Y \rightarrow X$ is finite, the quotient X is separated of finite type over S and there exists a Cartier divisor B of X such that the complement is U .

Proof. — It suffices to show the existence of B . By the assumption the sum $D = \sum_i D_i$ is stable by the action of G and $V = Y \setminus D$. The norm B of D is defined as a Cartier divisor of X since $\mathcal{O}_X \rightarrow \bar{f}_* \mathcal{O}_Y$ is injective. Since the inverse image of the complement $X \setminus B$ is V , we obtain $U = X \setminus B$. \square

Let $f: V \rightarrow U$ be a finite étale morphism of regular separated schemes of finite type over S . For a morphism $g': W \rightarrow V'$ of regular schemes of finite type over V , the pull-back map

$$(g' \times g')^*: \mathrm{Gr}_n^F G(V' \times_U V' \setminus V' \times_V V') \rightarrow \mathrm{Gr}_n^F G(W \times_U W \setminus W \times_V W)$$

by $g' \times g': W \times_S W \rightarrow V' \times_S V'$ is defined by Corollary 4.1.5 and Lemma 3.1.4.

Theorem 5.3.7. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular separated schemes of finite type over $S = \text{Spec } \mathcal{O}_K$ such that $U_K \rightarrow V_K$ is tamely ramified with respect to K . Let V' be a regular flat scheme of finite type over S and $V' \rightarrow V$ be a proper morphism over S . We assume that $\dim V_K = \dim V'_K$ and put $n = \dim V_K + 1$.*

Then, there exists a unique map

$$(5.3.7.1) \quad ((\cdot, \Delta_{V'})^{\log}) : \text{Gr}_n^F G(V' \times_U V' \setminus V' \times_V V') \rightarrow F_0 G(\partial_{V/U} V')_{\mathbf{Q}}$$

satisfying the following property:

For an object Y of $\mathcal{C}_{V/S}$, a finite family \mathcal{D} of Cartier divisors of Y such that $\Sigma_{V/U} Y = \Sigma_{V/U}^{\mathcal{D}} Y$ (Definition 2.4.1.1), an object $Y' \rightarrow Y$ of $\mathcal{C}_{V' \rightarrow V}$ and an object $\bar{g}' : Z \rightarrow Y'$ of $\mathcal{A}_{V' \rightarrow U}$ such that \bar{g}' is generically of constant degree $[W : V']$, the diagram

$$(5.3.7.2) \quad \begin{array}{ccc} \text{Gr}_n^F G(V' \times_U V' \setminus V' \times_V V') & \xrightarrow{((\cdot, \Delta_{V'})^{\log})} & F_0 G(\partial_{V/U} V')_{\mathbf{Q}} \\ (g' \times \bar{g}')^* \downarrow & & \downarrow \text{pr}_{Y'} \\ \text{Gr}_n^F G(W \times_U W \setminus W \times_V W) & & F_0 G(\Sigma_{V/U} Y \times_Y Y')_{\mathbf{Q}} \\ & \searrow ((\cdot, \Delta_Z)^{\log}) & \uparrow \frac{1}{[W:V']} \bar{g}'_* \\ & & F_0 G(\Sigma_{V/U} Y \times_Y Z) \end{array}$$

is commutative, where $g' : W = \bar{g}'^{-1}(V') \rightarrow V'$ is the restriction of $\bar{g}' : Z \rightarrow Y'$.

Proof. — By the remark after Definition 2.4.1, there exist an object Y of $\mathcal{C}_{V/S}$ and a finite family \mathcal{D} of Cartier divisors of Y such that $\Sigma_{V/U} Y = \Sigma_{V/U}^{\mathcal{D}} Y$. Further since $\mathcal{C}_{V' \rightarrow V}$ is non-empty, there exist an object $Y' \rightarrow Y$ of $\mathcal{C}_{V' \rightarrow V}$ and an object $\bar{g}' : Z \rightarrow Y'$ of $\mathcal{A}_{V' \rightarrow U}$ by Lemma 5.3.5.1. By the definition of $F_0 G(\partial_{V/U} V')_{\mathbf{Q}}$ as the projective limit, it suffices to show that the composition of the lower maps in the diagram (5.3.7.2) is independent of the choice of an object $\bar{g}' : Z \rightarrow Y'$ of $\mathcal{A}_{V' \rightarrow U}$ and that the compositions form an inverse system with respect to objects $Y' \rightarrow Y$ of $\mathcal{C}_{V' \rightarrow V}$.

The categories $\mathcal{C}_{V' \rightarrow V}$ and $\mathcal{A}_{V' \rightarrow U}$ are cofiltered by Lemmas 5.2.1.1 and 5.3.5.2. Hence, it suffices to show that the diagram

$$(5.3.7.3) \quad \begin{array}{ccc} \text{Gr}_n^F G(W \times_U W \setminus W \times_V W) & \xrightarrow{(\pi \times \pi)^*} & \text{Gr}_n^F G(W_1 \times_U W_1 \setminus W_1 \times_V W_1) \\ ((\cdot, \Delta_Z)^{\log}) \downarrow & & \downarrow ((\cdot, \Delta_{Z_1})^{\log}) \\ F_0 G(\Sigma_{V/U} Y \times_Y Z) & & F_0 G(\Sigma_{V/U} Y_1 \times_{Y_1} Z_1) \\ \frac{1}{[W:V']} \bar{g}'_* \downarrow & & \downarrow \frac{1}{[W_1:V']} \bar{g}'_{1*} \\ F_0 G(\Sigma_{V/U} Y \times_Y Y')_{\mathbf{Q}} & \xleftarrow{\varphi_*} & F_0 G(\Sigma_{V/U} Y_1 \times_{Y_1} Y'_1)_{\mathbf{Q}} \end{array}$$

is commutative for morphisms $(\varphi, \psi): (Y'_1 \rightarrow Y_1) \rightarrow (Y' \rightarrow Y)$ of $\mathcal{C}_{V' \rightarrow V}$ and $(\bar{\pi}, \varphi): (\bar{g}_1: Z_1 \rightarrow Y'_1) \rightarrow (\bar{g}: Z \rightarrow Y')$ of $\mathcal{A}_{V' \rightarrow U}$ where $\pi: W_1 \rightarrow W$ denotes the restriction of $\bar{\pi}: Z_1 \rightarrow Z$ on the inverse image of V' . By Corollary 5.3.4, the pull-back $\bar{\pi}^*: F_0G(\Sigma_{V/U}Y \times_Y Z) \rightarrow F_0G(\Sigma_{V/U}Y_1 \times_{Y_1} Z_1)$ makes the upper half of (5.3.7.3) into a commutative square. On the other hand, the push-forward $\bar{\pi}_*: F_0G(\Sigma_{V/U}Y_1 \times_{Y_1} Z_1) \rightarrow F_0G(\Sigma_{V/U}Y \times_Y Z)$ divided by the degree $[Z_1: Z]$ makes the lower half into a commutative square. The composition $\bar{\pi}_* \circ \bar{\pi}^*$ is equal to the multiplication by $[\mathbf{R}\bar{\pi}_* \mathcal{O}_{Z_1}]$ and induces the multiplication by $[Z_1: Z] = \text{rank}(\mathbf{R}\bar{\pi}_* \mathcal{O}_{Z_1})$ on $F_0G(\Sigma_{V/U}Y \times_Y Z)$. Hence, the assertion is proved. \square

Definition 5.3.8. — *Let the notation be as in Theorem 5.3.7. We call the map*

$$(5.3.8.1) \quad ((, \Delta_{V'})^{\log}: \text{Gr}_n^F G(V' \times_U V' \setminus V' \times_V V') \rightarrow F_0G(\partial_{V/U} V'))_{\mathbf{Q}}$$

the logarithmic localized intersection product with the diagonal. For an object $Y' \rightarrow Y$ of $\mathcal{C}_{V' \rightarrow V}$, we define

$$(5.3.8.2) \quad ((, \Delta_{V'})^{\log}: \text{Gr}_n^F G(V' \times_U V' \setminus V' \times_V V') \rightarrow F_0G(\Sigma_{V/U}Y \times_Y Y'))_{\mathbf{Q}}$$

as the composition of (5.3.8.1) with the projection $F_0G(\partial_{V/U} V')_{\mathbf{Q}} \rightarrow F_0G(\Sigma_{V/U}Y \times_Y Y')_{\mathbf{Q}}$ and call it also the logarithmic localized intersection product with the diagonal.

Since we assume that $V_K \rightarrow U_K$ is tamely ramified with respect to $\text{Spec } K$, the target group $F_0G(\partial_{V/U} V')_{\mathbf{Q}}$ is generated by the classes supported on the closed fibers. The assumption is always satisfied if K is of characteristic 0, by Corollary 2.4.5.

If V is finite étale over V , the graded piece $\text{Gr}_n^F G(V' \times_U V' \setminus V' \times_V V')$ is identified with the free abelian group $Z^0(V' \times_U V' \setminus V' \times_V V')$ generated by the irreducible components of $V' \times_U V' \setminus V' \times_V V'$. Thus, in this case, the maps (5.3.8.1) and (5.3.8.2) define

$$(5.3.8.3) \quad ((, \Delta_{V'})^{\log}: Z^0(V' \times_U V' \setminus V' \times_V V') \rightarrow F_0G(\partial_{V/U} V'))_{\mathbf{Q}},$$

$$(5.3.8.4) \quad ((, \Delta_{V'})^{\log}: Z^0(V' \times_U V' \setminus V' \times_V V') \rightarrow F_0G(\Sigma_{V/U}Y \times_Y Y'))_{\mathbf{Q}}.$$

Theorem 5.3.7 implies that, for an open and closed subscheme Γ of $V' \times_U V' \setminus V' \times_V V'$, the logarithmic localized intersection products $((\Gamma, \Delta_{V'})^{\log}$ for objects $Y' \rightarrow Y$ of $\mathcal{C}_{V' \rightarrow V}$ such that $\Sigma_{V/U}^+ Y_K = \emptyset$ form a projective system and defines an element of $F_0G(\partial_{V/U} V')_{\mathbf{Q}} = \varprojlim_{Y' \rightarrow Y} F_0G(\Sigma_{V/U}Y \times_Y Y')_{\mathbf{Q}}$.

Keep assuming $V' \rightarrow V$ is finite étale and let $Y' \rightarrow Y$ be an object of $\mathcal{C}_{V' \rightarrow V}$. Assume that Y' is regular and $V' \subset Y'$ is the complement of a divisor with simple normal crossings. Assume further that there exist a proper scheme X over S containing U as the complement of a Cartier divisor and a morphism $Y' \rightarrow X$ extending $V' \rightarrow U$. Then, the

identity $Y' \rightarrow Y$ is an object of $\mathcal{A}_{V' \rightarrow U}$ and, for a finite family \mathcal{D} of Cartier divisors of Y such that $\Sigma_{V'/U}^{\mathcal{D}} Y_{\mathbf{K}} = \emptyset$, the diagram

$$(5.3.8.5) \quad \begin{array}{ccc} Z^0(V' \times_U V' \setminus V' \times_V V') & \xrightarrow{((, \Delta_{V'})^{\log})} & F_0G(\partial_{V'/U} V') \\ ((, \Delta_{V'}^{\log})) \downarrow & & \downarrow \text{pr}_{V'} \\ F_0G(\Sigma_{V'/U}^{\mathcal{D}} Y \times_Y Y')_{\mathbf{Q}} & \longleftarrow & F_0G(\Sigma_{V'/U} Y \times_Y Y')_{\mathbf{Q}} \end{array}$$

is commutative. Consequently, if we assume resolution of singularities or if we assume $\dim Y_{\mathbf{K}} \leq 1$, we do not need to introduce denominator and an integral version

$$(5.3.8.6) \quad ((, \Delta_{V'})^{\log})_{\mathbf{Z}}: Z^0(V' \times_U V' \setminus V' \times_V V') \rightarrow F_0G(\partial_{V'/U} V')$$

is defined as the limit of $((, \Delta_{V'}^{\log}))$.

Let $f: V \rightarrow U$ be a finite étale morphism of smooth separated schemes of finite type over F and V' be a finite étale scheme over V . Then, similarly as above, slightly refining [27, Theorem 3.2.3], we define a map

$$(5.3.8.7) \quad ((, \Delta_{V'})^{\log}): Z^0(V' \times_U V' \setminus V' \times_U V) \rightarrow \text{CH}_0(\partial_{V'/U} V')_{\mathbf{Q}}.$$

We introduce a variant of the map (5.3.8.3) assuming \mathbf{K} is of characteristic 0. This variant is defined without removing the diagonal $\Delta_V \subset V \times_U V$.

Theorem 5.3.9. — *Assume \mathbf{K} is of characteristic 0. Let $V \rightarrow U$ be a finite étale morphism of smooth separated schemes of finite type over S and let Y be an object of $\mathcal{C}_{V/S}$. Then, there exists a unique map*

$$(5.3.9.1) \quad ((, \Delta_V))^{\log}: Z^0(V \times_U V) \rightarrow F_0G(\partial_F V)_{\mathbf{Q}}$$

satisfying the following property:

For an object Y of $\mathcal{C}_{V/S}$, a finite family \mathcal{D} of Cartier divisors of Y such that $\Sigma_{V'/U}^{\mathcal{D}} Y_{\mathbf{K}} = \emptyset$ and an object $\bar{g}: Z \rightarrow Y$ of $\mathcal{A}_{V \rightarrow U}$ such that \bar{g} is generically of constant degree $[W:V]$, there exists a map $((, \Delta_Z^{\log}))_{\mathbf{Z}}: \text{Gr}_n^F G(W \times_U W) \rightarrow F_0G(Z \times_S \text{Spec } F)$ that makes the diagram

$$(5.3.9.2) \quad \begin{array}{ccc} \text{Gr}_n^F G(V \times_U V) & \xrightarrow{((, \Delta_V^{\log}))} & F_0G(\partial_F V)_{\mathbf{Q}} \\ (g \times g)^* \downarrow & & \downarrow \text{pr}_Y \\ \text{Gr}_n^F G(W \times_U W) & & F_0G(Y \times_S \text{Spec } F)_{\mathbf{Q}} \\ \text{restriction} \uparrow & \searrow ((, \Delta_Z^{\log})) & \uparrow \frac{1}{[W:V]} \bar{g}^* \\ \text{Gr}_n^F G((Z \times_X Z)^{\sim}) & \xrightarrow{((, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}}} & F_0G(Z \times_S \text{Spec } F) \end{array}$$

commutative.

By the assumption that \mathbf{K} is of characteristic 0, in the notation of the proof of Proposition 5.3.3, the generic fiber $Z_{\mathbf{K}}$ is smooth over \mathbf{K} and the log product $(Z \times_X Z)^\sim \subset (Z \times_S Z)^\sim$ satisfies the condition (A') after Definition 4.3.2 with X replaced by Z . Except this remark, the proof is the same as that of Theorem 5.3.7.

We keep assuming that \mathbf{K} is of characteristic 0 and let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S . Then, the maps (5.3.8.3) and (5.3.9.1) are compatible in the sense that the diagram

$$(5.3.9.3) \quad \begin{array}{ccc} Z^0(V \times_U V \setminus \Delta_V) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0G(\partial_{V/U}V)_{\mathbf{Q}} \\ \cap \downarrow & & \downarrow (5.2.6.1) \\ Z^0(V \times_U V) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0G(\partial_F V)_{\mathbf{Q}} \end{array}$$

is commutative.

5.4. Elementary properties of the invariants of wild ramification. — The map (5.3.8.3) has the following compatibility.

Proposition 5.4.1. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular schemes over S such that the generic fiber $V_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to \mathbf{K} . Let V' be a finite étale scheme over V .*

1. *For a finite étale morphism $g: V'' \rightarrow V'$, the diagram*

$$(5.4.1.1) \quad \begin{array}{ccc} Z^0(V' \times_U V' \setminus V' \times_V V') & \xrightarrow{((\cdot, \Delta_{V'})^{\log}} & F_0G(\partial_{V'/U}V')_{\mathbf{Q}} \\ (g \times g)^* \downarrow & & \downarrow g^* \\ Z^0(V'' \times_U V'' \setminus V'' \times_V V'') & \xrightarrow{((\cdot, \Delta_{V''})^{\log}} & F_0G(\partial_{V''/U}V'')_{\mathbf{Q}} \end{array}$$

2. *Assume that the generic fiber $V'_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to \mathbf{K} . Then,*

$$(5.4.1.2) \quad \begin{array}{ccc} Z^0(V'' \times_U V'' \setminus V'' \times_V V'') & \xrightarrow{((\cdot, \Delta_{V''})^{\log}} & F_0G(\partial_{V''/U}V'')_{\mathbf{Q}} \\ \text{can} \downarrow & & \downarrow (5.2.5.1) \\ Z^0(V'' \times_U V'' \setminus V'' \times_{V'} V'') & \xrightarrow{((\cdot, \Delta_{V''})^{\log}} & F_0G(\partial_{V''/U}V'')_{\mathbf{Q}} \end{array}$$

is commutative.

3. For a finite étale morphism $U \rightarrow U'$ such that the generic fiber $V_K \rightarrow U'_K$ is tamely ramified with respect to K , the diagram

$$(5.4.1.3) \quad \begin{array}{ccc} Z^0(V' \times_U V' \setminus V' \times_V V') & \xrightarrow{((\cdot, \Delta_{V'})^{\log})} & F_0G(\partial_{V/U}V')_{\mathbf{Q}} \\ \text{can} \downarrow & & \downarrow (5.2.5.2) \\ Z^0(V' \times_{U'} V' \setminus V' \times_V V') & \xrightarrow{((\cdot, \Delta_{V'})^{\log})} & F_0G(\partial_{V/U'}V')_{\mathbf{Q}} \end{array}$$

is commutative.

Proof. — 1. By Lemma 5.2.4.2, the map g^* is injective. Hence, by replacing V'' if necessary, we may assume V'' is a G -torsor over V' for a finite group G . Since an object of $\mathcal{A}_{V'' \rightarrow U}$ define an object of $\mathcal{A}_{V' \rightarrow U}$, the square with the arrow g^* replaced by $|G|^{-1}g_*$ going the other way is commutative by the definition of the map (5.3.8.3). Since the images are in the G -fixed part, the assertion follows from Lemma 5.2.4.3.

The rest is clear from the definition and the remark after Proposition 5.3.3. \square

We show a compatibility with tame base change.

Corollary 5.4.2. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular schemes over S such that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K . Let $g: U' \rightarrow U$ be a finite étale morphism of regular schemes over S . Let $V' \subset V \times_U U'$ be an open and closed subscheme and $g': V' \rightarrow V$ denote the projection.*

Then, if $g: U' \rightarrow U$ is tamely ramified with respect to S , the diagram

$$(5.4.2.1) \quad \begin{array}{ccc} Z^0(V \times_U V \setminus \Delta_V) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0G(\partial_{V/U}V)_{\mathbf{Q}} \\ (g' \times g')^* \downarrow & & \downarrow g'^* \\ Z^0(V' \times_{U'} V' \setminus \Delta_{V'}) & \xrightarrow{\text{can} \circ ((\cdot, \Delta_{V'})^{\log})} & F_0G(\partial_{V/U}V')_{\mathbf{Q}} \end{array}$$

is commutative.

Proof. — By Lemma 2.1.3, the canonical map $F_0G(\partial_{V'/U'}V') \rightarrow F_0G(\partial_{V/U}V')$ is defined. By Proposition 5.4.1.1 applied to $V' \rightarrow V = V \rightarrow U$, the diagram (5.4.2.1) with $Z^0(V' \times_{U'} V' \setminus \Delta_{V'})$ replaced by $Z^0(V' \times_U V' \setminus V' \times_V V')$ is commutative.

By the assumption that $g: U' \rightarrow U$ is tamely ramified with respect to S , there exists an object X' of the category $\mathcal{C}_{U'/S}$ of compactifications of U' and a finite family \mathcal{D}' of Cartier divisors X' such that $\sum_{U'/U} \mathcal{D}'$ is empty. For an object $Y' \rightarrow X'$ of $\mathcal{C}_{V' \rightarrow U'}$, the closure of the inverse image $V' \times_U V' \setminus V' \times_{U'} V'$ of $U' \times_U U' \setminus \Delta_{U'}$ does not meet with the log diagonal $\Delta_{V'}$ in the log product $(Y' \times_S Y')_{\widetilde{\mathcal{D}'}}$ for the pull-back \mathcal{D}' of \mathcal{D} . Hence, we have $((\Gamma, \Delta_{V'})^{\log}) = 0$ in $F_0G(\partial_{V'/U'}V')_{\mathbf{Q}}$ if Γ is in $V' \times_U V' \setminus V' \times_{U'} V'$.

By the assumption $V' \subset V \times_U U'$, we have $(V' \times_{U'} V') \cap (V' \times_V V') = \Delta_{V'}$. Hence the assertion follows. \square

In the case where \mathbf{K} is of characteristic 0, the variant defined in Theorem 5.3.9 satisfies properties analogous to Proposition 5.4.1 and Corollary 5.4.2, by the same proof.

The logarithmic different and the logarithmic Lefschetz class defined in Section 5.1 are defined without assuming the existence of a regular model.

Definition 5.4.3. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S such that the generic fiber $V_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to \mathbf{K} .*

1. *We call*

$$(5.4.3.1) \quad D_{V/U}^{\log} = ((V \times_U V \setminus \Delta_V, \Delta_V))^{\log} \in F_0G(\partial_{V/U}V)_{\mathbf{Q}}$$

the logarithmic different of V over U . We call

$$(5.4.3.2) \quad d_{V/U}^{\log} = f_* D_{V/U}^{\log} \in F_0G(\partial_{V/U}U)_{\mathbf{Q}}$$

the logarithmic discriminant of V over U .

2. *Let σ be an automorphism of V over U such that the fixed part V^{σ} is empty and let $\Gamma_{\sigma} \subset V \times_U V$ be the graph of σ . We call*

$$(5.4.3.3) \quad ((\Gamma_{\sigma}, \Delta_V))^{\log} \in F_0G(\partial_{V/U}V)_{\mathbf{Q}}$$

the logarithmic Lefschetz class.

We show that the log different satisfies a chain rule and that, for a Galois covering, the logarithmic different is the sum of Lefschetz classes.

Lemma 5.4.4. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S such that the generic fiber $V_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to \mathbf{K} .*

1. *Let U' be a finite étale scheme over U such that the generic fiber $U'_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to \mathbf{K} and let $g: V \rightarrow U'$ be a finite étale morphism over U . Then, we have*

$$(5.4.4.1) \quad D_{V/U}^{\log} = D_{V/U'}^{\log} + g^* D_{U'/U}^{\log}$$

in $F_0G(\partial_{V/U}V)_{\mathbf{Q}}$.

2. *Assume that V is a G -torsor over U for a finite group G . Then we have*

$$(5.4.4.2) \quad D_{V/U}^{\log} = \sum_{\sigma \in G, \sigma \neq 1} ((\Gamma_{\sigma}, \Delta_V))^{\log}$$

in $F_0G(\partial_{V/U}V)_{\mathbf{Q}}$.

3. Assume that V is a G -torsor for a finite group G . Let $N \subset G$ be a normal subgroup of G and let $g: V \rightarrow V'$ be the corresponding N -torsor. Then, for an element $\sigma' \in G' = G/N$, $\neq 1$, we have

$$g^*((\Gamma_{\sigma'}, \Delta_{V'}))^{\log} = \sum_{\sigma \in G, \sigma \mapsto \sigma'} ((\Gamma_{\sigma}, \Delta_V))^{\log}$$

in $F_0G(\partial_{V/U}V)_{\mathbf{Q}}$.

Proof. — 1. It follows from $V \times_U V \setminus \Delta_V = (V \times_{U'} V \setminus \Delta_V) \amalg (g \times g)^{-1}(U' \times_U U' \setminus \Delta_{U'})$ and Proposition 5.4.1.1 applied to $U' \times_U U' \setminus \Delta_{U'}$.

2. Clear from $V \times_U V \setminus \Delta_V = \coprod_{\sigma \in G, \sigma \neq 1} \Gamma_{\sigma}$.

3. It follows from $(g \times g)^{-1}(\Gamma_{\sigma'}) = \coprod_{\sigma \mapsto \sigma'} \Gamma_{\sigma}$ and Proposition 5.4.1.1. \square

Corollary 5.4.5. — *Let the notation be as in Lemma 5.4.4.1. Let $f': U' \rightarrow U$ denote the morphism and assume that the map $g: V \rightarrow U'$ is of constant degree $[V:U']$. Then, for the discriminants defined in Definition 5.4.3.1, we have*

$$(5.4.5.1) \quad d_{V/U}^{\log} = f'_* d_{V/U'}^{\log} + [V:U'] \cdot d_{U'/U}^{\log}$$

in $F_0G(\partial_{V/U}U)_{\mathbf{Q}}$.

Proof. — We take the push-forward of (5.4.4.1). Then, similarly as Lemma 5.2.4.2, we obtain (5.4.5.1). \square

Conjecture 5.4.6. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S such that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K . Let σ be an automorphism of V over U such that the fixed part V^{σ} is empty.*

Then, for an integer i prime to the order of σ , we have

$$((\Gamma_{\sigma}, \Delta_V))^{\log} = ((\Gamma_{\sigma^i}, \Delta_V))^{\log}.$$

Proposition 5.4.7. — *Conjecture 5.4.6 is true if $\dim V_K \leq 1$.*

Proof. — By the resolution of singularity for two dimensional schemes, regular objects Y of the category $\mathcal{C}_{V/S}$ are cofinal in $\mathcal{C}_{V/S}$. Further, the regular objects Y such that the action of σ is extended to an admissible action on Y are cofinal in $\mathcal{C}_{V/S}$. Hence, the assertion of Conjecture 5.4.6 follows from Corollary 5.1.6. \square

6. Formulas for invariants of wild ramification

In this section, we establish formulas for the invariants of wild ramification defined in the previous section. We state and prove the results for the map (5.3.8.3). However, they also hold for the map (5.3.8.7) by the same argument.

We prove an excision formula Theorem 6.2.2 and a blow-up formula Proposition 6.2.1 in Section 6.2. We establish some preliminary formulas in Section 6.1. In Section 6.3, we prove a formula Proposition 6.3.2 for some semi-stable families applying the log Lefschetz trace formula Theorem 1.4.7, which will play a crucial role in the proof of the conductor formula.

6.1. Divisors and projective space bundles. — The results in this subsection will be used in the proof of the blow-up formula and of the excision formula in the next subsection. In Propositions 6.1.1 and 6.1.2, we compute the log localized intersection product of some classes supported on the inverse image of a divisor. In Proposition 6.1.3 and Lemma 6.1.4, we give formulas for a projective space bundle.

We keep the notation that $f: V \rightarrow U$ denotes a finite étale morphism of regular flat separated schemes of finite type over S and $n = \dim V_K + 1$ and the assumption that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K (Definition 2.4.1). Although we also state corresponding formulas for finite étale morphism $f: V \rightarrow U$ of smooth separated schemes of finite type over F , the proof is similar and easier and will be omitted.

We prepare to state Proposition 6.1.1. We consider a Cartesian diagram

$$(6.1.0.1) \quad \begin{array}{ccc} V & \xrightarrow{f} & U \\ g \downarrow & & \downarrow \\ V_0 & \xrightarrow{f_0} & U_0 \end{array}$$

of regular flat separated schemes of finite type over S where f_0 is finite étale and the vertical arrows are proper. We assume that the generic fiber $f_{0,K}: V_{0,K} \rightarrow U_{0,K}$ is tamely ramified with respect to K . If the map $g: V \rightarrow V_0$ is birational, the diagram

$$(6.1.0.2) \quad \begin{array}{ccc} \mathrm{Gr}_n^F \mathrm{G}(V \times_{U_0} V \setminus V \times_{V_0} V) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & \mathrm{F}_0 \mathrm{G}(\partial_{V_0/U_0} V)_{\mathbf{Q}} \\ (g \times g)^* \uparrow & & \downarrow g_* \\ \mathrm{Gr}_n^F \mathrm{G}(V_0 \times_{U_0} V_0 \setminus \Delta_{V_0}) & \xrightarrow{((\cdot, \Delta_{V_0}))^{\log}} & \mathrm{F}_0 \mathrm{G}(\partial_{V_0/U_0} V_0)_{\mathbf{Q}} \end{array}$$

is commutative.

Proposition 6.1.1. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S and $n = \dim V_K + 1$. Suppose that we have a Cartesian diagram (6.1.0.1) such that $V_{0,K} \rightarrow U_{0,K}$ is tamely ramified with respect to K .*

Let $U_1 \subset U$ be a regular divisor and i denote the immersion $U_1 \rightarrow U$ and its base changes. Assume that either U_1 is a scheme over K or a scheme over F . We put $V_1 = V \times_U U_1$. Then, for

$\Gamma_1 \in \mathrm{Gr}_n^F \mathbf{G}(V_1 \times_{U_0} V_1 \setminus V_1 \times_{V_0} V_1)$, we have

$$(6.1.1.1) \quad ((\Gamma_1, \Delta_V))^{\mathrm{log}} = \begin{cases} -i_*((\Gamma_1 \cdot c_1(\mathrm{pr}_2^* \mathbf{N}_{U_1/U}), \Delta_{V_1}))^{\mathrm{log}} & \text{if } V_1 = V_{1,K} \\ -i_*((\Gamma_1 \cdot c_1(\mathrm{pr}_2^* \mathbf{N}_{U_1/U}), \Delta_{V_1}))^{\mathrm{log}} & \text{if } V_1 = V_{1,F} \end{cases}$$

in $F_0 \mathbf{G}(\partial_{V_0/U_0} V)_{\mathbf{Q}}$.

Proof. — Let $Y \rightarrow Y_0$ be an object of $\mathcal{C}_{V \rightarrow V_0}$ and let \mathcal{D}_0 be a finite family of Cartier divisors such that $\Sigma_{V_0/U_0}^{\mathcal{D}_0} Y_0 = \Sigma_{V_0/U_0} Y_0$. Let $Y_1 \subset Y$ be the closure of V_1 . By replacing Y by the blow-up at Y_1 if necessary, we may assume Y_1 is a divisor of Y . We take an object $Z \rightarrow Y$ of $\mathcal{A}_{V \rightarrow U}$. Let $g^* V_1 = \sum_j e_j W_j$ be the decomposition by irreducible components and let Z_j denote the closure of W_j . Replacing Z if necessary, we may assume that $\sum_j Z_j$ has simple normal crossings and meets $D = Z \setminus W$ transversely.

Let $(Y \times_S Y)^\sim$ be the log product with respect to the pull-back \mathcal{D} of \mathcal{D}_0 and $(Z \times_S Z)^\sim$ be the log product with respect to the divisor D with simple normal crossings. We consider the map $(\bar{g} \times \bar{g})^\sim: (Z \times_S Z)^\sim \rightarrow (Y \times_S Y)^\sim$ and its restriction $g \times g: W \times_S W \rightarrow V \times_S V$. For an irreducible component Z_j , let $i_j: Z_j \rightarrow Z$ be the closed immersion and $\bar{g}_j: Z_j \rightarrow Y_1$ be the restrictions of $\bar{g}: Z \rightarrow Y$. Let $g_j: W_j = W \cap Z_j \rightarrow V_1$ be the restriction of \bar{g}_j . The intersection $D_j = Z_j \cap D$ is a divisor with simple normal crossings.

Let $(Z_i \times_S Z_j)^\sim$ denote the fiber product $(Z_i \times_S Z_j) \times_{Z \times_S Z} (Z \times_S Z)^\sim$. Define $(g_i \times g_j)^*: \mathrm{Gr}_n^F \mathbf{G}(V_1 \times_{U_0} V_1 \setminus V_1 \times_{V_0} V_1) \rightarrow \mathrm{Gr}_n^F \mathbf{G}(W_i \times_{U_0} W_j \setminus W_i \times_{V_0} W_j)$ as the pull-back by $g_i \times g_j: W_i \times_S W_j \rightarrow V_1 \times_S V_1$. Then, by Corollary 4.1.8.1, we obtain

$$(6.1.1.2) \quad (g \times g)^*(\Gamma_1) = \sum_{i,j} e_i e_j \cdot (g_i \times g_j)^*(\Gamma_1)$$

in $\mathrm{Gr}_n^F \mathbf{G}(W_1 \times_{U_0} W_1 \setminus W_1 \times_{V_0} W_1)$.

Let $A_0 \subset (Y_0 \times_S Y_0)_{\mathcal{D}_0}^\sim$ be the closure of $V_0 \times_{U_0} V_0 \setminus \Delta_{V_0}$ and let $A \subset (Z \times_S Z)^\sim$ be the intersection of the pull-back of A_0 with $(Z \times_S Z)^\sim$. For each i, j , we put $A_{ij} = A \cap (Z_i \times_S Z_j)^\sim \subset (Z \times_S Z)^\sim$. We have $A \cap (W \times_S W) = (W \times_{U_0} W) \setminus (W \times_{V_0} W)$ and $A_{ij} \cap (W_i \times_S W_j) = (W_i \times_{V_0} W_j) \setminus (W_i \times_{U_0} W_j)$. We take $\Gamma_{ij} \in \mathrm{Gr}_n^F \mathbf{G}(A_{ij})$ lifting $(g_i \times g_j)^*(\Gamma_1)$. Then, by (6.1.1.2), we have

$$(6.1.1.3) \quad ((\Gamma_1, \Delta_Y^{\mathrm{log}})) = \frac{1}{[W:V]} \sum_{i,j} e_i e_j \cdot \bar{g}_* ((\Gamma_{ij}, \Delta_Z^{\mathrm{log}}))_{(Z \times_S Z)^\sim}.$$

We continue the proof assuming $U_1 = U_{1,K}$. The proof of the other case $U_1 = U_{1,F}$ is similar and omitted. Let $\mathcal{I}_{Z_j} \subset \mathcal{O}_Z$ be the invertible ideal defining $Z_j \subset Z$. For each i, j , we show

$$(6.1.1.4) \quad \bar{g}_* ((\Gamma_{ij}, \Delta_Z^{\mathrm{log}}))_{(Z \times_S Z)^\sim} = -\bar{g}_* i_{i*} (((\Gamma_{ii}, \Delta_{Z_i}^{\mathrm{log}}))_{(Z_i \times_S Z_i)^\sim} \cdot c_1(\mathcal{I}_{Z_j}))$$

in $F_0G(\Sigma_{V_0/U_0} Y_0 \times_{Y_0} Y)_{\mathbf{Q}}$. If $i=j$, the equality (6.1.1.4) follows from Lemma 4.3.7.1. (In the case $U_1 = U_{1,F}$, we apply Lemma 4.3.7.2.)

We assume $Z_i \neq Z_j$. We put $Z_{ij} = Z_i \times_Z Z_j$, $W_{ij} = W_i \times_W W_j$, $A_{i,j} = A \cap (Z_i \times_S Z_{ij})^{\sim}$ and let $g_{ij}: W_{ij} \rightarrow V_1$ be the restriction of g . Then, the immersions $(Z_i \times_S Z_j)^{\sim} \rightarrow (Z \times_S Z)^{\sim}$ and $i_{ij}: Z_{ij} \rightarrow Z$ are regular immersions of codimension 2. Hence by Lemma 3.3.4, we obtain

$$((\Gamma_{ij}, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}} = i_{ij*}((\Gamma_{ij}, \Delta_{Z_{ij}}^{\log}))_{(Z_i \times_S Z_j)^{\sim}}.$$

Further, the immersion $(Z_i \times_S Z_{ij})^{\sim} \rightarrow (Z_i \times_S Z_j)^{\sim}$ is a regular immersion of codimension 1. Hence by Lemma 3.3.4, we obtain

$$((\Gamma_{ij}, \Delta_{Z_{ij}}^{\log}))_{(Z_i \times_S Z_j)^{\sim}} = (((\Gamma_{ij}, (Z_i \times_S Z_{ij})^{\sim}))_{(Z_i \times_S Z_j)^{\sim}}, \Delta_{Z_{ij}}^{\log})_{(Z_i \times_S Z_j)^{\sim}}.$$

Since both $(\Gamma_{ij}, (Z_i \times_S Z_{ij})^{\sim})_{(Z_i \times_S Z_j)^{\sim}}$ and $(\Gamma_{ii}, (Z_i \times_S Z_{ij})^{\sim})_{(Z_i \times_S Z_i)^{\sim}} \in \mathrm{Gr}_{n-1}^F G(A_{i,j})$ are liftings of $(g_i \times g_{ij})^*(\Gamma_1)$, we have

$$\begin{aligned} & (((\Gamma_{ij}, (Z_i \times_S Z_{ij})^{\sim}))_{(Z_i \times_S Z_j)^{\sim}}, \Delta_{Z_{ij}}^{\log})_{(Z_i \times_S Z_j)^{\sim}} \\ &= (((\Gamma_{ii}, (Z_i \times_S Z_{ij})^{\sim}))_{(Z_i \times_S Z_i)^{\sim}}, \Delta_{Z_{ij}}^{\log})_{(Z_i \times_S Z_j)^{\sim}} \end{aligned}$$

similarly as Proposition 4.3.5. By applying the associativity Lemmas 3.3.6 and 3.3.7 to the diagram

$$\begin{array}{ccc} (Z_i \times Z_{ij})^{\sim} & \longleftarrow & Z_{ij} \\ \downarrow & & \downarrow \\ (Z_i \times Z_i)^{\sim} & \longleftarrow & Z_i, \end{array}$$

we obtain

$$(((\Gamma_{ii}, (Z_i \times_S Z_{ij})^{\sim}))_{(Z_i \times_S Z_i)^{\sim}}, \Delta_{Z_{ij}}^{\log})_{(Z_i \times_S Z_j)^{\sim}} = (((\Gamma_{ii}, \Delta_{Z_i}^{\log}))_{(Z_i \times_S Z_i)^{\sim}}, Z_{ij})_{Z_i}.$$

Thus, we obtain

$$\begin{aligned} ((\Gamma_{ij}, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}} &= i_{ij*}(((\Gamma_{ii}, \Delta_{Z_i}^{\log}))_{(Z_i \times_S Z_i)^{\sim}}, Z_{ij})_{Z_i} \\ &= -i_{i*}(((\Gamma_{ii}, \Delta_{Z_i}^{\log}))_{(Z_i \times_S Z_i)^{\sim}} \cdot c_1(\mathcal{I}_{Z_j})) \end{aligned}$$

and the equality (6.1.1.4) is proved.

Therefore, the sum in the right hand side of (6.1.1.3) is equal to

$$(6.1.1.5) \quad - \sum_i e_i \cdot i_* \bar{g}_{i*} \left((((\Gamma_{ii}, \Delta_{Z_i}^{\log}))_{(Z_i \times_S Z_i)^{\sim}} \cdot \sum_j e_j c_1(\mathcal{I}_{Z_j})) \right).$$

Since $g^*V_1 = \sum_j e_j W_j$ as a divisor of W , the restriction $\sum_j e_j c_1(\mathcal{I}_{Z_j})|_{W_i}$ is equal to $c_1(\bar{g}_i^* N_{Y_1/Y})|_{W_i}$. Hence by Proposition 4.3.5, we have

$$\begin{aligned} & ((\Gamma_{\bar{u}}, \Delta_{Z_i}^{\log})_{(Z_i \times_S Z_i)^\sim} \cdot \sum_j e_j c_1(\mathcal{I}_{Z_j})) \\ &= \left(\left(\Gamma_{\bar{u}} \cdot \sum_j e_j c_1(\text{pr}_2^* \mathcal{I}_{Z_j}), \Delta_{Z_i}^{\log} \right)_{(Z_i \times_S Z_i)^\sim} \right) \\ &= ((\Gamma_{\bar{u}} \cdot \text{pr}_2^* c_1(\bar{g}_i^* N_{Y_1/Y}), \Delta_{Z_i}^{\log})_{(Z_i \times_S Z_i)^\sim}). \end{aligned}$$

If $Z_i \rightarrow Y_1$ is surjective, it defines an object of $\mathcal{A}_{V_1 \rightarrow U_1}$ and we have

$$\begin{aligned} & \bar{g}_{i*} \left((\Gamma_{\bar{u}} \cdot \text{pr}_2^* c_1(\bar{g}_i^* N_{Y_1/Y}), \Delta_{Z_i}^{\log})_{(Z_i \times_S Z_i)^\sim} \right) \\ &= [W_i : V_1] \cdot ((\Gamma_1 \cdot \text{pr}_2^* c_1(N_{Y_1/Y}), \Delta_{Y_1}^{\log})_{(Y_1 \times_S Y_1)^\sim}). \end{aligned}$$

We show that, if $Z_i \rightarrow Y_1$ is not surjective, the left hand side is 0. By replacing Z_i by an alteration, we may assume that there is an object $Z_0 \rightarrow Y_1$ of $\mathcal{A}_{V_1 \rightarrow U_1}$ and that $Z_i \rightarrow Y_1$ factors through $\pi : Z_i \rightarrow Z_0$. Since $\pi_* \pi^*$ is multiplication by $\text{rank } R\pi_* \mathcal{O}_{Z_i} = 0$ on $F_0 G(\Sigma_{V_0/U_0} Y \times_Y Z_0)$, the assertion follows from Corollary 5.3.4. Thus, by $\sum_i e_i [W_i : V_1] = [W : V]$, (6.1.1.5) is equal to the right hand side of (6.1.1.1). \square

We consider a regular divisor $U_1 \subset U$ as in Proposition 6.1.1. Let $(U \times_U U)^\sim$ denote the log product with respect to the Cartier divisor U_1 . It is the union of U with $E = \mathbf{G}_{m, U_1}$ meeting at U_1 . It is canonically identified with the fiber product $(U \times_S U)^\sim \times_{U \times_S U} \Delta_U$. The closed subscheme $E = \mathbf{G}_{m, U_1} \subset (U \times_U U)^\sim$ is the inverse image of $U_1 \subset U = \Delta_U$ by the canonical map $(U \times_U U)^\sim \rightarrow U$, as in Lemma 1.3.2.1. Let $(V \times_U V)^\sim = (V \times_U V) \times_U (U \times_U U)^\sim$ denote the log product with respect to $V_1 = V \times_U U_1$. Then, we define the localized log intersection product

$$(6.1.2.1) \quad ((, \Delta_V)^{\log} : \text{Gr}_n^F G((V \times_U V)^\sim \setminus (\Delta_V \times_U (U \times_U U)^\sim)) \rightarrow F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$$

similarly as follows, in order to state Proposition 6.1.2.

We consider an object Y of $\mathcal{C}_{V/S}$ and a finite family \mathcal{D} of Cartier divisors of Y such that $\Sigma_{U/V} Y = \Sigma_{U/V}^{\mathcal{D}} Y$. Let $Y_1 \subset Y$ denote the closure of V_1 and \mathcal{D}_1 denote the restriction of \mathcal{D} . By replacing Y by the blow-up at Y_1 if necessary, we assume Y_1 is a divisor of Y . We also consider an object $\bar{g} : Z \rightarrow Y$ of $\mathcal{A}_{V \rightarrow U}$. We assume that it also define an object of $\mathcal{A}_{(V \setminus V_1) \rightarrow (U \setminus U_1)}$. In particular, $W_0 = \bar{g}^{-1}(V \setminus V_1)$ is the complement of a divisor $D' \supset D$ of Z with simple normal crossings. Let $Z \rightarrow X$ be a proper morphism of schemes over S such that X contains U as the complement of a Cartier divisor B .

We define the log products $(Y \times_S Y)^\sim$ and $(Y \times_S Y)^\approx$ with respect to \mathcal{D} and to the union \mathcal{D}' of \mathcal{D} with Y_1 respectively. Similarly, we define the log products $(Z \times_S Z)^\sim$ and

$(Z \times_S Z)^\sim$ with respect to D and to D' . They form commutative diagrams

$$\begin{array}{ccccccc} (Y \times_S Y)^\sim & \xleftarrow{(\bar{g} \times \bar{g})^\sim} & (Z \times_S Z)^\sim & & (V \times_S V)^\sim & \xleftarrow{(g \times g)^\sim} & (W \times_S W)^\sim \\ \downarrow & & \downarrow & & \downarrow q & & \downarrow \\ (Y \times_S Y)^\sim & \longleftarrow & (Z \times_S Z)^\sim & & V \times_S V & \longleftarrow & W \times_S W \end{array}$$

where the right square is obtained by taking the base change to $V \times_S V$.

We define a closed subset $A \subset (Y \times_S Y)^\sim$ to be the closure of $(V \times_U V)^\sim \setminus (\Delta_V \times_U (U \times_U U)^\sim) \subset (V \times_S V)^\sim$ and put $A_Z = (\bar{g} \times \bar{g})^{\sim -1}(A) \cap (Z \times_X Z)^\sim$. Here $(Z \times_X Z)^\sim$ denote the log product defined with respect to D' and B . For $\Gamma \in \mathrm{Gr}_n^F G((V \times_U V)^\sim \setminus (\Delta_V \times_U (U \times_U U)^\sim))$, we take an element $\bar{\Gamma} \in \mathrm{Gr}_n^F G(A_Z)$ lifting the pull-back $(g \times g)^\sim{}^*(\Gamma)$ by $(g \times g)^\sim : (W \times_S W)^\sim \rightarrow (V \times_S V)^\sim$. Then, $((\Gamma, \Delta_Y^{\log}))$ is defined as $\bar{g}_*((\bar{\Gamma}, \Delta_Z^{\log}))_{(Z \times_S Z)^\sim}$ divided by $[Z : Y]$. The map (6.1.2.1) is defined as the projective limit.

By the associativity Lemma 3.3.7 applied to $(Z \times_S Z)^\sim \leftarrow (Z \times_S Z)^\sim \leftarrow Z$, we obtain a commutative diagram

$$(6.1.2.2) \quad \begin{array}{ccc} \mathrm{Gr}_n^F G(V \times_U V \setminus \Delta_V) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0 G(\partial_{V/U} V)_{\mathbf{Q}} \\ \downarrow q^* & & \parallel \\ \mathrm{Gr}_n^F G((V \times_U V)^\sim \setminus (\Delta_V \times_U (U \times_U U)^\sim)) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0 G(\partial_{V/U} V)_{\mathbf{Q}} \end{array}$$

for the pull-back q^* by the projection $q: (V \times_S V)^\sim \rightarrow V \times_S V$.

We put $U_0 = U \setminus U_1$, $V_0 = V \setminus V_1$ and we consider the diagram

$$(6.1.2.3) \quad \begin{array}{ccc} \mathrm{Gr}_n^F G(V \times_U V \setminus \Delta_V) & & \\ \downarrow & & \\ \mathrm{Gr}_n^F G((V \times_U V)^\sim \setminus (\Delta_V \times_U (U \times_U U)^\sim)) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0 G(\partial_{V/U} V)_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_n^F G(V_0 \times_{U_0} V_0 \setminus \Delta_{V_0}) & \xrightarrow{((\cdot, \Delta_{V_0}))^{\log}} & F_0 G(\partial_{V/U} V_0)_{\mathbf{Q}} \end{array}$$

where the upper left vertical arrow is induced by the immersion $V \times_U V \rightarrow (V \times_U V)^\sim$ and the lower left vertical arrow is the restriction. Since the log product $(Z \times_X Z)^\sim$ is defined with respect to B whose complement is U , the closed subset A_Z need *not* satisfy the condition (B) in Proposition 4.3.5 with respect to the components of $D' \setminus D$. Consequently, the square is *not* necessarily commutative. However, the compositions with the upper vertical arrow form a commutative diagram since the image of the composition $V \times_U V \rightarrow (V \times_U V)^\sim \rightarrow (X' \times_{X'} X')^\sim$ lies in the image of the log diagonal $X' \rightarrow (X' \times_{X'} X')^\sim$ for a compactification X' of U containing U_0 as the complement of a Cartier divisor B' extending U_1 and the log product $(X' \times_{X'} X')^\sim$ defined with respect to B' .

Proposition 6.1.2. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S and $n = \dim V_K + 1$. We assume that $V_K \rightarrow U_K$ is tamely ramified with respect to K .*

Let $U_1 \subset U$ be a regular divisor and $i: U_1 \rightarrow U$ denote the immersion. Assume either U_1 is a scheme over K or a scheme over F . Let $(V \times_S V)^\sim$ be the log product with respect to the Cartier divisor $V_1 = V \times_U U_1$ of V . Let $q_1: E = \mathbf{G}_{m, U_1} \rightarrow U_1$ be the projection and $q_1: E \times_{U_1} (V_1 \times_{U_1} V_1) \rightarrow V_1 \times_{U_1} V_1$ also denote the base change. We regard $E \times_{U_1} (V_1 \times_{U_1} V_1)$ as a closed subscheme of $(V \times_S V)^\sim$ as above.

Then, for $\Gamma_1 \in \mathrm{Gr}_{n-1}^F \mathbf{G}(V_1 \times_{U_1} V_1 \setminus \Delta_{V_1})$, the product (6.1.2.1) satisfies

$$(6.1.2.4) \quad ((q_1^* \Gamma_1, \Delta_V))^\log = \begin{cases} i_*((\Gamma_1, \Delta_{V_1}))^\log & \text{if } V_1 = V_{1,K} \\ i_*((\Gamma_1, \Delta_{V_1}))^\log & \text{if } V_1 = V_{1,F} \end{cases}$$

in $F_0 \mathbf{G}(\partial_{V/U} V)_\mathbf{Q}$.

Proof. — We keep the notation in the definition of (6.1.2.1) above. We put $g^* V_1 = \sum_{j \in J} e_j W_j$. For each irreducible component W_j , let Z_j be the closure, $i_j: Z_j \rightarrow Z$ be the closed immersion and $\bar{g}_j: Z_j \rightarrow Y_1$ be the restrictions of $\bar{g}: Z \rightarrow Y$. Let $g_j: W_j \rightarrow V_1$ be the restriction of \bar{g}_j . Let $\mathcal{D}_Z = (D_k)_{k \in I}$ and $\mathcal{D}'_Z = (D_k)_{k \in I'}$ be the families of irreducible components of $D = Z \setminus W \subset D' = Z \setminus W_0$ indexed by $I \subset I' = I \sqcup J$ respectively.

For $j \in J$, let $(Z_j \times_S Z_j)^\sim$ and $(Z_j \times_S Z_j)^\approx$ be the log product with respect to the families $\mathcal{D}_j = (D_k \cap Z_j)_{k \in I}$ and $\mathcal{D}'_j = (D_k \cap Z_j)_{k \in I', D_k \neq Z_j}$ respectively. Let $E_j \subset (Z \times_S Z)^\approx$ denote the inverse image of $Z_j \subset Z$ by either of the projections $(Z \times_S Z)^\approx \rightarrow Z$. Then, the canonical map $(Z_j \times_S Z_j)^\approx \rightarrow (Z_j \times_S Z_j)^\sim$ is of finite tor-dimension by Corollary 4.1.5 and E_j is flat over $(Z_j \times_S Z_j)^\approx$ by Lemma 1.3.2. Hence, the canonical map $\bar{q}_j: E_j \rightarrow (Z_j \times_S Z_j)^\sim$ is of finite tor-dimension. Let $q_j: E_j^\circ \rightarrow W_j \times_S W_j$ be the base change of \bar{q}_j . We consider the commutative diagram

$$\begin{array}{ccccc} E_1 = (V \times_S V)^\sim \times_{V \times_S V} (V_1 \times_S V_1) & \longleftarrow & E_j^\circ & \xrightarrow{c} & E_j \\ & & q_j \downarrow & & \downarrow \bar{q}_j \\ & & V_1 \times_S V_1 & \xleftarrow{g_j \times g_j} & W_j \times_S W_j \xrightarrow{c} (Z_j \times_S Z_j)^\sim \end{array}$$

where the right horizontal arrows are open immersions.

Let $(g_j \times g_j)^*: \mathrm{Gr}_{n-1}^F \mathbf{G}(V_1 \times_{U_1} V_1 \setminus \Delta_{V_1}) \rightarrow \mathrm{Gr}_{n-1}^F \mathbf{G}(W_j \times_{U_1} W_j \setminus W_j \times_{V_1} W_j)$ denote the pull-back by $g_j \times g_j: W_j \times_S W_j \rightarrow V_1 \times_S V_1$. Then, by Corollary 4.1.8.2, we obtain

$$(6.1.2.5) \quad (g \times g)^\sim{}^*(q_1^* \Gamma_1) = \sum_j e_j \cdot q_j^*(g_j \times g_j)^*(\Gamma_1)$$

in $\mathrm{Gr}_n^F \mathbf{G}((W_1 \times_{U_0} W_1)^\sim \setminus (W_1 \times_{V_0} W_1)^\sim)$.

Let $A_{\mathcal{D}} \subset (Y \times_S Y)^\sim_{\mathcal{D}}$ be the closure of $V \times_U V \setminus \Delta_V$ and let $A \subset (Z \times_S Z)^\approx$ be the intersection of the pull-back of $A_{\mathcal{D}}$ with $(Z \times_X Z)^\approx = (Z \times_S Z)^\approx \times_{(X \times_S X)^\sim} X$. For

each Z_j , let $A_j \subset (Z_j \times_S Z_j)^\sim$ be the intersection of the pull-back of A with $(Z_j \times_X Z_j)^\sim = (Z_j \times_S Z_j)^\sim \times_{(X \times_S X)^\sim} X$. We have $A \cap (W \times_S W)^\sim = (W \times_V W)^\sim \setminus (W \times_U W)^\sim$ and $A_j \cap (W_j \times_S W_j) = (W_j \times_{U_1} W_j) \setminus (W_j \times_{V_1} W_j)$. We take $\Gamma_j \in \mathrm{Gr}_n^F G(A_j)$ lifting $(g_j \times g_j)^*(\Gamma_1)$. Then, by (6.1.2.5), we have

$$(6.1.2.6) \quad ((q_1^* \Gamma_1, \Delta_V))^\mathrm{log} = \frac{1}{[W : V]} \sum_j e_j \cdot \bar{g}_* ((\bar{q}_j^* \Gamma_j, \Delta_Z^\mathrm{log}))_{(Z \times_S Z)^\approx}.$$

We continue the proof assuming $U_1 = U_{1,K}$. The proof of the other case $U_1 = U_{1,F}$ is similar using Lemma 4.3.4.2. By Lemma 4.3.4.1, we have $((\bar{q}_j^* \Gamma_j, \Delta_Z^\mathrm{log}))_{(Z \times_S Z)^\approx} = i_{j*} ((\bar{q}_j^* \Gamma_j, \Delta_{Z_j}^\mathrm{log}))_{(Z_j \times_S Z_j)^\sim, 1_{Z_j} \mathbf{G}_{m, Z_j}}$. Since $\bar{q}_j: E_j \rightarrow (Z_j \times_S Z_j)^\sim$ is of finite tor-dimension, we have $((\bar{q}_j^* \Gamma_j, \Delta_{Z_j}^\mathrm{log}))_{(Z_j \times_S Z_j)^\sim} = \bar{q}_j^* ((\Gamma_j, \Delta_{Z_j}^\mathrm{log}))_{(Z_j \times_S Z_j)^\sim}$ by the associativity Lemmas 3.3.6 and 3.3.7 where $\bar{q}_j: \mathbf{G}_{m, Z_j} \rightarrow Z_j$ in the right hand side denotes the restriction of $\bar{q}_j: E_j \rightarrow (Z_j \times_S Z_j)^\sim$. Thus we obtain $((\bar{q}_j^* \Gamma_j, \Delta_Z^\mathrm{log}))_{(Z \times_S Z)^\approx} = i_{j*} ((\Gamma_j, \Delta_{Z_j}^\mathrm{log}))_{(Z_j \times_S Z_j)^\sim}$ and

$$\bar{g}_* ((\bar{q}_j^* \Gamma_j, \Delta_Z^\mathrm{log}))_{(Z \times_S Z)^\approx} = \bar{g}_{j*} ((\Gamma_j, \Delta_{Z_j}^\mathrm{log}))_{(Z_j \times_S Z_j)^\sim}$$

for the right hand side of (6.1.2.6). The right hand side is equal to $[W_j : V_1] \cdot \mathrm{pr}_{Y_1}((\Gamma_1, \Delta_{V_1}))^\mathrm{log}$ if $Z_j \rightarrow Y_1$ is generically finite and is 0 if otherwise similarly as at the end of the proof of Proposition 6.1.1. Therefore, the sum of the right hand side of (6.1.2.6) is equal to

$$\sum_j e_j [W_j : V_1] \cdot \mathrm{pr}_{Y_1}((\Gamma_1, \Delta_{V_1}))^\mathrm{log}.$$

Since $\sum_j e_j [W_j : V_1] = [W : V]$, the assertion follows. \square

Proposition 6.1.3. — 1. *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S and $n = \dim V_K + 1$. We assume that $V_K \rightarrow U_K$ is tamely ramified with respect to K . Let \mathcal{E} be a locally free \mathcal{O}_U -module of rank c , $p: P = \mathbf{P}(\mathcal{E}) \rightarrow U$ be the associated \mathbf{P}^{c-1} -bundle and $P_V = P \times_U V$ be the base change. Let $\Gamma \subset V \times_U V \setminus \Delta_V$ be an open and closed subscheme and we regard $\Gamma_P = \Gamma \times_U P$ as an open and closed subscheme of $P_V \times_P P_V \setminus \Delta_{P_V} = (V \times_U V \setminus \Delta_V) \times_U P$.*

Then, we have

$$(6.1.3.1) \quad p_*((\Gamma_P, \Delta_{P_V}))^\mathrm{log} = c \cdot ((\Gamma, \Delta_V))^\mathrm{log}$$

in $F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$.

2. *Let the assumption be the same as in 1. except that we assume $f: V \rightarrow U$ is a finite étale morphism of smooth separated schemes of finite type over F and $n = \dim V$.*

Then, we have

$$(6.1.3.2) \quad p_*(\Gamma_P, \Delta_{P_V})^\mathrm{log} = c \cdot (\Gamma, \Delta_V)^\mathrm{log}$$

in $\mathrm{CH}_0(\partial_{V/U} V)_{\mathbf{Q}}$.

Proof. — 1. By the flattening theorem [36], there exist a proper scheme X over S containing U as a dense open subscheme and a locally free \mathcal{O}_X -module \mathcal{E}_X of rank c extending \mathcal{E} . Replacing X by a blow-up, we may assume U is the complement of a Cartier divisor B .

Let Y be an object of \mathcal{C}_V and \mathcal{D} be a finite family of Cartier divisors of Y such that $\Sigma_{U/V}Y = \Sigma_{U/V}^{\mathcal{D}}Y$. Let $Z \rightarrow Y$ be an object of $\mathcal{A}_{V \rightarrow U}$. Let $P_Z = Z \times_X P_X$ denote the base change of the projective space bundle $P_X = \mathbf{P}(\mathcal{E}_X)$. Let $A_{\mathcal{D}} \subset (Y \times_S Y)_{\mathcal{D}}^{\sim}$ be the closure of $V \times_U V \setminus \Delta_V$ and let $A \subset (Z \times_S Z)^{\sim}$ denote the intersection $(\bar{g} \times \bar{g})^{\sim^{-1}}(A_{\mathcal{D}}) \cap (Z \times_S Z)^{\sim}$ as in Proposition 5.3.3. We regard $A_P = A \times_X P_X$ as a closed subscheme of $(P_Z \times_S P_Z)^{\sim} = (Z \times_S Z)^{\sim} \times_{X \times_S X} (P_X \times_S P_X)$ by the diagonal maps $X \rightarrow X \times_S X$ and $P_X \rightarrow P_X \times_S P_X$. Then, $((\Gamma, \Delta_V))^{\log}$ is defined using the image of $((\tilde{\Gamma}, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}}$ by taking a lifting $\tilde{\Gamma} \in F_n G(A)$ of $(g \times g)^* \Gamma$. The product $((\Gamma_P, \Delta_{P_Z}))^{\log}$ is defined using the image of $((p^* \tilde{\Gamma}, \Delta_{P_Z}^{\log}))_{(P_Z \times_S P_Z)^{\sim}}$ where $p^*: F_n G(A) \rightarrow F_{n+c-1} G(A_P)$ denotes the pull-back.

We apply the associativity formula, Lemma 3.3.6, to $A_P \rightarrow (P_Z \times_S P_Z)^{\sim} \leftarrow P_Z \times_Z P_Z \leftarrow P_Z$. Since a projective space bundle P_Z is smooth over Z , the diagonal $P_Z \rightarrow P_Z \times_Z P_Z$ is a regular immersion and hence is of finite tor-dimension. By applying Lemma 3.3.6, we obtain

$$(6.1.3.3) \quad ((\Gamma_P, \Delta_{P_Z}^{\log}))_{(P_Z \times_S P_Z)^{\sim}} = (((\Gamma_P, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}}, \Delta_{P_Z})_{P_Z \times_Z P_Z}.$$

Since the projection $p_Z: P_Z \rightarrow Z$ is smooth, we have

$$((\Gamma_P, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}} = p_Z^*((\Gamma, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}}.$$

Since $(\Delta_{P_Z}, \Delta_{P_Z})_{P_Z \times_Z P_Z} = (-1)^{c-1} c_{c-1}(\Omega_{P_Z/Z}^1)$, the right hand side of (6.1.3.3) is equal to

$$p_Z^*((\Gamma, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}} \cdot (-1)^{c-1} c_{c-1}(\Omega_{P_Z/Z}^1)$$

by the excess intersection formula for the usual intersection product. Since $\deg(-1)^{c-1} \cdot c_{c-1}(\Omega_{P_Z/Z}^1) = c$, by the projection formula, we obtain

$$p_{Z*}((\Gamma_P, \Delta_{P_Z}^{\log}))_{(P_Z \times_S P_Z)^{\sim}} = c \cdot ((\Gamma, \Delta_Z^{\log}))_{(Z \times_S Z)^{\sim}}$$

and the assertion follows.

We also omit the similar and easier proof of 2. \square

Lemma 6.1.4. — 1. Let $f: V \rightarrow U$, \mathcal{E} , $p: P = \mathbf{P}(\mathcal{E}) \rightarrow U$, $\Gamma \subset V \times_U V \setminus \Delta_V$ etc. be the same as in Proposition 6.1.3. We consider the pull-back $(p \times p)^* \Gamma$ as an open and closed subscheme of $P_V \times_U P_V \setminus P_V \times_V P_V$. For an integer m , we put $c_{c,m} = \deg c_{c-1}(\Omega_{P^{c-1}}^1(m))$.

Then, we have

$$(6.1.4.1) \quad \begin{aligned} p_*(((p \times p)^* \Gamma \cdot c_{c-1}(\mathrm{pr}_1^* \Omega_{P/U}^1(m) \otimes \mathrm{pr}_2^* \mathcal{O}(m')), \Delta_{P_V}))^{\log} \\ = c_{c,m+m'} \cdot ((\Gamma, \Delta_V))^{\log} \end{aligned}$$

in $F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$.

2. Let the assumption be the same as in 1. except that $f: V \rightarrow U$ is a finite étale morphism of separated smooth schemes of finite type over F .

Then, we have

$$(6.1.4.2) \quad p_* \left((p \times p)^* \Gamma \cdot c_{c-1} \left(\text{pr}_1^* \Omega_{P/U}^1(m) \otimes \text{pr}_2^* \mathcal{O}(m') \right), \Delta_{P_V} \right)^{\log} = c_{c, m+m'} \cdot (\Gamma, \Delta_V)^{\log}$$

in $\text{CH}_0(\partial_{V/U} V)_{\mathbf{Q}}$.

An elementary calculation shows

$$(6.1.4.3) \quad c_{c,m} = \begin{cases} \frac{1}{m} \left((m-1)^c - (-1)^c \right) & \text{for } m \neq 0 \\ (-1)^{c-1} c & \text{for } m = 0. \end{cases}$$

Proof. — 1. We keep the notation X, \mathcal{E}_X, P_Z , etc. in the proof of Proposition 6.1.3 above. Since P_Z is smooth over Z , similarly as above, we obtain

$$\begin{aligned} & \left(((p \times p)^* \Gamma \cdot c_{c-1} \left(\text{pr}_1^* \Omega_{P/U}^1(m) \otimes \text{pr}_2^* \mathcal{O}(m') \right), \Delta_{P_Z}^{\log} \right)_{(P_Z \times_S P_Z)^\sim} \\ &= \left(((p \times p)^* \Gamma, \Delta_{P_Z}^{\log} \right)_{(P_Z \times_S P_Z)^\sim} \cdot c_{c-1} \left(\Omega_{P_Z/Z}^1(m+m') \right) \\ &= p_Z^* \left((\Gamma, \Delta_Z) \right)_{(Z \times_S Z)^\sim} \cdot c_{c-1} \left(\Omega_{P_Z/Z}^1(m+m') \right). \end{aligned}$$

By applying the projection formula, we obtain (6.1.4.1).

We also omit the similar and easier proof of 2. □

6.2. Excision formula. — We keep the notation $f: V \rightarrow U$ etc. as in the previous subsection. We prove the excision formula, Theorem 6.2.2. We begin with the following blow-up formula.

Proposition 6.2.1. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S such that $V_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to $\text{Spec } \mathbf{K}$. We put $n = \dim U_{\mathbf{K}} + 1$. Let $U_1 \subset U$ be a regular closed subscheme of codimension c and $p: U' \rightarrow U$ be the blow-up at U_1 . Assume either U_1 is a scheme over \mathbf{K} or a scheme over F . We consider the Cartesian diagram*

$$\begin{array}{ccccc} V' & \xrightarrow{p} & V & \xleftarrow{i} & V_1 \\ \downarrow & & \downarrow & & \downarrow \\ U' & \xrightarrow{p} & U & \xleftarrow{i} & U_1 \end{array}$$

where we use the same letters to denote the base change.

Let $\Gamma \subset V \times_U V \setminus \Delta_V$ be an open and closed subscheme and we regard $\Gamma' = \Gamma \times_U U'$ and $\Gamma_1 = \Gamma \times_U U_1$ as open and closed subschemes of $V' \times_{U'} V' \setminus \Delta_{V'}$ and of $V_1 \times_{U_1} V_1 \setminus \Delta_{V_1}$ respectively.

Then, we have

$$(6.2.1.1) \quad ((\Gamma, \Delta_V))^{\log} = p_*((\Gamma', \Delta_{V'}))^{\log} + (c-1) \begin{cases} i_*((\Gamma_1, \Delta_{V_1}))^{\log} & \text{if } U_1 = U_{1,K} \\ i_*(\Gamma_1, \Delta_{V_1})^{\log} & \text{if } U_1 = U_{1,F} \end{cases}$$

in $F_0G(\partial_{V/U}V)_{\mathbf{Q}}$.

Proof. — We consider the pull-back $(p \times p)^*\Gamma$ by $p \times p: V' \times_S V' \rightarrow V \times_S V$. We have

$$(6.2.1.2) \quad ((\Gamma, \Delta_V))^{\log} = p_*(((p \times p)^*\Gamma, \Delta_{V'}))^{\log},$$

by the commutative diagram (6.1.0.2). We compute $(p \times p)^*\Gamma$. Note that $p_1: V'_1 \rightarrow V_1$ is a \mathbf{P}^{c-1} -bundle. The morphism $U' \rightarrow U$ of regular scheme is locally of complete intersection. Since $(p \times p)^*[\Delta_U] = [\mathcal{O}_{U'} \otimes_{\mathcal{O}_U}^L \mathcal{O}_U \otimes_{\mathcal{O}_U}^L \mathcal{O}_{U'}] = \sum_i (-1)^i [\mathcal{T}or_i^{\mathcal{O}_U}(\mathcal{O}_{U'}, \mathcal{O}_{U'})]$, by applying Corollary 3.1.6 to the blow-up $U' \rightarrow U$, we obtain

$$(6.2.1.3) \quad (p \times p)^*[\Delta_U] - [\Delta_{U'}] = \sum_{i=1}^{c-1} (-1)^{i-1} \sum_{j=1}^i [\mathrm{pr}_1^* \Omega_{U'_1/U_1}^i(i) \otimes \mathrm{pr}_2^* N_{U'_1/U'}^{\otimes -j}]$$

in $\mathrm{Gr}_n^F G(U' \times_U U')$, where $\mathrm{pr}_k: U'_1 \times_{U_1} U'_1 \rightarrow U'_1$ denote the projections. Let Σ denote the right hand side of (6.2.1.3). We will use the computation

$$(6.2.1.4) \quad \begin{aligned} \Sigma \cdot c_1(\mathrm{pr}_2^* N_{U'_1/U'}) &= \sum_{i=0}^{c-1} (-1)^i [\mathrm{pr}_1^* \Omega_{U'_1/U_1}^i(i)] \otimes ([\mathrm{pr}_2^* N_{U'_1/U'}^{\otimes -i}] \\ &\quad - [\mathcal{O}_{U'_1 \times_{U_1} U'_1}]) \\ &= (-1)^{c-1} \cdot (c_{c-1}(\mathrm{pr}_1^* \Omega_{U'_1/U_1}^1(1) \otimes \mathrm{pr}_2^* \mathcal{O}(-1)) \\ &\quad - c_{c-1}(\mathrm{pr}_1^* \Omega_{U'_1/U_1}^1(1))) \end{aligned}$$

in $\mathrm{Gr}_{n-1}^F G(U'_1 \times_{U_1} U'_1)$ that follows from $N_{V'_1/V'} = \mathcal{O}(1)$.

By $(p \times p)^*[\Delta_U] - [\Delta_{U'}] = \Sigma$, we have $(p \times p)^*\Gamma = \Gamma' + (p_1 \times p_1)^*\Gamma_1 \cdot \Sigma$ since Γ is flat over U . Thus, by (6.2.1.2), we obtain

$$((\Gamma, \Delta_V))^{\log} = p_*((\Gamma', \Delta_{V'}))^{\log} + p_*(((p_1 \times p_1)^*\Gamma_1 \cdot \Sigma, \Delta_{V'}))^{\log}.$$

Let $i': V'_1 \rightarrow V'$ denote the immersion. If $U_1 = U_{1,K}$, then Proposition 6.1.1 gives us

$$(6.2.1.5) \quad (((p_1 \times p_1)^*\Gamma_1 \cdot \Sigma, \Delta_{V'}))^{\log} = -i'_*((p_1 \times p_1)^*\Gamma_1 \cdot \Sigma \cdot c_1(\mathrm{pr}_2^* N_{U'_1/U'}), \Delta_{V_1})^{\log}.$$

If $U_1 = U_{1,F}$, we replace the double parentheses in the right hand side by a single parentheses. By substituting (6.2.1.4) in (6.2.1.5) and applying Lemma 6.1.4.1 to the projective space bundle $p_1: V'_1 \rightarrow V_1$, we obtain

$$(6.2.1.6) \quad p_*(((p_1 \times p_1)^*\Gamma_1 \cdot \Sigma, \Delta_{V'}))^{\log} = i_*(-1)^{c-1} (c_{c,0} - c_{c,1}) ((\Gamma_1, \Delta_{V_1}))^{\log}$$

in $F_0G(\Sigma_{V/U}V)_{\mathbf{Q}}$. Thus, the assertion follows from $c_{c,0} - c_{c,1} = (-1)^{c-1}(c-1)$ (6.1.4.3). \square

We state and prove an *excision* formula.

Theorem 6.2.2. — *Let $f: V \rightarrow U$ be a finite étale morphism of regular flat separated schemes of finite type over S such that $V_{\mathbf{K}} \rightarrow U_{\mathbf{K}}$ is tamely ramified with respect to $\mathrm{Spec} \mathbf{K}$. Let $U_1 \subset U$ be a regular closed subscheme and we consider the Cartesian diagram*

$$\begin{array}{ccccc} V_1 & \xrightarrow{j} & V & \xleftarrow{i} & V_0 = V \setminus V_1 \\ \downarrow & & \downarrow & & \downarrow \\ U_1 & \xrightarrow{j} & U & \xleftarrow{i} & U_0 = U \setminus U_1 \end{array}$$

For an open and closed subscheme Γ of $V \times_U V$, we put $\Gamma_0 = \Gamma \times_U U_0$ and $\Gamma_1 = \Gamma \times_U U_1$.

Then, we have

$$(6.2.2.1) \quad ((\Gamma, \Delta_V))^{\log} = j_!((\Gamma_0, \Delta_{V_0}))^{\log} + \begin{cases} i_*((\Gamma_1, \Delta_{V_1}))^{\log} & \text{if } U_1 \text{ is flat over } S \\ i_*^*(\Gamma_1, \Delta_{V_1})^{\log} & \text{if } U_1 = U_{1,F} \end{cases}$$

in $F_0G(\partial_{V/U}V)_{\mathbf{Q}}$.

In the right hand side, $j_!((\Gamma_0, \Delta_{V_0}))^{\log}$ and $i_*((\Gamma_1, \Delta_{V_1}))^{\log}$ are elements of $F_0G(\partial_{V/U}V)_{\mathbf{Q}}$ defined as the image of the map (5.2.3.2).

Proof. — By a standard devissage, we may assume either $U_1 = U_{1,K}$ or $U_1 = U_{1,F}$. By Propositions 6.1.3 and 6.2.1, it suffices to prove the case where U_1 is a divisor of U . We put $V_1 = V \times_U U_1$. Let $(V \times_S V)^{\sim}$ denote the log product $(V \times_S V)^{\sim}_{V_1}$. We consider the pull-back $q^*\Gamma$ by the projection $q: (V \times_S V)^{\sim} \rightarrow V \times_S V$. In the notation of (6.1.2.1), we have

$$(6.2.2.2) \quad ((\Gamma, \Delta_V))^{\log} = ((q^*\Gamma, \Delta_V))^{\log},$$

by the commutative diagram (6.1.2.2). Let $\tilde{\Gamma} \subset (V \times_S V)^{\sim}$ denote the proper transform of Γ and we put $\Gamma_1 = \Gamma \times_V V_1$. We also have

$$(6.2.2.3) \quad ((\Gamma_0, \Delta_{V_0}))^{\log} = ((\tilde{\Gamma}, \Delta_V))^{\log}$$

by the commutative part of the diagram (6.1.2.3).

Let $q_1: E \rightarrow V_1 \times_S V_1$ be the base change of q and $q_1^*\Gamma_1$ be the pull-back. Since $q: (U \times_S U)^{\sim} \rightarrow U \times_S U$ is locally of complete intersection by Corollary 4.1.5 and since Γ is flat over U , we have an equality

$$(6.2.2.4) \quad q^*\Gamma = \tilde{\Gamma} + \tilde{i}_*(q_1^*\Gamma_1)$$

in $G((V \times_U V)^\sim \setminus (U \times_U U)^\sim)$ by Corollary 3.1.7. By Proposition 6.1.2, we have $((q_1^* \Gamma_1, \Delta_V))^{\log} = ((\Gamma_1, \Delta_{V_1}))^{\log}$ if $U_1 = U_{1,K}$ and $((q_1^* \Gamma_1, \Delta_V))^{\log} = (\Gamma_1, \Delta_{V_1})^{\log}$ if $U_1 = U_{1,F}$. Thus the assertion is proved. \square

Similarly and more easily, we have the following analogue of Theorem 6.2.2.

Proposition 6.2.3. — *Let $f: V \rightarrow U$ be a finite étale morphism of smooth separated schemes of finite type over F . Let $U_1 \subset U$ be a smooth closed subscheme and $U_0 = U \setminus U_1$ be the complement. For an open and closed subscheme Γ of $V \times_U V$, we put $\Gamma_0 = \Gamma \times_U U_0$ and $\Gamma_1 = \Gamma \times_U U_1$.*

Then, we have

$$(6.2.3.1) \quad (\Gamma, \Delta_V)^{\log} = (\Gamma_0, \Delta_{V_0})^{\log} + (\Gamma_1, \Delta_{V_1})^{\log}$$

in $\mathrm{CH}_0(\partial_{V/U} V)_{\mathbf{Q}}$.

We generalize the definition of the map (5.3.8.3) for not necessarily regular U . For a noetherian scheme X , let $\Gamma(X, \mathbf{Z})$ be the \mathbf{Z} -module of \mathbf{Z} -valued locally constant functions on X .

Corollary 6.2.4. — *For every finite étale morphism $f: V \rightarrow U$ of separated schemes of finite type over S such that $V_K \rightarrow U_K$ is tamely ramified with respect to K , there exists a unique way to attach a morphism*

$$((\cdot, \Delta_V))^{\log}: \Gamma(V \times_U V \setminus \Delta_V, \mathbf{Z}) \rightarrow F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$$

satisfying the following properties:

(1) *If U is regular and flat of dimension n over S , it is the composition*

$$\Gamma(V \times_U V \setminus \Delta_V, \mathbf{Z}) \longrightarrow \mathrm{Gr}_n^F G(V \times_U V \setminus \Delta_V) \xrightarrow{((\cdot, \Delta_V))^{\log}} F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$$

where the first arrow is the natural isomorphism.

If U is smooth of dimension n over F , it is the composition

$$\Gamma(V \times_U V \setminus \Delta_V, \mathbf{Z}) \longrightarrow \mathrm{CH}_n(V \times_U V \setminus \Delta_V) \xrightarrow{(\cdot, \Delta_V)^{\log}} F_0 G(\partial_{V/U} V)_{\mathbf{Q}}$$

where the first arrow is the natural isomorphism.

(2) *Assume $U = \coprod_i U_i$ is a finite decomposition by regular subschemes. Let $j_i: U_i \rightarrow U$ denote the immersion and put $V_i = V \times_U U_i$ for each i . Then, the diagram*

$$(6.2.4.1) \quad \begin{array}{ccc} \Gamma(V \times_U V \setminus \Delta_V, \mathbf{Z}) & \xrightarrow{((\cdot, \Delta_V))^{\log}} & F_0 G(\partial_{V/U} V)_{\mathbf{Q}} \\ \downarrow (j_i^*)_i & & \uparrow \Sigma_{j_i} \\ \bigoplus_i \Gamma(V_i \times_{U_i} V_i \setminus \Delta_{V_i}, \mathbf{Z}) & \xrightarrow{\bigoplus_i ((\cdot, \Delta_{V_i}))^{\log}} & \bigoplus_i F_0 G(\partial_{V_i/U_i} V_i)_{\mathbf{Q}} \end{array}$$

is commutative.

Proof. — The uniqueness is a consequence of the existence of a finite partition by regular subschemes. To show the existence, it suffices to compare the maps defined by taking partitions by regular subschemes. By taking a common refinement, it is reduced to verify the following. Let U be a regular scheme and $U = \coprod_i U_i$ be a finite stratification by regular subschemes. Then, the maps defined in (1) make the diagram (6.2.4.1) commutative. It follows from Theorem 6.2.2 and Proposition 6.2.3 by the induction on the maximum of the codimensions of U_i in U . \square

By the same argument, we have the following variant.

Corollary 6.2.5. — *Assume \mathbf{K} is of characteristic 0. For every finite étale morphism $f: V \rightarrow U$ of separated schemes of finite type over \mathbf{K} , there exists a unique way to attach a morphism*

$$((, \Delta_V))^{\log}: \Gamma(V \times_U V, \mathbf{Z}) \rightarrow F_0G(\partial_F V)_{\mathbf{Q}}$$

satisfying the following properties:

(1) *If U is regular and flat of dimension n over S , it is the composition*

$$\Gamma(V \times_U V, \mathbf{Z}) \longrightarrow \mathrm{Gr}_n^F G(V \times_U V) \xrightarrow{((, \Delta_V))^{\log}} F_0G(\partial_F V)_{\mathbf{Q}}$$

where the first arrow is the natural isomorphism.

(2) *Assume $U = \coprod_i U_i$ is a finite decomposition by smooth subschemes. Let $j_i: U_i \rightarrow U$ denote the immersion and put $V_i = V \times_U U_i$ for each i . Then, the diagram*

$$(6.2.5.1) \quad \begin{array}{ccc} \Gamma(V \times_U V, \mathbf{Z}) & \xrightarrow{((, \Delta_V))^{\log}} & F_0G(\partial_F V)_{\mathbf{Q}} \\ \downarrow (j_i^*)_i & & \uparrow \sum j_i! \\ \bigoplus_i \Gamma(V_i \times_{U_i} V_i, \mathbf{Z}) & \xrightarrow{\bigoplus_i ((, \Delta_{V_i}))^{\log}} & \bigoplus_i F_0G(\partial_F V_i)_{\mathbf{Q}} \end{array}$$

is commutative.

6.3. A semi-stable case. — In this subsection, we establish a crucial step in the proof of the conductor formula. Namely, in Proposition 6.3.2, we compare the log localized intersection products $((T, \Delta_V))^{\log}$ and $((\Gamma, \Delta_{V'}))^{\log}$ for a morphism $f: V' \rightarrow V$ using the Lefschetz trace formula Theorem 1.4.7, assuming among other things that f is extended to a weakly semi-stable morphism of compactifications.

We consider a commutative diagram

$$(6.3.1.1) \quad \begin{array}{ccc} U' & \longleftarrow & V' \\ \downarrow & & \downarrow f \\ U & \longleftarrow & V \end{array}$$

of separated schemes of finite type over S where the horizontal arrows are finite étale and the vertical arrows are smooth. We further consider a commutative diagram

$$(6.3.1.2) \quad \begin{array}{ccc} \Gamma & \longrightarrow & V' \times_{U'} V' \\ \downarrow & & \downarrow \\ T & \longrightarrow & V \times_U V \end{array}$$

where $T \subset V \times_U V$ and $\Gamma \subset V' \times_{U'} V'$ are open and closed subschemes.

Let $V_T^{(1)}$ and $V_T^{(2)}$ denote the base change $V' \times_V T$ with respect to the first and the second projections respectively. We identify the fiber product $V_T^{(1)} \times_T V_T^{(2)}$ with an open and closed subscheme $(V' \times_U V') \times_{V \times_U V} T$ of $V' \times_U V'$. Then, (6.3.1.2) implies that Γ is a closed subscheme of $V_T^{(1)} \times_T V_T^{(2)}$.

We compare the elements $((T, \Delta_V))^{\log}$ and $f((\Gamma, \Delta_{V'}))^{\log}$ of $F_0G(\partial_F V)_{\mathbf{Q}}$, assuming that we have a commutative diagram

$$(6.3.1.3) \quad \begin{array}{ccc} U' & \xrightarrow{c} & X' \supset B' \\ \uparrow & & \uparrow \\ V' & \xrightarrow{c} & Y' \supset D \\ f \downarrow & & \downarrow \bar{f} \\ V & \xrightarrow{c} & Y \supset E_j \end{array}$$

of separated schemes of finite type over S satisfying the following conditions:

- (6.3.1.4) The schemes Y, Y' and X' are proper over S and Y is the disjoint union of irreducible components. The scheme V is the complement in Y of a finite family $\mathcal{E} = (E_j)_{j \in J}$ of Cartier divisors and U' is the complement of a Cartier divisor B' of X' . The morphism $\bar{f}: Y' \rightarrow Y$ is proper weakly semi-stable of relative dimension d such that $Y'_V = Y' \times_Y V \rightarrow V$ is smooth. The subscheme D is a divisor of Y' over Y with simple normal crossings relatively to Y and $V' = Y'_V \setminus D_V$.

Let D_1, \dots, D_m be the irreducible components of D . Let $Y_T^{(1)}, D_1^{(1)}, \dots, D_m^{(1)}$ denote the base change of Y', D_1, \dots, D_m over Y by the composition $T \rightarrow V \times_U V \rightarrow V \rightarrow Y$ of the first projection. Similarly, we define $Y_T^{(2)}, D_1^{(2)}, \dots, D_m^{(2)}$ as the base change using the second projections. Let $(Y_T^{(1)} \times_T Y_T^{(2)})^{\sim}$ denote the log product with respect to the families of Cartier divisors $(D_1^{(1)}, \dots, D_m^{(1)})$ and $(D_1^{(2)}, \dots, D_m^{(2)})$.

Let K' be a finite extension of K and $\gamma': \text{Spec } K' \rightarrow T$ be a morphism over K . By the valuative criterion, the compositions $\text{Spec } K' \rightarrow T \rightarrow Y$ with the two projections are extended to $S' = \text{Spec } \mathcal{O}_{K'} \rightarrow Y$. Let $Y_{S'}^{(1)}$ and $Y_{S'}^{(2)}$ denote the base change with respect to the two morphisms respectively.

We regard Y as a log scheme with the log structure defined by the finite family of Cartier divisors $\mathcal{E} = (E_j)_{j \in J}$. We assume that the restrictions of $S' \rightarrow Y$ to the closed point $s' \in S'$ define the same log point $s' \rightarrow Y$. Then, we have a canonical isomorphism $\iota_{s'}: Y_{s'}^{(1)} \rightarrow Y_{s'}^{(2)}$. Hence, if the second projection $\Gamma \rightarrow V_T^{(2)}$ is proper, the alternating sum $\mathrm{Tr}((\gamma'^* \Gamma)^*: H_c^*(V'_{\bar{K}}, \mathbf{Q}_\ell))$ (1.4.2.3) is defined for a prime number ℓ invertible on S .

Proposition 6.3.2. — *Let the notations be as in (6.3.1.1)–(6.3.1.4) and assume either of (5.3.2.1a) or (5.3.2.1b) is satisfied. Let $T \subset V \times_U V \setminus \Delta_V$ be an open and closed subscheme and let $\bar{\Gamma}$ be a closed subscheme of $(Y_T^{(1)} \times_T Y_T^{(2)})^\sim$ flat over T such that $\Gamma = \bar{\Gamma} \cap (V_T^{(1)} \times_T V_T^{(2)})$ is an open subscheme of $V' \times_U V'$. We regard Y as a log scheme with the log structure defined by $\mathcal{E} = (E_j)_{j \in J}$. Let \mathcal{E}' be the finite family of Cartier divisors of Y' consisting of the pull-back of \mathcal{E} and the irreducible components of D . We assume that the generic fibers $\Sigma_{V/U}^{\mathcal{E}} Y_K$ and $\Sigma_{V'/U'}^{\mathcal{E}'} Y'_K$ are empty.*

Then, there exist a finite family $(\mathbf{K}_i)_{i \in I}$ of finite extensions of \mathbf{K} , a family $(\gamma_i: \mathrm{Spec} \mathbf{K}_i \rightarrow T)_{i \in I}$ of morphisms over S and rational numbers $(r_i)_{i \in I}$ satisfying the following properties:

Let $s_i \in S_i = \mathrm{Spec} \mathcal{O}_{\mathbf{K}_i}$ denote the closed point for $i \in I$. Then, for each $i \in I$, the log points $\bar{\gamma}_i: s_i \rightarrow Y$ defined by the unique maps $S_i \rightarrow Y$ extending the composition $\mathrm{Spec} \mathbf{K}_i \rightarrow T \rightarrow Y$ with the first and the second projections are equal to each other. Further, for a prime number ℓ invertible on S , we have $\mathrm{Tr}((\gamma_i^ \Gamma)^*: H_c^*(V'_{\bar{K}_i}, \mathbf{Q}_\ell)) \in \mathbf{Q}$ and, for the logarithmic product (5.3.8.4),*

$$(6.3.2.1) \quad ((T, \Delta_Y^{\log})) = \sum_i r_i [\bar{\gamma}_i(s_i)],$$

$$(6.3.2.2) \quad \bar{f}_*((\Gamma, \Delta_Y^{\log})) = \sum_i r_i \mathrm{Tr}((\gamma_i^* \Gamma)^*: H_c^*(V'_{\bar{K}_i}, \mathbf{Q}_\ell)) \cdot [\bar{\gamma}_i(s_i)]$$

in $F_0 G(\Sigma_{V/U} Y)_{\mathbf{Q}}$.

Proof. — We take an object $\bar{g}: Z \rightarrow Y$ of the category $\mathcal{A}_{V \rightarrow U}$ of alterations. Since the conditions (5.3.2.1a) and (5.3.2.1b) are stable by the base change, we may assume that we have a log blow-up $Z' \rightarrow Z \times_Y Y'$ as in the conclusion of Lemma 5.3.2. Hence, the inverse image $W' = V' \times_{Y'} Z'$ is the complement D' of a divisor with simple normal crossings. By the assumption on the upper square in (6.3.1.3), $\bar{g}': Z' \rightarrow Y'$ defines an object of $\mathcal{A}_{V' \rightarrow U'}$.

Let $g: W \rightarrow V$ be the restriction of $\bar{g}: Z \rightarrow Y$. We put $(g \times g)^*(T) = \sum_j m_j T_j \in F_n G((W \times_S W) \times_{V \times_S V} T)$ and, for each j , let $\bar{T}_j \subset (Z \times_S Z)^\sim$ be the schematic closure. Then, we have

$$(6.3.2.3) \quad ((T, \Delta_Y^{\log})) = \frac{1}{[W:V]} \sum_j m_j \cdot \bar{g}_*((\bar{T}_j, \Delta_Z^{\log}))_{(Z \times_S Z)^\sim}.$$

We define the log products $(Y \times_S Y)^\sim$ and $(Y' \times_S Y')^\sim$ with respect to \mathcal{E} and \mathcal{E}' . The log product $(Y_T^{(1)} \times_T Y_T^{(2)})^\sim$ defined with respect to the pull-backs of the irreducible components of D is canonically identified with $(Y' \times_S Y')^\sim \times_{(Y \times_S Y)^\sim} T$. We define

$(Z' \times_S Z')^\sim$ to be the log product with respect to the irreducible components of a divisor $Z' \setminus (Z' \times_Z W)$ with simple normal crossings and the pull-backs of the irreducible components of $D \subset Y'$. The inverse image of $W \times_S W$ by $(Z' \times_S Z')^\sim \rightarrow (Z \times_S Z)^\sim$ is canonically identified with $(Y' \times_S Y')^\sim \times_{(Y \times_S Y)^\sim} (W \times_S W)$.

For T_j as above, both $(Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})^\sim \times_T T_j$ and $(Z' \times_S Z')^\sim \times_{(Z \times_S Z)^\sim} T_j$ are identified with $(Y' \times_S Y')^\sim \times_{(Y \times_S Y)^\sim} T_j$. We regard $\tilde{\Gamma}_j = \tilde{\Gamma} \times_T T_j$ as a closed subscheme of $(Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})^\sim \times_T T_j = (Z' \times_S Z')^\sim \times_{(Z \times_S Z)^\sim} T_j$. By the assumption that $\tilde{\Gamma} \subset (Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})^\sim$ is flat over T and by the flattening theorem [36], there exists a proper modification $\bar{q}_j: \bar{T}'_j \rightarrow \bar{T}_j$ for each \bar{T}_j satisfying the following conditions: The map $\bar{q}_j: \bar{T}'_j \rightarrow \bar{T}_j$ induces the identity on the dense open subscheme T_j and the schematic closure $\bar{\Gamma}'_j$ of $\tilde{\Gamma}_j \subset (Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})^\sim \times_T T_j = (Z' \times_S Z')^\sim \times_{(Z \times_S Z)^\sim} T_j$ in the base change $(Z' \times_S Z')^\sim \times_{(Z \times_S Z)^\sim} \bar{T}'_j$ is flat over \bar{T}'_j . Then, similarly as (6.3.2.3), we have

$$(6.3.2.4) \quad ((\Gamma, \Delta_{Y'}^{\log})) = \frac{1}{[W:V]} \sum_j m_j \cdot \bar{q}_{j*} \bar{q}_{j*}((\bar{\Gamma}'_j, \Delta_{Z'}^{\log}))_{(Z' \times_S Z')^\sim}.$$

For each j , we put

$$(6.3.2.5) \quad ((\bar{T}'_j, \Delta_Z^{\log}))_{(Z \times_S Z)^\sim} = \sum n_i [s'_i]$$

in $F_0G(\bar{T}'_j \times_{(Z \times_S Z)^\sim} \Delta_Z^{\log})$. Since $s'_i \in \bar{T}'_j$ is a closed point and T_j is dense in \bar{T}'_j , there exist a discrete valuation field K_i and a map $\gamma_i: S_i = \text{Spec } \mathcal{O}_{K_i} \rightarrow \bar{T}_j$ extending $\text{Spec } K_i \rightarrow T_j$ such that s'_i is the image of the closed point S_i of S_i . Since the image of s_i in $(Y \times_S Y)^\sim$ is in the log diagonal, the log points $s_i \rightarrow Y$ defined by the two projections are equal to each other. Thus, by (6.3.2.3), we obtain (6.3.2.1).

We prove the equality (6.3.2.2). We fix j and let $p_j: \bar{\Gamma}'_j \rightarrow \bar{T}'_j$ denote the projection. First we show

$$(6.3.2.6) \quad p_{j*}((\bar{\Gamma}'_j, \Delta_{Z'}^{\log}))_{(Z' \times_S Z')^\sim} = \sum_i n_i \deg(\bar{\Gamma}'_{j, s'_i}, \Delta_{Z'_i}^{\log})_{(Z'_i \times_{S'_i} Z'_i)^\sim} \cdot [s'_i]$$

in $F_0G(\bar{T}'_j \times_{(Z \times_S Z)^\sim} \Delta_Z^{\log})$. Since $Z' \rightarrow Z$ is log smooth, the morphism $(Z' \times_S Z')^\sim \rightarrow (Z \times_S Z)^\sim$ is smooth and the log diagonal map $Z' \rightarrow (Z' \times_Z Z')^\sim = (Z' \times_S Z')^\sim \times_{(Z \times_S Z)^\sim} Z$ is a regular immersion. By applying the associativity Lemma 3.3.6 to $(Z' \times_S Z')^\sim \leftarrow (Z' \times_Z Z')^\sim \leftarrow Z'$, we obtain

$$((\bar{\Gamma}'_j, \Delta_{Z'}^{\log}))_{(Z' \times_S Z')^\sim} = (((\bar{\Gamma}'_j, (Z' \times_Z Z')^\sim))_{(Z' \times_S Z')^\sim}, \Delta_{Z'}^{\log})_{(Z' \times_Z Z')^\sim}.$$

Since $(Z' \times_S Z')^\sim$ is smooth over $(Z \times_S Z)^\sim$, it is tor-independent with Z . Hence by applying the associativity Lemma 3.3.7 to $(Z \times_S Z)^\sim \leftarrow (Z' \times_S Z')^\sim \leftarrow \bar{\Gamma}'_j$ as $X \leftarrow X' \leftarrow W$, we obtain

$$((\bar{\Gamma}'_j, (Z' \times_Z Z')^\sim))_{(Z' \times_S Z')^\sim} = ((\Delta_Z^{\log}, \bar{\Gamma}'_j))_{(Z \times_S Z)^\sim}.$$

Since $\rho_j: \bar{\Gamma}'_j \rightarrow \bar{\Gamma}'_j$ is flat, by further applying the associativity Lemma 3.3.6 to $(Z \times_S Z)^\sim \leftarrow \bar{\Gamma}'_j \leftarrow \bar{\Gamma}'_j$, we obtain

$$\left((\Delta_Z^{\log}, \bar{\Gamma}'_j) \right)_{(Z \times_S Z)^\sim} = \rho_j^* \left((\bar{\Gamma}'_j, \Delta_Z^{\log}) \right)_{(Z \times_S Z)^\sim}.$$

Thus, we obtain

$$\left((\bar{\Gamma}'_j, \Delta_{Z'}^{\log}) \right)_{(Z' \times_S Z)^\sim} = \left(\rho_j^* \left((\bar{\Gamma}'_j, \Delta_Z^{\log}) \right)_{(Z \times_S Z)^\sim}, \Delta_{Z'}^{\log} \right)_{(Z' \times_Z Z)^\sim}.$$

Substituting (6.3.2.5), we see that the right hand side is equal to

$$\sum_i n_i \left(\rho_j^* [s'_i], \Delta_{Z'}^{\log} \right)_{(Z' \times_Z Z)^\sim} = \sum_i n_i \left(\bar{\Gamma}'_{j, s'_i}, \Delta_{Z'_i}^{\log} \right)_{(Z'_i \times_{S_i} Z'_i)^\sim}$$

in $F_0 G(\bar{\Gamma}'_j \times_{(Z \times_S Z)^\sim} \Delta_Z^{\log})$. Thus the equality (6.3.2.6) is proved.

For the right hand side of (6.3.2.6), we show

$$(6.3.2.7) \quad \deg \left(\bar{\Gamma}'_{j, s'_i}, \Delta_{Z'_i}^{\log} \right)_{(Z'_i \times_{S_i} Z'_i)^\sim} = \text{Tr} \left((\gamma_i^* \Gamma)^* : H_c^*(V'_{\bar{K}_i}, \mathbf{Q}_\ell) \right)$$

by applying Theorem 1.4.7 to the base changes $Z' \times_Z S_i \rightarrow S_i = \text{Spec } \mathcal{O}_{K_i}$. To apply it, we verify that the assumptions are satisfied. The log blow-up of the product $(Y'_T)^{(1)} \times_T Y'^{(2)}$ with respect to the families of Cartier divisors $(D_1^{(1)}, \dots, D_m^{(1)})$ and $(D_1^{(2)}, \dots, D_m^{(2)})$ contains the log product $(Y'_T)^{(1)} \times_T Y'^{(2)}$ as the complement of the proper transforms $(D_T^{(1)} \times_T Y_T^{(2)})'$ and $(Y_T^{(1)} \times_T D_T^{(1)})'$. Let Γ' be the closure of $\tilde{\Gamma}$ in the product $(Y'_T)^{(1)} \times_T Y'^{(2)}$. We show

$$(6.3.2.8) \quad \Gamma' \cap (D_T^{(1)} \times_T Y_T^{(2)})' = \Gamma' \cap (Y_T^{(1)} \times_T D_T^{(1)})'.$$

Let $(Y'_U \times_U Y'_U)'$ denote the log product with respect to the families of Cartier divisors $(D_{1,U}, \dots, D_{m,U})$. We have an open immersion $(Y'_T)^{(1)} \times_T Y'^{(2)} \rightarrow (Y'_U \times_U Y'_U)'$ as the base change of $T \rightarrow V \times_U V$. Since $\Gamma \subset V' \times_{U'} V'$, the image of Γ' by $(Y'_T)^{(1)} \times_T Y'^{(2)} \subset (Y'_U \times_U Y'_U)'$ is in the diagonal X'_U . Since $U' \subset X'$ is assumed to be the complement of a Cartier divisor B' , we have $\Gamma' \cap (D_U \times_U Y'_U)' = \Gamma' \cap (Y'_U \times_U D_U)'$ by [27, Proposition 1.1.6.2]. Thus, we obtain (6.3.2.8). Consequently, the base change to K_i satisfies the inclusion (1.4.7.1).

We construct a commutative diagram (1.4.2.4) of monoids for the two base changes $Z' \times_Z S_i \rightarrow S_i = \text{Spec } \mathcal{O}_{K_i}$ satisfying the condition (P) loc. cit. Since the log structures of Z and Z' are defined by divisors with simple normal crossings, we have a commutative diagram

$$(6.3.2.9) \quad \begin{array}{ccc} \mathbf{N}^m & \longrightarrow & \mathbf{N}^{m'} \\ \downarrow & & \downarrow \\ \Gamma(Z, \bar{\mathcal{M}}_Z) & \longrightarrow & \Gamma(Z', \bar{\mathcal{M}}_{Z'}) \end{array}$$

of morphisms of monoids, locally lifted to charts. Since the image of the closed point by the composition $S_i = \text{Spec } \mathcal{O}_{K_i} \rightarrow \bar{T}_j \rightarrow (Z \times_S Z)^\sim$ lies in the log diagonal, the compositions with the two propositions define the same log points $s_i \rightarrow Z$. Hence there exists one morphism $\mathbf{N}^m \rightarrow \mathbf{N}$ of monoids that makes the diagram

$$(6.3.2.10) \quad \begin{array}{ccc} \mathbf{N} & \longleftarrow & \mathbf{N}^m \\ \downarrow & & \downarrow \\ \Gamma(S_i, \bar{\mathcal{M}}_{S_i}) & \longleftarrow & \Gamma(Z, \bar{\mathcal{M}}_Z). \end{array}$$

commutative for the compositions with the two projections. We define a monoid \mathbf{P} to be the saturated sum $\mathbf{N} +_{\mathbf{N}^m}^{\text{sat}} \mathbf{N}^{m'}$. Then, the commutative diagrams (6.3.2.9) and (6.3.2.10) induces a morphism $\mathbf{P} \rightarrow \Gamma(Z' \times_Z S_i, \bar{\mathcal{M}}_{Z' \times_Z S_i})$, locally lifted to charts. It defines a commutative diagram (1.4.2.4) of monoids satisfying the condition (P). Thus we may apply Theorem 1.4.7 and we obtain (6.3.2.7).

Therefore the equality (6.3.2.2) follows from (6.3.2.4), (6.3.2.6) and (6.3.2.7). \square

By the same argument, in the case where \mathbf{K} is of characteristic 0, we have the following.

Proposition 6.3.3. — *Assume that \mathbf{K} is of characteristic 0 and let the notations be as in (6.3.1.1)–(6.3.1.4). We assume that either (5.3.2.1a) or (5.3.2.1b) is satisfied. Let $\mathbf{T} \subset \mathbf{V} \times_{\mathbf{U}} \mathbf{V}$ be an open and closed subscheme and let $\tilde{\Gamma}$ be a closed subscheme of the log product $(\mathbf{Y}_{\mathbf{T}}^{(1)} \times_{\mathbf{T}} \mathbf{Y}_{\mathbf{T}}^{(2)})^\sim$ flat over \mathbf{T} such that $\Gamma = \tilde{\Gamma} \cap (\mathbf{V}_{\mathbf{T}}^{(1)} \times_{\mathbf{T}} \mathbf{V}_{\mathbf{T}}^{(2)})$ is an open subscheme of $\mathbf{V}' \times_{\mathbf{U}'} \mathbf{V}'$. We regard \mathbf{Y} as a log scheme with the log structure defined by a finite family of Cartier divisors $\mathcal{E} = (\mathbf{E}_j)_{j \in \mathbf{J}}$ satisfying $\mathbf{V} = \mathbf{Y} \setminus \bigcup_{j \in \mathbf{J}} \mathbf{E}_j$.*

Then, there exist a finite family $(\mathbf{K}_i)_{i \in \mathbf{I}}$ of finite extensions of \mathbf{K} , a family $(\gamma_i: \text{Spec } \mathbf{K}_i \rightarrow \mathbf{T})_{i \in \mathbf{I}}$ of morphisms over \mathbf{S} and rational numbers $(r_i)_{i \in \mathbf{I}}$ satisfying the following properties:

Let $s_i \in S_i = \text{Spec } \mathcal{O}_{K_i}$ denote the closed point for $i \in \mathbf{I}$. Then, for each $i \in \mathbf{I}$, the log points $\bar{\gamma}_i: s_i \rightarrow \mathbf{Y}$ defined by the unique maps $S_i \rightarrow \mathbf{Y}$ extending the composition $\text{Spec } \mathbf{K}_i \rightarrow \mathbf{T} \rightarrow \mathbf{Y}$ with the first and the second projections are equal to each other. Further for a prime number ℓ invertible on \mathbf{S} , we have

$$(6.3.3.1) \quad ((\mathbf{T}, \Delta_{\mathbf{Y}}^{\log})) = \sum_i r_i [\bar{\gamma}_i(s_i)],$$

$$(6.3.3.2) \quad \bar{f}_*((\Gamma, \Delta_{\mathbf{V}'}^{\log})) = \sum_i r_i \text{Tr}((\gamma_i^* \Gamma)^*: \mathbf{H}_c^*(\mathbf{V}'_{\bar{K}_i}, \mathbf{Q}_\ell)) \cdot [\bar{\gamma}_i(s_i)]$$

in $\mathbf{F}_0 \mathbf{G}(\mathbf{Y} \times_{\mathbf{S}} \mathbf{F})_{\mathbf{Q}}$.

7. The Swan class and a conductor formula

We keep the notation that \mathbf{K} is a complete discrete valuation ring and $\mathbf{S} = \text{Spec } \mathcal{O}_{\mathbf{K}}$ as in the previous sections. We fix a prime number ℓ different from the characteristic p of

the residue field F of K . First, we define the Swan character classes for Galois coverings in Section 7.1. We define the Swan class of a locally constant $\overline{\mathbf{F}}_\ell$ -sheaf in Section 7.2. We extend the definition of the Swan class to a constructible sheaf in Section 7.4 using the excision formula Proposition 7.2.5.2, assuming K is of characteristic 0. We prove a conductor formula for some relative curves in Section 7.3 and derive the general case in Section 7.5. In an equal characteristic case, more elementary proof is found in [42, Corollaries 5.12, 5.13].

In this paper, we state and prove results for $\overline{\mathbf{F}}_\ell$ -sheaves. The corresponding results for $\overline{\mathbf{Q}}_\ell$ -sheaves are obtained simply by taking reduction modulo ℓ .

7.1. Swan character classes. — We define the Swan character class for a Galois covering.

Definition 7.1.1. — Let U be a separated regular flat scheme of finite type over S and $f: V \rightarrow U$ be a finite étale G -torsor for a finite group G such that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K . Then, for an element $\sigma \in G$, we define the Swan character class $s_{V/U}(\sigma) \in F_0G(\partial_{V/U}V)_{\mathbf{Q}}$ by

$$(7.1.1.1) \quad s_{V/U}(\sigma) = \begin{cases} D_{V/U}^{\log} & \text{for } \sigma = 1 \\ -((\Gamma_\sigma, \Delta_V))^{\log} & \text{for } \sigma \neq 1. \end{cases}$$

By Corollaries 5.1.3 and 5.1.8, for a finite Galois extension L of K of Galois group G and $U = \text{Spec } K$, $V = \text{Spec } L$, we have

$$(7.1.1.2) \quad s_{V/U}(\sigma) = \begin{cases} \text{length}_{\mathcal{O}_L} \Omega_{\mathcal{O}_L/\mathcal{O}_K}^1(\log/\log) & \text{for } \sigma = 1 \\ -\text{length}_{\mathcal{O}_L} \mathcal{O}_L/J_\sigma & \text{for } \sigma \neq 1 \end{cases}$$

in $F_0G(\partial_{V/U}V) = \mathbf{Z}$, where J_σ is the ideal of \mathcal{O}_L generated by $\sigma(a) - a$ for $a \in \mathcal{O}_L$ and $\sigma(b)/b - 1$ for $b \in L^\times$.

Lemma 7.1.2. — Let the notation be as in Definition 7.1.1. Then, the following hold:

1. We have

$$\sum_{\sigma \in G} s_{V/U}(\sigma) = 0.$$

2. Let H be a subgroup of G and $g: V \rightarrow U'$ be the corresponding H -torsor. Then, for $\sigma \in H$, we have

$$s_{V/U}(\sigma) = \begin{cases} s_{V/U'}(1) + g^* D_{U'/U}^{\log} & \text{if } \sigma = 1 \\ s_{V/U'}(\sigma) & \text{if } \sigma \neq 1. \end{cases}$$

3. Let N be a normal subgroup of G and $G' = G/N$ be the quotient. Let $g: V \rightarrow V'$ be the corresponding N -torsor. Then, for $\sigma' \in G'$, we have

$$g^*_{s_{V'/U}}(\sigma') = \sum_{\sigma \in G, \bar{\sigma} = \sigma'} s_{V/U}(\sigma).$$

Proof. — 1. Clear from Lemma 5.4.4.2.

2. For $\sigma = 1$, it follows from Lemma 5.4.4.1. For $\sigma \neq 1$, it is clear from the definition.

3. Clear from Lemma 5.4.4.3. □

Corollary 7.1.3. — *If the order of $\sigma \in G$ is not a power of p , we have*

$$s_{V/U}(\sigma) = 0.$$

Proof. — By Lemma 7.1.2.2, we may assume that G is the cyclic group generated by σ . Assume the order of σ is not a power of p . Let $N \subset G$ be the p -Sylow subgroup and $U' \rightarrow U$ be the corresponding $G' = G/N$ -torsor. Then, since the order of G' is prime to p , the finite étale morphism $U' \rightarrow U$ is tamely ramified with respect to S by Corollary 2.4.5. Hence, it follows from Corollary 5.4.2 applied to $V' = V$. □

For an element $\sigma \in G$ of order a power of p and an integer i prime to p , Conjecture 5.4.6 predicts

$$s_{V/U}(\sigma) = s_{V/U}(\sigma^i).$$

Corollary 7.1.4. — *Let the notation be as in Definition 7.1.1. Let X be a normal proper scheme over S containing U as a dense open subscheme and let Y be the normalization of X in V . Let $\sigma \in G$ be an element not contained in any conjugate of a p -Sylow group of the inertia group $I_{\bar{y}} \subset G$ for any geometric point \bar{y} of Y . Then, we have*

$$s_{V/U}(\sigma) = 0.$$

Proof. — By Corollary 7.1.3, it suffices to consider the case where the order of σ is a power of p . As in the proof of Corollary 7.1.3, we may assume that G is the cyclic group generated by σ . Let $N \subset G$ be the unique maximal proper subgroup generated by σ^p . Let $V' \rightarrow U$ be the corresponding $G' = G/N$ -torsor and let Y' be normalization of X in V' . Then, by the assumption, the inertia group at every geometric point is a subgroup of N and hence $Y' \rightarrow X$ is étale. Hence the assertion follows from Corollary 5.4.2. □

7.2. *Swan class of a locally constant sheaf.* — We briefly recall the Brauer trace of an ℓ -regular element [18]. Let G be a finite group and ℓ be a prime number. An element $\sigma \in G$ is called an ℓ -regular element if the order of σ is prime to ℓ . Let $G^{(\ell)}$ denote the

subset of G consisting of ℓ -regular elements. For an element σ of a pro-finite group, we say that σ is ℓ -regular if it is a projective limit of ℓ -regular elements.

Let M be an $\overline{\mathbf{F}}_\ell$ -vector space of finite dimension n and σ be an automorphism of M of order prime to ℓ . Then the Brauer trace $\mathrm{Tr}^{\mathrm{Br}}(\sigma : M) \in \mathbf{Z}_\ell^{\mathrm{ur}}$ is defined as follows. Let $\alpha_1, \dots, \alpha_n \in \overline{\mathbf{F}}_\ell^\times$ be the eigenvalues of σ on M counted with multiplicities and $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in \mathbf{Z}_\ell^{\mathrm{ur}\times}$ be the liftings of finite orders prime to ℓ . Then the Brauer trace is defined by $\mathrm{Tr}^{\mathrm{Br}}(\sigma : M) = \sum_{i=1}^n \tilde{\alpha}_i$.

Let $f : V \rightarrow U$ be a finite étale G -torsor for a finite group G such that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K . By Corollary 7.1.3, we have $s_{V/U}(\sigma) = 0$, if the order of σ is not a power of p . In the following, let $G_{(p)}$ denote the subset of G consisting of elements of order a power of p . For $\ell \neq p$, we have $G_{(p)} \subset G^{(\ell)}$. We put $F_0G(\partial_{V/U}V)_{\mathbf{Q}(\zeta_{p^\infty})} = F_0G(\partial_{V/U}V)_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta_{p^\infty})$, $F_0G(\partial_{V/U}V)_{\mathbf{Z}[\zeta_{p^\infty}]} = F_0G(\partial_{V/U}V) \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^\infty}]$ etc.

Definition 7.2.1. — *Let U be a regular flat separated scheme of finite type over $S = \mathrm{Spec} \mathcal{O}_K$ and let \mathcal{F} be a locally constant constructible $\overline{\mathbf{F}}_\ell$ -sheaf on U . Let $f : V \rightarrow U$ be a finite étale G -torsor for a finite group G such that $f^*\mathcal{F}$ is a constant sheaf on V . We assume that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K . Let M be the $\overline{\mathbf{F}}_\ell$ -representation of G corresponding to \mathcal{F} .*

Then, we define the Swan class $\mathrm{Sw}_{V/U}\mathcal{F} \in F_0G(\partial_{V/U}V)_{\mathbf{Q}(\zeta_{p^\infty})}$ by

$$(7.2.1.1) \quad \mathrm{Sw}_{V/U}\mathcal{F} = \sum_{\sigma \in G_{(p)}} \mathrm{Tr}^{\mathrm{Br}}(\sigma : M) \cdot s_{V/U}(\sigma).$$

By Lemma 7.1.2.1 and $\mathrm{Tr}^{\mathrm{Br}}(1 : M) = \dim M$, the defining equality (7.2.1.1) is equivalent to the following:

$$(7.2.1.2) \quad \mathrm{Sw}_{V/U}\mathcal{F} = \sum_{\sigma \in G_{(p)}, \neq 1} (\dim M - \mathrm{Tr}^{\mathrm{Br}}(\sigma : M)) \cdot ((\Gamma_\sigma, \Delta_V))^{\log}.$$

Recall that in [27], the Swan class is defined similarly for a locally constant sheaf on a smooth scheme over a perfect field and is called the naive Swan class and is denoted by Sw' . Modifying the notation, we remove “'”. If we assume Conjecture 5.4.6 asserting that $s_{V/U}(\sigma) = s_{V/U}(\sigma^i)$ for an integer i prime to p , the Swan class $\mathrm{Sw}_{V/U}\mathcal{F}$ is in fact defined in the subspace $F_0G(\partial_{V/U}V)_{\mathbf{Q}} \subset F_0G(\partial_{V/U}V)_{\mathbf{Q}(\zeta_{p^\infty})}$.

If we assume a strong form of resolution of singularity, the Swan character class is defined integrally (5.3.8.6) and hence the Swan class $\mathrm{Sw}_{V/U}\mathcal{F}$ is defined as an element of $F_0G(\partial_{V/U}V)_{\mathbf{Z}[\zeta_{p^\infty}]}$. Further, if we assume Conjecture 5.4.6, the Swan class $\mathrm{Sw}_{V/U}\mathcal{F}$ is in fact defined integrally in the subgroup $F_0G(\partial_{V/U}V) \subset F_0G(\partial_{V/U}V)_{\mathbf{Z}[\zeta_{p^\infty}]}$. When, we emphasize that it is defined integrally, we write $\mathrm{Sw}_{V/U}^{\mathbf{Z}}\mathcal{F}$ and call it the integral Swan class. Note that Conjecture 5.4.6 itself is a consequence of a strong form of equivariant resolution of singularities.

Similarly as [27, Lemma 4.3.10], we have the following analogue of [18, Théorème 2.1].

Proposition 7.2.2. — (cf. [43, Corollaire 3.4]) *Let U be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and \mathcal{F}_1 and \mathcal{F}_2 be locally constant constructible sheaves of $\bar{\mathbf{F}}_\ell$ -modules on U . Let $f: V \rightarrow U$ be a finite étale G -torsor for a finite group G such that $f^*\mathcal{F}_1$ and $f^*\mathcal{F}_2$ are constant on V and that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K .*

Let X be a proper normal scheme over S containing U as a dense open subscheme. Assume that, for every geometric point \bar{x} of X , the restriction to a p -Sylow subgroup of the inertia group $I_{\bar{x}}$ of the representations M_1 and M_2 of G corresponding to \mathcal{F}_1 and \mathcal{F}_2 are isomorphic. Then, we have $\text{Sw}_{V/U}\mathcal{F}_1 = \text{Sw}_{V/U}\mathcal{F}_2$.

Proof. — It follows from (7.2.1.1) and Corollary 7.1.4. □

Lemma 7.2.3. — *Let U, V, G and \mathcal{F} be as in Definition 7.2.1. Let $f': V' \rightarrow U$ be a finite étale G' -torsor for a finite group G' such that $f'^*\mathcal{F}$ is a constant sheaf on V' . We assume that $V'_K \rightarrow U_K$ is tamely ramified with respect to $\text{Spec } K$. Let $g: V' \rightarrow V$ be a morphism over U compatible with a group homomorphism $G' \rightarrow G$. Then, we have*

$$\text{Sw}_{V'/U}\mathcal{F} = g^*\text{Sw}_{V/U}\mathcal{F}$$

in $F_0G(\partial_{V/U}V')_{\mathbf{Q}(\zeta_{p^\infty})}$.

Proof. — It follows from the definition and Lemma 7.1.2.3. □

By Lemma 7.2.3, the Swan class $\text{Sw}_{V/U}\mathcal{F}$ is G -invariant and hence $\frac{1}{|G|}f_*\text{Sw}_{V/U}\mathcal{F} \in F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_{p^\infty})}$ is independent of the choice of a Galois covering V trivializing \mathcal{F} . Thus the following definition makes sense.

Definition 7.2.4. — *Let U be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and \mathcal{F} be a locally constant constructible sheaf of $\bar{\mathbf{F}}_\ell$ -modules on U .*

1. *We say that \mathcal{F} is tamely ramified on the generic fiber if there exists a finite étale surjective morphism $f: V \rightarrow U$ such that $f^*\mathcal{F}$ is constant on V and that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K .*

2. *Assume that \mathcal{F} is tamely ramified on the generic fiber and that U is connected. Let $f: V \rightarrow U$ be a finite étale G -torsor for a finite group G such that $f^*\mathcal{F}$ is constant on V and that \mathcal{F} corresponds to a faithful $\bar{\mathbf{F}}_\ell$ -representation of G .*

Then, we put

$$F_0G(\partial_{\mathcal{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})} = F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_{p^\infty})}$$

and define the Swan class

$$\text{Sw}_U\mathcal{F} = \frac{1}{|G|}f_*\text{Sw}_{V/U}\mathcal{F}$$

to be the image by the isomorphism from the G -fixed part

$$\frac{1}{|G|}f_* : F_0G(\partial_{V/U}V)_{\mathbf{Q}(\zeta_{p^\infty})}^G \rightarrow F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_{p^\infty})} = F_0G(\partial_{\mathcal{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})}.$$

If U is not connected, we define the Swan class componentwise.

For the G -torsor V in Definition 7.2.4.2, we have $\mathrm{Sw}_{V/U}\mathcal{F} = f^*\mathrm{Sw}_U\mathcal{F}$ by Lemma 5.2.5.2. If K is of characteristic 0, every locally constant sheaf on U is tamely ramified on the generic fiber. In the case where $U = \mathrm{Spec} K$ and \mathcal{F} is wildly ramified, the Swan class $\mathrm{Sw}_U\mathcal{F} \in F_0G(\partial_{\mathcal{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})} = \mathbf{Q}(\zeta_{p^\infty})$ is nothing but the Swan conductor

$$\mathrm{Sw}_K\mathcal{F} = \frac{1}{|G|} \sum_{\sigma \in G(\rho)} \mathrm{Tr}^{\mathrm{Br}}(\sigma : M) \cdot f_{*\mathcal{S}_{L/K}}(\sigma)$$

by (7.1.1.2), known to be an integer ≥ 1 .

The Swan classes satisfy the following additivity and the excision formula.

Proposition 7.2.5. — *Let U be a regular flat separated scheme of finite type over $S = \mathrm{Spec} \mathcal{O}_K$ and \mathcal{F} be a locally constant constructible sheaf of $\bar{\mathbf{F}}_\ell$ -modules on U . We assume that \mathcal{F} is tamely ramified on the generic fiber.*

1. *For an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of locally constant constructible sheaves of $\bar{\mathbf{F}}_\ell$ -modules on U , we have*

$$(7.2.5.1) \quad \mathrm{Sw}_U\mathcal{F} = \mathrm{Sw}_U\mathcal{F}' + \mathrm{Sw}_U\mathcal{F}''$$

in $F_0G(\partial_{\mathcal{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})}$.

2. *Let $U_1 \subset U$ be a regular closed subscheme and $U_0 = U \setminus U_1$ be the complement. For the immersions $i : U_1 \rightarrow U$ and $j : U_0 \rightarrow U$, we have*

$$(7.2.5.2) \quad \mathrm{Sw}_U\mathcal{F} = j_!\mathrm{Sw}_{U_0}\mathcal{F}|_{U_0} + i_*\mathrm{Sw}_{U_1}\mathcal{F}|_{U_1}$$

in $F_0G(\partial_{\mathcal{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})}$.

Proof. — 1. Clear from (7.2.1.1).

2. It follows from (7.2.1.1) and the excision formula Theorem 6.2.2. \square

For a smooth $\bar{\mathbf{Q}}_\ell$ -sheaf \mathcal{F} on U , its Swan class $\mathrm{Sw}_U\mathcal{F}$ is defined as the Swan class $\mathrm{Sw}_U\bar{\mathcal{F}}$ of the reduction $\bar{\mathcal{F}} = \mathcal{F}_0/\lambda\mathcal{F}_0$ modulo ℓ . Though the $\bar{\mathbf{F}}_\ell$ -sheaf $\bar{\mathcal{F}}$ itself depend on the choice of a lattice \mathcal{F}_0 , is defined as the Swan class $\mathrm{Sw}_U\bar{\mathcal{F}}$ is well-defined by Proposition 7.2.5.1.

We prove an induction formula for the Swan classes. The following results are regarded as the relative conductor formula in the case of relative dimension 0.

Proposition 7.2.6. — *Let U be a regular flat separated scheme of finite type over S and $f: V \rightarrow U$ be a finite étale G -torsor for a finite group G such that the generic fiber $V_K \rightarrow U_K$ is tamely ramified with respect to K . Let $H \subset G$ be a subgroup and $g: V \rightarrow U'$ be the corresponding H -torsor. Let \mathcal{F} be a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on U' such that $g^*\mathcal{F}$ is a constant sheaf on V and let $h: U' \rightarrow U$ denote the canonical map.*

Let $T \subset G$ be a complete set of representatives of G/H . Then, we have

$$(7.2.6.1) \quad \mathrm{Sw}_{V/U} h_* \mathcal{F} = \sum_{\tau \in T} \tau_* (\mathrm{Sw}_{V/U'} \mathcal{F} + \mathrm{rank} \mathcal{F} \cdot g^* D_{U'/U}^{\log})$$

in $F_0 G(\partial_{V/U} V)_{\mathbf{Q}(\zeta_{\rho^\infty})}$.

Proof. — Let M be the representation of H corresponding to \mathcal{F} . Then, we have $\mathrm{Tr}^{\mathrm{Br}}(\sigma: \mathrm{Ind}_H^G M) = \sum_{\tau \in T, \tau^{-1}\sigma\tau \in H} \mathrm{Tr}^{\mathrm{Br}}(\tau^{-1}\sigma\tau: M)$. Hence, by the definition of the Swan class, the left hand side of (7.2.6.1) is equal to

$$(7.2.6.2) \quad \begin{aligned} & \sum_{\sigma \in G^{(\rho)}} \mathrm{Tr}^{\mathrm{Br}}(\sigma: \mathrm{Ind}_H^G M) \cdot s_{V/U}(\sigma) \\ &= \sum_{\sigma \in G^{(\rho)}} \sum_{\tau \in T, \tau^{-1}\sigma\tau \in H} \mathrm{Tr}^{\mathrm{Br}}(\tau^{-1}\sigma\tau: M) \cdot s_{V/U}(\sigma) \\ &= \sum_{\tau \in T} \tau_* \left(\sum_{\rho \in H^{(\rho)}} \mathrm{Tr}^{\mathrm{Br}}(\rho: M) \cdot s_{V/U}(\rho) \right). \end{aligned}$$

Thus, it follows from Lemma 7.1.2.2. □

Corollary 7.2.7. — *Let $f: U \rightarrow V$ be a finite étale morphism of regular flat schemes of finite type over S such that the generic fiber $U_K \rightarrow V_K$ is tamely ramified with respect to K . Let \mathcal{F} be a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on U such that there exists a finite étale morphism $g: U' \rightarrow U$ over S such that $g^*\mathcal{F}$ is constant on U' and that $U'_K \rightarrow V_K$ is tamely ramified with respect to K .*

Then, we have

$$(7.2.7.1) \quad \mathrm{Sw}_V f_* \mathcal{F} = f_* \mathrm{Sw}_U \mathcal{F} + \mathrm{rank} \mathcal{F} \cdot d_{U/V}^{\log}.$$

In particular, for $\mathcal{F} = \overline{\mathbf{F}}_\ell$, we obtain

$$(7.2.7.2) \quad \mathrm{Sw}_U f_* \overline{\mathbf{F}}_\ell = d_{U/V}^{\log}.$$

Proof. — It follows immediately from (7.2.1.1), (5.4.3.2), Proposition 7.2.6 and the remark on the Galois closure after Definition 2.4.1. □

We expect the following generalization of the Hasse-Arf theorem to hold.

Conjecture 7.2.8. — *The Swan class $\mathrm{Sw}_{\mathbf{U}}\mathcal{F}$ is in the image of the map $\mathrm{F}_0\mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}}\mathbf{U}) \rightarrow \mathrm{F}_0\mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}}\mathbf{U})_{\mathbf{Q}(\xi_p^\infty)}$.*

Note that Conjecture 7.2.8 is much stronger than the statement that the Swan class $\mathrm{Sw}_{\mathbf{V}/\mathbf{U}}\mathcal{F}$ is in the image of the map $\mathrm{F}_0\mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}}\mathbf{V}) \rightarrow \mathrm{F}_0\mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}}\mathbf{V})_{\mathbf{Q}(\xi_p^\infty)}$, which is, as we have seen above, a consequence of a strong form of equivariant resolution of singularities. We will prove Conjecture 7.2.8 in the case $\dim \mathbf{U}_{\mathbf{K}} \leq 1$ later at Corollary 8.3.8.

Conjecture 7.2.8 is related to the following conjecture of Serre.

Conjecture 7.2.9 ([41, Section 6]). — *Let \mathbf{A} be a regular local noetherian ring and \mathbf{G} be a finite group of automorphisms of \mathbf{A} . Assume that the fixed part $\mathbf{A}^{\mathbf{G}}$ is noetherian and that for every $\sigma \in \mathbf{G}$, $\sigma \neq 1$, the quotient $\mathbf{A}/\mathbf{I}_\sigma$ by the ideal $\mathbf{I}_\sigma = \langle \sigma(a) - a; a \in \mathbf{A} \rangle$ is of finite length. Then the \mathbf{Z} -valued function $a_{\mathbf{G}}$ of \mathbf{G} defined by*

$$a_{\mathbf{G}}(\sigma) = \begin{cases} \text{length } \mathbf{A}/\mathbf{I}_\sigma & \text{if } \sigma \neq 1 \\ -\sum_{\tau \in \mathbf{G}, \tau \neq 1} a_{\mathbf{G}}(\tau) & \text{if } \sigma = 1 \end{cases}$$

is a character of \mathbf{G} .

We prove Conjecture 7.2.9 in the case where $\dim \mathbf{A} = 2$ at the end of Section 8.3.

Lemma 7.2.10. — *Assume that the fraction field of \mathbf{A} is of characteristic 0 and that the residue field \mathbf{F} of \mathbf{A} is of characteristic $p > 0$. Then, Conjecture 7.2.8 for \mathbf{U} such that $n = \dim \mathbf{U}_{\mathbf{K}} + 1$ implies Conjecture 7.2.9 for \mathbf{A} of dimension n .*

Proof. — First, we consider the following special case. Let \mathbf{Y} be a regular flat separated scheme of finite type over $\mathbf{S} = \mathrm{Spec} \mathcal{O}_{\mathbf{K}}$ and $y \in \mathbf{Y}$ be a closed point in the closed fiber as in Proposition 5.1.7. Let \mathbf{G} be a finite group of automorphisms of \mathbf{Y} over \mathbf{S} such that, for every $\sigma \in \mathbf{G}$, $\sigma \neq 1$, the fixed part \mathbf{Y}^σ is equal to $\{y\}$ set-theoretically. We assume that the quotient $\mathbf{X} = \mathbf{Y}/\mathbf{G}$ exists as a scheme of finite type over \mathbf{S} . We put $\mathbf{V} = \mathbf{Y} \setminus \{y\}$ and $f: \mathbf{V} \rightarrow \mathbf{U} = \mathbf{V}/\mathbf{G} \subset \mathbf{X}$.

We show that Conjecture 7.2.8 for $f: \mathbf{V} \rightarrow \mathbf{U}$ and \mathbf{G} implies Conjecture 7.2.8 for $\mathbf{A} = \mathcal{O}_{\mathbf{Y}, y}$ and \mathbf{G} . We consider the image $s_{\mathbf{G}}(\sigma) \in \mathbf{Q}$ of $s_{\mathbf{V}/\mathbf{U}}(\sigma) \in \mathrm{F}_0\mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}}\mathbf{V})_{\mathbf{Q}}$ by $\mathrm{F}_0\mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}}\mathbf{V})_{\mathbf{Q}} \rightarrow \mathrm{F}_0\mathbf{G}(\Sigma_{\mathbf{V}/\mathbf{U}}\mathbf{Y})_{\mathbf{Q}} \rightarrow \mathrm{F}_0\mathbf{G}(\{y\})_{\mathbf{Q}} = \mathbf{Q}$. Conjecture 7.2.8 implies that the function $s_{\mathbf{G}}(\sigma)$ is a character of \mathbf{G} . By Proposition 5.1.7, we have $a_{\mathbf{G}} = r_{\mathbf{G}} - u_{\mathbf{G}} + s_{\mathbf{G}}$ where $r_{\mathbf{G}}$ and $u_{\mathbf{G}}$ denote the characters of the regular and the unit representations of \mathbf{G} respectively. Hence, the assertion is proved in this case.

We reduce the general case to the special case above similarly as in the proof of [28, Lemma (5.3)]. By replacing \mathbf{A} by the completion, we may assume \mathbf{A} and hence $\mathbf{A}^{\mathbf{G}}$ are complete. Let \mathbf{C} be a complete valuation ring such that p is a prime of \mathbf{C} and the residue field k is the same as that of $\mathbf{A}^{\mathbf{G}}$. Then, by [12, Chapitre 0, Théorème 19.8.8 (ii)] there exists a finite injection $\mathbf{C}[[t_1, \dots, t_{n-1}]] \rightarrow \mathbf{A}^{\mathbf{G}}$. Let $\mathbf{W}(k)$ be the ring of Witt vectors and

we take a local homomorphism $C \rightarrow W$. By replacing A by a factor of the completion $A \hat{\otimes}_C W$, we may assume k is algebraically closed and $C = W$. Then, the rest of the argument is the same as that in the proof of [28, Lemma (5.3)] by replacing $k[[t_1, \dots, t_{n-1}]]$ by $W[[t_1, \dots, t_{n-1}]]$ and $k\{t_1, \dots, t_{n-1}\}$ by the strict localization $W\{t_1, \dots, t_{n-1}\}$. \square

Conjecture 5.4.6 implies the weaker statement that the Swan class $\text{Sw}_U \mathcal{F}$ is in the image of the map $F_0 G(\partial_{V/U} U)_{\mathbf{Q}} \rightarrow F_0 G(\partial_{V/U} U)_{\mathbf{Q}(\zeta_{p^\infty})}$. Since we do not know the proof of Conjecture 5.4.6 in general, we make the following definition. Let $\text{pr}_{\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}} : F_0 G(\partial_{V/U} U)_{\mathbf{Q}(\zeta_{p^\infty})} \rightarrow F_0 G(\partial_{V/U} U)_{\mathbf{Q}}$ be the projection induced by $\lim_{\rightarrow n} \frac{1}{[\mathbf{Q}(\zeta_{p^n}):\mathbf{Q}]} \text{Tr}_{\mathbf{Q}(\zeta_{p^n})/\mathbf{Q}}$.

Definition 7.2.11. — *The rationalized Swan class $\text{Sw}_U^{\mathbf{Q}} \mathcal{F} \in F_0 G(\partial_{V/U} U)_{\mathbf{Q}}$ is the image of the Swan class $\text{Sw}_U \mathcal{F}$ by the projection $\text{pr}_{\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}} : F_0 G(\partial_{V/U} U)_{\mathbf{Q}(\zeta_{p^\infty})} \rightarrow F_0 G(\partial_{V/U} U)_{\mathbf{Q}}$.*

In [27], the rationalized Swan class is called the Swan class and is denoted by Sw . By modifying the notation there we write the Swan class by Sw and the rationalized Swan class by $\text{Sw}^{\mathbf{Q}}$.

7.3. Conductor formula for a relative curve. — Let $f : U' \rightarrow U$ be a smooth morphism of separated regular flat schemes of finite type over S and \mathcal{F} be a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on U' . Let $\pi' : V' \rightarrow U'$ be a finite étale morphism such that the pull-back $\pi'^* \mathcal{F}$ is a constant sheaf on V' . We assume that the following conditions are satisfied:

- (7.3.0.1) There exists a proper smooth scheme X' over U containing U' as the complement $U' = X' \setminus D$ of a divisor D with simple normal crossing relatively to U . The finite étale morphism $\pi' : V' \rightarrow U'$ is tamely ramified with respect to X' .

Then, by [17], the higher direct images $R^q f_* \mathbf{F}_\ell$ and $R^q f_* \mathcal{F}$ are locally constant sheaves on U . Further, for the alternating sum of ranks, we have

$$(7.3.0.2) \quad \text{rank } R^q f_* \mathcal{F} = \text{rank } \mathcal{F} \cdot \text{rank } R^q f_* \overline{\mathbf{F}}_\ell.$$

We assume that the locally constant sheaves $R^q f_* \mathcal{F}$ on U are tamely ramified on the generic fiber and that \mathcal{F} on U' is tamely ramified on the generic fiber. Then the Swan class

$$\text{Sw}_U R^q f_* \mathcal{F} = \sum_q (-1)^q \text{Sw}_U R^q f_* \mathcal{F}$$

$\in F_0 G(\partial_{\mathcal{F}} U)_{\mathbf{Q}(\zeta_{p^\infty})}$ is also defined as the alternating sum and the Swan class $\text{Sw}_U \mathcal{F}$ is defined. A conductor formula should express the Swan class $\text{Sw}_U R^q f_* \mathcal{F}$ in terms of the class $\text{Sw}_U \mathcal{F}$. We prove a conductor formula for a relative curve under a certain assumption in Corollary 7.3.6 and in general in Section 7.5, assuming K is of characteristic 0.

First, we give a general formalism Proposition 7.3.3 to prove a conductor formula. We consider a commutative diagram

$$(7.3.0.3) \quad \begin{array}{ccc} U' & \xleftarrow{\pi'} & V' \\ f \downarrow & & \downarrow g \\ U & \xleftarrow{\pi} & V \end{array}$$

of regular flat separated schemes of finite type over S satisfying the following condition:

- (7.3.0.4) The horizontal arrows are finite étale, V is a G -torsor over U and V' is a G' -torsor over U' for finite groups G and G' . The arrow g is compatible with a morphism $\varphi: G' \rightarrow G$ of finite groups in the sense that, for $\sigma \in G'$ and $\tau = \varphi(\sigma) \in G$, we have $g \circ \sigma = \tau \circ g$.

Lemma 7.3.1. — We consider a commutative diagram (7.3.0.3) of separated regular flat schemes of finite type over S satisfying the conditions (7.3.0.1) and (7.3.0.4). Assume that the higher direct image $R^q g_* \mathbf{F}_\ell$ is a constant sheaf on V for every $q \geq 0$.

Let $\sigma \in G'$ be an element and we put $\tau = \varphi(\sigma) \in G$. Let $\bar{\eta}$ be a geometric point of V and $\bar{\tau}$ be an automorphism of $\bar{\eta}$ compatible with τ . Let $\sigma^* \circ \bar{\tau}^*$ denote the automorphisms of $H_c^q(V'_\sigma, \mathbf{Q}_\ell)$ and $H_c^q(V'_\sigma, \mathbf{F}_\ell)$ defined by the pull-back by $\sigma \times \bar{\tau}$ on $V'_\sigma = V' \times_V \bar{\eta}$.

1. The automorphism $\sigma^* \circ \bar{\tau}^*$ of $H_c^q(V'_\sigma, \mathbf{F}_\ell)$ is independent of the choice of $\bar{\tau}$.
2. Assume that σ and $\bar{\tau}$ are ℓ -regular. Let σ^* denote the automorphism $\sigma^* \circ \bar{\tau}^*$ of $H_c^*(V'_\sigma, \mathbf{F}_\ell)$ independent of $\bar{\tau}$. Then, the alternating sum $\mathrm{Tr}(\sigma^* \circ \bar{\tau}^*: H_c^*(V'_\sigma, \mathbf{Q}_\ell))$ is independent of $\bar{\tau}$ and is equal to the alternating sum $\mathrm{Tr}^{\mathrm{Br}}(\sigma^*, H_c^*(V'_\sigma, \mathbf{F}_\ell))$ of Brauer traces.

Proof. — 1. By the assumption that $R^q g_* \mathbf{F}_\ell$ is constant, the action of $\sigma \circ \bar{\tau}$ on $H_c^q(V'_\sigma, \mathbf{F}_\ell)$ is independent of the lifting $\bar{\tau}$ of τ .

2. Since σ and $\bar{\tau}$ are assumed ℓ -regular, the actions of $\sigma^* \circ \bar{\tau}^*$ on $H_c^*(V'_\sigma, \mathbf{Q}_\ell)$ and on $H_c^*(V'_\sigma, \mathbf{F}_\ell)$ are of finite order prime-to- ℓ . Hence the assertion follows. \square

Lemma 7.3.2. — We consider a commutative diagram (7.3.0.3) of separated regular flat schemes of finite type over S satisfying the conditions (7.3.0.1) and (7.3.0.4). Let \mathcal{F} be a locally constant constructible sheaf of $\bar{\mathbf{F}}_\ell$ -modules on U' such that the pull-back $\pi'^* \mathcal{F}$ is a constant sheaf on V' . Assume that the higher direct image $R^q g_* \mathbf{F}_\ell$ and the pull-back $\pi^* R^q f_* \mathcal{F}$ are constant sheaves on V for every $q \geq 0$.

Let $\tau \in G$ be an ℓ -regular element. Let $\bar{\xi}$ be a geometric point of U and lift it to a geometric point $\bar{\eta}$ of V . Let M be the $\bar{\mathbf{F}}_\ell$ -representation of G' corresponding to \mathcal{F} . Then, we have

$$(7.3.2.1) \quad \mathrm{Tr}^{\mathrm{Br}}(\tau, H_c^*(U'_\xi, \mathcal{F})) = \frac{1}{|G'|} \sum_{\substack{\sigma \in G'^{(\ell)}, \rho \in G; \\ \varphi(\sigma) = \rho \tau \rho^{-1}}} \mathrm{Tr}^{\mathrm{Br}}(\sigma^*: H_c^*(V'_\sigma, \mathbf{F}_\ell)) \cdot \mathrm{Tr}^{\mathrm{Br}}(\sigma^*: M).$$

Proof. — By the assumption that $\pi^* \mathbf{R}^q f_* \mathcal{F}$ are constant on V , we may regard $H_c^q(U'_\xi, \mathcal{F})$ as an \mathbf{F}_ℓ -representation of G and the Brauer trace $\mathrm{Tr}^{\mathrm{Br}}(\tau, H_c^*(U'_\xi, \mathcal{F}))$ is defined.

We consider $H_c^q(V'_\xi, \bar{\mathbf{F}}_\ell)$ as an $\bar{\mathbf{F}}_\ell$ -representation of G' by the pull-back action of G' on $V'_\xi = V' \times_U \bar{\xi}$. By [18, Lemma 2.2], the alternating sum $H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)$ defines an element $[H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)]$ of the Grothendieck group $P_\ell(G')$ of the exact category of finitely generated projective $\bar{\mathbf{F}}_\ell[G']$ -modules ([40, Partie III 1.3]).

Since τ is assumed ℓ -regular, we have a decomposition

$$H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell) = \bigoplus_{\chi \in \langle \tau \rangle} H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)_\chi$$

by characters of the cyclic subgroup $\langle \tau \rangle \subset G$ and an equality

$$[H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)] = \sum_{\chi \in \langle \tau \rangle} [H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)_\chi]$$

in $P_\ell(G')$. Similarly as [18, Lemma 2.2], we obtain

$$(7.3.2.2) \quad \dim H_c^*(U'_\xi, \mathcal{F})_\chi = \dim([H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)_\chi] \cdot [\mathbf{M}])^{G'}.$$

Thus, we have

$$(7.3.2.3) \quad \mathrm{Tr}^{\mathrm{Br}}(\tau, H_c^*(U'_\xi, \mathcal{F})) = \sum_{\chi \in \langle \tau \rangle} \chi(\tau) \cdot \dim([H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)_\chi] \cdot [\mathbf{M}])^{G'}.$$

Similarly as [18, Lemma 2.3], the right hand side of (7.3.2.3) is equal to

$$\begin{aligned} & \sum_{\chi \in \langle \tau \rangle} \chi(\tau) \cdot \frac{1}{|G'|} \left(\sum_{\sigma \in G'^{(\ell)}} \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)_\chi) \cdot \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : \mathbf{M}) \right) \\ &= \frac{1}{|G'|} \sum_{\sigma \in G'^{(\ell)}} \left(\sum_{\chi \in \langle \tau \rangle} \chi(\tau) \cdot \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_\xi, \bar{\mathbf{F}}_\ell)_\chi) \right) \cdot \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : \mathbf{M}) \\ &= \frac{1}{|G'|} \sum_{\sigma \in G'^{(\ell)}} \mathrm{Tr}^{\mathrm{Br}}(\sigma^* \times \bar{\tau}^* : H_c^*(V'_\xi, \mathbf{F}_\ell)) \cdot \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : \mathbf{M}). \end{aligned}$$

For $\rho \in G$, let $V'_{\rho(\bar{\eta})}$ denote the geometric fiber of $V' \rightarrow V$ by the composition of $\bar{\eta} \rightarrow V$ with $\rho : V \rightarrow V$. Then, the geometric fiber $V'_\xi = V' \times_U \bar{\xi}$ is the disjoint union $\bigsqcup_{\rho \in G} V'_{\rho(\bar{\eta})}$ and we have

$$\mathrm{Tr}^{\mathrm{Br}}(\sigma^* \times \bar{\tau}^* : H_c^*(V'_\xi, \mathbf{F}_\ell)) = \sum_{\rho \in G; \varphi(\sigma) = \rho \tau \rho^{-1}} \mathrm{Tr}^{\mathrm{Br}}(\sigma^* \circ \bar{\tau}^* : H_c^*(V'_{\rho(\bar{\eta})}, \mathbf{F}_\ell)).$$

We have $\mathrm{Tr}^{\mathrm{Br}}(\sigma^* \circ \bar{\tau}^* : H_c^*(V'_{\rho(\bar{\eta})}, \mathbf{F}_\ell)) = \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_{\bar{\eta}}, \mathbf{F}_\ell))$ since $R^q g_! \mathbf{F}_\ell$ are assumed constant on V . \square

Proposition 7.3.3. — *We consider a commutative diagram (7.3.0.3) of separated regular flat schemes of finite type over S satisfying the conditions (7.3.0.1) and (7.3.0.4). Assume that U, U', V and V' are irreducible and that $V_K \rightarrow U_K$ and $V'_K \rightarrow U'_K$ are tamely ramified with respect to K .*

Let ℓ be a prime number invertible on S and \mathcal{F} be a locally constant constructible sheaf of $\bar{\mathbf{F}}_\ell$ -modules on U' . Assume further that the pull-backs $\pi^ R^q f_! \pi'_* \mathbf{F}_\ell$ and $\pi^* R^q f_! \mathcal{F}$ are constant on V . Let $\bar{\xi}$ be a geometric point of U and $\bar{\eta}$ be a geometric point of V above $\bar{\xi}$. For an ℓ -regular element $\sigma \in G'$, define $\mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_{\bar{\eta}}, \mathbf{F}_\ell))$ as in Lemma 7.3.1. Then, we have the following.*

1. *Assume that, for each non-trivial ℓ -regular element $\sigma \in G'$ and $\tau = \varphi(\sigma) \in G$, we have*

$$(7.3.3.1) \quad \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_{\bar{\eta}}, \mathbf{F}_\ell)) \cdot ((\Gamma_\tau, \Delta_V))^{\log} = g_!((\Gamma_\sigma, \Delta_{V'}))^{\log}$$

in $F_0G(\partial_{U/V}V)_{\mathbf{Q}}$. Then, we have

$$(7.3.3.2) \quad \mathrm{Sw}_U R f_! \mathcal{F} - \mathrm{rank} \mathcal{F} \cdot \mathrm{Sw}_U R f_! \bar{\mathbf{F}}_\ell = g_! \mathrm{Sw}_{U'} \mathcal{F}$$

in $F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_\ell^\infty)}$.

2. *Assume K is of characteristic 0 and suppose U and hence V, U', V' are schemes over K . Assume that, for each ℓ -regular element $\sigma \in G'$ and $\tau = \varphi(\sigma) \in G$, we have*

$$(7.3.3.3) \quad \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_{\bar{\eta}}, \mathbf{F}_\ell)) \cdot ((\Gamma_\tau, \Delta_V))^{\log} = g_!((\Gamma_\sigma, \Delta_{V'}))^{\log}$$

in $F_0G(\partial_F V)_{\mathbf{Q}}$. Then, we have

$$(7.3.3.4) \quad \mathrm{Sw}_U R f_! \bar{\mathbf{F}}_\ell = -f_!((\Delta_{U'}, \Delta_{U'}))^{\log} + \chi_c(U'_{\bar{\xi}}) \cdot ((\Delta_U, \Delta_U))^{\log}$$

in $F_0G(\partial_F U)_{\mathbf{Q}(\zeta_\ell^\infty)}$.

Proof. — 1. Let M denote the representation of G' corresponding to the locally constant $\bar{\mathbf{F}}_\ell$ -sheaf \mathcal{F} on U' . By (7.3.0.2) and by the definition of the Swan class (7.2.1.2), the equality (7.3.3.2) is equivalent to the following:

$$(7.3.3.5) \quad \begin{aligned} & \frac{1}{|G|} \sum_{\tau \in G^{(\ell)}, \tau \neq 1} (\mathrm{Tr}^{\mathrm{Br}}(\tau, H_c^*(U'_{\bar{\xi}}, \mathcal{F})) - \mathrm{rank} \mathcal{F} \cdot \mathrm{Tr}^{\mathrm{Br}}(\tau, H_c^*(U'_{\bar{\xi}}, \mathbf{F}_\ell))) \\ & \quad \times \pi_*((\Gamma_\tau, \Delta_V))^{\log} \\ & = \frac{1}{|G'|} \sum_{\sigma \in G^{(\ell)}, \sigma \neq 1} (\mathrm{Tr}^{\mathrm{Br}}(\sigma, M) - \dim M) \cdot f_! \pi'_*((\Gamma_\sigma, \Delta_{V'}))^{\log}. \end{aligned}$$

Substituting (7.3.2.1), we see that the left hand side of (7.3.3.5) is

$$\begin{aligned}
(7.3.3.6) \quad & \frac{1}{|G|} \sum_{\tau \in G^{(\ell)}, \tau \neq 1} \frac{1}{|G'|} \sum_{\substack{\sigma \in G'^{(\ell)}, \sigma \neq 1, \rho \in G; \\ \varphi(\sigma) = \rho \tau \rho^{-1}}} \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_\eta, \mathbf{F}_\ell)) \\
& \times (\mathrm{Tr}^{\mathrm{Br}}(\sigma, M) - \dim M) \cdot \pi_*((\Gamma_\tau, \Delta_V))^{\log} \\
& = \frac{1}{|G'|} \sum_{\sigma \in G'^{(\ell)}, \sigma \neq 1} (\mathrm{Tr}^{\mathrm{Br}}(\sigma, M) - \dim M) \\
& \times \frac{1}{|G|} \sum_{\rho \in G} \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_\eta, \mathbf{F}_\ell)) \cdot \pi_*((\Gamma_{\rho^{-1}\varphi(\sigma)\rho}, \Delta_V))^{\log}.
\end{aligned}$$

By the assumption (7.3.3.1), each term in the second summation in the second line of (7.3.3.6) is

$$\pi_* \rho_* g_i((\Gamma_\sigma, \Delta_{V'})^{\log}) = f_i \pi'_*((\Gamma_\sigma, \Delta_{V'}))^{\log}.$$

Thus the equality (7.3.3.5) follows.

2. Similarly as above, the left hand side of the equality (7.3.3.4) is equal to

$$\begin{aligned}
(7.3.3.7) \quad & - \frac{1}{|G|} \sum_{\tau \in G^{(\ell)}} \mathrm{Tr}^{\mathrm{Br}}(\tau, H_c^*(U'_\xi, \mathbf{F}_\ell)) \cdot \pi_*((\Gamma_\tau, \Delta_V))^{\log} \\
& + \frac{1}{|G|} \sum_{\tau \in G^{(\ell)}} \dim H_c^*(U'_\xi, \mathbf{F}_\ell) \cdot \pi_*((\Gamma_\tau, \Delta_V))^{\log}.
\end{aligned}$$

Substituting (7.3.2.1), we see that the first term in (7.3.3.7) is equal to

$$\begin{aligned}
& \frac{1}{|G'|} \sum_{\sigma \in G'^{(\ell)}} \mathrm{Tr}^{\mathrm{Br}}(\sigma : 1) \cdot \frac{1}{|G|} \sum_{\rho \in G} \mathrm{Tr}^{\mathrm{Br}}(\sigma^* : H_c^*(V'_\eta, \mathbf{F}_\ell)) \\
& \times \pi_*((\Gamma_{\rho^{-1}\varphi(\sigma)\rho}, \Delta_V))^{\log}.
\end{aligned}$$

Hence, similarly as above, by the assumption (7.3.3.3), it is further equal to

$$\begin{aligned}
\frac{1}{|G'|} \sum_{\sigma \in G'^{(\ell)}} f_i \pi'_*((\Gamma_\sigma, \Delta_{V'}))^{\log} & = f_i \frac{1}{|G'|} \pi'_* \pi'^{*/}((\Delta_{U'}, \Delta_{U'}))^{\log} \\
& = f_i((\Delta_{U'}, \Delta_{U'}))^{\log}.
\end{aligned}$$

Similarly, the second term in (7.3.3.7) is equal to $\chi_c(U'_\xi)$ times $\frac{1}{|G|} \sum_{\tau \in G^{(\ell)}} \pi_*((\Gamma_\tau, \Delta_V))^{\log} = ((\Delta_U, \Delta_U))^{\log}$ and the equality (7.3.3.4) follows. \square

We derive from the crucial step Proposition 6.3.2 a sufficient condition on a relative curve for the assumptions (7.3.3.1) and (7.3.3.3) of Proposition 7.3.3 to be satisfied.

Proposition 7.3.4. — *Let $f: U' \rightarrow U$ be a smooth morphism of relative dimension 1 of separated regular flat schemes of finite type over S . Let ℓ be a prime number invertible on S and \mathcal{F} be a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on U' . We consider a commutative diagram (7.3.0.3) satisfying the conditions (7.3.0.1) and (7.3.0.4). Assume that the pull-back $\pi'^*\mathcal{F}$ is constant on V' and that $\pi^*\mathbf{R}^q f_* \pi'^*\mathbf{F}_\ell$ is constant on V for every $q \geq 0$.*

We assume that V and V' are integral. Let η denote the generic point of V and let $\bar{\eta}$ be a geometric point above η . Let $\sigma \in G'$ and $\tau = \varphi(\sigma) \in G$ be non-trivial ℓ -regular elements.

We also consider a commutative diagram

$$(7.3.4.1) \quad \begin{array}{ccc} V' & \xrightarrow{c} & Y' \supset D \\ \downarrow g & & \downarrow \bar{g} \\ V & \xrightarrow{c} & Y \end{array}$$

of separated schemes of finite type over S and a finite family \mathcal{E} of Cartier divisors Y satisfying the following condition:

(7.3.4.2) *The schemes Y and Y' are proper over S . There exist d sections $(t_k: Y \rightarrow Y')_k$ such that the pair $(Y', (t_k))$ is a d pointed stable curve of genus g over Y and that $D \subset Y'$ is the disjoint union of the sections $t_k(Y)$. The open subscheme $V \subset Y$ is the complement of the union of \mathcal{E} . The restriction $Y'_V = Y' \times_Y V \rightarrow V$ is smooth and $V' \subset Y'_V$ is the complement of D_V . The action of G on V is extended to an admissible action on Y and on \mathcal{E} , in the sense that the quotient $X = Y/G$ exists as a scheme.*

Let \mathcal{E}' denote the family of Cartier divisors of Y' defined as the union of the pull-back of \mathcal{E} and the sections $(t_k(Y))_k$. Then, $\Sigma_{V/U}^{\mathcal{E}} Y_K$ and $\Sigma_{V'/U'}^{\mathcal{E}'} Y'_K$ are empty.

Then, we have

$$(7.3.4.3) \quad \bar{g}_* \left((\Gamma_\sigma, \Delta_{Y'}^{\log}) \right) = \mathrm{Tr}^{\mathrm{Br}} \left(\sigma^*, H_c^*(V'_{\bar{\eta}}, \mathbf{F}_\ell) \right) \cdot \left((\Gamma_\tau, \Delta_Y^{\log}) \right)$$

in $F_0 G(\Sigma_{V/U} Y)_{\mathbf{Q}}$,

Proof. — Since the smooth compactification $Y'_V \supset V$ is unique, the action of G' on V' is extended uniquely to that on Y'_V compatible with the action of G on V . Further, since an extension $Y' \supset Y'_V$ is unique, the action of G' on Y'_V is extended uniquely to that on Y' compatible with the admissible action of G on Y . Since $Y' \rightarrow Y$ is projective and G' acts on the relatively ample sheaf $\Omega_{Y'/Y}^1(\log D)$, the action of G' on Y' is admissible in the sense that the quotient $X' = Y'/G'$ exists as a scheme. Hence, by Lemma 5.3.6, the quotients $X = Y/G$ and $X' = Y'/G'$ contains U and U' as the complements of Cartier

divisors respectively. In particular, we have a commutative diagram (6.3.1.3) satisfying (6.3.1.4).

Let $T \subset V' \times_V V'$ be the graph Γ_τ of τ . The base changes $Y'_T{}^{(1)}$ and $Y'_T{}^{(2)}$ defined as in Proposition 6.3.2 are isomorphic to Y'_V . We define the log product and the log blow-up $(Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})^\sim \subset (Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})'$ with respect to the log structure defined by the sections $(t_k(T))_k$. Since the automorphism σ of $Y'_V = Y'_T{}^{(1)}$ permutes the sections $(t_k(T))_k$ of $Y'_T{}^{(1)}$, the intersection $t_k(T) \cap \sigma(t_k(T))$ is either empty or equal to a divisor $t_k(T)$ of $Y'_T{}^{(1)}$. Hence by the universality of blow-up, the closed immersion $\gamma = (1, \sigma): Y'_T{}^{(1)} \rightarrow Y'_T{}^{(1)} \times_T Y'_T{}^{(2)}$ is uniquely lifted to a closed immersion $\gamma': Y'_T{}^{(1)} \rightarrow (Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})'$. Let $\Gamma' \subset (Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})'$ denote the image of γ' and $\tilde{\Gamma} = \Gamma' \cap (Y'_T{}^{(1)} \times_T Y'_T{}^{(2)})^\sim$ be the intersection. Then, Γ' and hence $\tilde{\Gamma}$ are flat over T .

We regard Y as a log scheme with the log structure defined by \mathcal{E} and define the log product $(Y \times_S Y)^\sim$. By applying Proposition 6.3.2 to $Y' \rightarrow Y$, we obtain a finite family $(K_i)_{i \in I}$ of finite extensions of K , a family $(\gamma_i: \text{Spec } K_i \rightarrow T)_{i \in I}$ of morphisms over S extended to $(\bar{\gamma}_i: \text{Spec } \mathcal{O}_{K_i} \rightarrow (Y \times_S Y)^\sim)_{i \in I}$ such that the image of the closed points $\bar{\gamma}_i(s_i)$ are in the log diagonal $\Delta_Y^{\log} \subset (Y \times_S Y)^\sim$ and a family $(r_i)_{i \in I}$ of rational numbers satisfying $((T, \Delta_Y^{\log})) = \sum_i r_i [\bar{\gamma}_i(s_i)]$ and

$$(7.3.4.4) \quad \bar{f}'_*((\Gamma, \Delta_{Y'}^{\log})) = \sum_i r_i \text{Tr}((\gamma_i^* \Gamma)^*: H_c^*(V'_{\bar{K}_i}, \mathbf{Q}_\ell)) \cdot [\bar{\gamma}_i(s_i)].$$

Thus, it suffices to show

$$(7.3.4.5) \quad \text{Tr}^{\text{Br}}(\sigma^*, H_c^*(V'_\eta, \mathbf{F}_\ell)) = \text{Tr}((\gamma_i^* \Gamma)^*: H_c^*(V'_{\bar{K}_i}, \mathbf{Q}_\ell))$$

for each i .

We regard the composition $s_i \rightarrow (Y \times_S Y)^\sim \rightarrow Y$ as a morphism of log scheme. Since $\bar{\gamma}_i(s_i) \in (Y \times_S Y)^\sim$ are in the log diagonal, the composition $s_i \rightarrow Y \xrightarrow{\tau} Y$ of the morphisms of log schemes is the same as the original morphism $s_i \rightarrow Y$ of log schemes. Let \bar{s}_i be a log geometric point above s_i and let $\tilde{Y}_{\bar{s}_i}$ be the log strict localization. Let $\tilde{\tau}$ be the automorphism of $\tilde{Y}_{\bar{s}_i}$ induced by τ . We lift the generic geometric point $\bar{\eta} \rightarrow Y$ to a geometric point $\bar{\eta} \rightarrow \tilde{Y}_{\bar{s}_i}$ dominating the generic point $\tilde{\eta} \in \tilde{Y}_{\bar{s}_i}$. Let k_0 be the fixed subfield of $\kappa(\tilde{\eta})$ by an automorphism $\tilde{\tau}$ of order prime to ℓ . We take an ℓ -regular lifting $\bar{\tau} \in \text{Gal}(\bar{\eta}/k_0)$ of $\tilde{\tau} \in \text{Gal}(\tilde{\eta}/k_0) = \langle \tilde{\tau} \rangle$. By applying Lemma 7.3.1.2., we obtain

$$\text{Tr}^{\text{Br}}(\sigma^*, H_c^*(V'_\eta, \mathbf{F}_\ell)) = \text{Tr}(\sigma^* \circ \bar{\tau}^*, H_c^*(V'_\eta, \mathbf{Q}_\ell)).$$

We deduce (7.3.4.5) from Proposition 1.6.2. We show that the assumption of Proposition 1.6.2 is satisfied. The intersection $\tilde{\Gamma} \cap (V'_T{}^{(1)} \times_T V'_T{}^{(2)})$ is the graph Γ_σ of σ . Hence, the second projection $\tilde{\Gamma} \cap (V'_T{}^{(1)} \times_T V'_T{}^{(2)}) = \Gamma_\sigma \rightarrow V'_T{}^{(2)}$ is proper and Γ_σ is flat over T . Thus Γ_σ satisfies the conditions in Proposition 1.6.2. We define a map $Y_{\bar{s}_i} \rightarrow S_0$ to a regular noetherian scheme satisfying the condition (1.6.2.2) for $Y_{\bar{s}_i} \rightarrow S$ in the notation there. We consider the map $Y \rightarrow \mathcal{M}_{g,d}$ to the moduli of d pointed stable curves of

genus g defined by the pointed stable curve $(X, (t_k))$. Let S_0 be the strict localization of $\overline{\mathcal{M}}_{g,d}$ at the geometric point \bar{s}_i . Then, the map $Y_{\bar{s}_i} \rightarrow S_0$ satisfies the condition (1.6.2.2).

Since $\bar{\tau}$ is compatible with $\tilde{\tau}$, we may apply Proposition 1.6.2 and we obtain

$$\mathrm{Tr}(\sigma^* \circ \bar{\tau}^*, H_c^*(V'_{\bar{\eta}}, \mathbf{Q}_\ell)) = \mathrm{Tr}((\gamma_i^* \Gamma)^* : H_c^*(U'_{\bar{K}_i}, \mathbf{Q}_\ell)).$$

Thus, the equality (7.3.4.5) is proved. \square

Similarly, if \mathbf{K} is of characteristic 0, the same argument gives us the following variant.

Proposition 7.3.5. — *We assume \mathbf{K} is of characteristic 0. Let the assumption be the same as in Proposition 7.3.4 except that \mathbf{U} is a scheme over \mathbf{K} and that we do not assume σ or τ be non-trivial. Then, the equality (7.3.4.3) holds in $F_0G(Y_F)_{\mathbf{Q}}$.*

We derive a conductor formula for relative curves from Proposition 7.3.3 assuming $\mathrm{char} \mathbf{K} = 0$.

Corollary 7.3.6. — *Assume that \mathbf{K} is of characteristic 0. Let $f: U' \rightarrow U$ be a smooth morphism or relative dimension 1 of separated regular flat schemes of finite type over S and let ℓ be a prime number invertible on S . We assume that U and U' are connected. Let $f: X' \rightarrow U$ be a proper smooth curve with geometrically connected fibers of genus g and let D be a divisor of X' finite étale of degree d over U such that $U' = X' \setminus D$ and $2g - 2 + d > 0$.*

1. *Let \mathcal{F} be a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on U' . Then, there exists a finite étale covering $\pi: V \rightarrow U$ such that we have an equality*

$$(7.3.6.1) \quad \mathrm{Sw}_U Rf_! \mathcal{F} - \mathrm{rank} \mathcal{F} \cdot \mathrm{Sw}_U Rf_! \overline{\mathbf{F}}_\ell = f_! \mathrm{Sw}_{U'} \mathcal{F}$$

in $F_0G(\partial_{V/U}U)_{\mathbf{Q}(\zeta_\ell^\infty)}$.

2. *Assume that the schemes U and U' are schemes over \mathbf{K} . Then, we have*

$$(7.3.6.2) \quad \mathrm{Sw}_U Rf_! \overline{\mathbf{F}}_\ell = -f_!((\Delta_{U'}, \Delta_{U'}))^{\log} + \chi_c(U'_{\bar{\xi}}) \cdot ((\Delta_U, \Delta_U))^{\log}$$

in $F_0G(\partial_F U)_{\mathbf{Q}(\zeta_\ell^\infty)}$ for a geometric point $\bar{\xi}$ of U .

Proof. — By the assumption $\mathrm{char} \mathbf{K} = 0$, the locally constant sheaf \mathcal{F} on U' and the locally constant sheaves $R^q f_! \mathcal{F}$ and $R^q f_! \overline{\mathbf{F}}_\ell$ on U are tamely ramified on the generic fiber.

We define a commutative diagram (7.3.0.3) satisfying the condition (7.3.0.4). Since we assume \mathbf{K} is of characteristic 0, the sheaf \mathcal{F} is tamely ramified along D by Abhyankar's lemma [37, Proposition 5.5]. Hence, we may take a G' -torsor $\pi: V' \rightarrow U'$ for a finite group G' that is tamely ramified along D such that the pull-back $\pi^* \mathcal{F}$ is a constant sheaf on U' . Let Y' be the normalization of X' in U' . Then, since $V' \rightarrow U'$ is tamely ramified

along D , the proper curve Y' is smooth over U and $V' \subset Y'$ is the complement of a divisor D' finite étale over U by Lemma 7.3.7 below.

Since $Y' \rightarrow U$ is proper smooth, its Stein factorization $D'' \rightarrow U$ is finite étale [11, Remarque (7.8.10)]. Let $\pi': V \rightarrow U$ be a finite étale G -torsor trivializing the Stein factorization of the finite étale coverings D' and D'' . Replacing V' by a connected component of $U' \times_U V$ and G' by the stabilizer in $G' \times H$, we obtain a commutative diagram (7.3.0.3) satisfying the condition (7.3.0.4).

The proper smooth curve $Y' \rightarrow V$ has geometrically connected fibers of genus g' and D'_V is the union of d' disjoint sections. By the assumption that $2g - 2 + d > 0$, we have $2g' - 2 + d' > 0$. Hence, with an ordering of sections $V \rightarrow D'$, the pair (Y', D') defines a d' pointed smooth stable curve of genus g' . Further replacing V if necessary, we may assume that $\pi'^*R^q f_i \mathbf{F}_\ell$, $\pi'^*R^q f_i \mathcal{F}$ and $\pi'^*R^q f_i \mathbf{Z}/n\mathbf{Z}$ are constant on V for some integer $n \geq 3$ invertible on S .

Let Y be a proper scheme over S containing V as the complement of a family \mathcal{E} of Cartier divisors such that $\Sigma_{V/U} Y = \Sigma_{V/U}^{\mathcal{E}} Y$. Let $V \rightarrow \mathcal{M}_{g',d',n}$ be the morphism to the moduli space defined by the d' pointed smooth curve Y' of genus g' over V . By replacing Y by the schematic closure of the graph of the map $V \rightarrow \mathcal{M}_{g',d',n}$ in $Y \times_{\mathbf{Z}} \bar{\mathcal{M}}_{g',d',n}$, we may assume that $V \rightarrow \mathcal{M}_{g',d',n}$ is extended to a morphism $Y \rightarrow \bar{\mathcal{M}}_{g',d',n}$. Further by replacing Y if necessary, we may and do assume that the action of G on Y is admissible in the sense that the quotient Y/G exists as a scheme and that \mathcal{E} carries an action of G . The pull-back of the universal family by the map $Y \rightarrow \bar{\mathcal{M}}_{g',d',n}$ is a pointed stable curve over Y and satisfies the condition (7.3.4.2).

Thus, the assumptions in Propositions 7.3.4 and 7.3.5 are satisfied. By Propositions 7.3.4 and 7.3.5, the assumptions (7.3.3.1) and (7.3.3.3) in Proposition 7.3.3 are satisfied respectively. Thus the assertion follows. \square

Lemma 7.3.7. — *Let S be a normal scheme and X be a smooth curve over S . Let D be a divisor of X étale over S and $U = X \setminus D$ be the complement. Let $V \rightarrow U$ be a finite étale morphism tamely ramified along D and Y be the normalization of X in V . Then, Y is smooth over S and V is the complement of a divisor E of Y étale over S .*

Proof. — Let \bar{x} be a geometric point of X and t be a function on a neighborhood defining D . Let \bar{y} be a geometric point of Y above \bar{x} . Then, by Abhyankar's lemma [37, Proposition 5.5], Y is étale locally isomorphic to $X[T]/(T^n - t)$ for an integer $n \geq 1$ invertible at \bar{x} on a neighborhood of \bar{y} . Hence the assertion follows. \square

7.4. Swan class of a constructible sheaf. — In the rest of this section, we assume that the characteristic of K is 0. We define the Swan class for a constructible sheaf on a scheme over K .

For a separated scheme U of finite type over K , let $K(U, \bar{\mathbf{F}}_\ell)$ be the Grothendieck group of constructible $\bar{\mathbf{F}}_\ell$ -sheaves on the étale site of U . More precisely, it is the quotient

of the free abelian group generated by the isomorphism classes $[\mathcal{F}]$ of constructible $\overline{\mathbf{F}}_\ell$ -sheaves \mathcal{F} on the étale site of U divided by the relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}']$ for exact sequences $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$.

Lemma 7.4.1. — *The abelian group $\mathbf{K}(U, \overline{\mathbf{F}}_\ell)$ is generated by the classes $[i_*\mathcal{F}]$ where $i : Z \rightarrow U$ is a locally closed immersion of smooth subscheme and \mathcal{F} is a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on Z . The relations are given by*

$$(7.4.1.1) \quad [i_*\mathcal{F}] = [i_*\mathcal{F}'] + [i_*\mathcal{F}']$$

for exact sequences $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of locally constant constructible $\overline{\mathbf{F}}_\ell$ -modules on Z and

$$(7.4.1.2) \quad [i_*\mathcal{F}] = [i_{0!}\mathcal{F}|_{Z_0}] + [i_{1!}\mathcal{F}|_{Z_1}]$$

for smooth locally closed subschemes $Z_1 \subset Z$ where $i_0 : Z_0 = Z \setminus Z_1 \rightarrow U$ and $i_1 : Z_1 \rightarrow U$ are the immersions.

Proof. — We consider the free abelian group generated by the classes $[i_*\mathcal{F}]$ where $i : Z \rightarrow U$ are locally closed immersions of smooth subschemes and \mathcal{F} are locally constant constructible $\overline{\mathbf{F}}_\ell$ -modules on Z . Let \mathbf{K}' denote its quotient by the relations (7.4.1.1) and (7.4.1.2). Clearly, we have a canonical map $\mathbf{K}' \rightarrow \mathbf{K}(U, \overline{\mathbf{F}}_\ell)$. The inverse is defined as follows.

For a constructible sheaf \mathcal{F} on U , there exists a finite partition $U = \coprod_i U_i$ by smooth schemes such that $\mathcal{F}|_{U_i}$ are locally constant. It follows from (7.4.1.2) that the sum $\sum_i [i_*\mathcal{F}|_{U_i}]$ is independent of the partition. Thus, the class $[\mathcal{F}] = \sum_i [i_*\mathcal{F}|_{U_i}] \in \mathbf{K}'$ is well-defined. Further, the equalities (7.4.1.1) and (7.4.1.2) implies $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}']$ for exact sequences $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$. Thus, the map $\mathbf{K}(U, \overline{\mathbf{F}}_\ell) \rightarrow \mathbf{K}'$ is well-defined and is the inverse of the map $\mathbf{K}' \rightarrow \mathbf{K}(U, \overline{\mathbf{F}}_\ell)$ above. \square

Proposition 7.4.2. — *For separated schemes U of finite type over \mathbf{K} , there exists a unique way to attach morphisms*

$$(7.4.2.1) \quad \mathrm{Sw}_U : \mathbf{K}(U, \overline{\mathbf{F}}_\ell) \rightarrow \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_p, \infty)}$$

satisfying the following properties:

- (1) *If U is smooth over \mathbf{K} and if \mathcal{F} is a locally constant constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules on U , we have $\mathrm{Sw}_U([\mathcal{F}]) = \mathrm{Sw}_U\mathcal{F}$.*
- (2) *For an immersion $i : U' \rightarrow U$, the diagram*

$$(7.4.2.2) \quad \begin{array}{ccc} \mathbf{K}(U, \overline{\mathbf{F}}_\ell) & \xrightarrow{\mathrm{Sw}_U} & \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_p, \infty)} \\ \uparrow i & & \uparrow i \\ \mathbf{K}(U', \overline{\mathbf{F}}_\ell) & \xrightarrow{\mathrm{Sw}_{U'}} & \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U')_{\mathbf{Q}(\zeta_p, \infty)} \end{array}$$

is commutative.

Proof. — By (1) and (2), the map Sw_U is characterized by $\mathrm{Sw}_U([i_*\mathcal{F}]) = i_*\mathrm{Sw}_{U'}\mathcal{F}$ for a locally constant constructible sheaf \mathcal{F} on a regular subscheme U' and the immersion $i: U' \rightarrow U$. Hence the uniqueness follows from Lemma 7.4.1. Further by Lemma 7.4.1, the existence follows from Proposition 7.2.5. \square

We define a modification of the map Sw_U with stronger compatibility for push-forward. For a separated scheme U of finite type over \mathbf{K} , let $\mathrm{Const}(U)$ be the \mathbf{Z} -module of constructible \mathbf{Z} -valued functions on U . For a constructible $\overline{\mathbf{F}}_\ell$ -sheaf \mathcal{F} on U , let $\mathrm{rank}\mathcal{F} \in \mathrm{Const}(U)$ be the constructible function defined by $\mathrm{rank}\mathcal{F}(x) = \dim \mathcal{F}_{\bar{x}}$. Let $\mathrm{rank}: \mathbf{K}(U, \overline{\mathbf{F}}_\ell) \rightarrow \mathrm{Const}(U)$ be the homomorphism sending the class $[\mathcal{F}]$ to the function $\mathrm{rank}\mathcal{F}$.

Similarly as Proposition 7.4.2, we have a map $\mathrm{Ch}_U: \mathrm{Const}(U) \rightarrow \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}}$ characterized as follows.

Proposition 7.4.3. — *For separated schemes U of finite type over \mathbf{K} , there exists a unique way to attach morphisms*

$$(7.4.3.1) \quad \mathrm{Ch}_U: \mathrm{Const}(U) \rightarrow \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}}$$

satisfying the following properties:

- (1) *If U is smooth over \mathbf{K} , for the constant function 1_U , we have $\mathrm{Ch}_U(1_U) = ((\Delta_U, \Delta_U))^{\mathrm{log}}$.*
- (2) *For an immersion $i: U' \rightarrow U$, the diagram*

$$(7.4.3.2) \quad \begin{array}{ccc} \mathrm{Const}(U) & \xrightarrow{\mathrm{Ch}_U} & \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}} \\ \uparrow i & & \uparrow i \\ \mathrm{Const}(U') & \xrightarrow{\mathrm{Ch}_{U'}} & \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U')_{\mathbf{Q}} \end{array}$$

is commutative.

Proof. — It follows from the excision formula Theorem 6.2.2. \square

Definition 7.4.4. — *Let U be a separated scheme of finite type over \mathbf{K} . For a constructible $\overline{\mathbf{F}}_\ell$ -sheaf \mathcal{F} on U , we define the total Swan class $\overline{\mathrm{Sw}}_U\mathcal{F} \in \mathrm{F}_0\mathrm{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_\ell, \infty)}$ by*

$$(7.4.4.1) \quad \overline{\mathrm{Sw}}_U\mathcal{F} = \mathrm{Sw}_U\mathcal{F} - \mathrm{Ch}_U(\mathrm{rank}\mathcal{F}).$$

Corollary 7.4.5. — *1. Assume U is smooth and \mathcal{F} is a locally constant constructible $\overline{\mathbf{F}}_\ell$ -sheaf of constant rank on U . Let $f: V \rightarrow U$ be a finite étale \mathbf{G} -torsor for a finite group \mathbf{G} such that $\pi^*\mathcal{F}$ is a constant sheaf. Then, we have*

$$(7.4.5.1) \quad \overline{\mathrm{Sw}}_U\mathcal{F} = -\frac{1}{|\mathbf{G}|} \sum_{\sigma \in \mathbf{G}} \mathrm{Tr}^{\mathrm{Br}}(\sigma: \mathbf{M}) \cdot f_*((\Gamma_\sigma, \Delta_V))^{\mathrm{log}}.$$

2. For separated schemes U of finite type over \mathbf{K} , the collection of the maps $\overline{\text{Sw}}_U: \mathbf{K}(U, \overline{\mathbf{F}}_\ell) \rightarrow \mathbf{F}_0\mathbf{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})}$ is characterized by the following properties:

(1) Under the assumption in 1, we have

$$(7.4.5.2) \quad \overline{\text{Sw}}_U([\mathcal{F}]) = -\frac{1}{|G|} \sum_{\sigma \in G} \text{Tr}^{\text{Br}}(\sigma: M) \cdot f_*((\Gamma_\sigma, \Delta_V))^{\log}.$$

(2) For an immersion $i: U' \rightarrow U$, the diagram

$$(7.4.5.3) \quad \begin{array}{ccc} \mathbf{K}(U, \overline{\mathbf{F}}_\ell) & \xrightarrow{\overline{\text{Sw}}_U} & \mathbf{F}_0\mathbf{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})} \\ \uparrow i & & \uparrow i \\ \mathbf{K}(U', \overline{\mathbf{F}}_\ell) & \xrightarrow{\overline{\text{Sw}}_{U'}} & \mathbf{F}_0\mathbf{G}(\partial_{\mathbf{F}}U')_{\mathbf{Q}(\zeta_{p^\infty})} \end{array}$$

is commutative.

Proof. — 1. We have

$$(7.4.5.4) \quad \overline{\text{Sw}}_U \mathcal{F} = \text{Sw}_U \mathcal{F} - \text{rank } \mathcal{F} \cdot ((\Delta_U, \Delta_U))^{\log}.$$

Thus, the equality (7.4.5.1) follows from definition (7.2.1.2) of the Swan class and the equality $|G| \cdot ((\Delta_U, \Delta_U))^{\log} = \sum_{\sigma \in G} f_*((\Gamma_\sigma, \Delta_V))^{\log}$.

2. We define the map $\overline{\text{Sw}}_U$ by $\overline{\text{Sw}}_U = \text{Sw}_U - \text{Ch}_U \circ \text{rank}$. Then, the commutative diagram (7.4.5.3) follows from (7.4.2.2) and (7.4.3.2). The uniqueness is clear from Lemma 7.4.1. \square

7.5. Conductor formula. — We keep the assumption that \mathbf{K} is of characteristic 0. We show that the diagram (7.4.5.3) is commutative for arbitrary morphisms over \mathbf{K} . Changing the notation, $f: U \rightarrow V$ denotes an arbitrary morphism over \mathbf{K} of separated schemes of finite type over \mathbf{K} .

Theorem 7.5.1. — *Let $f: U \rightarrow V$ be a morphism of separated schemes of finite type over \mathbf{K} . Then, the diagram*

$$(7.5.1.1) \quad \begin{array}{ccc} \mathbf{K}(U, \overline{\mathbf{F}}_\ell) & \xrightarrow{\overline{\text{Sw}}_U} & \mathbf{F}_0\mathbf{G}(\partial_{\mathbf{F}}U)_{\mathbf{Q}(\zeta_{p^\infty})} \\ \downarrow f & & \downarrow f \\ \mathbf{K}(V, \overline{\mathbf{F}}_\ell) & \xrightarrow{\overline{\text{Sw}}_V} & \mathbf{F}_0\mathbf{G}(\partial_{\mathbf{F}}V)_{\mathbf{Q}(\zeta_{p^\infty})} \end{array}$$

is commutative.

Proof. — It suffices to show the equality

$$(7.5.1.2) \quad \overline{\mathrm{Sw}}_V f_! [\mathcal{F}] = f_! \overline{\mathrm{Sw}}_U [\mathcal{F}]$$

for a constructible sheaf \mathcal{F} on U . We prove this by induction on the dimensions of U and V . By a standard devissage using the excision formula, it suffices to show that there exist dense open subschemes $U' \subset U, V' \subset V$ such that $f(U') \subset V'$ and that we have

$$\overline{\mathrm{Sw}}_{V'} f|_{U'} [\mathcal{F}|_{U'}] = f_! \overline{\mathrm{Sw}}_{U'} [\mathcal{F}|_{U'}].$$

Hence, we may assume the following condition is satisfied.

- The sheaf \mathcal{F} is locally constant and the scheme V is smooth.

The formula (7.5.1.2) is compatible with the composition of morphisms. Hence, by the induction on relative dimension, we may further assume the following.

(7.5.1.3) The morphism $f: U \rightarrow V$ is smooth of relative dimension ≤ 1 .

Since we are allowed to shrink V , we may assume that V is connected and that there exists a proper smooth curve X over V containing U as the complement of a divisor D finite étale over V . Since the formula (7.5.1.2) is proved for a finite étale morphism in Corollary 7.2.7, by replacing V by the Stein factorization of $X \rightarrow V$, we may assume that the relative dimension is 1 and that the geometric fibers of $X \rightarrow V$ is connected. Further shrinking U and V , we may replace the condition (7.5.1.3) by the following.

(7.5.1.4) There exist a proper smooth and geometrically connected curve $\bar{f}: X \rightarrow V$ of genus g and an open immersion $U \rightarrow X$ such that U is the complement of a divisor $D \subset X$ finite and étale over V of degree d such that $2g - 2 + d > 0$.

Then, applying Corollary 7.3.6, we obtain the equalities (7.3.6.1) and (7.3.6.2). The equality (7.5.1.2) follows from them together with (7.4.5.4). \square

We derive some consequences of Theorem 7.5.1.

Corollary 7.5.2. — *Let $f: U \rightarrow V$ be a smooth morphism of smooth separated schemes of finite type over \mathbf{K} . Assume that $\mathbf{R}^q f_! \mathbf{F}_\ell$ is locally constant for every $q \geq 0$.*

1. *Let \mathcal{F} be a constructible sheaf of $\overline{\mathbf{F}}_\ell$ -modules of constant rank on U . Assume that $\mathbf{R}^q f_! \mathcal{F}$ is locally constant for every $q \geq 0$. Then, we have*

$$(7.5.2.1) \quad \mathrm{Sw}_V \mathbf{R} f_! \mathcal{F} = f_! \mathrm{Sw}_U \mathcal{F} + \mathrm{rank} \mathcal{F} \cdot \mathrm{Sw}_V \mathbf{R} f_! \overline{\mathbf{F}}_\ell$$

in $\mathbf{F}_0(\partial_F V)_{\mathbf{Q}(\zeta_p^\infty)}$.

2. *We have*

$$(7.5.2.2) \quad \mathrm{Sw}_V \mathbf{R} f_! \overline{\mathbf{F}}_\ell = \mathrm{rank} \mathbf{R} f_! \overline{\mathbf{F}}_\ell \cdot ((\Delta_V, \Delta_V))^{\log} - f_! ((\Delta_U, \Delta_U))^{\log}$$

in $\mathbf{F}_0(\partial_F V)_{\mathbf{Q}(\zeta_p^\infty)}$.

Proof. — 1. Since $\text{rank } Rf_i\mathcal{F} = \text{rank } \mathcal{F} \cdot \text{rank } Rf_i\overline{\mathbf{F}}_\ell$, it suffices to apply Theorem 7.5.1 to $[\mathcal{F}] - \text{rank } \mathcal{F} \cdot [\mathbf{F}_\ell]$.

2. It suffices to apply Theorem 7.5.1 to $[\mathbf{F}_\ell]$. \square

If there exist a proper smooth scheme $\bar{f}: X \rightarrow V$ and a divisor D of X with normal crossings relatively to V such that U is the complement $X \setminus D$, the assumption of Corollary 7.5.2 is satisfied. Further, if d denotes the relative dimension of X over V we have

$$\text{rank } Rf_i\overline{\mathbf{F}}_\ell = (-1)^d \deg c_d(\Omega_{X/V}^1(\log D)).$$

In particular, for $V = \text{Spec } \mathbf{K}$, we have the following.

Corollary 7.5.3. — *Let U be a smooth separated scheme of finite type over \mathbf{K} and \mathcal{F} be a smooth $\overline{\mathbf{F}}_\ell$ -sheaf of constant rank on U . Then, we have*

$$(7.5.3.1) \quad \text{Sw}_{\mathbf{K}} \text{R}\Gamma_c(U_{\overline{\mathbf{K}}}, \mathcal{F}) = \deg \text{Sw}_U \mathcal{F} + \text{rank } \mathcal{F} \cdot \text{Sw}_{\mathbf{K}} \text{R}\Gamma_c(U_{\overline{\mathbf{K}}}, \overline{\mathbf{F}}_\ell),$$

$$(7.5.3.2) \quad \text{Sw}_{\mathbf{K}} \text{R}\Gamma_c(U_{\overline{\mathbf{K}}}, \overline{\mathbf{F}}_\ell) = -\deg((\Delta_U, \Delta_U))^{\log}.$$

The equality (7.5.3.2) implies the conductor formula of Bloch in the case proved in [26] as follows. We assume that U is proper smooth over \mathbf{K} and X is a proper regular flat scheme over $S = \text{Spec } \mathcal{O}_{\mathbf{K}}$ such that $U = X_{\mathbf{K}}$ and the reduced closed fiber $X_{F, \text{red}}$ is a divisor with simple normal crossings. Then, by Proposition 4.3.8 and [26, (5.4.2.6)], we have $((\Delta_U, \Delta_U))_{(U \times_S U)^\sim}^{\log} = ((\Delta_U, \Delta_U))_{(U \times_S U)^\sim}^{\log} = (-1)^d c_d(\Omega_{X/S}^1(\log / \log))_{X_F}$. Thus, in this case, the equality (7.5.3.2) is equivalent to [26, Theorem 6.2.5] and hence to the conductor formula of Bloch [3]. This proof of the conductor formula of Bloch uses the same tools including the localized intersection product. However, the excision formula allows us to reduce the proof to relative curves.

8. A computation in the rank 1 case

We state Conjecture 8.3.1 comparing the Swan class of a sheaf of rank 1 with the cycle class defined in [24, Section 5.1] and prove it in Theorem 8.3.7 assuming $\dim U_{\mathbf{K}} \leq 1$. Using it, we prove the integrality conjecture Conjecture 7.2.8 under the assumption $\dim U_{\mathbf{K}} \leq 1$. In Section 8.2 and in the second half of Section 8.3, we will assume that \mathbf{K} is of characteristic 0.

8.1. Ramification of characters. — We briefly recall the theory of ramification of characters of Galois groups in [21]. For a field \mathbf{K} , let $X_{\mathbf{K}}$ denote the dual group $H^1(\mathbf{K}, \mathbf{Q}/\mathbf{Z}) = H^2(\mathbf{K}, \mathbf{Z})$ of the abelian quotient $G_{\mathbf{K}}^{\text{ab}}$ of the absolute Galois group $G_{\mathbf{K}} = \text{Gal}(\overline{\mathbf{K}}/\mathbf{K})$. The cup-product defines a canonical pairing

$$(8.1.1.1) \quad (,)_{\mathbf{K}}: X_{\mathbf{K}} \times \mathbf{K}^\times = H^2(\mathbf{K}, \mathbf{Z}) \times H^0(\mathbf{K}, \mathbf{G}_m) \rightarrow \text{Br}(\mathbf{K}) = H^2(\mathbf{K}, \mathbf{G}_m).$$

Assume \mathbf{K} is a henselian discrete valuation field and let F be the residue field of characteristic $p > 0$. In this subsection, we drop the assumption that F is perfect. We briefly recall the definition of the exact sequence

$$0 \rightarrow \Omega_F \rightarrow \Omega_F(\log) \xrightarrow{\text{res}} F \rightarrow 0$$

of F -vector spaces. A canonical map $d\log: \mathcal{O}_K^\times \rightarrow \Omega_F = \Omega_{F/F^p}$ is defined by $a \mapsto \bar{a}^{-1} d\bar{a}$. The F -vector space $\Omega_F(\log)$ is defined as the amalgamate sum of Ω_F with $F \otimes_{\mathbf{Z}} \mathbf{K}^\times$ over $F \otimes_{\mathbf{Z}} \mathcal{O}_K^\times$ with respect to the map $d\log: \mathcal{O}_K^\times \rightarrow \Omega_F$ and the inclusion $\mathcal{O}_K^\times \rightarrow \mathbf{K}^\times$. The valuation $\mathbf{K}^\times \rightarrow \mathbf{Z}$ induces the residue map $\text{res}: \Omega_F(\log) \rightarrow F$. The map $d\log: \mathcal{O}_K^\times \rightarrow \Omega_F$ is canonically extended to $d\log: \mathbf{K}^\times \rightarrow \Omega_F(\log)$.

We identify a character $\chi \in X_F$ with the corresponding unramified character $\chi \in X_K$ and regard X_F as a subgroup of X_K . For $a \in F$, let $\chi_a \in X_F$ be the character defined by the Artin-Schreier equation $T^p - T = a$. We define a map $\chi: F \rightarrow X_K$ by sending $a \in F$ to $\chi_a \in X_F \subset X_K$. In [21, (1.4)], it is shown that there exists a unique map $\lambda_K: \Omega_F(\log) \rightarrow \text{Br}(\mathbf{K})$ that makes the diagram

$$(8.1.1.2) \quad \begin{array}{ccc} F \times \mathbf{K}^\times & \xrightarrow{(a,b) \mapsto a \cdot d\log b} & \Omega_F(\log) \\ \chi \times 1 \downarrow & & \downarrow \lambda_K \\ X_K \times \mathbf{K}^\times & \xrightarrow{(\cdot, \cdot)_K} & \text{Br}(\mathbf{K}) \end{array}$$

commutative.

The main construction in [21, Definition (2.1)] is the increasing filtration F_\bullet of X_K indexed by $r \in \mathbf{N}$. We have $X_K = \bigcup_{r \geq 0} F_r X_K$. The subgroup $F_0 X_K$ consists of the characters at most tamely ramified. For $r \geq 1$, we put $U_K^r = 1 + \mathfrak{m}_K^r \subset \mathbf{K}^\times$. Then, the pairing $(\cdot, \cdot)_K: X_K \times \mathbf{K}^\times$ maps $F_r X_K \times U_K^r$ for $r \geq 1$ and $F_0 X_K \times \mathbf{K}^\times$ to $\text{Im } \lambda_K \subset \text{Br}(\mathbf{K})$. For an extension L of henselian discrete valuation field such that $\mathcal{O}_K = \mathbf{K} \cap \mathcal{O}_L$ and $\mathfrak{m}_K \mathcal{O}_L = \mathfrak{m}_L^e$, the canonical map $X_K \rightarrow X_L$ sends $F_r X_K$ to $F_{re} X_L$.

For $r \geq 1$, we put $\text{Gr}_r^F X_K = F_r X_K / F_{r-1} X_K$. A canonical injection

$$(8.1.1.3) \quad \text{rs}_{r,K}: \text{Gr}_r^F X_K \rightarrow \text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log))$$

is defined in [21, Corollary (5.2)]. It is characterized by the following properties:

(1) For $\chi \in F_r X_K$ and $c \in \mathfrak{m}_K^r$, we have

$$(8.1.1.4) \quad (\chi, 1 - c)_K = \lambda_K(\text{rs}_{r,K}(\chi)(\bar{c})).$$

(2) Let L be an arbitrary extension of henselian discrete valuation field such that $\mathcal{O}_K = \mathbf{K} \cap \mathcal{O}_L$ and $\mathfrak{m}_K \mathcal{O}_L = \mathfrak{m}_L^e$. Let F_L denote the residue field of L . Then,

the diagram

$$(8.1.1.5) \quad \begin{array}{ccc} \mathrm{Gr}_r^{\mathbf{F}} \mathbf{X}_{\mathbf{K}} & \xrightarrow{\mathrm{rsw}_{r,\mathbf{K}}} & \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}_{\mathbf{K}}^r / \mathfrak{m}_{\mathbf{K}}^{r+1}, \Omega_{\mathbf{F}}(\log)) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{e'}^{\mathbf{F}} \mathbf{X}_{\mathbf{L}} & \xrightarrow{\mathrm{rsw}_{e',\mathbf{L}}} & \mathrm{Hom}_{\mathbf{F}_{\mathbf{L}}}(\mathfrak{m}_{\mathbf{L}}^{e'} / \mathfrak{m}_{\mathbf{L}}^{e'+1}, \Omega_{\mathbf{F}_{\mathbf{L}}}(\log)) \end{array}$$

is commutative.

For an element $\chi \in \mathrm{F}_r \mathbf{X}_{\mathbf{K}} \setminus \mathrm{F}_{r-1} \mathbf{X}_{\mathbf{K}}$, the injection

$$(8.1.1.6) \quad \mathrm{rsw}_{r,\mathbf{K}}(\chi): \mathfrak{m}_{\mathbf{K}}^r / \mathfrak{m}_{\mathbf{K}}^{r+1} \rightarrow \Omega_{\mathbf{F}}(\log)$$

is called the refined Swan conductor of χ and will be denoted by $\mathrm{rsw} \chi$.

We compute the refined Swan conductor of a Kummer character of degree p explicitly. Assume that \mathbf{K} is of characteristic 0 and the residue field \mathbf{F} is of characteristic p . Assume further that \mathbf{K} contains a primitive p -th root ζ_p of 1. We identify $\mathbf{Z}/p\mathbf{Z} = \mu_p$ by ζ_p and the p -torsion part $\mathbf{X}_{\mathbf{K}}[p] = \mathrm{H}^1(\mathbf{K}, \mathbf{Z}/p\mathbf{Z})$ with $\mathbf{K}^{\times}/\mathbf{K}^{\times p} = \mathrm{H}^1(\mathbf{K}, \mu_p)$ by the isomorphism $\theta: \mathbf{K}^{\times}/\mathbf{K}^{\times p} \rightarrow \mathbf{X}_{\mathbf{K}}[p]$ of Kummer theory. For $a \in \mathbf{K}^{\times}$, let $\theta_a \in \mathbf{X}_{\mathbf{K}}[p]$ denote the corresponding character.

We put $z = \zeta_p - 1$. Then, we have $z^p + pz \equiv (z+1)^p - 1 = 0 \pmod{pz^2}$ and $\mathrm{ord} z^p = \mathrm{ord} pz > \mathrm{ord} pz^2$. Hence, for an element $a \in \mathcal{O}_{\mathbf{K}}$, the reduction of the Kummer equation $(1-zt)^p = 1-az^p$ gives the Artin-Schreier equation $t^p - t = \bar{a}$ and the unramified character $\chi_{\bar{a}} \in \mathbf{X}_{\mathbf{K}}[p]$ is identified with $1-az^p \in \mathbf{K}^{\times}/\mathbf{K}^{\times p}$. In particular, we have $1+az^p \mathfrak{m}_{\mathbf{K}} \subset \mathbf{K}^{\times p}$. Consequently, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{K}^{\times}/\mathbf{K}^{\times p} & \xrightarrow{\theta} & \mathbf{X}_{\mathbf{K}}[p] \\ \bar{a} \mapsto 1-az^p \uparrow & & \uparrow \\ \mathbf{F} & \xrightarrow{\chi} & \mathbf{X}_{\mathbf{F}}[p]. \end{array}$$

Proposition 8.1.2. — *Let \mathbf{K} be a henselian discrete valuation field of mixed characteristic $(0, p)$ containing a primitive p -th root ζ_p of 1. We put $e' = p \cdot \mathrm{ord}_{\mathbf{K}}(\zeta_p - 1) = \mathrm{ord}_{\mathbf{K}} z^p$. We define a decreasing filtration \mathbf{F}^{\bullet} on $\mathbf{K}^{\times}/\mathbf{K}^{\times p}$ by $\mathbf{F}^m(\mathbf{K}^{\times}/\mathbf{K}^{\times p}) = \mathrm{Image} \mathrm{U}_{\mathbf{K}}^m$ for $m \geq 1$ and by $\mathbf{F}^0(\mathbf{K}^{\times}/\mathbf{K}^{\times p}) = \mathbf{K}^{\times}/\mathbf{K}^{\times p}$.*

1. ([21, Proposition 4.1]) *The isomorphism $\theta: \mathbf{K}^{\times}/\mathbf{K}^{\times p} \rightarrow \mathbf{X}_{\mathbf{K}}[p]$ induces an isomorphism*

$$(8.1.2.1) \quad \mathbf{F}^m(\mathbf{K}^{\times}/\mathbf{K}^{\times p}) \rightarrow \mathrm{F}_r \mathbf{X}_{\mathbf{K}}[p]$$

for $0 \leq m = e' - r \leq e'$. In particular, we have $\mathrm{F}_{e'} \mathbf{X}_{\mathbf{K}}[p] = \mathbf{X}_{\mathbf{K}}[p]$.

2. *For $a \in \mathbf{K}^{\times}$ such that $d \log a \neq 0$ in $\Omega_{\mathbf{F}}(\log)$, the map*

$$(8.1.2.2) \quad \mathrm{rsw}_{e',\mathbf{K}}(\theta_a): \mathfrak{m}_{\mathbf{K}}^{e'}/\mathfrak{m}_{\mathbf{K}}^{e'+1} \rightarrow \Omega_{\mathbf{F}}(\log)$$

sends $c \cdot z^\rho$ to $-c \cdot d \log a$ for $c \in \mathcal{O}_K$.

3. For $1 \leq m = e' - r < e'$ and $a = 1 - b \in \mathbf{U}_K^m, \notin \mathbf{U}_K^{m+1}$, the map

$$(8.1.2.3) \quad \text{rsw}_{r,K}(\theta_a) : \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \rightarrow \Omega_F(\log)$$

sends $c \cdot z^\rho / b$ to $c \cdot d \log b$ for $c \in \mathcal{O}_K$.

Proof. — We identify the p -torsion part $\text{Br}(\mathbf{K})[p] = \text{H}^2(\mathbf{K}, \mu_p)$ with $\text{H}^2(\mathbf{K}, \mu_p^{\otimes 2})$ by ζ_p . Then, for $a, b \in \mathbf{K}^\times$, the cup-product $(\theta_a, b)_K \in \text{Br}(\mathbf{K})$ is identified with the Galois symbol $\{a, b\}$ defined as $\theta_a \cup \theta_b \in \text{H}^2(\mathbf{K}, \mu_p^{\otimes 2})$. Let $a \in \mathbf{K}^\times$ and let L be an arbitrary extension of henselian discrete valuation field. Then, for $c \in \mathcal{O}_L$, we have

$$(8.1.2.4) \quad (\theta_a, 1 - z^\rho c)_L = \{a, 1 - z^\rho c\} = -\{1 - z^\rho c, a\} = -(\chi_{\bar{c}}, a)_L = -\lambda_L(\bar{c} \cdot d \log a).$$

By the characterization of $\text{rsw}_{r,K}$, the equality (8.1.2.4) implies that the map $\text{rsw}_{r,K}$ is the zero-map for $r > e'$. Hence, by the injectivity of $\text{rsw}_{r,K}$, we have $\text{Gr}_r^F \mathbf{X}_K[p] = 0$ for $r > e'$. Thus by $\mathbf{X}_K = \bigcup_{r \geq 0} \text{F}_r \mathbf{X}_K$, we obtain $\text{F}_\rho \mathbf{X}_K[p] = \mathbf{X}_K[p]$. Now, the equality (8.1.2.4) implies the assertion 2.

To show the remaining assertions, we use the following elementary lemma on the symbol map.

Lemma 8.1.3 ([20, Lemma 6]). — For $b, c \in \mathbf{K}^\times \setminus \{1\}$ such that $bc \neq 1$, we have

$$(8.1.3.1) \quad \{1 - b, 1 - c\} = \{1 - bc, -b\} + \{1 - bc, 1 - c\} - \{1 - bc, 1 - b\}.$$

Proof of Lemma. — Since $\{x, y\} = 0$ for $x, y \in \mathbf{K}^\times$ satisfying $x + y = 1$, we have

$$\left\{ 1 - b, 1 - \frac{1 - bc}{1 - b} \right\} = \left\{ 1 - bc, 1 - \frac{1 - bc}{1 - b} \right\}.$$

Since $1 - \frac{1 - bc}{1 - b} = \frac{-b(1 - c)}{1 - b}$, the right hand side is equal to that of (8.1.3.1). Further, since $\frac{1}{1 - b} + \frac{-b}{1 - b} = 1$, the left hand side is equal to that of (8.1.3.1). \square

We go back to the proof of Proposition 8.1.2. Let $1 \leq m = e' - r < e'$ be an integer, $b \in \mathfrak{m}_K^m$ and put $a = 1 - b$. Let L be an arbitrary extension of discrete valuation field and $c \in \mathfrak{m}_K^{-m} \mathcal{O}_L$ be an arbitrary element. Since $\mathbf{U}_L^{e'_L+1} \subset L^{\times p}$ for $e'_L = \text{ord}_L z^\rho$, we have $\{\mathbf{U}_L^{e'_L}, \mathbf{U}_L^1\} = 0$ by Lemma 8.1.3. Hence, by $b, z^\rho c \in \mathfrak{m}_L$, we have

$$(8.1.3.2) \quad \begin{aligned} (\theta_a, 1 - z^\rho c)_L &= \{1 - b, 1 - z^\rho c\} = \{1 - bc z^\rho, -b\} \\ &= (\chi_{\bar{bc}}, -b)_L = \lambda_L(\bar{bc} \cdot d \log(-b)) = \lambda_L(\bar{bc} \cdot d \log b) \end{aligned}$$

further by Lemma 8.1.3.

Similarly as above, the equality (8.1.3.2) together with the characterization and the injectivity of $\text{rsw}_{n,K}$ shows that θ maps U_K^m to $F_r X_K[\rho]$, by induction on $1 \leq m = \ell' - r \leq \ell'$. Further the equality (8.1.3.2) implies the assertion 3. Hence, the composition of the map

$$\begin{aligned} \text{Gr}_F^m(K^\times / K^{\times \rho}) &= \begin{cases} \mathfrak{m}_K^m / \mathfrak{m}_K^{m+1} & \text{if } p \nmid m \\ \mathfrak{m}_K^m / (\mathfrak{m}_K^n)^\rho \mathfrak{m}_K^{m+1} & \text{if } m = n\rho \end{cases} \\ &\xrightarrow{\text{Gr}\theta} \text{Gr}_r^F X_K \xrightarrow{\text{rsw}_{r,K}} \text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log)) \end{aligned}$$

sends $1 - b$ to the map $c \mapsto bc/z^\rho \cdot d \log b$ and is an injection. Since $\theta : K^\times / K^{\times \rho} \rightarrow X_K[\rho]$ is an isomorphism, we conclude $\theta(U_K^m) = F_r X_K[\rho]$. \square

8.2. Kummer covering of degree p . — We apply the theory recalled in the previous section to the following geometric situation. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field F of characteristic $p > 0$. Let X be a regular flat separated scheme of finite type over $S = \text{Spec } \mathcal{O}_K$ and D be a divisor with simple normal crossings. Let D_1, \dots, D_n be the irreducible components of D . For an irreducible component D_i , let K_i be the fraction field of the completion $\widehat{\mathcal{O}_{X,\xi_i}}$ of the local ring at the generic point ξ_i of D_i . The residue field $F_i = \kappa(\xi_i)$ of the complete discrete valuation field K_i is the function field of D_i . The fiber $\Omega_{X/S}^1(\log D)_{\xi_i} \otimes_{\mathcal{O}_{X,\xi_i}} F_i$ is identified with the F_i -vector space $\Omega_{F_i}(\log)$ in the notation of the last subsection.

Let $\chi \in H^1(U, \mathbf{Q}/\mathbf{Z})$ be a character. Then, for each local field K_i , the restriction defines a character $\chi_i \in X_{K_i}$. By the ramification theory recalled in Section 8.1, the Swan conductor $r_i = \text{sw}_{K_i}(\chi_i) \geq 0$ is defined for each K_i . We define the Swan divisor of χ by $D_\chi = \sum_{i=1}^n r_i D_i$. Let $E = \sum_{i,r_i > 0} D_i$ be the support of D_χ . For each irreducible component D_i such that $r_i > 0$, the refined Swan conductor $\text{rsw}_{K_i}(\chi_i)$ defines a non-zero map

$$\mathcal{O}_X(-D_\chi)_{\xi_i} \otimes_{\mathcal{O}_{X,\xi_i}} F_i = \mathfrak{m}_{K_i}^{r_i} / \mathfrak{m}_{K_i}^{r_i+1} \rightarrow \Omega_{X/S}^1(\log D)_{\xi_i} \otimes_{\mathcal{O}_{X,\xi_i}} F_i = \Omega_{F_i}(\log).$$

In [21, Theorem (7.1), Proposition (7.3)], it is shown that there exists an \mathcal{O}_E -linear injection

$$(8.2.0.1) \quad \text{rsw } \chi : \mathcal{O}_X(-D_\chi) \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E$$

inducing $\text{rsw}_{K_i}(\chi_i)$ at each generic point.

Definition 8.2.1. — We say that χ is clean with respect to X if the map

$$\text{rsw } \chi : \mathcal{O}_X(-D_\chi) \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E$$

is a locally splitting injection.

Assume χ is clean with respect to X . Then, we say that χ is s -clean with respect to X if, for each irreducible component D_i of E , the composition

$$\mathcal{O}_X(-D_\chi) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} \xrightarrow{\text{rsw}_i(\chi)} \Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} \xrightarrow{\text{res}} \mathcal{O}_{D_i}$$

is either an isomorphism or the zero-map, depending on D_i .

It is conjectured in [24] that for any χ , there exists a proper modification X' of X such that χ is clean with respect to X' , see Lemma 8.2.6.3.

We compute the Swan divisor D_χ and the map $\text{rsw } \chi$ (8.2.0.1) for a Kummer character χ of order p explicitly.

Lemma 8.2.2. — *Let A be a regular local ring such that the fraction field is of characteristic 0 and the residue field is of characteristic $p > 0$. Let $t_1, \dots, t_n \in A$ be a part of regular system of parameters. Assume that A contains a primitive p -th root ζ_p of 1 and that t_1, \dots, t_n divide p . Let K_i be the fraction field of the completion \mathcal{O}_{K_i} of the discrete valuation ring $A_{(t_i)}$ for each i .*

Let $m = (m_1, \dots, m_n)$ be a family of integers satisfying $0 \leq m_i \leq e'_i = p \cdot \text{ord}_{K_i}(\zeta_p - 1)$ and let $F^m A^\times$ denote the subgroup $1 + t_1^{m_1} \cdots t_n^{m_n} A$ for $m \neq 0$ and $F^0 A^\times = A^\times$. Then, the inverse image of $\bigoplus_i F^{m_i}(K_i^\times/K_i^{\times p}) \subset \bigoplus_i K_i^\times/K_i^{\times p}$ by the canonical map $A^\times/A^{\times p} \rightarrow \bigoplus_i K_i^\times/K_i^{\times p}$ is the image of $F^m A^\times$.

Proof. — First, we show the case where $n = 1$. We prove it by induction on $m = m_1$. It is obvious for $m = 0$. Assume $m = 1$ and that the image of $a \in A^\times$ in $K_1^\times/K_1^{\times p}$ is in $F^1(K_1^\times/K_1^{\times p})$. Let F_1 denote the residue field of K_1 . Then, we have $\bar{a} \in F_1^{\times p}$. We put $\bar{a} = b^p$ for $b \in F_1^\times$. Since b is integral over the normal ring $A/t_1 A$, we have $b \in A/t_1 A$ and $b \in (A/t_1 A)^\times$. Take a unit $c \in A^\times$ lifting b . Then, a/c^p is in $F^1 A^\times = 1 + t_1 A$.

Assume $m \geq 1$ and that the image of $a \in F^m A^\times = 1 + t_1^m A$ is in $F^{m+1}(K_1^\times/K_1^{\times p})$. If $p \nmid m$, we have $a \in 1 + t_1^{m+1} \mathcal{O}_{K_1}$. Since $A \cap t_1^{m+1} \mathcal{O}_{K_1} = t_1^{m+1} A$, we have $a \in F^{m+1} A^\times$. Assume $p \mid m$ and we put $a = 1 + t_1^m b$. Then, we have $\bar{b} \in F_1^p$. Similarly as above, there is an element $c \in A$ such that $b \equiv c^p \pmod{t_1}$. Then, $a/(1 + t_1^{m/p} c)^p$ is in $F^{m+1} A^\times = 1 + t_1^{m+1} A$.

We prove the general case. Assume that the image of $a \in A^\times$ is in $\bigoplus_i F^{m_i}(K_i^\times/K_i^{\times p})$. Then, for each i , there exists $a_i \in A^\times$ such that $a/a_i^p \in F^{m_i} A^\times$. Let $m'_i \geq 0$ be the smallest integer satisfying $p \cdot m'_i \geq m_i$. Then, since the p -th power map $(A/t_i^{m'_i})^\times \rightarrow (A/t_i^{m_i})^\times$ is injective, the class $\bar{a}_i \in (A/t_i^{m'_i})^\times$ is uniquely determined by the condition $a/a_i^p \in F^{m_i} A^\times$. Further, for $i \neq j$, the p -th power map $(A/(t_i^{m'_i}, t_j^{m'_j}))^\times \rightarrow (A/(t_i^{m_i}, t_j^{m_j}))^\times$ is also injective. Hence, there exists a unique element $b \in (A/t_1^{m'_1} \cdots t_n^{m'_n} A)^\times$ satisfying $b \equiv a_i \pmod{t_i^{m'_i}}$. Let $c \in A^\times$ be a unit lifting b . Then, we have a/c^p is in $F^m A^\times$. \square

Corollary 8.2.3. — *Let $\chi \in H^1(U, \mathbf{Z}/p\mathbf{Z})$ be a character of order p and $x \in D$ be a point. Let D_1, \dots, D_n be the irreducible components of D containing x and put $D_\chi = \sum_i r_i D_i$ on a neighborhood of x . We put $A = \mathcal{O}_{x,x}$ and, for each irreducible component D_i , we put $e'_i = p \cdot \text{ord}_{D_i}(\zeta_p - 1)$ and $m_i = e'_i - r_i$.*

1. *On a neighborhood of x , there exists an element $a \in \Gamma(U, \mathcal{O}_U^\times)$ such that χ is defined by $t^p = a$ and satisfying one of the following conditions:*

$$(8.2.3.1) \quad \text{ord}_{D_i} a \text{ is prime to } p \text{ for at least one } D_i.$$

(8.2.3.2) a is a unit at x and its image in A^\times is in $F^m A^\times$ for $m = (m_1, \dots, m_n)$ in the notation of Lemma 8.2.2.

2. Assume $D_\chi = \sum_i e'_i D_i$ and let a be as in 1. Then, the map

$$\text{rsw } \chi : \mathcal{O}_E(-D_\chi) \rightarrow \Omega_{X/S}^1(\log D) \otimes \mathcal{O}_E$$

sends z^\flat to $-d \log a$ where $z = \zeta_p - 1$.

3. Assume $D_\chi < \sum_i e'_i D_i$. Then, the condition (8.2.3.2) holds. Let b be a basis of the invertible sheaf $\mathcal{O}_X(-\sum_i m_i D_i)$ on a neighborhood of x . We put $a = 1 - bc \in \mathcal{O}_X^\times$ where $c \in \mathcal{O}_X$ on a neighborhood of x as in (8.2.3.2). Then, the map

$$\text{rsw } \chi : \mathcal{O}_E(-D_\chi) \rightarrow \Omega_{X/S}^1(\log D) \otimes \mathcal{O}_E$$

sends $a \cdot z^\flat / b$ to $c \cdot d \log b + dc$.

Proof. — 1. Let a be a rational function on X such that χ is defined by $t^\flat = a$ on the generic point. The regular local ring $A = \mathcal{O}_{X,x}$ is a UFD. Hence, we may assume $0 \leq \text{ord}_y a < p$ for every discrete valuation defined by a point $y \in X$ of codimension 1, after dividing a by the p -th power of a rational function and shrinking X if necessary. For a point $y \in X$ of codimension 1, if the valuation of a at y is not divisible by p , then χ is ramified at y . Hence, a is a unit on U . Further, if the condition (8.2.3.1) is not satisfied, then a is a unit at x . Then, by Lemma 8.2.2, after dividing a by the p -th power of a rational function, the condition (8.2.3.2) is satisfied.

2. 3. Clear from Proposition 8.1.2 and the equality $-da/b = c \cdot d \log b + dc$. \square

We give a condition for character χ of order p to be clean.

Proposition 8.2.4. — *Let $\chi \in H^1(U, \mathbf{Z}/p\mathbf{Z})$ be a character of order p and $x \in D$ be a point. Assume that χ is not tamely ramified at x and that \mathbf{K} is of characteristic 0 and contains a primitive p -th root ζ_p of 1. Let D_1, \dots, D_n be the irreducible components of D containing x and let C denote the intersection $D_1 \cap \dots \cap D_n$. We put $D_\chi = \sum_i r_i D_i > 0$ as in Corollary 8.2.3.*

For each irreducible component D_i , we put $e'_i = p \cdot \text{ord}_{D_i}(\zeta_p - 1)$ and $m_i = e'_i - r_i$. Let $t_i \in \Gamma(X, \mathcal{O}_X)$ be an element defining D_i .

1. Assume $D_\chi = \sum_i e'_i D_i$. Then, χ is clean at x if and only if on a neighborhood of x , there exists an element $a \in \Gamma(U, \mathcal{O}_U^\times)$ such that χ is defined by $t^\flat = a$ and satisfying one of the following conditions:

(8.2.4.1) $\text{ord}_{D_i} a$ is prime to p for at least one D_i .

(8.2.4.2) a is a unit at x and $da|_C$ has no zero at x .

2. Assume $D_\chi < \sum_i e'_i D_i$. Then, χ is clean at x if and only if, on a neighborhood of x , there exists an element $a \in \Gamma(U, \mathcal{O}_U^\times)$ such that χ is defined by $t^\flat = a$ and satisfying one of the following conditions:

(8.2.4.3) $a = 1 - u \cdot t_1^{m_1} \cdots t_n^{m_n}$ for a unit u at x and, for at least one D_i , the integer m_i is prime to p .

(8.2.4.4) $a = 1 - c \cdot t_1^{m_1} \cdots t_n^{m_n}$ for a regular function c at x such that $dc|_C$ has no zero at x .

Proof. — We have an exact sequence $0 \rightarrow \Omega_C^1 \rightarrow \Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_C \rightarrow \bigoplus_i \mathcal{O}_C \rightarrow 0$ and $(d \log t_i)_i$ defines a splitting. Hence, if the condition (8.2.3.1) holds, then χ is clean and we have $D_\chi = \sum_i e'_i D_i$. Thus, it suffices to consider the case where (8.2.3.2) holds.

1. Assume $D_\chi = \sum_i e'_i D_i$ and hence $m_i = 0$ for every i . Then, χ is clean if and only if $da|_C$ has no zero at x .

2. Assume $D_\chi < \sum_i e'_i D_i$ and hence $m_i > 0$ for some i . Then, by Corollary 8.2.3.1, there exists a regular function c at x such that χ is defined by $t^p = a$ for $a = 1 - c \cdot t_1^{m_1} \cdots t_n^{m_n}$. By Corollary 8.2.3.3 and by the local splitting above, χ is clean if and only if either $c \cdot (m_i)_i \in \bigoplus_i \mathcal{O}_C$ or $dc \in \Omega_C^1$ has no zero at x . The condition that $c \cdot (m_i)_i \in \bigoplus_i \mathcal{O}_C$ has no zero at x means that c is a unit at x and one of m_i is prime to p . The second condition is equivalent to (8.2.4.4). \square

Corollary 8.2.5. — *Let the assumption be as in Proposition 8.2.4.*

1. Assume $D_\chi = \sum_i e'_i D_i$. Then, χ is clean at x if and only if χ is s -clean at x .

2. Assume $D_\chi < \sum_i e'_i D_i$. Then, χ is s -clean at x if and only if, on a neighborhood of x , there exists an element $a \in \Gamma(U, \mathcal{O}_U^\times)$ such that χ is defined by a and satisfying either the condition (8.2.4.3) or the following condition:

(8.2.4.4') $a = 1 - c \cdot t_1^{m_1} \cdots t_n^{m_n}$ for a regular function c at x such that $dc|_C$ has no zero at x and, for every D_i , the integer m_i is divisible by p .

Proof. — 1. If the condition (8.2.4.1) or (8.2.4.2) is satisfied, then χ is s -clean at x .

2. If the condition (8.2.4.3) is satisfied, then χ is s -clean at x . Assume the condition (8.2.4.4) is satisfied. Then, in the notation of the proof of Proposition 8.2.4.2, χ is s -clean at x if and only if either $c \cdot (m_i)_i \in \bigoplus_i \mathcal{O}_C$ has no zero at x or $c|_C \cdot (m_i)_i = 0$. The first condition is equivalent to (8.2.4.3). By the condition (8.2.4.4), we have $c|_C \neq 0$. Hence, the second condition $c|_C \cdot (m_i)_i = 0$ is equivalent to that m_i is divisible by p for every i . \square

We recall the main result from [24] and prove a complement.

Lemma 8.2.6. — *Let the assumption be as in Proposition 8.2.4. Assume $\dim X_K + 1 = 2$ and let $\Sigma \subset \Sigma_s \subset D$ be the sets of points where \mathcal{F} is not clean and not s -clean with respect to X respectively.*

1. *The subsets Σ and Σ_s consist of finitely many closed points of D .*

2. [24, Remark 4.13] *Let x be a closed point of D and $f: X' \rightarrow X$ be the blow-up at x . If χ is clean at x , then χ is clean on a neighborhood of $f^{-1}(x)$.*

3. [24, Theorem 4.1] *There exists a successive blow-up $f: X' \rightarrow X$ at Σ such that χ is clean with respect to X' .*

4. *There exists a successive blow-up $f: X' \rightarrow X$ at Σ_s such that χ is s -clean with respect to X' .*

Proof. — 1. Clear from the definition.

2 and 3. See [24, Remark 4.13] and [24, Theorem 4.1] respectively.

4. By 3, we may assume χ is clean with respect to X . Let $x \in \Sigma_s$. If x is a singular point of D , then χ is s -clean at x . Hence, x is a smooth point of D and we may assume D is irreducible. We put $D_\chi = rD$ and prove the assertion by induction on $r > 0$. By [24, Corollary 4.9] and [21, Theorem (8.1)], we have $e = 1$ in the notation of [24, Corollary 4.9] for the blow-up $X' \rightarrow X$ at x . Hence, we have $r' = r - e = r - 1 < r$ and the assertion follows by induction. \square

We expect that Lemma 8.2.6.3 holds in arbitrary dimension.

For an s -clean character of order p , we give a local description of the normalization in the corresponding cyclic covering of degree p .

Proposition 8.2.7 (Cf. [38, Lemma 1]). — *Let X be a regular flat separated scheme of finite type over \mathcal{O}_K and $U = X \setminus D$ be the complement of a divisor D with simple normal crossing. Assume that K is of characteristic 0 and contains a primitive p -th root ζ_p of 1.*

Let $\chi \in H^1(U, \mathbf{Z}/p\mathbf{Z})$ be a character of order p . Let $V \rightarrow U$ be the cyclic Galois covering of order p trivializing χ and Y be the integral closure of X in V .

Assume that χ is s -clean with respect to X . Then, there exists an fs log structure \mathcal{M}_Y on Y such that (Y, \mathcal{M}_Y) is log regular [25] and that V is the maximum open subscheme where \mathcal{M}_Y is trivial. We have $\mathcal{M}_Y = j_ \mathcal{O}_V^\times \cap \mathcal{O}_Y$ where $j: V \rightarrow Y$ denotes the open immersion.*

Proof. — The proof is similar to that of [38, Lemma 1]. Since the question is local, we may assume that $X = \text{Spec } A$ is affine and that the log structure on X is defined by the chart $P = \prod_i \mathbf{N}e_i \rightarrow A$ sending the basis e_i to t_i defining irreducible components D_i of D . By Corollary 8.2.5, it suffices to consider each case (8.2.4.1)–(8.2.4.3) and (8.2.4.4'). We take the notation in Proposition 8.2.4.

(8.2.4.1) Assume $a = u \prod_i t_i^{m_i}$ where $u \in A^\times$ and $p \nmid m_i$ for at least one i . We put $Q_0 = P \times \mathbf{Z}e_u$, $e_i = \frac{1}{p}(\sum_i m_i e_i + e_u)$ and $Q_1 = Q_0 + \langle e_t \rangle$. Let $Q \subset Q_1^{\text{gp}}$ be the saturation of Q_1 . We put $B_1 = A[t]/(t^p - u \prod_i t_i^{m_i})$ and define a monoid homomorphism $Q_1 \rightarrow B_1$ by $e_u \mapsto u$ and $e_i \mapsto t$. We define $B = B_1 \otimes_{A[Q_1]} A[Q]$, $Y = \text{Spec } B$ and define a log structure \mathcal{M}_Y on Y by $Q \rightarrow B$.

We show that Y is log regular and is equal to the normalization of X in V . By the assumption that there exists m_i prime to p , the quotient $Q_1^{\text{gp}}/Q_1^\times = (P^{\text{gp}} \times \mathbf{Z}e_t)/\mathbf{Z}e_u$ is torsion free. The quotient B_1/I_1 by the ideal $I_1 \subset B_1$ generated by $Q_1 \setminus Q_1^\times$ is equal to $\bar{A} = A/(t_i; i = 1, \dots, n)$ and is regular. Since B_1 is flat over A , we have $\dim B_1 = \dim A = \dim \bar{A} + n = \dim B_1/I_1 + \text{rank } Q_1^{\text{gp}}/Q_1^\times$. Hence, similarly as the proof of [38, Claim], the log scheme Y is regular by [25, Proposition (12.2)]. Since the normal scheme Y is finite over X and $Y \times_X U = V$, it is the normalization of X in V .

(8.2.4.2) Assume $a \in A^\times$ and $da|_C$ has no zero. We put $B = A[t]/(t^p - a)$, $\bar{A} = A/(t_1, \dots, t_n)$ and $\bar{B} = \bar{A}[t]/(t^p - \bar{a})$. By the assumption that $d\bar{a}$ is non-vanishing, the ring \bar{B} is regular and hence $B = A[t]/(t^p - a)$ is the normalization on a neighborhood of x . The open subscheme V of $Y = \text{Spec } B$ is the complement of a divisor with simple normal crossing defined by $(t_1 \cdots t_n)$.

(8.2.4.3) Let $u \in A^\times$ be a unit, $0 \leq m_i \leq e'_i$ be integers and we put $b = u \prod_i t_i^{m_i}$ and $a = 1 - b$. We assume that $p \nmid m_i < e'_i$ for at least one i . We put $z = \zeta_p - 1$ as above and $t = 1 - z/s$. Then, by an elementary computation, the equation $t^p = a$ is equivalent to

$$s^p = \frac{1}{b} (s^p - (s - z)^p).$$

We define a polynomial $f \in \mathcal{O}_K[S]$ of degree $p - 1$ by $f = (S^p - (S - z)^p)/z^p$. We have $f \equiv 1 - S^{p-1} \pmod{z}$. We put $B_1 = A[s]/(s^p - \frac{z^p}{b} \cdot f(s))$. We put $r_i = e'_i - m_i \geq 0$ and define a unit $w \in A^\times$ by $z^p/b = w \cdot \prod_i t_i^{r_i}$. The assumption $m_i < e'_i$ for at least one i means that $z^p/b = w \cdot \prod_i t_i^{r_i}$ is in the maximal ideal at x . Hence, shrinking X if necessary, we may assume $f(s)$ is a unit of B .

We put $Q_0 = \mathbf{P} \times \mathbf{Z}e_w$, $e_s = \frac{1}{p}(\sum_i r_i e_i + e_w)$ and $Q_1 = Q_0 + \langle e_s \rangle$. Let $Q \subset Q_1^{\text{gp}}$ be the saturation of Q_1 . We define a monoid homomorphism $Q_1 \rightarrow B_1$ by $e_w \mapsto w \cdot f(s)$ and $e_s \mapsto s$. We define $B = B_1 \otimes_{A[Q_1]} A[Q]$, $Y = \text{Spec } B$ and define a log structure \mathcal{M}_Y on Y by $Q \rightarrow B$. Then similarly as in the case (8.2.4.1), we see that Y is log regular and is equal to the normalization of X in V .

(8.2.4.4') Let $c \in A$, $0 \leq m_i \leq e'_i$ be integers and we put $b = c \prod_i t_i^{m_i}$ and $a = 1 - b$. We assume that $m_i < e'_i$ for at least one i and p divides $m_i = p \cdot m'_i$ for every i . We also assume that $dc|_C$ is not 0 at x . We put $t = 1 - \prod_i t_i^{m'_i} s$. Then, the equation $t^p = a$ is equivalent to $(1 - \prod_i t_i^{m'_i} s)^p = 1 - \prod_i t_i^{m_i} c$. We define a polynomial $g \in A[S]$ of degree p by $g = (S - \prod_i t_i^{-m'_i})^p + \prod_i t_i^{-m_i}$. Then, as in the case (8.2.4.2), $B = A[s]/(g(s) + c)$ is the normalization of A . The open subscheme V of $Y = \text{Spec } B$ is the complement of a divisor with simple normal crossing.

Since (Y, \mathcal{M}_Y) is log regular, we have $\mathcal{M}_Y = j_* \mathcal{O}_V^\times \cap \mathcal{O}_Y$. □

We consider the sheaf of differential forms $\Omega_{Y, \mathcal{M}_Y/S}^1$ defined with respect to the log structure \mathcal{M}_Y and the trivial log structure on S .

Corollary 8.2.8 (Cf. [38, Lemma 1]). — *Let the notation and the assumption be as in Proposition 8.2.7 and σ be a generator of $\text{Gal}(V/U)$. Let \mathcal{I}_σ denote the ideal sheaf defining the fixed part $Y^\sigma \subset Y$. Then, we have the following.*

1. *For each geometric point \bar{y} of the fixed part Y^σ , the action of σ on the stalk $\mathcal{M}_{Y, \bar{y}}/\mathcal{O}_{Y, \bar{y}}^\times$ is trivial.*
2. *We define an ideal sheaf \mathcal{J}_σ of \mathcal{O}_Y to be that generated by \mathcal{I}_σ and $\sigma(b)/b - 1$ for $b \in \mathcal{M}_Y$. Then, \mathcal{J}_σ is an invertible ideal defined by an effective Cartier divisor D_σ . Further, we have*

$$(8.2.8.1) \quad \pi^* D_\chi = p D_\sigma.$$

3. We put $D' = (p-1)D_\sigma$. Then, the coherent \mathcal{O}_Y -module $\Omega_{Y/X}^1(\log/\log)$ defined by $\Omega_{Y/X}^1(\log/\log) = \text{Coker}(\pi^*\Omega_{X/S}^1(\log D) \rightarrow \Omega_{Y, \mathcal{M}_Y/S}^1)$ is an invertible $\mathcal{O}_{D'}$ -module.

4. Define an \mathcal{O}_Y -linear surjection $\varphi_\sigma : \Omega_{Y, \mathcal{M}_Y/S}^1 \rightarrow \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$ by $\varphi_\sigma(da) = \sigma(a) - a$ and $\varphi_\sigma(d \log b) = \sigma(b)/b - 1$. Then, it induces an isomorphism $\Omega_{Y/X}^1(\log/\log) \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_\sigma} \rightarrow \mathcal{J}_\sigma/\mathcal{J}_\sigma^2 = \mathcal{O}_{D_\sigma}(-D_\sigma)$.

5. We put $E_Y = (\pi^*E)_{\text{red}}$ where $E = \sum_{i, r_i > 0} D_i$ denotes the support of D_χ . Then, the sequence

$$(8.2.8.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_{E_Y}(-\pi^*D_\chi) & \xrightarrow{\text{rsw}_\chi} & \pi^*\Omega_X^1(\log D) \otimes_{\mathcal{O}_Y} \mathcal{O}_{E_Y} & & \\ & & & & \longrightarrow & \Omega_{Y, \mathcal{M}_Y}^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_{E_Y} & \xrightarrow{\varphi_\sigma} \mathcal{O}_{E_Y}(-D_\sigma) \rightarrow 0 \end{array}$$

is exact.

Proof. — Since the assertions are local, it suffices to consider the cases (8.2.4.1)–(8.2.4.3) or (8.2.4.4') respectively.

1. Let $\pi : Y \rightarrow X$ be the canonical map. Since the canonical map $\bar{\mathcal{M}}_{X, \pi(\bar{y})} \rightarrow \bar{\mathcal{M}}_{Y, \bar{y}}$ induces an isomorphism $\bar{\mathcal{M}}_{X, \pi(\bar{y})}^{\text{gp}} \otimes \mathbf{Q} \rightarrow \bar{\mathcal{M}}_{Y, \bar{y}}^{\text{gp}} \otimes \mathbf{Q}$, the assertion follows.

2 and 3. We take the notation in the proof of Proposition 8.2.7.

(8.2.4.1) The ideal \mathcal{J}_σ is generated by $\sigma(x)/x - 1$ for $x \in \mathbf{Q}$. Hence, it is generated by $\zeta_p - 1$. We have $D_\sigma = \text{div}(\zeta_p - 1)$ and $D_\chi = p \cdot \text{div}(\zeta_p - 1)$ by Proposition 8.2.7.

The \mathcal{O}_Y -module $\Omega_{Y/X}^1(\log/\log)$ is generated by $d \log t$ and the annihilator is (p) . We have $\text{div } p = (p-1) \cdot \text{div}(\zeta_p - 1)$.

(8.2.4.2) The ideal \mathcal{J}_σ is generated by $\sigma(t) - t = (\zeta_p - 1)t$. Since t is a unit, it is generated by $\zeta_p - 1$. We also have $D_\sigma = \text{div}(\zeta_p - 1)$ and $D_\chi = p \cdot \text{div}(\zeta_p - 1)$ by Proposition 8.2.7.

The \mathcal{O}_Y -module $\Omega_{Y/X}^1(\log/\log)$ is generated by dt and the annihilator is (p) . We have $\text{div } p = (p-1) \cdot \text{div}(\zeta_p - 1)$.

(8.2.4.3) The ideal \mathcal{J}_σ is generated by $\sigma(s)/s - 1$. By $s = z/(1-t)$, we have $\sigma(s)/s - 1 = (1-t)/(1-\zeta_p t) - 1 = (\zeta_p - 1)t/(1-\zeta_p t) = zt/(1-t-zt)$. Since $1-t = z/s$, it is further equal to $zt/(z/s - zt) = st/(1-st)$. Since $1-st$ and t are unit, the ideal \mathcal{J}_σ is generated by (s) . Since $(s^p) = (z^p/b)$, we have $p \cdot D_\sigma = \pi^*D_\chi$.

We put $g = S^p - \frac{z^p}{b}f(S) \in \mathbf{A}[S]$. Then, the \mathcal{O}_Y -module $\Omega_{Y/X}^1(\log/\log)$ is generated by $d \log s$ and the relation is given by $p \cdot d \log s = d \log f(s)$ and $g'(s) \cdot s \cdot d \log s = 0$. Since $g'(s) = ps^{p-1} - \frac{z^p}{b}f'(s) = s^{p-1}(p - s \cdot f'(s)/f(s))$, the annihilator is $(p - s \cdot f'(s)/f(s))$. Since $g'(s) = \prod_{i=1}^{p-1} (s - \sigma^i(s))$ and $\text{div}(1 - \sigma^i(s)/s) = D_\sigma$ for each i , we have $\text{div}(p - s \cdot f'(s)/f(s)) = (p-1) \cdot D_\sigma$.

(8.2.4.4') The ideal \mathcal{J}_σ is generated by $\sigma(s) - s$. By $t = 1 - \prod_i t_i^{m_i} s$, we have $\sigma(s) - s = (\zeta_p - 1)t / \prod_i t_i^{m_i}$. Since t is a unit, the ideal \mathcal{J}_σ is generated by $(z / \prod_i t_i^{m_i})$. Thus, we have $p \cdot D_\sigma = \pi^*D_\chi$.

The \mathcal{O}_Y -module $\Omega_{Y/X}^1(\log/\log)$ is generated by ds and the annihilator is $(g'(s))$. Since $g'(s) = \prod_{i=1}^{p-1} (s - \sigma^i(s))$ and $\text{div}(s - \sigma^i(s)) = D_\sigma$ for each i , we have $\text{div } g'(s) = (p-1) \cdot D_\sigma$.

4. By the assertion 3, the map $\varphi_\sigma : \Omega_{Y/X}^1(\log/\log) \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_\sigma} \rightarrow \mathcal{O}_{D_\sigma}(-D_\sigma)$ is a surjection of invertible \mathcal{O}_{D_σ} -modules and hence is an isomorphism.

5. It follows from 4 that the sequence (8.2.8.2) is exact at $\Omega_{Y, \mathcal{M}_Y}^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_{E_Y}$ and at $\mathcal{O}_{E_Y}(-D_\sigma)$. Hence, the kernel of the map in the middle is an invertible \mathcal{O}_{E_Y} -module. By the assumption that θ is clean, the map $\text{rsw } \theta$ is a locally split injection. The composition $\mathcal{O}_{E_Y}(-\pi^*D_\chi) \rightarrow \pi^*\Omega_X^1(\log D) \otimes_{\mathcal{O}_Y} \mathcal{O}_{E_Y} \rightarrow \Omega_{Y, \mathcal{M}_Y}^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_{E_Y}$ is the 0-map by (8.1.1.5). Hence, the assertion follows. \square

8.3. *A computation in the rank 1 case.* — Let X and $U = X \setminus D$ over $S = \text{Spec } \mathcal{O}_K$ be as in the previous subsection. We briefly recall the definition of the 0-cycle class $c_{\mathcal{F}}$ in [24] for a smooth sheaf \mathcal{F} of $\bar{\mathbf{F}}_\ell$ -vector spaces of rank 1 on U . Let D_1, \dots, D_n be the irreducible components of D and let $E = \sum_{r_i > 0} D_i \subset X$ be the support of the Swan divisor $D_\chi = \sum_i r_i D_i$. We put $n = \dim X_K + 1$. The divisor E is supported on the closed fiber X_F . Hence, the coherent \mathcal{O}_E -module $\Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E$ is locally free of rank n and the bivariant Chern class $c(\Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E)$ is defined as an operator $\text{CH}^*(E \rightarrow E)$.

Assume χ is clean with respect to X . Then, we define the 0-cycle class $c_\chi \in \text{CH}_0(E)$ by

$$\begin{aligned} (8.3.0.1) \quad c_\chi &= \{c(\Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E)^* \cap (1 + D_\chi)^{-1} \cap D_\chi\}_{\dim 0} \\ &= (-1)^{n-1} \sum_{i=1}^m r_i \cdot c_{n-1}(\text{Coker}(\text{rsw}_i(\chi))) \cap [D_i]. \end{aligned}$$

By [24, Theorem 5.2], the cycle classes c_χ define an element of $F_0G(\partial_{V/U}U)$ for the finite étale Galois covering $V \rightarrow U$ trivializing χ , if $\dim U_K \leq 1$.

We fix an isomorphism $\bar{\mathbf{F}}_\ell^\times \rightarrow \mathbf{Q}/\mathbf{Z}[\frac{1}{\ell}] \subset \mathbf{Q}/\mathbf{Z}$. For a character $\chi \in H^1(U, \mathbf{Q}/\mathbf{Z}[\frac{1}{\ell}])$ of order prime to ℓ , let \mathcal{F}_χ denote the corresponding locally constant constructible sheaf of $\bar{\mathbf{F}}_\ell$ -vector spaces of rank 1 on U .

Conjecture 8.3.1. — *Let X be a regular flat separated scheme of finite type over S and $U = X \setminus D$ be the complement of a divisor with simple normal crossings. Let $f: V \rightarrow U$ be the étale cyclic covering trivializing χ . Assume that χ is clean with respect to X and that χ is tamely ramified on the generic fiber.*

Then, we have

$$(8.3.1.1) \quad \text{Sw}_{V/U} \mathcal{F}_\chi = f^* c_\chi$$

in $F_0G(\partial_{V/U}V)_{\mathbf{Q}(\zeta_{p^\infty})}$.

We prove a refinement of Conjecture 8.3.1 assuming $\dim U_K \leq 1$ in Theorem 8.3.7 at the end of this section. Similarly as [27, Lemma 5.1.2], Conjecture 8.3.1 implies Conjecture 7.2.8, by Brauer induction.

We show that the class c_χ also satisfies an excision formula.

Lemma 8.3.2. — *Let X be a regular flat separated scheme of finite type over S and $U = X \setminus D$ be the complement of a divisor with simple normal crossings. Let X_1 be a regular divisor meeting D transversely. We put $U_0 = X \setminus (D \cap X_1)$ and $U_1 = U \cap X_1$.*

Let χ be a character on U and let $\chi_0 = \chi|_{U_0}$ and $\chi_1 = \chi|_{U_1}$ be the restrictions. Assume that both χ and χ_0 are clean with respect to X . Then χ_1 is also clean with respect to X_1 and we have

$$c_\chi = c_{\chi_0} + c_{\chi_1}.$$

Proof. — The union $D' = D \cup X_1$ is a divisor of X with simple normal crossings and the intersection $D_1 = D \cap X_1$ is a divisor of X_1 with simple normal crossings. Let D_i be an irreducible component of E and we put $C_i = D_i \cap X_1$. The image of the map $\Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C_i} \rightarrow \Omega_{X/S}^1(\log D') \otimes_{\mathcal{O}_X} \mathcal{O}_{C_i}$ is canonically identified with $\Omega_{X_1/S}^1(\log D_1) \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{C_i}$. Hence, if both χ and χ_0 are clean, then χ is strongly clean on a neighborhood of C_i in the terminology of [21, Definition (7.4)]. Thus, by [21, Theorem (9.1)], χ_1 is strongly clean with respect to X_1 and D_{X_1} is the pull-back of D_X .

We put $E_1 = E \cap X_1$. Then, by the exact sequence $0 \rightarrow \Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \Omega_{X/S}^1(\log D') \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \mathcal{O}_{E_1} \rightarrow 0$, the difference $c_\chi - c_{\chi_0}$ is equal to

$$\begin{aligned} & \left\{ c(\Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E)^* \cap (1 + D_X)^{-1} \cap D_X \right\}_{\dim 0} \\ & \quad - \left\{ c(\Omega_{X/S}^1(\log D') \otimes_{\mathcal{O}_X} \mathcal{O}_E)^* \cap (1 + D_X)^{-1} \cap D_X \right\}_{\dim 0} \\ & = \left\{ c(\Omega_{X/S}^1(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E)^* \cap (1 + D_X)^{-1} \cap (X_1 \cap D_X) \right\}_{\dim 0}. \end{aligned}$$

By $D_{X_1} = X_1 \cap D_X$ and by $c(\Omega_{X/S}^1(\log D))^* \cap X_1 = c(\Omega_{X_1/S}^1(\log D_1))^*$, the right hand side is equal to c_{χ_1} . \square

Let $U' \rightarrow U$ be a finite étale morphism tamely ramified with respect to X and let X' be the normalization of X in U' . Then, X' has a natural log structure such that U' is the maximum open subscheme where the log structure is trivial and the map $X' \rightarrow X$ is log étale with respect to this log structure. By taking a regular proper subdivision of the associated fan [25, Section 10], we obtain a log blow-up $X'' \rightarrow X'$ such that X'' contains U' as the complement of a divisor with simple normal crossings.

Lemma 8.3.3. — *Let X be a regular flat separated scheme of finite type over S and $U = X \setminus D$ be the complement of a divisor with simple normal crossings. Assume $\dim X_K = 1$. Let $g: U' \rightarrow U$ be a finite étale morphism tamely ramified with respect to X . Let X' be a log blow-up of the normalization of X and $\bar{g}: X' \rightarrow X$ be the canonical map.*

Let χ be a character on \mathbf{U} and let χ' be the pull-back to \mathbf{U}' . Assume that χ is clean with respect to \mathbf{X} . Then χ' is also clean with respect to \mathbf{X}' and we have

$$c_{\chi'} = \bar{g}^* c_{\chi}.$$

Proof. — We may assume that the subdivision defining \mathbf{X}' induces a subdivision of the fan associated to \mathbf{X} and defines a log blow-up $\mathbf{X}_1 \rightarrow \mathbf{X}$. The induced map $\mathbf{X}' \rightarrow \mathbf{X}_1$ is finite. Since $\bar{g}: \mathbf{X}' \rightarrow \mathbf{X}$ is log étale, the map $\bar{g}^* \Omega_{\mathbf{X}/S}^1(\log \mathbf{D}) \rightarrow \Omega_{\mathbf{X}'/S}^1(\log \mathbf{D}')$ is an isomorphism. At each singular point of \mathbf{D} , χ is strongly clean with respect to \mathbf{X} . Hence, by [21, Theorem (8.1)], the divisor \mathbf{D}_{χ_1} is the pull-back of \mathbf{D}_{χ} and the divisor $\mathbf{D}_{\chi'}$ is also the pull-back of \mathbf{D}_{χ} . Further by [21, Theorem (8.1)], χ' is clean with respect to \mathbf{X}' and we have $c_{\chi'} = \bar{g}^* c_{\chi}$. \square

In the rest of this section, we assume that \mathbf{K} is of characteristic 0. The case where \mathbf{K} is of characteristic $p > 0$ is studied similarly as in [27] and in [38].

We first show that the computation in the previous subsection implies Conjecture 8.3.1 for a character of order p under a slightly stronger assumption. Since $(\mathbf{Y}, \mathcal{M}_{\mathbf{Y}})$ in Proposition 8.2.7 is log regular, by [25], there exists a log blow-up $\tilde{\mathbf{Y}} \rightarrow \mathbf{Y}$ satisfying the following property: The map $\tilde{\mathbf{Y}} \rightarrow \mathbf{Y}$ induces an isomorphism over \mathbf{V} and the scheme $\tilde{\mathbf{Y}}$ is regular and contains \mathbf{V} as the complement of a divisor $\mathbf{D}_{\tilde{\mathbf{Y}}}$ with simple normal crossings. We regard $\tilde{\mathbf{Y}}$ as a log scheme with the log structure defined by $\mathbf{D}_{\tilde{\mathbf{Y}}}$. Then, the map $\tilde{\mathbf{Y}} \rightarrow \mathbf{Y}$ is log étale.

Proposition 8.3.4. — *Let \mathbf{X} be a regular flat separated scheme of finite type over $\mathbf{S} = \text{Spec } \mathcal{O}_{\mathbf{K}}$ and $\mathbf{U} = \mathbf{X} \setminus \mathbf{D}_{\mathbf{X}}$ be the complement of a divisor with simple normal crossings. Assume that \mathbf{K} contains a primitive p -th root ζ_p of 1. Let θ be a character of order p s -clean with respect to \mathbf{X} .*

Let $\tilde{\mathbf{Y}} \rightarrow \mathbf{Y}$ be a log blow-up as above and assume that the action of $\mathbf{G} = \text{Gal}(\mathbf{V}/\mathbf{U}) \simeq \mathbf{Z}/p\mathbf{Z}$ is extended to an action on $\tilde{\mathbf{Y}}$. Let σ be a generator of \mathbf{G} and assume that σ is an admissible automorphism (Definition 5.1.4.2) of $\tilde{\mathbf{Y}}$.

1. *We put $c_{\sigma} = ((\Gamma_{\sigma}, \Delta_{\tilde{\mathbf{Y}}})_{(\tilde{\mathbf{Y}} \times_{\mathbf{S}} \tilde{\mathbf{Y}})^{\sim}} \in \mathbf{F}_0 \mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}} \tilde{\mathbf{Y}})$. Then, we have*

$$c_{\sigma} = \left\{ c^* (\Omega_{\tilde{\mathbf{Y}}/S}^1(\log \mathbf{D}_{\tilde{\mathbf{Y}}})) \cdot (1 + \mathbf{D}_{\sigma})^{-1} \cdot \mathbf{D}_{\sigma} \right\}_{\dim 0}.$$

2. *Let $\pi: \tilde{\mathbf{Y}} \rightarrow \mathbf{X}$ be the canonical map. Then, we have*

$$(8.3.4.1) \quad \text{Sw}_{\mathbf{V}/\mathbf{U}, \tilde{\mathbf{Y}}}^{\mathbf{Z}} \mathcal{F}_{\theta} = \pi^* c_{\theta}$$

in $\mathbf{F}_0 \mathbf{G}(\partial_{\mathbf{V}/\mathbf{U}} \tilde{\mathbf{Y}})$.

Proof. — 1. This is Lemma 5.1.5.2.

2. Since $\tilde{\mathbf{Y}} \rightarrow \mathbf{Y}$ is log étale, we have an exact sequence $0 \rightarrow \mathcal{O}_{\mathbf{E}_{\tilde{\mathbf{Y}}}}(-\tilde{\pi}^* \mathbf{D}_{\theta}) \rightarrow \Omega_{\mathbf{X}/S}(\log \mathbf{D}_{\mathbf{X}}) \otimes \mathcal{O}_{\mathbf{E}_{\tilde{\mathbf{Y}}}} \rightarrow \Omega_{\tilde{\mathbf{Y}}/S}(\log \mathbf{D}_{\tilde{\mathbf{Y}}}) \otimes \mathcal{O}_{\mathbf{E}_{\tilde{\mathbf{Y}}}} \rightarrow \mathcal{O}_{\mathbf{E}_{\tilde{\mathbf{Y}}}}(-\mathbf{D}_{\sigma}) \rightarrow 0$ by Corollary 8.2.8.2. Hence the ratio of the total Chern classes $c(\Omega_{\tilde{\mathbf{Y}}/S}(\log \mathbf{D}_{\tilde{\mathbf{Y}}}) \otimes \mathcal{O}_{\mathbf{E}_{\tilde{\mathbf{Y}}}}) \cdot c(\Omega_{\mathbf{X}/S}(\log \mathbf{D}_{\mathbf{X}}) \otimes \mathcal{O}_{\mathbf{E}_{\tilde{\mathbf{Y}}}})$

$\mathcal{O}_E))^{-1}$ is equal to $(1 - D_\sigma)(1 - \pi^*D_\theta)^{-1}$. Thus, by the equality $\pi^*D_\theta = pD_\sigma$ (8.2.8.1), we have

$$\begin{aligned}\pi^*c_\theta &= \pi^*\left\{c^*\left(\Omega_{X/S}^1(\log D_X)\right) \cdot (1 + \pi^*D_\theta)^{-1} \cdot D_\theta\right\}_{\dim 0} \\ &= \left\{c^*\left(\Omega_{\tilde{Y}/S}^1(\log D_{\tilde{Y}})\right) \cdot (1 + \pi^*D_\sigma)^{-1} \cdot p \cdot D_\sigma\right\}_{\dim 0}.\end{aligned}$$

By 1, the right hand side is equal to $p \cdot c_\sigma$.

Since $c_{\sigma^i} = c_\sigma$ for every $i \in (\mathbf{Z}/p\mathbf{Z})^\times$, the integral Swan class $\mathrm{Sw}_{V/U, \tilde{Y}}^{\mathbf{Z}} \mathcal{F}_\theta$ is equal to $p \cdot s_{V/U}(\sigma) = -p \cdot c_\sigma$. Thus the assertion follows. \square

We recall an induction step from [38], which will be used in the proof of Theorem 8.3.7 below.

Lemma 8.3.5 (Cf. [38, Lemma 2]). — *Let X be a regular scheme and U be the complement of a divisor with simple normal crossings. Let $\chi, \theta \in H^1(U, \mathbf{Q}/\mathbf{Z})$ be characters clean with respect to X . Assume θ is of order p and s -clean. Let $V \rightarrow U$ be the cyclic covering of degree p trivializing θ and Y be the normalization of X in V with the log structure defined by V . Let $\tilde{Y} \rightarrow Y$ be the log blow-up defined by a regular proper subdivision of the fan of Y and $\pi: \tilde{Y} \rightarrow X$ be the canonical map.*

Assume χ is clean with respect to X and the pull-back $\chi' = \pi^\chi$ is clean with respect to \tilde{Y} . Assume further that the following condition is satisfied:*

$$(8.3.5.1) \quad \text{For } D_\chi = \sum_i r_i D_i \text{ and } D_\theta = \sum_i s_i D_i, \text{ the condition } r_i = 0 \text{ implies } s_i = 0, \text{ and the condition } r_i > 0 \text{ implies } r_i > s_i.$$

Then, we have

$$(8.3.5.2) \quad \pi^*c_\chi = c_{\chi'} + D_{U_1/V, Y_1}^{\log}$$

in $\mathrm{CH}_0(E_{\tilde{Y}})$.

The proof is the same as [38, Lemma 2] by using the exact sequence (8.2.8.2) and we omit it.

Corollary 8.3.6. — *We keep the notation and the assumptions in Lemma 8.3.5 except that we do not assume (8.3.5.1). Assume further that χ is of order $n = mp$ and $\theta = m \cdot \chi$. Assume that the Swan class $\mathrm{Sw}_{V/U_1, Y}^{\mathbf{Z}} \mathcal{F}_{\chi'}$ is defined integrally and that we have*

$$(8.3.6.1) \quad \mathrm{Sw}_{V/U_1, Y}^{\mathbf{Z}} \mathcal{F}_{\chi'} = \bar{g}^* c_{\chi'}$$

Then, the Swan class $\mathrm{Sw}_{V/U, Y}^{\mathbf{Z}} \mathcal{F}_\chi$ is also defined integrally and we have

$$(8.3.6.2) \quad \mathrm{Sw}_{V/U, Y}^{\mathbf{Z}} \mathcal{F}_\chi = \bar{f}^* c_\chi.$$

Proof. — By the definition of the Swan class and by $s_{V/U}(\sigma^i) = s_{V/U}(\sigma)$ for an integer prime to p , we obtain

$$\mathrm{Sw}_{V/U,Y}^{\mathbf{Z}} \mathcal{F}_\chi - D_{V/U,Y}^{\log} = \mathrm{Sw}_{V/U_1,Y}^{\mathbf{Z}} \mathcal{F}_{\chi_1} - D_{V/U_1,Y}^{\log}.$$

By $\theta = m \cdot \chi$, the assumption (8.3.5.1) is satisfied. Hence, by applying Lemma 8.3.5, we obtain $\bar{f}^* c_\chi = \bar{g}^* c_{\chi_1} + \bar{g}^* D_{U_1/V,Y_1}^{\log}$. Thus, it follows from the assumption (8.3.6.1) and the chain rule $D_{U/V,Y}^{\log} = D_{U/U_1,Y}^{\log} + \bar{g}^* D_{U_1/V,Y_1}^{\log}$. \square

In the rest of the paper, we consider the case $\dim U_{\mathbf{K}} = 1$. In this case, the strong form of resolution of singularity is known and consequently the Swan class $\mathrm{Sw}_{V/U}^{\mathbf{Z}} \mathcal{F}$ is defined integrally as an element of $F_0 G(\partial_{V/U} V)$.

Theorem 8.3.7. — *Assume \mathbf{K} is of characteristic 0. Let U be a regular flat separated scheme of finite type over S such that $\dim U_{\mathbf{K}} = 1$. Let $\mathcal{F} = \mathcal{F}_\chi$ be a locally constant constructible sheaf of $\bar{\mathbf{F}}_\ell$ -vector spaces of rank 1 and $\chi \in H^1(U, \mathbf{Q}/\mathbf{Z})$ be the corresponding character. Let $f: V \rightarrow U$ be the cyclic covering trivializing χ . Then, we have*

$$(8.3.7.1) \quad [\mathbf{K}(\zeta_p) : \mathbf{K}] \cdot \mathrm{Sw}_{V/U}^{\mathbf{Z}} \mathcal{F}_\chi = [\mathbf{K}(\zeta_p) : \mathbf{K}] \cdot f^* c_\chi$$

in $F_0 G(\partial_{V/U} V)$.

Proof. — Let χ' be the p -primary part of χ and $V' \rightarrow U$ be the cyclic covering trivializing χ' and let $\pi: V \rightarrow V'$ be the canonical map. Then, since $\mathrm{Sw}_{V/U}^{\mathbf{Z}} \mathcal{F}_\chi = \pi^* \mathrm{Sw}_{V'/U}^{\mathbf{Z}} \mathcal{F}_{\chi'}$ and $c_\chi = c_{\chi'}$, we may assume that the order of χ is a power of p .

We show that we may assume $U = U_{\mathbf{K}}$. Let X be a proper regular flat scheme over S containing U as the complement of a divisor D with simple normal crossings. By blowing up some closed points in the closed fiber of X , we may assume that $U_{\mathbf{K}} \subset X$ is the complement of a divisor with simple normal crossings. We show the claim by the induction on the number of irreducible components of $U_{\mathbf{F}}$.

If the number is 0, then $U_{\mathbf{F}}$ is empty and there is nothing to prove. Let C be an irreducible component of $U_{\mathbf{F}}$. Let χ_0 be the restriction of χ to $U_0 = U \setminus C$. By blowing up X at the boundary of C , we may assume that both χ and χ_0 are clean with respect to X . Then, by the excision formulas Proposition 7.2.5.2 and Lemma 8.3.2, the equality (8.3.1.1) for χ_0 is equivalent to that for χ . Thus, by the induction, the claim is proved.

We assume $U = U_{\mathbf{K}}$. By taking the base change to $\mathbf{K}(\zeta_p)$ and by applying Corollary 5.4.2 and Lemma 8.3.3, we may assume that \mathbf{K} contains a primitive p -th root of 1.

Assume χ is of order p . Then, by Lemma 8.2.6.4, we may assume χ is s -clean with respect to X by replacing X by a blow-up. Then, it follows from Proposition 8.3.4.

Assume χ is of order p^n and we prove the assertion by induction on $n \geq 1$. Similarly as above, we may assume that $\theta = p^{n-1} \chi$ is s -clean. Let $U_1 \rightarrow U$ be the cyclic covering of degree p and $g: V \rightarrow U_1$ be the canonical map. Let Y_1 be the normalization of X in U_1

and $\tilde{Y}_1 \rightarrow Y_1$ be a blow-up as in Corollary 8.3.6. Let $\pi : \tilde{Y}_1 \rightarrow X$ be the canonical map. Then, further similarly as above, we may assume that $\chi' = \pi^* \chi$ is clean with respect to \tilde{Y} . Then, by the induction hypothesis, we have $\mathrm{Sw}_{\mathbb{V}/\mathbb{U}_1, \tilde{Y}}^{\mathbb{Z}} \mathcal{F}_{\chi'} = g^* c_{\chi'}$. Thus it follows from Corollary 8.3.6. \square

We deduce the integrality of the Swan classes and the conjecture of Serre under the assumption $\dim \mathbb{U}_K \leq 1$.

Corollary 8.3.8. — 1. Let \mathbb{U} be a regular flat separated scheme of finite type over \mathcal{O}_K . If $\dim \mathbb{U}_K \leq 1$, Conjecture 7.2.8 is true.

2. For a regular local ring A of $\dim A \leq 2$, Conjecture 7.2.9 is true.

Proof. — 1. By the Brauer induction and by the induction formula Proposition 7.2.6, we may assume $\mathrm{rank} \mathcal{F} = 1$. Then, it follows from Theorem 8.3.7.

2. Since the positive characteristic case is proved in [28], it follows from 1 and Lemma 7.2.10. \square

As in the classical ramification theory, our proof of the integrality Conjecture 7.2.8 is by the reduction to the rank 1 case using Brauer induction.

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