A FINITE ELEMENT DISCRETIZATION OF THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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Abstract. We consider a variational formulation of the three-dimensional Navier–Stokes equations with mixed boundary conditions and prove that the variational problem admits a solution provided that the domain satisfies a suitable regularity assumption. Next, we propose a finite element discretization relying on the Galerkin method and establish *a priori* and *a posteriori* error estimates.

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1. INTRODUCTION

Let Ω be a bounded connected domain in $\mathbb{R}^3$, with a Lipschitz-continuous boundary $\partial\Omega$. We consider a partition without overlap of $\partial\Omega$ into two connected parts Γ_m and Γ and introduce the unit outward normal vector $\mathbf{n}$ to Ω on $\partial\Omega$. We are interested in the finite element discretization of the Navier–Stokes equations, for a positive constant viscosity ν,

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{grad}\,P &= \mathbf{f} && \text{in } \Omega, \\
\operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma, \\
\mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \Gamma_m, \\
(\mathbf{curl}\,\mathbf{u})\times\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_m,
\end{aligned}
\tag{1.1}
$$

where the unknowns are the velocity $\mathbf{u}$ and the pressure P of the fluid. This type of mixed boundary conditions appears in a large number of physical situations, the simplest one being a tank closed by a membrane on a part of its boundary (the index "m" in Γ_m means membrane). An other example is the flow in a bifurcating pipe [8].

The Navier–Stokes equations with mixed boundary conditions have been considered for a long time. We refer to [4] for a pioneering paper on this subject. However, in a number of works concerning these mixed conditions, the boundary of the domain Ω is assumed to be piece-wise smooth. Here, we prove the existence of a solution

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to problem (1.1) with minimal regularity assumptions on the boundary. We refer to [2] for similar results which, however, mainly concern the two-dimensional situation.

In order to discretize the Navier–Stokes equations provided with the boundary conditions on Γ_m, a new formulation has been recently introduced in [20] (see also [11,12]) and has been extended to mixed conditions in [1]: the vorticity of the fluid is considered as a third independent unknown. However, we prefer to keep the formulation with two unknowns, in order to make use of standard finite element results. We propose several finite element discretizations of our problem and prove optimal *a priori* estimates provided some regularity assumptions are satisfied. These conditions seem to be not too restrictive. In addition, we establish *a posteriori* error estimates which are optimal except in a neighbourhood of the re-entrant corners and edges in Γ_m. Despite this lack of optimality, we think that the discretizations that we propose lead to efficient simulations of incompressible viscous fluids in realistic situations.

An outline of the paper is as follows:

- In Section 2, we give a variational formulation of the problem and prove the existence of a solution.
- Section 3 is devoted to the description of the finite element discretization.
- In Section 4, we prove the existence of a solution for the discrete problem and derive *a priori* error estimates.
- In Section 5, finally, we present *a posteriori* error estimates based on residual error indicators.
- A numerical experiment is described in Section 6.

2. The continuous problem

From now on, we assume for simplicity that Ω is simply-connected and has a connected boundary, and also that $\partial\Gamma_m = \partial\Gamma$ is a Lipschitz-continuous submanifold of $\partial\Omega$. For the variational formulation of problem (1.1), we first observe by using the formulas

$$-\Delta\mathbf{u} = \mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) - \mathbf{grad}\,(\mathrm{div}\,\mathbf{u})$$

and

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\mathbf{curl}\,\mathbf{u})\times\mathbf{u} + \frac{1}{2}\mathbf{grad}\,|\mathbf{u}|^2$$

that this problem can equivalently be written as, for $\varepsilon = 0$ and 1,

$$
\begin{aligned}
\nu\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) - \varepsilon\nu\,\mathbf{grad}\,(\mathrm{div}\,\mathbf{u}) & \\
+ (\mathbf{curl}\,\mathbf{u})\times\mathbf{u} + \mathbf{grad}\,p &= \mathbf{f} \quad \text{in } \Omega, \\
\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u}\times\mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma, \\
\mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on } \Gamma\cup\Gamma_m, \\
(\mathbf{curl}\,\mathbf{u})\times\mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_m,
\end{aligned}
\tag{2.1}
$$

where the new unknown

$$p = P + \frac{1}{2}|\mathbf{u}|^2$$

represents the dynamic pressure. The reason for choosing this modified form is that the last boundary condition, namely $(\mathbf{curl}\,\mathbf{u})\times\mathbf{n} = \mathbf{0}$ on Γ_m, can now be treated as a natural boundary condition. Note also that, thanks to the equation $\mathrm{div}\,\mathbf{u} = 0$ in Ω, the two formulations for $\varepsilon = 0$ and $\varepsilon = 1$ are equivalent in the sense of distributions; the reason for considering both of them is explained later on (see Rem. 3.3).

We consider the full scale of Sobolev spaces $H^s(\Omega)$, $s \geq 0$, provided with the usual norm $\|\cdot\|_{H^s(\Omega)}$ and semi-norm $|\cdot|_{H^s(\Omega)}$, together with their subspaces $H_0^s(\Omega)$. We introduce the domain $H(\text{div}\,,\Omega)$ of the divergence operator, namely

$$H(\text{div}\,,\Omega) = \big\{\mathbf{v} \in L^2(\Omega)^d; \text{div}\,\mathbf{v} \in L^2(\Omega)\big\}\cdot$$

Since the normal trace operator $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}$ is defined from $H(\text{div},\Omega)$ onto $H^{-\frac{1}{2}}(\partial\Omega)$, see [14], Chapter I, Theorem 2.5, we also consider its kernel

$$H_0(\text{div}\,,\Omega) = \big\{\mathbf{v} \in H(\text{div},\Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\big\}\cdot$$

Similarly, we introduce the domain of the **curl** operator

$$H(\mathbf{curl}\,,\Omega) = \big\{\mathbf{v} \in L^2(\Omega)^3; \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3\big\}\cdot$$

The tangential trace operator $\mathbf{v} \mapsto \mathbf{v} \times \mathbf{n}$ is defined on $H(\mathbf{curl}\,,\Omega)$ with values in $H^{-\frac{1}{2}}(\partial\Omega)^3$, see [14], Chapter I, Theorem 2.11. It can also be checked that its restriction to Γ maps $H(\mathbf{curl}\,,\Omega)$ into the dual space $H_{00}^{\frac{1}{2}}(\Gamma)'$ of $H_{00}^{\frac{1}{2}}(\Gamma)$ (see [17], Chapter I, Thm. 11.7, for the definition of the last space). So, in view of the first boundary condition in (2.1), we define the space

$$H_*(\mathbf{curl}\,,\Omega) = \big\{\mathbf{v} \in H(\mathbf{curl}\,,\Omega); \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\big\}\cdot$$

We set

$$\mathbb{X}(\Omega) = H_0(\text{div},\Omega) \cap H_*(\mathbf{curl}\,,\Omega).$$

From now on, this space is equipped with the semi-norm

$$\|\mathbf{v}\|_{\mathbb{X}(\Omega)} = \big(\|\text{div}\,\mathbf{v}\|_{L^2(\Omega)}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{L^2(\Omega)^3}^2\big)^{\frac{1}{2}}. \tag{2.2}$$

Since Ω is simply-connected, we recall from [3], Corollary 3.16, that this quantity is a norm, which is equivalent to the graph norm of $H(\text{div},\Omega) \cap H(\mathbf{curl}\,,\Omega)$, i.e., that there exists a constant c only depending on Ω such that

$$\forall\mathbf{v} \in \mathbb{X}(\Omega), \quad \|\mathbf{v}\|_{L^2(\Omega)^3} \leq c\|\mathbf{v}\|_{\mathbb{X}(\Omega)}.$$

We denote by $L_0^2(\Omega)$ the space of functions in $L^2(\Omega)$ with a zero mean-value on Ω.

We now assume that the data $\mathbf{f}$ belong to the dual space $\mathbb{X}(\Omega)'$ of $\mathbb{X}(\Omega)$ and consider the variational problem:

Find $(\mathbf{u}, p)$ in $\mathbb{X}(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned}
\forall\mathbf{v} \in \mathbb{X}(\Omega), \quad & a_\varepsilon(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle\mathbf{f}, \mathbf{v}\rangle, \\
\forall q \in L_0^2(\Omega), \quad & b(\mathbf{u}, q) = 0,
\end{aligned} \tag{2.3}$$

where the bilinear forms $a_\varepsilon(\cdot, \cdot)$ and $b(\cdot, \cdot)$ and the trilinear form $c(\cdot, \cdot, \cdot)$ are defined by

$$\begin{aligned}
a_\varepsilon(\mathbf{u}, \mathbf{v}) &= \nu \int_\Omega (\mathbf{curl}\,\mathbf{u})(\mathbf{x}) \cdot (\mathbf{curl}\,\mathbf{v})(\mathbf{x})\,d\mathbf{x} \\
&\quad + \varepsilon\nu \int_\Omega (\text{div}\,\mathbf{u})(\mathbf{x})(\text{div}\,\mathbf{v})(\mathbf{x})\,d\mathbf{x}, \\
b(\mathbf{v}, q) &= -\int_\Omega (\text{div}\,\mathbf{v})(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}, \\
c(\mathbf{w}, \mathbf{u}, \mathbf{v}) &= \int_\Omega \big((\mathbf{curl}\,\mathbf{w})(\mathbf{x}) \times \mathbf{u}(\mathbf{x})\big) \cdot \mathbf{v}(\mathbf{x})\,d\mathbf{x}.
\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathbb{X}(\Omega)$ and its dual space. The reason for the introduction of this problem is stated in the next proposition.

Proposition 2.1. *Any solution of problem* (2.3) *is a solution of problem* (2.1) *where the first two equations are satisfied in the sense of distributions.*

Proof. Let $(\mathbf{u}, p)$ be a solution of problem (2.3). Denoting by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with a compact support in Ω, we first take $\mathbf{v}$ in $\mathcal{D}(\Omega)^3$ in the first line of problem (2.3). This gives the first equation in problem (2.1). Next, it is readily checked from the Stokes formula that the second line of problem (2.3) is also satisfied when q is a constant, hence for all q in $L^2(\Omega)$. Thus, we take q in $\mathcal{D}(\Omega)$, which yields the second equation in problem (2.1). It also follows from the definition of $\mathbb{X}(\Omega)$ that the first two boundary conditions in problem (2.1) hold. Finally, introducing an infinitely differentiable function $\boldsymbol{\varphi}$ with a compact support in Γ_m and choosing $\mathbf{v}$ as a lifting in $\mathbb{X}(\Omega) \cap H^1(\Omega)^3$ of the extension of $\boldsymbol{\varphi} \times \mathbf{n}$ by zero to $\partial\Omega$ gives the last boundary condition of problem (2.1). $\qquad\square$

Note that the converse property, *i.e.*, the fact that any solution of problem (2.1) is a solution of problem (2.3), would require the density of $\mathcal{D}(\Omega)^3$ in $\mathbb{X}(\Omega)$ which is unlikely when Γ_m has a positive measure.

Next, we observe that the forms $a_\varepsilon(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on $\mathbb{X}(\Omega) \times \mathbb{X}(\Omega)$ and $\mathbb{X}(\Omega) \times L_0^2(\Omega)$, respectively. Moreover the following ellipticity property holds for $\varepsilon = 1$:

$$\forall \mathbf{v} \in \mathbb{X}(\Omega), \quad a_1(\mathbf{v}, \mathbf{v}) \geq \nu \|\mathbf{v}\|_{\mathbb{X}(\Omega)}^2. \tag{2.4}$$

On the other hand, the kernel

$$V = \left\{\mathbf{v} \in \mathbb{X}(\Omega); \, \forall q \in L_0^2(\Omega), \, b(\mathbf{v}, q) = 0\right\}$$

is a closed subspace of $\mathbb{X}(\Omega)$ and coincides with

$$V = \left\{\mathbf{v} \in \mathbb{X}(\Omega); \, \text{div } \mathbf{v} = 0 \text{ in } \Omega\right\}\cdot$$

This result together with the definition (2.2) of the norm of $\mathbb{X}(\Omega)$ yields the following ellipticity property for $\varepsilon = 0$ and 1:

$$\forall \mathbf{v} \in V, \quad a_\varepsilon(\mathbf{v}, \mathbf{v}) \geq \nu \|\mathbf{v}\|_{\mathbb{X}(\Omega)}^2. \tag{2.5}$$

Finally, the inf-sup condition for $b(\cdot,\cdot)$ is a direct consequence of its analogue with $\mathbb{X}(\Omega)$ replaced by $H_0^1(\Omega)^3$ which can be found in [14], Chapter I, Corollary 2.4, for instance: there exists a constant $\beta > 0$ such that

$$\forall q \in L_0^2(\Omega), \quad \sup_{\mathbf{v} \in \mathbb{X}(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbb{X}(\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}. \tag{2.6}$$

Combining all this with [14], Chapter I, Corollary 4.1, yields the well-posedness of the Stokes problem with the same boundary conditions as in problem (2.1).

Lemma 2.2. *For any data* $\mathbf{f}$ *in* $\mathbb{X}(\Omega)'$, *the Stokes problem*

  Find $(\mathbf{u}, p)$ *in* $\mathbb{X}(\Omega) \times L_0^2(\Omega)$ *such that*

$$\begin{aligned}
\forall \mathbf{v} \in \mathbb{X}(\Omega), \quad a_\varepsilon(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle, \\
\forall q \in L_0^2(\Omega), \quad b(\mathbf{u}, q) &= 0,
\end{aligned} \tag{2.7}$$

has a unique solution $(\mathbf{u}, p)$ *in* $\mathbb{X}(\Omega) \times L_0^2(\Omega)$. *Moreover, this solution satisfies*

$$\nu\|\mathbf{u}\|_{\mathbb{X}(\Omega)} + \beta\|p\|_{L^2(\Omega)} \leq 3\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}. \tag{2.8}$$

Let $\mathcal{S}$ denote the Stokes operator, *i.e.*, the operator which associates with any $\mathbf{f}$ in $\mathbb{X}(\Omega)'$ the part $\mathbf{u}$ of the solution $(\mathbf{u}, p)$ of problem (2.7). Lemma 2.2 states that this operator is well-defined. We now prove some further regularity properties.

Lemma 2.3. *The operator $\mathcal{S}$ maps $L^2(\Omega)^3$ into $H^s(\Omega)^3$ for $s \leq \frac{1}{2}$ in the general case and $s \leq 1$ when Ω is convex.*

Proof. This follows from the embedding of $H_0(\mathrm{div}\,, \Omega) \cap H(\mathbf{curl}\,, \Omega)$ into $H^{\frac{1}{2}}(\Omega)^3$ in the general case [9] and into $H^1(\Omega)^3$ when Ω is convex [3], Theorem 2.17. □

It can also be noted that, for any data $\mathbf{f}$ in $L^2(\Omega)^3$, the restrictions of $\mathcal{S}\mathbf{f}$ to any subdomain of Ω which does not intersect a neighbourhood of Γ or a neighbourhood of Γ_m is more regular. However, the singularities that appear near $\overline{\Gamma} \cap \overline{\Gamma}_m$ seem presently unknown. On the other hand, it is likely that, if Ω is a convex polyhedron, $\mathcal{S}$ maps $L^2(\Omega)^3$ into $H^s(\Omega)^3$ for some $s > 1$. However, proving this result is beyond the scope of this work.

To study the non-linear part of the problem, we suppose that:

Assumption 2.4. *The space $\mathbb{X}(\Omega)$ is compactly embedded in $L^4(\Omega)^3$.*

Assumption 2.4 is always satisfied when Ω has a $\mathcal{C}^{1,1}$ boundary or is convex: indeed, it is proved in [13] or in [3], Theorem 2.17 that, in these cases, the space $H_0(\mathrm{div}\,, \Omega) \cap H(\mathbf{curl}\,, \Omega)$ is contained in $H^1(\Omega)^3$, hence in $L^6(\Omega)^3$. This of course yields the desired embedding. In the next lemma, we make more precise the situations where we can prove that this hypothesis is satisfied.

Lemma 2.5. *Assume that Ω is a polyhedron and denote by $\mathcal{V}$ a neighbourhood of the re-entrant corners of Ω inside Γ_m. Then, the space of restrictions of functions of $\mathbb{X}(\Omega)$ to $\Omega \setminus \overline{\mathcal{V}}$ is compactly embedded in $L^4(\Omega \setminus \overline{\mathcal{V}})^3$.*

Proof. Since this property is local, it suffices to check it for the different parts of $\Omega \setminus \overline{\mathcal{V}}$. For each such part Ω_k, we implicitly use a regular function α_k which is equal to 1 on Ω_k and vanishes outside a (small enough) neighbourhood of Ω_k in $\Omega \setminus \overline{\mathcal{V}}$.

(1) Let Ω_1 be a convex subset of $\Omega \setminus \overline{\mathcal{V}}$. Then, the previous arguments yield the desired property with $\Omega \setminus \overline{\mathcal{V}}$ replaced by Ω_1.

(2) Let Ω_2 be a subset of $\Omega \setminus \overline{\mathcal{V}}$ such that $\partial\Omega_2 \cap \partial\Omega$ does not intersect $\overline{\Gamma}_m$. Since the space of functions in $H(\mathrm{div}\,, \Omega_2) \cap H(\mathbf{curl}\,, \Omega_2)$ with zero traces on $\partial\Omega_2$ coincides with $H_0^1(\Omega_2)^3$, we have the desired result on Ω_2.

(3) Let Ω_3 be a neighbourhood in $\Omega \setminus \overline{\mathcal{V}}$ of part of a re-entrant edge E in $\overline{\Gamma}_m$. It follows from [10], equation (2.1) that each function in $H_0(\mathrm{div}\,, \Omega) \cap H(\mathbf{curl}\,, \Omega)$ can be written as the sum of a function in $H^1(\Omega_3)^3 \cap H_0(\mathrm{div}\,, \Omega_3)$ and of the gradient of a function Φ_3. Moreover, this function Φ_3 can be written as (see [10], Sect. 3.E)

$$\Phi_3(r, \theta, z) = S(r, z)\, r^{\frac{\pi}{\omega}}\, \varphi(\theta),$$

where

- z stands for the tangential coordinate on E;
- r and θ denote the polar coordinates in the dihedron with axis E such that its two faces intersect $\partial\Omega_3$;
- ω is the angle of this dihedron;
- S is a smooth cut-off function and φ a smooth function on $[0, \omega]$ which satisfies the appropriate boundary conditions.

Thus, it is readily checked that, since $\pi < \omega < 2\pi$, the function $\mathbf{grad}\,\Phi_3$ belongs to a compact subset of $L^4(\Omega_3)^3$, whence the desired embedding follows.

(4) Let Ω_4 be a neighbourhood of the re-entrant corners or edges in $\overline{\Gamma} \cap \overline{\Gamma}_m$, and let Γ_4 denote the intersection $\Gamma \cap \partial\Omega_4$. Using once more [10], equation (2.1), we observe that the restriction to Ω_4 of any function $\mathbf{v}$ in $H_0(\mathrm{div}\,, \Omega) \cap H(\mathbf{curl}\,, \Omega)$ can be written as

$$\mathbf{v}_{|\Omega_4} = \mathbf{v}_4 + \mathbf{grad}\,\Phi_4,$$

where $\mathbf{v}_4$ belongs to $H^1(\Omega_4)^3 \cap H_0(\operatorname{div}, \Omega_4)$ and Φ_4 is the product of a smooth cut-off function by a harmonic function φ_4. If, moreover, $\mathbf{v}$ belongs to $\mathbb{X}(\Omega)$, the following boundary conditions hold on Γ_4

$$\mathbf{v}_4 + \mathbf{grad}\, \Phi_4 = \mathbf{0} \quad \text{on } \Gamma_4.$$

By using the harmonic lifting of $\mathbf{v}_4$ in $H^1(\Omega_4)^3$, we can easily assume that $\mathbf{grad}\, \Phi_4$ vanishes on Γ_4. Thus, the function φ_4 satisfies, up to a constant,

$$-\Delta\varphi_4 = 0 \quad \text{in } \Omega_4, \qquad \varphi_4 = \partial_n\varphi_4 = 0 \quad \text{on } \Gamma_4.$$

Thus, according to Holmgren's principle (see [21], Sect. 21, for instance), it is zero, and the function $\mathbf{v}_4$ belongs to $H^1(\Omega_4)^3$.

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

It follows from the previous lemma that Assumption 2.4 holds for any polyhedron without re-entrant corners inside Γ_m. This assumption seems necessary to prove the continuity of the form $c(\cdot,\cdot,\cdot)$.

Lemma 2.6. *If Assumption 2.4 is satisfied, the form $c(\cdot,\cdot,\cdot)$ is continuous on $\mathbb{X}(\Omega) \times \mathbb{X}(\Omega) \times \mathbb{X}(\Omega)$.*

Proof. Owing to Assumption 2.4, this a direct consequence of Hölder's inequality

$$|c(\mathbf{w}, \mathbf{u}, \mathbf{v})| \le \|\mathbf{curl}\, \mathbf{w}\|_{L^2(\Omega)^3} \|\mathbf{u}\|_{L^4(\Omega)^3} \|\mathbf{v}\|_{L^4(\Omega)^3}. \qquad\qquad\qquad\qquad\square$$

Moreover, the following antisymmetry property is readily checked:

$$\forall \mathbf{w} \in \mathbb{X}(\Omega), \ \forall \mathbf{v} \in \mathbb{X}(\Omega), \quad c(\mathbf{w}, \mathbf{v}, \mathbf{v}) = 0. \tag{2.9}$$

Thus, we are in a position to prove the existence of a solution to problem (2.3). We proceed in three steps and begin with an *a priori* estimate of the solution.

Proposition 2.7. *If Assumption 2.4 is satisfied, for any data $\mathbf{f}$ in $\mathbb{X}(\Omega)'$, any solution $(\mathbf{u}, p)$ of problem (2.3) satisfies*

$$\begin{aligned}
\|\mathbf{u}\|_{\mathbb{X}(\Omega)} &\le \frac{1}{\nu} \|\mathbf{f}\|_{\mathbb{X}(\Omega)'}; \\
\|p\|_{L^2(\Omega)} &\le c \|\mathbf{f}\|_{\mathbb{X}(\Omega)'} \big(1 + \|\mathbf{f}\|_{\mathbb{X}(\Omega)'}\big).
\end{aligned} \tag{2.10}$$

Proof. When taking $\mathbf{v}$ equal to $\mathbf{u}$ in problem (2.3) and using (2.5) and property (2.9), we obtain the estimate for $\mathbf{u}$ in (2.10) with the desired constant. The estimate for p is then easily derived from the inf-sup condition (2.6). $\quad\square$

Proposition 2.8. *If Assumption 2.4 is satisfied, there exists a real number $\nu_0 > 0$ such that, for $\nu \ge \nu_0$ and for any data $\mathbf{f}$ in $\mathbb{X}(\Omega)'$, problem (2.3) has a unique solution $(\mathbf{u}, p)$.*

Proof. With any function $\mathbf{u}$ in $\mathbb{X}(\Omega)$, we associate the function

$$\Phi(\mathbf{u}) = \mathcal{S}\big(\mathbf{f} - (\mathbf{curl}\, \mathbf{u}) \times \mathbf{u}\big).$$

(1) Taking $\mathbf{v}$ equal to $\Phi(\mathbf{u})$ in the equation defining $\Phi(\mathbf{u})$, we derive from inequality (2.5) and Assumption 2.4 that

$$\|\Phi(\mathbf{u})\|_{\mathbb{X}(\Omega)} \le \frac{1}{\nu} \|\mathbf{f}\|_{\mathbb{X}(\Omega)'} + \frac{c_0^2}{\nu} \|\mathbf{u}\|_{\mathbb{X}(\Omega)}^2,$$

where c_0 stands for the norm of the embedding of $\mathbb{X}(\Omega)$ into $L^4(\Omega)^3$. We now set:

$$\nu_0 > 2c_0\sqrt{\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}},$$
$$R = \frac{\nu}{2c_0^2}\left(1 - \sqrt{1 - \frac{4c_0^2}{\nu^2}\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}}\right).$$

It thus follows from the previous lines that, for $\nu \geq \nu_0$, Φ maps the ball with radius R into itself.

(2) Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two functions in the ball with radius R. The same arguments previously used yield that

$$\|\Phi(\mathbf{u}_1) - \Phi(\mathbf{u}_2)\|_{\mathbb{X}(\Omega)} \leq \frac{1}{\nu}\sup_{\mathbf{v}\in\mathbb{X}(\Omega)}\frac{c(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1, \mathbf{v}) + c(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{v})}{\|\mathbf{v}\|_{\mathbb{X}(\Omega)}}$$
$$\leq \frac{2c_0^2 R}{\nu}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbb{X}(\Omega)}.$$

When $\nu \geq \nu_0$ the choice of R implies that the quantity $\frac{2c_0^2 R}{\nu}$ is less than 1. Hence, the mapping Φ is a contraction of the ball with radius R.

(3) Combining the first two parts of the proof with Banach's fixed point theorem yields the existence of a function $\mathbf{u}$ in $\mathbb{X}(\Omega)$ such that $\Phi(\mathbf{u}) = \mathbf{u}$. Denoting by p the function associated with this $\Phi(\mathbf{u})$ in problem (2.7), we observe that the pair $(\mathbf{u}, p)$ is a solution of problem (2.3).

(4) Let $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ be two solutions of problem (2.3). The same arguments as in part (2) of the proof combined with the estimates for $\mathbf{u}_1$ and $\mathbf{u}_2$ established in Proposition 2.7 lead to

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbb{X}(\Omega)} \leq \frac{2c_0^2}{\nu^2}\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbb{X}(\Omega)}.$$

When $\nu \geq \nu_0$, this inequality implies that $\mathbf{u}_1$ is equal to $\mathbf{u}_2$. Next, by applying the inf-sup condition (2.6), it is readily checked that p_1 and p_2 coincide. This gives the uniqueness of the solution. $\square$

Proposition 2.9. *If Assumption* 2.4 *is satisfied, for all* $\nu > 0$ *and for any data* $\mathbf{f}$ *in* $\mathbb{X}(\Omega)'$, *problem* (2.3) *has at least a solution* $(\mathbf{u}, p)$.

Proof. Since the result for $\nu \geq \nu_0$ is proven in Proposition 2.8, we assume that $0 < \nu < \nu_0$. Denoting by $\mathcal{S}_*$ the Stokes operator $\mathcal{S}$ for the viscosity ν equal to 1 and setting $\mu = \frac{1}{\nu}$, we observe that problem (2.3) can equivalently be written (up to a modification of the pressure) as

$$\mathbf{u} + \mu\,\mathcal{S}_*\big((\mathbf{curl\,u}) \times \mathbf{u} - \mathbf{f}\big). \tag{2.11}$$

We now intend to prove the existence of a solution for this problem, for μ in the interval $[\mu_0, \mu_1]$ with $\mu_0 = \frac{1}{\nu_0}$, $\mu_1 = \frac{1}{\nu}$ by using the Leray–Schauder theory (see [16], Chapter 2, Sect. 5, for instance).

(1) Let B denote the open ball of $\mathbb{X}(\Omega)$ with radius $2\mu_1\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}$. It follows from Proposition 2.7 that, for any $\mu \in [\mu_0, \mu_1]$, equation (2.11) has no solution $\mathbf{u}$ such that $\|\mathbf{u}\|_{\mathbb{X}(\Omega)} = 2\mu_1\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}$.

(2) Let Ψ be the mapping $(\mu, \mathbf{v}) \mapsto \mathbf{v} + \mu\,\mathcal{S}_*\big((\mathbf{curl\,v}) \times \mathbf{v} - \mathbf{f}\big)$. The mapping Ψ is continuously differentiable on $[\mu_0, \mu_1] \times \mathbb{X}(\Omega)$, and it follows from Assumption 2.4, more precisely from the compactness of the embedding of $\mathbb{X}(\Omega)$ into $L^4(\Omega)^3$, that the mapping $\mathrm{Id} - \Psi$ maps $[\mu_0, \mu_1] \times \overline{B}$ into a compact subset of $\mathbb{X}(\Omega)$.

(3) Proposition 2.8 yields that, for $\mu = \mu_0$, equation (2.11) has a unique solution $\mathbf{u}$. Then, this solution is nonsingular in the sense that the Fréchet derivative of Ψ with respect to $\mathbf{v}$ at $(\mu_0, \mathbf{u})$ is an isomorphism of $\mathbb{X}(\Omega)$. Thus the topological degree of $\Psi(\mu_0, \cdot)$ with respect to B is not zero (we skip the technical argument which is needed to check this result and which relies on the fact that the compact operator $D(\mathrm{Id} - \Psi)$ is the limit of operators with finite-dimensional range).

(4) Since the topological degree is invariant under homotopy, the degree of $\Psi(\mu_1, \cdot)$ with respect to B is not zero. This yields the existence of a solution $\mathbf{u}$ of equation (2.11) for $\mu = \mu_1$.

(5) Using the inf-sup condition (2.6), we easily derive the existence of a p in $L_0^2(\Omega)$ such that $(\mathbf{u}, p)$ is a solution of problem (2.3). □

Remark 2.10. By combining Lemma 2.3 with a boot-strap argument, we easily derive that any solution $(\mathbf{u}, p)$ of problem (2.3) satisfies the same regularity properties as the solution of problem (2.7). For instance, for any data $\mathbf{f}$ in $L^2(\Omega)^3$, the velocity $\mathbf{u}$ belongs to $H^1(\Omega)^3$ when Ω is convex.

To conclude, we note that all results of this section hold for both $\varepsilon = 0$ and $\varepsilon = 1$. Working with $\varepsilon = 0$ seems simpler. But, as we will see in the next section, the formulation with $\varepsilon = 1$ is advantageous for the discretization. Therefore we work from now on with $\varepsilon = 1$. For simplicity, we write $a(\cdot, \cdot)$ instead of $a_1(\cdot, \cdot)$ in what follows.

3. THE DISCRETE PROBLEM

From now on, we assume that Ω is a polyhedron. We introduce a regular family of triangulations $(\mathcal{T}_h)_h$ by tetrahedra, in the usual sense that:

- For each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- The intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge or a whole face of both of them;
- The ratio of the diameter h_K of any element K of $\mathcal{T}_h$ to the diameter of its inscribed sphere is smaller than a constant σ independent of h.

Further we make the non restrictive assumption that $\overline{\Gamma}_m$ is the union of whole faces of elements of $\mathcal{T}_h$.

As usual, h denotes the maximum of the diameters h_K, $K \in \mathcal{T}_h$. For each K in $\mathcal{T}_h$ and each nonnegative integer k, we denote by $\mathcal{P}_k(K)$ the spaces of restrictions to K of polynomials with three variables and total degree at most k.

In what follows, c, $c' \ldots$ stand for generic constants which may vary from line to line but are always independent of h. From now on, we call finite element space associated with $\mathcal{T}_h$ a space of functions such that their restrictions to any element K of $\mathcal{T}_h$ belong to a space of polynomials of fixed degree.

For each h, we associate with $\mathcal{T}_h$ two finite element spaces $\mathbb{X}_h$ and $\mathbb{M}_h$ which are contained in $\mathbb{X}(\Omega)$ and $L_0^2(\Omega)$, respectively, and such that the following inf-sup condition holds for a constant $\beta_* > 0$:

$$\forall q_h \in \mathbb{M}_h, \quad \sup_{\mathbf{v}_h \in \mathbb{X}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbb{X}(\Omega)}} \geq \beta_* \|q_h\|_{L^2(\Omega)}. \tag{3.1}$$

Indeed, there exist many examples of finite element spaces satisfying these conditions (the inf-sup condition being usually proved with $\mathbb{X}_h$ replaced by $\mathbb{X}_h \cap H_0^1(\Omega)^3$), see [14], Chapter II. We give two examples of them, the second one dealing with continuous discrete pressures.

Example 3.1. Spaces associated with the Bernardi–Raugel elements (see *e.g.* [14], Sect. II.2.1)

$$\mathbb{X}_h = \left\{ \mathbf{v}_h \in \mathbb{X}(\Omega); \forall K \in \mathcal{T}_h, \mathbf{v}_{h|K} \in \mathcal{P}(K) \right\},$$

where $\mathcal{P}(K)$ stands for the space spanned by the restrictions to K of functions in $\mathcal{P}_1(K)^3$ and the normal bubble functions $\psi_e \, \mathbf{n}_e$ (for each face e of K, ψ_e denotes the bubble function on e equal to the product of the barycentric coordinates associated with the vertices of e and $\mathbf{n}_e$ stands for the unit outward normal vector on e),

$$\mathbb{M}_h = \left\{ q_h \in L_0^2(\Omega); \forall K \in \mathcal{T}_h, q_{h|K} \in \mathcal{P}_0(K) \right\}.$$

Example 3.2. Spaces associated with the Taylor–Hood elements (see *e.g.* [14], Sect. II.4.2)

$$\mathbb{X}_h = \big\{\mathbf{v}_h \in \mathbb{X}(\Omega); \, \forall K \in \mathcal{T}_h, \, \mathbf{v}_{h|K} \in \mathcal{P}_2(K)^3\big\},$$

$$\mathbb{M}_h = \big\{q_h \in L^2_0(\Omega) \cap H^1(\Omega); \, \forall K \in \mathcal{T}_h, \, q_{h|K} \in \mathcal{P}_1(K)\big\}.$$

The discrete problem is obtained from problem (2.3) (with $\varepsilon = 1$) by the Galerkin method. It reads:

Find $(\mathbf{u}_h, p_h)$ in $\mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned}
\forall \mathbf{v}_h \in \mathbb{X}_h, \quad & a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle, \\
\forall q_h \in \mathbb{M}_h, \quad & b(\mathbf{u}_h, q_h) = 0.
\end{aligned} \tag{3.2}$$

It follows from the choice of $\mathbb{X}_h$ and $\mathbb{M}_h$ that the discretization is fully conforming. As a consequence, the forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and – if Assumption 2.4 holds – $c(\cdot, \cdot, \cdot)$ are continuous on $\mathbb{X}_h \times \mathbb{X}_h$, $\mathbb{X}_h \times \mathbb{M}_h$ and $\mathbb{X}_h \times \mathbb{X}_h \times \mathbb{X}_h$, respectively, with norms bounded independently of h. Moreover, an immediate consequence of the ellipticity property (2.4) is its discrete analogue

$$\forall \mathbf{v}_h \in \mathbb{X}_h, \quad a(\mathbf{v}_h, \mathbf{v}_h) \geq \nu \|\mathbf{v}_h\|^2_{\mathbb{X}(\Omega)}. \tag{3.3}$$

Remark 3.3. As usual, we denote by V_h the kernel

$$V_h = \big\{\mathbf{v}_h \in \mathbb{X}_h; \, \forall q_h \in \mathbb{M}_h, \, b(\mathbf{v}_h, q_h) = 0\big\}.$$

We have no examples of finite elements such that the space V_h consists of exactly divergence-free functions, *i.e.*, it coincides with $\mathbb{X}_h \cap V$. Indeed, this seems in contradiction with the inf-sup condition (3.1) (see [14], Chapter II, or [6], Chapter VI, for more comments). For this reason, the discrete analogue of the ellipticity property (2.5) does not hold, and we are led to work with $\varepsilon = 1$.

The antisymmetry property (2.9) also holds for all functions $\mathbf{w}_h$ and $\mathbf{v}_h$ in $\mathbb{X}_h$. So using the same arguments as in the proof of Proposition 2.7 combined with inequalities (3.1) and (3.3), we obtain an *a priori* estimate for the solutions of problem (3.2).

Proposition 3.4. *For any data* $\mathbf{f}$ *in* $\mathbb{X}(\Omega)'$, *any solution* $(\mathbf{u}_h, p_h)$ *of problem* (3.2) *satisfies*

$$\|\mathbf{u}_h\|_{\mathbb{X}(\Omega)} \leq \frac{1}{\nu}\|\mathbf{f}\|_{\mathbb{X}(\Omega)'},$$

$$\|p_h\|_{L^2(\Omega)} \leq c\|\mathbf{f}\|_{\mathbb{X}(\Omega)'}\big(1 + \|\mathbf{f}\|_{\mathbb{X}(\Omega)'}\big).$$

4. Existence of a solution and *a priori* error estimates

Our approach for proving the existence of a solution to problem (3.2) relies on the theory of Brezzi *et al.* [7]. So we first investigate the discretization of the linear problem (2.7).

For this, we need a further assumption concerning the approximation properties of the spaces $\mathbb{X}_h$ and $\mathbb{M}_h$.

Assumption 4.1. *There exists an integer* $k \geq 1$ *such that the following approximation properties hold for any function* $\mathbf{v}$ *in* $\mathbb{X}(\Omega) \cap H^s(\Omega)^3$ *and* q *in* $L^2_0(\Omega) \cap H^{s-1}(\Omega)$, $1 \leq s \leq k+1$,

$$\begin{aligned}
\inf_{\mathbf{v}_h \in \mathbb{X}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbb{X}(\Omega)} &\leq c\,h^{s-1}\,\|\mathbf{v}\|_{H^s(\Omega)^3}, \\
\inf_{q \in \mathbb{M}_h} \|q - q_h\|_{L^2(\Omega)} &\leq c\,h^{s-1}\,\|q\|_{H^{s-1}(\Omega)}.
\end{aligned} \tag{4.1}$$

It can be observed that this assumption holds with $k = 1$ for Example 3.1 and with $k = 2$ for Example 3.2. For any data $\mathbf{f}$ in $\mathbb{X}(\Omega)'$, we consider the discrete Stokes problem:

Find $(\mathbf{u}_h, p_h)$ in $\mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned}
\forall \mathbf{v}_h \in \mathbb{X}_h, \quad & a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle, \\
\forall q_h \in \mathbb{M}_h, \quad & b(\mathbf{u}_h, q_h) = 0.
\end{aligned} \tag{4.2}$$

It follows from inequalities (3.1) and (3.3) that this problem has a unique solution.

We denote by $\mathcal{S}_h$ the operator which associates with any $\mathbf{f}$ in $\mathbb{X}(\Omega)'$ the part $\mathbf{u}_h$ of the solution $(\mathbf{u}_h, p_h)$ of problem (4.2). The first property of $\mathcal{S}_h$ is easily derived from inequality (3.3).

Lemma 4.2. *The following stability property holds for any $\mathbf{f}$ in $\mathbb{X}(\Omega)'$:*

$$\|\mathcal{S}_h \mathbf{f}\|_{\mathbb{X}(\Omega)} \leq \frac{1}{\nu} \|\mathbf{f}\|_{\mathbb{X}(\Omega)'}.$$

We also need an *a priori* error estimate between the solutions $\mathbf{u}$ of problem (2.7) and $\mathbf{u}_h$ of problem (4.2).

Lemma 4.3. *If Assumption 4.1 holds, for any $\mathbf{f}$ in $\mathbb{X}(\Omega)' \cap H^{s-2}(\Omega)^3$ such that $\mathcal{S}\mathbf{f}$ belongs to $H^s(\Omega)^3$ with $1 \leq s \leq k+1$, the error estimate*

$$\|(\mathcal{S} - \mathcal{S}_h)\mathbf{f}\|_{\mathbb{X}(\Omega)} \leq c h^{s-1} \left(\|\mathcal{S}\mathbf{f}\|_{H^s(\Omega)^3} + \|\mathbf{f}\|_{H^{s-2}(\Omega)^3} \right)$$

is fulfilled.

Proof. Let p be the pressure associated with $\mathbf{u} = \mathcal{S}\mathbf{f}$ in problem (2.7). The functions $\mathbf{u}$ and $\mathbf{u}_h = \mathcal{S}_h \mathbf{f}$ satisfy

$$\begin{aligned}
\forall \mathbf{v}_h \in V_h, \forall q_h \in \mathbb{M}_h, \quad & a(\mathbf{u}, \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle - b(\mathbf{v}_h, p - q_h), \\
\forall \mathbf{v}_h \in V_h, \quad & a(\mathbf{u}_h, \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle.
\end{aligned}$$

It thus follows from the ellipticity property (3.3) that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} \leq c \left(\inf_{\mathbf{w}_h \in V_h} \|\mathbf{u} - \mathbf{w}_h\|_{\mathbb{X}(\Omega)} + \inf_{q \in \mathbb{M}_h} \|p - q_h\|_{L^2(\Omega)} \right).$$

Next, combining the inf-sup condition (3.1) with [14], Chapter II, equation (1.16) gives

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} \leq c \left(\inf_{\mathbf{w}_h \in \mathbb{X}_h} \|\mathbf{u} - \mathbf{w}_h\|_{\mathbb{X}(\Omega)} + \inf_{q \in \mathbb{M}_h} \|p - q_h\|_{L^2(\Omega)} \right).$$

Finally, it follows from the regularity assumptions on $\mathbf{f}$ and $\mathbf{u}$ that p belongs to $H^{s-1}(\Omega)$ and satisfies

$$\|p\|_{H^{s-1}(\Omega)} \leq c \left(\|\mathcal{S}\mathbf{f}\|_{H^s(\Omega)^3} + \|\mathbf{f}\|_{H^{s-2}(\Omega)^3} \right).$$

Thus, the desired estimate follows from the approximation properties stated in Assumption 4.1. □

We now set

$$\mathcal{G}(\mathbf{u}) = (\mathbf{curl}\, \mathbf{u}) \times \mathbf{u} - \mathbf{f},$$

and observe that problem (2.3) can equivalently be written as

$$\mathbf{u} + \mathcal{S}\mathcal{G}(\mathbf{u}) = \mathbf{0}. \tag{4.3}$$

Similarly, problem (3.2) can equivalently be written as

$$\mathbf{u}_h + \mathcal{S}_h \mathcal{G}(\mathbf{u}_h) = \mathbf{0}. \tag{4.4}$$

This new formulation is needed to apply the Brezzi–Rappaz–Raviart theorem. We now consider a solution of problem (2.3). Denoting by D the Fréchet derivative assume that:

Assumption 4.4. *The solution* $(\mathbf{u}, p)$ *of problem* (2.3) *is such that*

(1) *it belongs to* $H^s(\Omega)^3 \times H^{s-1}(\Omega)$ *and* $\mathbf{curl}\,\mathbf{u}$ *belongs to* $H^{s-\frac{3}{4}}(\Omega)^3$ *for some* $s > \frac{3}{2}$;

(2) *the operator* $\mathrm{Id} + \mathcal{S}D\mathcal{G}(\mathbf{u})$ *is an isomorphism of* $\mathbb{X}(\Omega)$.

Part (1) of this assumption states a regularity result for the solution $(\mathbf{u}, p)$ of our problem. Even if the regularity results for the Stokes problem with mixed boundary conditions in a three-dimensional domain are presently unknown, this property seems likely for a convex polyhedron (see [18] for the first results in dimension 2) and can easily be extended to the nonlinear case. On the other hand, part (2) means that the solution $\mathbf{u}$ is locally unique, which is much weaker than the global uniqueness, see Proposition 2.8.

In view of the next lemma, we also make a further assumption. It is likely satisfied when Ω is convex; however, we think that it is weaker.

Assumption 4.5. *The operator* $\mathcal{S}$ *maps* $L^2(\Omega)^3$ *into* $H^t(\Omega)^3$ *for some* $t > 1$.

Lemma 4.6. *If Assumptions* 2.4, 4.1, 4.4 *and* 4.5 *are satisfied, there exists a real number* $h_0 > 0$ *such that, for all* $h \leq h_0$, *the operator* $\mathrm{Id} + \mathcal{S}_h D\mathcal{G}(\mathbf{u})$ *is an isomorphism of* $\mathbb{X}(\Omega)$. *Moreover, the norm of its inverse is bounded independently of* h.

Proof. We take advantage of the expansion

$$\mathrm{Id} + \mathcal{S}_h D\mathcal{G}(\mathbf{u}) = \mathrm{Id} + \mathcal{S}D\mathcal{G}(\mathbf{u}) - \left(\mathcal{S} - \mathcal{S}_h\right)D\mathcal{G}(\mathbf{u}).$$

Therefore, owing to Part (2) of Assumption 4.4, it suffices to check that $\left(\mathcal{S} - \mathcal{S}_h\right)D\mathcal{G}(\mathbf{u})$ tends to zero in the norm of endomorphisms of $\mathbb{X}(\Omega)$. To prove this, we observe that for any $\mathbf{z}$ in $\mathbb{X}(\Omega)$,

$$D\mathcal{G}(\mathbf{u}) \cdot \mathbf{z} = (\mathbf{curl}\,\mathbf{u}) \times \mathbf{z} + (\mathbf{curl}\,\mathbf{z}) \times \mathbf{u}.$$

Using Assumptions 2.4 and 4.4 and noting that the embeddings of $H^s(\Omega)$ and $H^{s-\frac{3}{4}}(\Omega)$ into $L^\infty(\Omega)$ and $L^4(\Omega)$, respectively, are compact, we conclude that the mapping $\mathbf{z} \mapsto D\mathcal{G}(\mathbf{u}) \cdot \mathbf{z}$ maps the unit sphere of $\mathbb{X}(\Omega)$ into a compact subset of $L^2(\Omega)^3$. Owing to Assumption 4.5, the operator $\mathcal{S}$ maps this compact set into a compact subset of $H^t(\Omega)^3$. The desired convergence property then is a direct consequence of Lemma 4.3. $\square$

From now on we denote by $\mathcal{L}(\mathbb{X}(\Omega))$ the space of endomorphisms of $\mathbb{X}(\Omega)$.

Lemma 4.7. *If Assumption* 2.4 *is satisfied, there exists a real number* $L > 0$ *such that the following Lipschitz property holds*

$$\forall \mathbf{w} \in \mathbb{X}(\Omega), \quad \|\mathcal{S}_h\left(D\mathcal{G}(\mathbf{u}) - D\mathcal{G}(\mathbf{w})\right)\|_{\mathcal{L}(\mathbb{X}(\Omega))} \leq L\|\mathbf{u} - \mathbf{w}\|_{\mathbb{X}(\Omega)}.$$

Proof. We now write

$$D\mathcal{G}(\mathbf{u}) \cdot \mathbf{z} - D\mathcal{G}(\mathbf{w}) \cdot \mathbf{z} = \left(\mathbf{curl}\,(\mathbf{u} - \mathbf{w})\right) \times \mathbf{z} + (\mathbf{curl}\,\mathbf{z}) \times (\mathbf{u} - \mathbf{w}).$$

Owing to Assumption 2.4, this gives

$$\|D\mathcal{G}(\mathbf{u}) \cdot \mathbf{z} - D\mathcal{G}(\mathbf{w}) \cdot \mathbf{z}\|_{\mathbb{X}(\Omega)'} \leq c_0^2\|\mathbf{u} - \mathbf{w}\|_{\mathbb{X}(\Omega)}\|\mathbf{z}\|_{\mathbb{X}(\Omega)}.$$

Combining this with Lemma 4.2 yields the desired property. $\square$

Lemma 4.8. *If Assumptions* 4.1 *and* 4.4 *are satisfied, the following estimate holds*

$$\|\mathbf{u} + \mathcal{S}_h\mathcal{G}(\mathbf{u})\|_{\mathbb{X}(\Omega)} \leq ch^{s^*-1}\left(\|\mathbf{u}\|_{H^s(\Omega)^3} + \|\mathbf{f}\|_{H^{s-2}(\Omega)^3}\right), \tag{4.5}$$

with $s^* = \min\{s, k+1\}$.

Proof. We have

$$\|\mathbf{u} + \mathcal{S}_h \mathcal{G}(\mathbf{u})\|_{\mathbb{X}(\Omega)} = \|(\mathcal{S} - \mathcal{S}_h)\mathcal{G}(\mathbf{u})\|_{\mathbb{X}(\Omega)}.$$

Estimate (4.5) then is a direct consequence of equation (4.3), Assumption 4.4 and Lemma 4.3. $\square$

We now state and prove the main result of this section.

Proposition 4.9. *Suppose that Assumptions* 2.4, 4.1 *and* 4.5 *hold and that the solution* $(\mathbf{u}, p)$ *satisfies Assumption* 4.4. *Then, there exists a neighbourhood* $\mathcal{U}$ *of* $\mathbf{u}$ *in* $\mathbb{X}(\Omega)$ *such that problem* (3.2) *has a unique solution* $(\mathbf{u}_h, p_h)$ *with* $\mathbf{u}_h$ *in* $\mathcal{U}$. *Moreover, this solution satisfies the following* a priori *error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq c h^{s^* - 1}\big(\|\mathbf{u}\|_{H^s(\Omega)^3} + \|p\|_{H^{s-1}(\Omega)}\big), \tag{4.6}$$

with $s^* = \min\{s, k + 1\}$.

Proof. By combining the Brezzi–Rappaz–Raviart theorem [7] (see also [14], Chapter IV, Thm. 3.1) with Lemmas 4.6 to 4.8, we derive the existence of a neighbourhood $\mathcal{U}$ of $\mathbf{u}$ such that problem (4.4) has a unique solution $\mathbf{u}_h$ in $\mathcal{U}$, together with the estimate for $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)}$. On the other hand, it follows from the inf-sup condition (3.1) that there exists a unique p_h in $\mathbb{M}_h$ such that

$$\forall \mathbf{v}_h \in \mathbb{X}_h, \quad b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle - a(\mathbf{u}_h, \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h).$$

Thus, the pair $(\mathbf{u}_h, p_h)$ is a solution of problem (3.2). Moreover, by subtracting problem (3.2) from problem (2.3) and noting that the norms of both $\mathbf{u}$ and $\mathbf{u}_h$ are bounded independently of h (see Props. 2.7 and 3.4), we obtain

$$\|p - p_h\|_{L^2(\Omega)} \leq c\big(\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} + \inf_{q_h \in \mathbb{M}_h} \|p - q_h\|_{L^2(\Omega)}\big).$$

Thus, the estimate for $\|p - p_h\|_{L^2(\Omega)}$ follows from the estimate for $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)}$ and the approximation properties of the space $\mathbb{M}_h$. $\square$

Estimate (4.6) is fully optimal, and the regularity which is required for $(\mathbf{u}, p)$ seems reasonable. However, Assumptions 2.4 and 4.5 apparently add limitations to the geometry of the domain.

5. A *POSTERIORI* ERROR ESTIMATES

We first recall some standard notation: for each K in $\mathcal{T}_h$, $\mathcal{E}_K$ denotes the set of all faces of K which are not contained in $\partial\Omega$ and $\mathcal{E}_K^m$ the set of faces of K which are contained in $\overline{\Gamma}_m$. Next, we introduce an approximation $\mathbf{f}_h$ of the data $\mathbf{f}$ which is constant on each element of $\mathcal{T}_h$.

We are thus in a position to define the error indicators. However, as in Section 2, we propose two families associated with the values 0 and 1 of the parameter ε. For ε equal to 0 and 1 and for each K in $\mathcal{T}_h$, the indicator η_K^ε is given by

$$\begin{aligned}
\eta_K^\varepsilon = {} & h_K \|\mathbf{f}_h - \nu\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}_h) + \varepsilon\nu\,\mathbf{grad}\,(\mathrm{div}\,\mathbf{u}_h) - (\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{u}_h - \mathbf{grad}\,p_h\|_{L^2(K)^3} \\
& + \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \|[\nu(\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{n} + \varepsilon\nu(\mathrm{div}\,\mathbf{u}_h)\mathbf{n} - p_h\mathbf{n}]_e\|_{L^2(e)^3} \\
& + \sum_{e \in \mathcal{E}_K^m} h_e^{\frac{1}{2}} \|\nu(\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{n}\|_{L^2(e)^3} + \|\mathrm{div}\,\mathbf{u}_h\|_{L^2(K)}, \tag{5.1}
\end{aligned}$$

where

- for each e in $\mathcal{E}_K$ or $\mathcal{E}_K^m$, h_e stands for the diameter of e;
- for each e in $\mathcal{E}_K$, $[\cdot]_e$ denotes the jump across e in the direction of $\mathbf{n}$ which is a unit vector orthogonal to e.

In view of the discrete problem (3.2), it seems more natural to use the indicators η_K^1. However, triangle inequalities combined with the standard inverse inequality (see [5], Sect. XI.2, for instance)

$$h_K \| \nu \operatorname{\mathbf{grad}} (\operatorname{div} \mathbf{u}_h) \|_{L^2(K)^3} + \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \| [\nu(\operatorname{div} \mathbf{u}_h)\mathbf{n}]_e \|_{L^2(e)^3} \leq c \, \| \operatorname{div} \mathbf{u}_h \|_{L^2(K)} \qquad (5.2)$$

yield that the indicators η_K^0 and η_K^1 are fully equivalent.

In order to prove an upper bound of the error as a function of the indicators, we follow the approach of [19] and [22], Chapter 2. We first establish the residual equations associated with any solutions $(\mathbf{u}, p)$ of problem (2.3) and $(\mathbf{u}_h, p_h)$ of problem (3.2). To this end we observe that, for any $\mathbf{v}$ in $\mathbb{X}(\Omega)$ and $\mathbf{v}_h$ in $\mathbb{X}_h$,

$$a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - a(\mathbf{u}_h, \mathbf{v}) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h)$$
$$= \langle \mathbf{f}, \mathbf{v} - \mathbf{v}_h \rangle - a(\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v} - \mathbf{v}_h) - b(\mathbf{v} - \mathbf{v}_h, p_h).$$

By integrating by parts the right-hand side on each K in $\mathcal{T}_h$, we obtain

$$a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - a(\mathbf{u}_h, \mathbf{v}) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = \langle \mathbf{f} - \mathbf{f}_h, \mathbf{v} - \mathbf{v}_h \rangle + \langle R, \mathbf{v} - \mathbf{v}_h \rangle, \quad (5.3)$$

where the residual R is given by

$$\langle R, \mathbf{v} \rangle = \sum_{K \in \mathcal{T}_h} \Bigg(\int_K \big(\mathbf{f}_h - \nu \operatorname{\mathbf{curl}} (\operatorname{\mathbf{curl}} \mathbf{u}_h) + \nu \operatorname{\mathbf{grad}} (\operatorname{div} \mathbf{u}_h) $$
$$- (\operatorname{\mathbf{curl}} \mathbf{u}_h) \times \mathbf{u}_h - \operatorname{\mathbf{grad}} p_h \big)(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x}$$
$$- \sum_{e \in \mathcal{E}_K} \int_e \big(\nu(\operatorname{\mathbf{curl}} \mathbf{u}_h) \times \mathbf{n} + \nu(\operatorname{div} \mathbf{u}_h)\mathbf{n} - p_h \mathbf{n} \big)(\boldsymbol{\tau}) \cdot \mathbf{v}(\boldsymbol{\tau}) \, d\boldsymbol{\tau}$$
$$- \sum_{e \in \mathcal{E}_K^m} \int_e \big(\nu(\operatorname{\mathbf{curl}} \mathbf{u}_h) \times \mathbf{n} \big)(\boldsymbol{\tau}) \cdot \mathbf{v}(\boldsymbol{\tau}) \, d\boldsymbol{\tau} \Bigg).$$

On the other hand, it is readily checked that, for all q in $L^2(\Omega)$,

$$b(\mathbf{u} - \mathbf{u}_h, q) = \int_\Omega (\operatorname{div} \mathbf{u}_h)(\mathbf{x}) q(\mathbf{x}) \, d\mathbf{x}. \qquad (5.4)$$

In what follows we consider solutions of problem (2.3) which satisfy a hypothesis weaker than Assumption 4.4:

Assumption 5.1. *The solution $(\mathbf{u}, p)$ of problem (2.3) is such that the operator* $\operatorname{Id} + \mathcal{SDG}(\mathbf{u})$ *is an isomorphism of $\mathbb{X}(\Omega)$.*

Notation 5.2. Let $\mathcal{V}$ be a neighbourhood of the re-entrant corners and edges in Γ_m. For each re-entrant edge E_i with angle ω_i, $\pi < \omega_i < 2\pi$, we consider a neighbourhood $\mathcal{V}_i$ of a part of E_i in $\mathcal{V}$ which does not contain any re-entrant corner and does not intersect any other $\mathcal{V}_j$. Next, with each K in $\mathcal{T}_h$, we associate the quantity γ_K:

- larger than $1 - \frac{\pi}{\omega_i}$ if K is contained in $\mathcal{V}_i$;
- equal to $\frac{1}{2}$ if K intersects $\mathcal{V}$ but is not contained in any $\mathcal{V}_i$;
- equal to 0 otherwise.

We are now in a position to prove the upper bound for the error. It requires a further assumption on the space $\mathbb{X}_h$ that is satisfied by all examples of spaces that we have in mind, and in particular by the spaces $\mathbb{X}_h$ introduced in Examples 3.1 and 3.2.

Proposition 5.3. *Suppose that the data* $\mathbf{f}$ *belong to* $L^2(\Omega)^3$*, that* $\mathbb{X}_h$ *contains the space*

$$\overline{\mathbb{X}}_h = \big\{ \mathbf{v}_h \in \mathbb{X}(\Omega); \, \forall K \in \mathcal{T}_h, \, \mathbf{v}_{h|K} \in \mathcal{P}_1(K)^3 \big\},$$

and that the solution $(\mathbf{u}, p)$ *satisfies Assumption 5.1. Then, there exists a neighbourhood* $\mathcal{U}^*$ *of* $\mathbf{u}$ *in* $\mathbb{X}(\Omega)$ *such that any solution* $(\mathbf{u}_h, p_h)$ *of problem* (3.2) *with* $\mathbf{u}_h$ *in* $\mathcal{U}^*$ *satisfies the following* a posteriori *error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq c \Big(\sum_{K \in \mathcal{T}_h} h_K^{-2\gamma_K} \big((\eta_K^\varepsilon)^2 + h_K^2 \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^3}^2 \big) \Big)^{\frac{1}{2}}. \tag{5.5}$$

Proof. We only establish estimate (5.5) for $\varepsilon = 1$, since it implies the estimate for $\varepsilon = 0$ owing to (5.2). It is performed in two steps.

(1) The same arguments as in the proof of Lemma 4.7 combined with inequality (2.8) yield that the mapping $\mathbf{w} \mapsto \mathrm{Id} + \mathcal{SDG}(\mathbf{w})$ is Lipschitz-continuous from $\mathbb{X}(\Omega)$ into $\mathcal{L}(\mathbb{X}(\Omega))$. Thus, owing to Assumption 5.1, it follows from [19] (see also [22], Prop. 2.1) that, for any $\mathbf{u}_h$ in an appropriate neighbourhood $\mathcal{U}^*$ of $\mathbf{u}$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} \leq c \|\mathbf{u}_h + \mathcal{SG}(\mathbf{u}_h)\|_{\mathbb{X}(\Omega)}.$$

From equations (4.3), (5.3) and (5.4), we obtain

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)} \quad \leq \quad c \Bigg(& \sup_{\mathbf{v} \in \mathbb{X}(\Omega)} \inf_{\mathbf{v}_h \in \mathbb{X}_h} \frac{\langle \mathbf{f} - \mathbf{f}_h, \mathbf{v} - \mathbf{v}_h \rangle + \langle R, \mathbf{v} - \mathbf{v}_h \rangle}{\|\mathbf{v}\|_{\mathbb{X}(\Omega)}} \\
& + \sup_{q \in L_0^2(\Omega)} \frac{\int_\Omega (\mathrm{div}\, \mathbf{u}_h)(\mathbf{x}) q(\mathbf{x}) \, \mathrm{d}\mathbf{x}}{\|q\|_{L^2(\Omega)}} \Bigg).
\end{aligned}
\tag{5.6}
$$

The next steps rely on the existence of a suitable Clément regularization operator mapping $L^2(\Omega)^3$ into $\mathbb{X}_h$. For the standard no-slip boundary conditions this operator is constructed in [5], Theorem IX.3.11 and Corollary IX.3.12. By straightforward but technical arguments this construction can easily be modified to take the present boundary conditions into account. This operator has the following local approximation properties: when $\frac{1}{2} < t \leq 2$, for each K in $\mathcal{T}_h$ and each face e of K,

$$
\begin{aligned}
\|\mathbf{v} - \mathcal{C}_h \mathbf{v}\|_{L^2(K)^3} &\leq c h_K^t \|\mathbf{v}\|_{H^t(\Delta_K)^3}, \\
\|\mathbf{v} - \mathcal{C}_h \mathbf{v}\|_{L^2(e)^3} &\leq c h_e^{t-\frac{1}{2}} \|\mathbf{v}\|_{H^t(\Delta_e)^3},
\end{aligned}
\tag{5.7}
$$

where Δ_K and Δ_e stand for the unions of all elements K of $\mathcal{T}_h$ that intersect K and e, respectively. On the other hand, by combining the arguments in the proof of Lemma 2.5 with [3], Remark 2.16, we obtain that, for the $\mathcal{V}$ introduced in Notation 5.2, the spaces of restrictions of functions of $\mathbb{X}(\Omega)$ to $\Omega \setminus \overline{\mathcal{V}}$ and to $\mathcal{V}$ are embedded in $H^1(\Omega \setminus \overline{\mathcal{V}})^3$ and $H^t(\mathcal{V})^3$, $t > \frac{1}{2}$ (this follows from [10], equation (2.1) since Ω is a polyhedron). Moreover, it also follows from [10] that the space of restrictions of functions of $\mathbb{X}(\Omega)$ to $\mathcal{V}_i$ for the $\mathcal{V}_i$ also introduced in Notation 5.2 are embedded in $H^s(\mathcal{V}_i)^3$ for all $s < \frac{\pi}{\omega_i}$. Hence, taking $\mathbf{v}_h$ equal to $\mathcal{C}_h \mathbf{v}$ in inequality (5.6) and using estimates (5.7) combined with these embeddings, we obtain the desired estimate for $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\Omega)}$.

(2) By computing $b(\mathbf{v}, p - p_h)$ from equation (5.3), using the inf-sup condition (2.6) and the fact that the norms of both $\mathbf{u}$ and $\mathbf{u}_h$ are bounded independently of h, we derive from this estimate the bound of $\|p - p_h\|_{L^2(\Omega)}$ stated in (5.5). $\qquad \square$

Proving the upper bounds for the indicators is standard. Therefore, we only give an abridged proof of the next statement.

Proposition 5.4. *Suppose that the data* **f** *belong to* $L^2(\Omega)^3$. *Each indicator* η_K^ε, $K \in \mathcal{T}_h$, *as defined in equation* (5.1) *satisfies for any solution* **u** *of problem* (2.3)

$$\eta_K^\varepsilon \leq c\big(\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{X}(\omega_K)} + \|p - p_h\|_{L^2(\omega_K)} + \sum_{\substack{\kappa \in \mathcal{T}_h \\ \kappa \subset \omega_K}} h_\kappa \|\mathbf{f} - \mathbf{f}_h\|_{L^2(\kappa)^3}\big), \tag{5.8}$$

where ω_K *stands for the union of elements of* $\mathcal{T}_h$ *that share a least a face with* K.

Proof. For the same reasons as above, we only give the proof for $\varepsilon = 1$. In formula (5.3), we take $\mathbf{v}_h$ equal to zero, and we choose successively $\mathbf{v}$ equal to

(1) $\mathbf{v}_K$, with

$$\mathbf{v}_K = \begin{cases} (\mathbf{f}_h - \nu\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}_h) + \nu\,\mathbf{grad}\,(\mathrm{div}\,\mathbf{u}_h) \\ \quad -(\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{u}_h - \mathbf{grad}\,p_h)\psi_K & \text{on } K, \\ \mathbf{0} & \text{on } \Omega\backslash K, \end{cases}$$

where ψ_K is the bubble function on K, equal to the product of the barycentric coordinates associated with the vertices of K;

(2) $\mathbf{v}_e$ for each e in $\mathcal{E}_K$, with

$$\mathbf{v}_e = \begin{cases} \mathcal{L}_{e,K}([\nu(\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{n} \\ \qquad + \nu(\mathrm{div}\,\mathbf{u}_h)\mathbf{n} - p_h\mathbf{n}]_e)\psi_e & \text{on } K, \\ \mathcal{L}_{e,K'}([\nu(\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{n} \\ \qquad + \nu(\mathrm{div}\,\mathbf{u}_h)\mathbf{n} - p_h\mathbf{n}]_e)\psi_e & \text{on } K', \\ \mathbf{0} & \text{on } \Omega\backslash(K \cup K'), \end{cases}$$

where ψ_e is the bubble function on e, K' denotes the other element of $\mathcal{T}_h$ adjacent to e and $\mathcal{L}_{e,\kappa}$ is a lifting operator from e into κ constructed by affine transformation from a fixed lifting operator on the reference element;

(3) $\mathbf{v}_e^m$ for each e in $\mathcal{E}_K^m$, with

$$\mathbf{v}_e^m = \begin{cases} \mathcal{L}_{e,K}(\nu(\mathbf{curl}\,\mathbf{u}_h) \times \mathbf{n})\psi_e & \text{on } K, \\ \mathbf{0} & \text{on } \Omega \backslash K, \end{cases}$$

with the same notation as above.

Thus, standard inverse inequalities [22], Lemma 3.3 (see also [5], Sect. XI.2) lead to the desired bound for the first three terms of η_K^1. Similarly, in formula (5.4), we take q equal to q_K, with

$$q_K = \begin{cases} \mathrm{div}\,\mathbf{u}_h & \text{on } K, \\ 0 & \text{on } \Omega\backslash K. \end{cases}$$

This yields the bound for the fourth term of η_K^1. □

Estimate (5.8) is fully optimal and the bound for each η_K^ε is local, in the sense that it only involves the error in a "small" neighbourhood of K. In contrast, estimate (5.5) is optimal only when Ω has neither re-entrant corners nor re-entrant edges in Γ_m. However, the lack of optimality for a general domain only concerns the elements that intersect a neighbourhood of these re-entrant corners or edges.
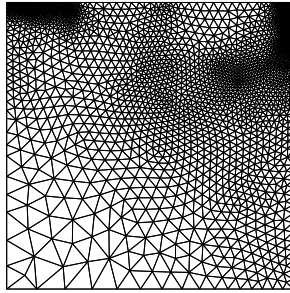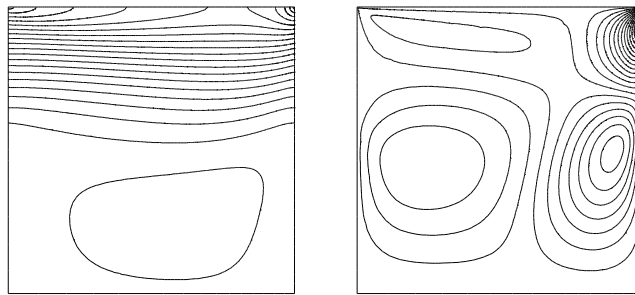
FIGURE 1. The adapted mesh.

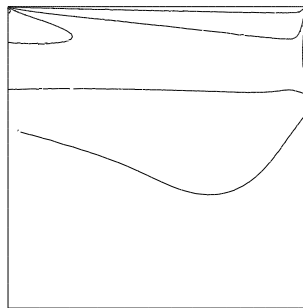FIGURE 2. The isovalues of the two components of the velocity.

FIGURE 3. The isovalues of the vorticity.

6. A NUMERICAL EXPERIMENT

The numerical simulation that we now present has been performed on the code FreeFem++ due to Hecht and Pironneau, see [15]. We only consider a model domain, the square $\Omega =]-1, 1[^2$, and take Γ_m equal to the upper edge of Ω, namely $\Gamma_m =]-1, 1[\times \{1\}$.

It must be noted that, in dimension 2, $\mathbf{curl\,u}$ is a scalar function, so that the condition $(\mathbf{curl\,u}) \times \mathbf{n} = \mathbf{0}$ on Γ_m must be replaced by $\mathbf{curl\,u} = \mathbf{0}$. Up to these modifications, the analysis of the problem and of its discretization is the same as in dimension 3. We also replace the data $\mathbf{f}$ by a boundary data $\mathbf{g}$: more precisely $\mathbf{f}$ is equal to zero and the third line in system (1.1) is replaced by

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma. \tag{6.1}$$

The Navier–Stokes system is solved *via* the Newton–Raphson method, the initial value of the velocity being computed as the solution of the Stokes problem.

We work with the parameter ε equal to 1 and with Taylor–Hood finite elements as described in Example 3.2. We take the viscosity ν equal to 10^{-2}. The datum **g** has its second component equal to zero on Γ, while its first component g_x is given by

$$g_x(\pm 1, y) = \begin{cases} 0, & -1 \leq y \leq 0, \\ y^2, & 0 \leq y \leq 1, \end{cases}$$

$$g_x(x, -1) = 0, \qquad -1 \leq x \leq 1.$$

(6.2)

Figure 1 presents the final adapted mesh. It can be noted that **curl u**$_h$ is discontinuous at the vertices $(\pm 1, 1)$, whence the refinement of the final mesh near these vertices.

Figure 2 presents the curves of isovalues of the two components of the velocity. Figure 3 presents the curve of isovalues of **curl u**$_h$. The boundary conditions that are enforced on these unknowns are perfectly taken into account in the computation.

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