ESAIM: M2AN 43 (2009) 429–444
DOI: 10.1051/m2an/2009003

**ESAIM: Mathematical Modelling and Numerical Analysis**
www.esaim-m2an.org

# A MULTILEVEL PRECONDITIONER FOR THE MORTAR METHOD FOR NONCONFORMING $P_1$ FINITE ELEMENT*

Talal Rahman[1,2] and Xuejun Xu[3]

**Abstract.** A multilevel preconditioner based on the abstract framework of the auxiliary space method, is developed for the mortar method for the nonconforming $P_1$ finite element or the lowest order Crouzeix-Raviart finite element on nonmatching grids. It is shown that the proposed preconditioner is quasi-optimal in the sense that the condition number of the preconditioned system is independent of the mesh size, and depends only quadratically on the number of refinement levels. Some numerical results confirming the theory are also provided.

**Mathematics Subject Classification.** 65F10, 65N30, 65N55.

Received April 3rd, 2007.
Published online February 7, 2009.

## 1. Introduction

The mortar method is a special domain decomposition methodology, which appears to be very attractive to the scientific computing community since it can handle situations where meshes on different subdomains need not align across the interfaces. The matching between the discretizations on adjacent subdomains is only enforced weakly. In [6], Bernardi *et al.* first introduced the basic concept for mortar methods, and applied it for coupling spectral elements with finite elements. Since then, the methodology has been extensively used and analyzed by many authors. In [4], Ben Belgacem studied the mortar method under a primal hybrid finite element formulation. Meanwhile, some extensions to the three dimensional problem, and to using dual basis for the Lagrange multiplier space were considered, *cf.* [5,7,16,28]. Recently, much work has been devoted towards constructing efficient iterative solvers for the discrete system resulting from the mortar finite element discretization. The first approaches were based on the iterative substructuring method, see for instance [1–3,12]. Multigrid methods for the mortar finite element have also been considered. Gopalakrishnan and Pasciak [14] presented a variable $V$-cycle multigrid, while Braess *et al.* [8], and Wohlmuth [29] established a $W$-cycle multigrid based on the hybrid formulation which gives rise to a saddle point problem. We note that Braess *et al.* [9] have

*Keywords and phrases.* Crouzeix-Raviart FE, mortar method, multilevel preconditioner, auxiliary space method.

* *X. Xu was supported by the National Basic Research Program of China (Grant No. 2005CB321701) and the National Natural Science Foundation of China (Grant No. 10731060).*

1 Department of Mathematics, University of Bergen, c/o Center for Integrated Petroleum Research, Allegt. 41, 5007 Bergen, Norway. talal.rahman@math.uib.no

2 Present address: Faculty of Engineering, Bergen University College, 5020 Bergen, Norway.

3 LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, P.R. China. xxj@lsec.cc.ac.cn

recently constructed a subspace cascadic multigrid method for the mortar finite element based on a saddle point formulation.

There have been some interests in the construction and implementation of the mortar method for the lowest order Crouzeix-Raviart (CR) finite element or the nonconforming $P_1$ finite element. Marcinkowski [17] first presented the standard mortar method for the CR finite element. This has been further extended by Rahman *et al.* in their recent work, *cf.* [23], where the standard mortar condition has been replaced by a new approximate mortar condition. Based on the first approach, Xu and Chen [33] introduced an optimal W-cycle multigrid method for the discrete system, with a convergence rate which is independent of the mesh size and the level of refinements. Recently developed domain decomposition methods for elliptic problems with discontinuous coefficients using the CR mortar finite element can be found in [18,19,22,23].

Multilevel preconditioning methods have also received many researchers' attention for solving large algebraic systems resulting from finite element approximation of partial differential equations [31,34]. The objective of this paper is to propose an effective multilevel preconditioner for the CR mortar finite element. Using the so-called auxiliary space technique developed in [20,32], also see [10,21,27], we propose a multilevel preconditioner for the CR mortar finite element. We choose the conforming $P_1$ mortar finite element space as the auxiliary space. A recently developed effective multilevel preconditioner for the conforming $P_1$ mortar finite element, *cf.* [13], is used as the preconditioner for the auxiliary space. The new multilevel preconditioner is shown to have the same quasi-optimal convergence behavior as the auxiliary preconditioner. The condition number of the preconditioned system is independent of the mesh size, and only quadratically dependent on the number of refinement levels.

The rest of this paper is organized as follows. In Section 2, we introduce our discrete problem, in Section 3 we describe the multilevel preconditioner for the conforming $P_1$ mortar finite element. In Section 4, we construct our multilevel preconditioner for the CR mortar finite element. Condition number estimate of the multilevel preconditioner will be given in Section 5. In the last section, some numerical results supporting the theory will be presented.

## 2. Mortar method for the nonconforming $P_1$ finite element

For simplicity, we consider the following model problem

$$\begin{cases} -\triangle u &= f \quad \text{in} \quad \Omega, \\ \quad u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a bounded polygonal domain, and $f \in L^2(\Omega)$. The variational formulation of the problem (2.1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where the bilinear form $a(\cdot,\cdot)$ is given as

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \quad \forall u,v \in H^1(\Omega),$$

and $(f,v) = \int_\Omega fv \, \mathrm{d}x$.

**Remark.** It is not difficult to extend the results of this paper to a more general second order elliptic problem.

We now introduce the mortar finite element method of [17] for solving (2.1), where the nonconforming $P_1$ finite element is used for the discretization. Let $\Omega$ be partitioned into a set of nonoverlapping polygonal subdomains $\{\Omega_i\}$ such that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. We consider only the case where the partition is geometrically conforming, that is, the subdomains are arranged so that the intersection $\overline{\Omega}_i \cap \overline{\Omega}_j$ for $i \neq j$ is either an empty set, an edge or a vertex. This intersection, if it is an edge, is called an interface and will have two sides each being an edge of one of the two neighboring subdomains. The skeleton $\Gamma = \bigcup_{i=1}^N \partial\Omega_i \backslash \partial\Omega$

is decomposed into a set of disjoint open straight segments $\gamma_m$ $(1 \leq m \leq M)$ (edges of subdomains) called the mortars, such that

$$\Gamma = \bigcup_{m=1}^{M} \bar{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset, \text{ if } m \neq n.$$

If $\gamma_m$ is an open edge of a subdomain $\Omega_i$, we refer to it as a mortar side of $\Omega_i$, and we denote it by $\gamma_{m(i)}$. Consequently, the other side of $\gamma_{m(i)}$, which is an edge of another subdomain say $\Omega_j$ occupying the same geometrical space as that of $\gamma_{m(i)}$, is referred to as the corresponding nonmortar side of $\Omega_j$, and is denoted by $\delta_{m(j)}$.

For $i = 1, \ldots, N$, let $\mathcal{T}_{1,i}$ be the initial shape regular triangulation of $\Omega_i$ with the mesh size $h_i^1$. The overall triangulation generally does not match at the subdomain interfaces. Let the global mesh $\bigcup_i \mathcal{T}_{1,i}$ be denoted by $\mathcal{T}_1$ and the corresponding global mesh size denoted by $h_1 = \max_i h_i^1$. We refine the triangulation $\mathcal{T}_1$ to produce $\mathcal{T}_2$ by joining the edge midpoints of the triangles in $\mathcal{T}_1$. The mesh size $h_i^2$ in the triangulation $\mathcal{T}_2$ is then given by $h_i^2 = h_i^2/2$. Repeating the process for $l$ times, $l = 1, \ldots, N$, we get an $l$ times refined triangulation $\mathcal{T}_l$ with the mesh size $h_i^l = h_i^l 2^{-l}$ $(l = 1, \ldots, L)$. Let the global mesh size on the level $l$ be $h_l = \max_i h_i^l$ $(l = 1, \ldots, L)$.

At the finest level $L$, locally in each subdomain $\Omega_i$, we use the nonconforming $P_1$ or the lowest order CR finite element space, $V_{L,i}$, whose functions are piecewise linear on the triangulation $\mathcal{T}_{L,i}$, determined uniquely by their values at the edge midpoints, and vanishing at the edge midpoints of the boundary $\partial\Omega$. The sets of edge midpoints, also referred to as the nonconforming $P_1$ or the CR nodal points, those belonging to $\overline{\Omega}_i$, $\partial\Omega_i$ and $\partial\Omega$ are denoted by $\Omega_{L,i}^{CR}$, $\partial\Omega_{L,i}^{CR}$ and $\partial\Omega_L^{CR}$, respectively. Consequently, the functions of $V_{L,i}$ are piecewise linear on each triangle of $\mathcal{T}_{L,i}$, continuous at the CR nodes of $\Omega_{L,i}^{CR} \backslash \partial\Omega_{L,i}^{CR}$, and equals to zero at the CR nodes of $\partial\Omega_{L,i}^{CR} \cap \partial\Omega_L$.

Let

$$\tilde{V}_L = \prod_{i=1}^{N} V_{L,i} = \{v_L |\ v_{L|_{\Omega_i}} = v_{L,i} \in V_{L,i}\}\cdot$$

Due to nonmatching meshes along subdomain interfaces, each interface $\gamma_m$, where $\gamma_m = \gamma_{m(i)} = \delta_{m(j)}$, $1 \leq m \leq M$, inherits two independent 1D triangulations $\mathcal{T}_L(\gamma_{m(i)})$ and $\mathcal{T}_L(\delta_{m(j)})$. Consequently, there are two sets of CR nodes belonging to $\gamma_m$, the midpoints of the elements belonging to $\mathcal{T}_L(\gamma_{m(i)})$ and $\mathcal{T}_L(\delta_{m(j)})$, we denote them by $\gamma_{L,m(i)}^{CR}$ and $\delta_{L,m(j)}^{CR}$, respectively. Additionally, we introduce an auxiliary test space $S_L(\delta_{m(j)})$ defined as

$$\begin{aligned} S_L(\delta_{m(j)}) := \{v \quad | \quad & v \in L^2(\delta_{m(j)}), \text{ and } v \text{ is piecewise constant on the} \\ & \text{elements of the nonmortar triangulation } \mathcal{T}_L(\delta_{m(j)})\}\cdot \end{aligned} \tag{2.3}$$

The dimension of $S_L(\delta_{m(j)})$ is equal to the number of midpoints on the $\delta_{m(j)}$, *i.e.* to the number of elements on $\delta_{m(j)}$. For each nonmortar edge $\delta_{m(j)}$, let $Q_{L,\delta_{m(j)}} : L^2(\gamma_m) \to S_L(\delta_{m(j)})$ be the $L^2$-projection operator defined by

$$(Q_{L,\delta_{m(j)}} v, w)_{L^2(\delta_{m(j)})} = (v, w)_{L^2(\delta_{m(j)})} \quad \forall w \in S_L(\delta_{m(j)}), \tag{2.4}$$

where $(\cdot,\cdot)_{L^2(\delta_{m(j)})}$ denotes the $L^2$ inner product in the space $L^2(\delta_{m(j)})$.

We now define the following mortar finite element space for the nonconforming $P_1$ finite element, associated with the finest level $L$:

$$V_L = \left\{ v_L |\ v_L \in \tilde{V}_L,\ Q_{L,\delta_{m(j)}} \left( v_{L|_{\delta_{m(j)}}} \right) = Q_{L,\delta_{m(j)}} \left( v_{L|_{\gamma_{m(i)}}} \right), \text{ for } \forall \gamma_m \in \Gamma \right\}\cdot$$

Let

$$a_{L,i}(u,v) := \sum_{K \in \mathcal{T}_{L,i}} \int_K \nabla u \cdot \nabla v \mathrm{d}x \quad \forall u, v \in V_{L,i},$$

and

$$a_L(u,v) := \sum_{i=1}^{N} a_{L,i}(u,v).$$

The nonconforming $P_1$ mortar finite element approximation of the problem (2.2) then is to find $u_L \in V_L$ such that

$$a_L(u_L, v_L) = (f, v_L) \quad \forall v_L \in V_L, \tag{2.5}$$

where $(f, v_L) = \sum_{i=1}^{N} \int_{\Omega_i} f v_L \mathrm{d}x$.

We define

$$\|v\|_{L,i}^2 := a_{L,i}(v,v) \quad \text{and} \quad \|v\|_L^2 := \sum_{i=1}^{N} \|v\|_{L,i}^2, \quad \forall v \in V_L.$$

From [17], we know that the discrete problem (2.5) has a unique solution. Moreover, the following error estimate can be found in [17]: Let $u$ and $u_L$ be the solutions of (2.2) and (2.5), respectively, then

$$\|u - u_L\|_L^2 \le C h_L^2 \sum_{i=1}^{N} |u|_{H^2(\Omega_i)}^2.$$

Next we define the operator $A_L : V_L \to V_L$ as follows:

$$(A_L v_L, w_L) = a_L(v_L, w_L) \quad \forall v_L, w_L \in V_L.$$

Then (2.5) can be rewritten as:

$$A_L u_L = f_L, \tag{2.6}$$

where $f_L = Q_L f$, and $Q_L$ is the $L^2$-projection from the space $L^2(\Omega)$ to $V_L$. In the following two sections, we design our multilevel preconditioner for (2.6).

## 3. A multilevel preconditioner for the conforming $P_1$ mortar FE

In this section, we briefly describe the mortar method for the conforming $P_1$ finite element, *cf.* [6], and then formulate the multilevel preconditioner for the method, developed in [13]. In the following section, we will use this preconditioner to construct our multilevel preconditioner for the discrete system (2.6).

Let $\tilde{W}_{l,i}$ be the continuous piecewise linear finite element space over the triangulation $\mathcal{T}_{l,i}$, whose functions have zero trace on $\partial\Omega$. Let

$$\tilde{W}_l = \prod_{i=1}^{N} \tilde{W}_{l,i},$$

for all $l = 1, \ldots, L$. Obviously, the subspaces $\{\tilde{W}_l\}$ are nested, that is, we have

$$\tilde{W}_1 \subseteq \cdots \subseteq \tilde{W}_L.$$

We now describe the conforming $P_1$ mortar finite element method on the finest level $L$. In order to specify the mortar interface condition, we need to introduce some trace spaces. Let $\bar{M}_L(\gamma_{m(i)})$ and $\bar{M}_L(\delta_{m(j)})$ be the continuous piecewise linear function space corresponding to the triangulation $\mathcal{T}_L(\gamma_{m(i)})$ and $\mathcal{T}_L(\delta_{m(j)})$, respectively. In addition, we define an auxiliary test space $M_L(\delta_{m(j)})$ as a subspace of the space $\bar{M}_L(\delta_{m(j)})$ such that its functions are constants on elements that intersect the ends of $\delta_{m(j)}$. Based on the above preparation, we can now define the following conforming $P_1$ mortar finite element space

$$W_L = \{v_L \in \tilde{W}_L \mid \forall \delta_{m(j)} \subset \Gamma,\ \int_{\delta_{m(j)}} (v_{L,i} - v_{L,j})\varphi \mathrm{d}s = 0,\ \forall \varphi \in M_L(\delta_{m(j)})\}.$$

The conforming $P_1$ mortar finite element approximation of the problem (2.2) on the finest level $L$, is to find $u_L \in W_L$ such that

$$\hat{a}_L(u_L, v_L) = (f, v_L), \quad \forall v_L \in W_L, \tag{3.1}$$

where

$$\hat{a}_L(u_L, v_L) := \sum_{i=1}^{N} \int_{\Omega_i} \nabla u_L \cdot \nabla v_L \mathrm{d}x.$$

It is shown in [6] that (3.1) has a unique solution.

Let $\hat{Q}_{L,\delta_{m(j)}} : L^2(\gamma_m) \to M_L(\delta_{m(j)})$ be the $L^2$-projection, *i.e.*,

$$(\hat{Q}_{L,\delta_{m(j)}} v, w_L) = (v, w_L) \quad \forall w_L \in M_L(\delta_{m(j)}).$$

It is known that, *cf.* [8],

$$\|(I - \hat{Q}_{L,\delta_{m(j)}})v\|_{L^2(\gamma_m)} \leq Ch_L^{\frac{1}{2}} |v|_{H^{\frac{1}{2}}(\gamma_m)} \quad \forall v \in H^{\frac{1}{2}}(\gamma_m). \tag{3.2}$$

Define the operator $\hat{A}_L : W_L \to W_L$ as follows:

$$(\hat{A}_L v_L, w_L) = \hat{a}_L(v_L, w_L) \quad \forall v_L,\ w_L \in W_L.$$

In [13], an effective multilevel preconditioner $\hat{B}_L$ for $\hat{A}_L$ has been designed. In the following, we briefly describe this preconditioner.

First, we define the space $\hat{S}_l(\delta_{m(j)})$ by

$$\begin{aligned} \hat{S}_l(\delta_{m(j)}) = \{v \quad | \quad & v \text{ is continuous piecewise linear on } \mathcal{T}_l(\delta_{m(j)}) \\ & \text{and vanishes at the endpoints of } \delta_{m(j)}\}. \end{aligned}$$

Accordingly, we define a projection operator $\Pi_{L,\delta_{m(j)}} : L^2(\gamma_m) \to \hat{S}_L(\delta_{m(j)})$ as follows [6,14]:

$$\int_{\delta_{m(j)}} (\Pi_{L,\delta_{m(j)}} v)\chi \mathrm{d}s = \int_{\delta_{m(j)}} v\chi \mathrm{d}s, \quad \forall \chi \in M_L(\delta_{m(j)}).$$

This projection is known to be stable in $L^2(\gamma_m)$ and $H_{00}^{1/2}(\gamma_m)$ [6,8], that is

$$\begin{aligned} \|\Pi_{L,\delta_{m(j)}} v\|_{L^2(\delta_{m(j)})} &\leq C\|v\|_{L^2(\gamma_{m(i)})}, \\ \|\Pi_{L,\delta_{m(j)}} v\|_{H_{00}^{1/2}(\delta_{m(j)})} &\leq C\|v\|_{H_{00}^{1/2}(\gamma_{m(i)})}. \end{aligned}$$

As in [13], we introduce an extension operator $Z_{\gamma_m} : L^2(\gamma_m) \to \tilde{W}_{L,j}$ on each interface $\gamma_m$ as follows:

$$Z_{\gamma_m} v := \sum_{l=1}^{L} F_{l,\delta_{m(j)}} (P_{l,\delta_{m(j)}} - P_{l-1,\delta_{m(j)}}) \Pi_{L,\delta_{m(j)}} v, \quad \forall v \in L^2(\gamma_m),$$

where $F_{l,\delta_{m(j)}}$ is the trivial zero extension operator on the level $l$ with respect to the nonmortar subdomain, and $P_{l,\delta_{m(j)}}$ is the $L^2$ projection operator from $\hat{S}_L(\delta_{m(j)})$ to $\hat{S}_l(\delta_{m(j)})$, where we set $P_{0,\delta_{m(j)}} := 0$.

Next we define an intergrid transfer operator $Z_i : \tilde{W}_{l,i} \to W_L$ in terms of $Z_{\gamma_m}$ by means of

$$Z_i v := \begin{cases} v - \sum_{\delta_{m(i)} \subset \partial\Omega_i} Z_{\delta_{m(i)}} v_{|\bar{\Omega}_i \cap \delta_{m(i)}} & \text{on} \quad \bar{\Omega}_i, \\ Z_{\gamma_{m(i)}} v_{|\bar{\Omega}_i \cap \gamma_{m(i)}} & \text{on} \quad \bar{\Omega}_{s(\gamma_{m(i)})}, \\ 0 & \text{elsewhere}, \end{cases} \tag{3.3}$$

where $\bar{\Omega}_{s(\gamma_{m(i)})}$ denotes the nonmortar subdomains with $\gamma_{m(i)} \subset \partial\Omega_i$ as mortar edges. It is easy to check that $Z_i v \in W_L$. Based on the intergrid transfer operator, we can provide a decomposition of the space $W_L$, that is (*cf.* [13])

$$W_L = \sum_{l=1}^{L} \sum_{i=1}^{N} Z_i \tilde{W}_{l,i}.$$

We now introduce an inexact bilinear form $b_{l,i}(\cdot,\cdot) : \tilde{W}_{l,i} \times \tilde{W}_{l,i} \to R$ as

$$b_{l,i}(v_{l,i}, w_{l,i}) := \sum_{x \in \mathcal{N}_{l,i}} v_{l,i}(x) w_{l,i}(x), \quad v_{l,i}, w_{l,i} \in \tilde{W}_{l,i},$$

where $\mathcal{N}_{l,i}$ is the set of conforming $P_1$ nodal points of the triangulation $\mathcal{T}_{l,i}$ in $\bar{\Omega}_i \setminus \partial\Omega$. The corresponding projection like operator $\tilde{T}_{l,i} : W_L \to \tilde{W}_{l,i}$ can be defined by

$$b_{l,i}(\tilde{T}_{l,i} v, v_{l,i}) := \hat{a}_L(v, Z_i v_{l,i}), \quad v_{l,i} \in \tilde{W}_{l,i}.$$

Let

$$T_{l,i} := Z_i \tilde{T}_{l,i} : W_L \to W_L.$$

Then the preconditioned system can be expressed by

$$T := \hat{B}_L \hat{A}_L := \sum_{l=1}^{L} \sum_{i=1}^{N} T_{l,i}.$$

The multilevel preconditioner $\hat{B}_L$ has the following algebraic form (*cf.* [13] for details)

$$\hat{\mathcal{B}}_L := \sum_{l=1}^{L} \sum_{i=1}^{N} \mathcal{Z}_i \mathcal{R}_{l,i} (\mathcal{Z}_i \mathcal{R}_{l,i})^T,$$

where $\mathcal{Z}_i$ is the algebraic representation of $Z_i$, and $\mathcal{R}_{l,i}$ is the prolongation matrix from $\tilde{W}_{l,i} \to \tilde{W}_{L,i}$ (*cf.* [13] for details). It states in [13] that the condition number is proportional to the square of the number of refinement levels.

**Theorem 3.1.** *There holds that* [13]

$$cH^2 \hat{a}_L(v,v) \le \hat{a}_L(\hat{B}_L \hat{A}_L v, v) \le CL^2 \hat{a}_L(v,v), \quad \forall v \in W_L.$$

By adding a coarse space solver, based on a continuous vertex basis function for each subdomain vertex, we could eliminate the dependence of $H$ in the above theorem [13].

**Remark 3.2.** For the case of two subdomains, we have [13]

$$c\hat{a}_L(v,v) \le \hat{a}_L(\hat{B}_L \hat{A}_L v, v) \le C\hat{a}_L(v,v), \quad \forall v \in W_L.$$

As shown in [13], the algorithm can be efficiently implemented. One application of the preconditioner involves applying the BPX preconditioner once within each subdomain, and the matrices $\mathcal{Z}_i$ and $\mathcal{Z}_i^t$ once on the whole domain. The cost of applying the BPX preconditioner is $O(h^{-2})$, and that of $\mathcal{Z}_i$ or $\mathcal{Z}_i^t$ is $O(h^{-1})$ resulting in a total of $O(h^{-2})$ operations ($h$ is the mesh size).

## 4. A multilevel preconditioner for the nonconforming $P_1$ mortar FE

In this section, we will use the idea from [20,32] to construct our multilevel preconditioner for the mortar finite element method in Section 2. In doing so, we make an assumption which follows.

As in a standard mortar finite element method, *e.g.* the conforming $P_1$ mortar finite element, the nodal values on the nonmortar sides are determined by the nodal values on the neighboring mortar sides. In case of the nonconforming $P_1$ mortar finite element, however, the nodal values on a nonmortar side may depend on nodal values from several mortar sides. This complicates the design of an algorithm, see [22,23] for illustrations.

We assume therefore that nodal values on a nonmortar side will depend only on the nodal values on the corresponding mortar side. There are several ways to achieve this. The first way is to avoid having corner triangles in the triangulation of each subdomain. A corner triangle is a triangle with two of its sides lying on the subdomain boundary and sharing a subdomain vertex. The second way is to make sure that the triangle edge touching a subdomain vertex on the mortar side, is smaller than the corresponding triangle edge touching the vertex on the nonmortar side. The third way is to replace the exact mortar condition with an approximate one as the one given in a recent paper [23]. However, this was not the only reason why an approximate mortar condition was introduced in that paper, see later in this section.

We now move into constructing our multilevel preconditioner. We start by introducing a transfer operator from the space $W_L$ to $V_L$. Define an operator $\Xi_{L,\delta_{m(j)}} : \tilde{V}_L \to \tilde{V}_L$ by

$$(\Xi_{L,\delta_{m(j)}}(v))(x) = \begin{cases} (Q_{L,\delta_{m(j)}}(v_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}}))(x) & x \in \delta^{CR}_{L,m(j)}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then for any $v \in \tilde{W}_L \subset \tilde{V}_L$, let

$$v^* = v + \sum_{m=1}^{M} \Xi_{L,\delta_{m(j)}}(v).$$

It is easy to check that $v^* \in V_L$. Based on this observation, we define the following transfer operator $I_L : W_L \to V_L$, which will appear in the following multilevel algorithm:

$$I_L v = v + \sum_{m=1}^{M} \Xi_{L,\delta_{m(j)}}(v), \quad \forall v \in W_L. \tag{4.2}$$

For the operator $I_L$, we have:

**Lemma 4.1.** *For any $v \in W_L$, it holds that*

$$\|v - I_L v\|_{L^2(\Omega)} \leq C h_L \|v\|_L, \tag{4.3}$$
$$\|I_L v\|_L \leq C\|v\|_L. \tag{4.4}$$

*Proof.* At first, we prove (4.3). It is easy to see that

$$\|v - I_L v\|^2_{L^2(\Omega)} \leq C \sum_{m=1}^{M} \|\Xi_{L,\delta_{m(j)}}(v)\|^2_{L^2(\Omega)}. \tag{4.5}$$

For each nonmortar edge $\delta_{m(j)}$, we can derive

$$\begin{aligned}
\|\Xi_{L,\delta_{m(j)}}(v)\|^2_{L^2(\Omega)} &\leq Ch_L^2 \sum_{x\in\delta^{CR}_{L,m(j)}} \left(\Xi_{L,\delta_{m(j)}}(v)(x)\right)^2 \\
&= Ch_L^2 \sum_{x\in\delta^{CR}_{L,m(j)}} \left(Q_{L,\delta_{m(j)}}\left(v_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}}\right)\right)^2(x) \\
&\leq Ch_L \left\|Q_{L,\delta_{m(j)}}\left(v_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}}\right)\right\|^2_{L^2(\gamma_m)} \\
&\leq Ch_L \left\|v_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}}\right\|^2_{L^2(\gamma_m)}.
\end{aligned} \tag{4.6}$$

On the other hand, owing to $v \in W_L$ and (3.2), we have

$$\begin{aligned}
\left\|v_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}}\right\|^2_{L^2(\gamma_m)} &\leq 2\left(\left\|v_{|\gamma_{m(i)}} - \hat{Q}_{L,\delta_{m(j)}}\left(v_{|\gamma_{m(i)}}\right)\right\|^2_{L^2(\gamma_m)}\right. \\
&\qquad \left. + \left\|v_{|\delta_{m(j)}} - \hat{Q}_{L,\delta_{m(j)}}\left(v_{|\delta_{m(j)}}\right)\right\|^2_{L^2(\gamma_m)}\right) \\
&\leq Ch_L\left(\left|v_{|\gamma_{m(i)}}\right|^2_{H^{\frac{1}{2}}(\gamma_m)} + \left|v_{|\delta_{m(j)}}\right|^2_{H^{\frac{1}{2}}(\gamma_m)}\right) \\
&\leq Ch_L\left(|v|^2_{H^1(\Omega_i)} + |v|^2_{H^1(\Omega_j)}\right).
\end{aligned} \tag{4.7}$$

Combining the above three inequalities, (4.5)–(4.7), gives (4.3). We now prove (4.4). By using the definition of the operator $\Xi_{L,\delta_{m(j)}}$ and a similar argument as in the proof of (4.3), we can derive

$$\|v - I_L v\|_L \leq C\|v\|_L,$$

which, together with the triangle inequality, yields (4.4). □

Next we describe a basis for the space $V_L$. Let $\{\tilde{\phi}^k_L | k = 1, \ldots, n_L\}$ be the nodal basis of $\tilde{V}_L$. By the definition of the operator $\Xi_{L,\delta_{m(j)}}$, the basis of $V_L$ consists of functions of the form:

$$\phi^k_L = \tilde{\phi}^k_L + \sum_{m=1}^{M} \Xi_{L,\delta_{m(j)}}(\tilde{\phi}^k_L). \tag{4.8}$$

From the above definition, we can see that there exist two kinds of basis functions of the space $V_L$:

- **Case 1.** $\phi^k_L$ and $\tilde{\phi}^k_L$ corresponding to all nodal points in the interior of each subdomain, except those belonging to the triangles having an edge on a mortar side, are identical.
- **Case 2.** $\phi^k_L$ corresponding to all nodal points on each mortar edge $\gamma_m \subset \Gamma$, and all nodal points in the interior of each subdomain, those belonging to the triangles having an edge on a mortar side, are defined by (4.8).

The above classification is based on the following. A basis function of the first case has nonzero support only inside the subdomain it belongs to and zero support across interfaces. A basis function of the second case, on the other hand, may have nonzero support across interfaces. By definition, the basis functions corresponding to the subdomain interior nodal points those lying closest to the mortar sides may have nonzero supports on the neighboring nonmortar side. Those nodal points have therefore been removed from the first case, and have been included in the second case. However, this is easily avoided if the standard mortar condition is modified in the way shown in [23]. It is easy to check that $\phi_L^k$ corresponding to all nodal points on each nonmortar edge $\delta_{m(j)} \subset \Gamma$ are equal to zero. Consequently, one can see that these $\phi_L^k$ from Case 1 and Case 2 form a basis of $V_L$. Moreover, by the definition of mortar space and (4.8), we know that the basis function $\phi_L^k$ has local support as the basis function of a standard finite element space, say for instance the conforming $P_1$ finite element. This is not the case for the conforming $P_1$ mortar finite element [14].

We now define a smoothing operator $R_L$ which corresponds to the Richardson iteration, as the following:

$$R_L v = \sum_{k=1}^{n_L} (v, \phi_L^k)\phi_L^k \quad \forall v \in V_L. \tag{4.9}$$

**Lemma 4.2.** *It holds that*

$$c h_L^2 (v, v) \le (R_L v, v) \le C h_L^2 (v, v) \quad \forall v \in V_L.$$

*Proof.* We only need to prove

$$\|\phi_L^k\|_{L^2(\Omega)} = O(h_L^2). \tag{4.10}$$

Indeed, for the basis function $\phi_L^k$ which are in the interior of each subdomains, using a standard scaling argument, it is easy to check that $\|\phi_L^k\|_{L^2(\Omega)} = O(h_L^2)$. For the second case basis function, by the definition (4.4), we know

$$\begin{aligned} \|\phi_L^k\|_{L^2(\Omega)} &\le \|\tilde{\phi}_L^k\|_{L^2(\Omega)} + \|\Xi_{L,\delta_{m(j)}}(\tilde{\phi}_L^k)\|_{L^2(\Omega)} \\ &\le C h_L^2 + C h_L \|Q_{L,\delta_{m(j)}}(\tilde{\phi}_L^k)\|_{L^2(\delta_{m(j)})} \\ &\le C h_L^2 + C h_L \|\tilde{\phi}_L^k\|_{L^2(\gamma_{m(i)})} \\ &\le C h_L^2. \end{aligned}$$

Then for all basis function in $V_L$, (4.10) is true. Using a similar argument as in [30,31], we know that Lemma 4.2 is valid. □

Finally we can define our multilevel preconditioner for the mortar-type CR element method as follows:

$$B_L = R_L + I_L \hat{B}_L I_L^t, \tag{4.11}$$

where $I_L^t : V_L \to W_L$ is given by:

$$(I_L^t v, w) = (v, I_L w) \quad \forall v \in V_L,\ w \in W_L.$$

## 5. Condition number estimate

By the Lemmas 4.1 and 4.2 and the abstract theorem developed in [15,32], we know that the following theorem is true.

**Theorem 5.1.** *Let $B_L$ be defined as in (4.11), and assume that there exists a linear operator $J_L : V_L \to W_L$ such that*

$$\|J_L v\|_L \le C\|v\|_L, \quad \forall v \in V_L, \tag{5.1}$$

*and*

$$\|v - I_L J_L v\|_{L^2(\Omega)} \le C h_L \|v\|_L \quad \forall v \in V_L. \tag{5.2}$$

*Then, we have*

$$c H^2 a_L(v, v) \le a_L(B_L A_L v, v) \le C L^2 a_L(v, v) \quad \forall v \in V_L.$$

*For the case of two subdomains we have*

$$c a_L(v, v) \le a_L(B_L A_L v, v) \le C a_L(v, v) \quad \forall v \in V_L.$$

*Proof.* The proof follows immediately from the abstract framework of the auxiliary space method [32], Lemmas 4.1, 4.2, and Theorem 3.1. □

We note here that the proposed multilevel preconditioner for the CR mortar finite element has the same quasi optimality as the multilevel preconditioner for the conforming $P_1$ mortar finite element.

It follows from Theorem 5.1 that, if we can show (5.1) and (5.2) for the spaces $V_L$ and $W_L$, then $B_L$ will be a good multilevel preconditioner for the CR mortar finite element.

At first, we introduce a transfer operator $E_L$ from $\tilde{V}_L$ to $\tilde{W}_L$, which is similar to the one constructed in [33]. On each subdomain, we define an operator $E_{L,i} : V_{L,i} \to W_{L,i}$ as follows:

- **Case 1.** If $x \in \Omega^P_{L,i}$, and $x \notin \partial\Omega$, then

$$(E_{L,i}v)(x) = \frac{1}{q(x)} \sum_{K_i} v|_{K_i}(x)$$

  where $\Omega^P_{L,i}$ is the set of the vertices of the triangulation $\mathcal{T}_{L,i}$ that are in $\bar{\Omega}_i$, and the sum is taken over all triangles $K \in \mathcal{T}_{L,i}$ having $x$ as their common vertex, and $q(x)$ being the number of those triangles.
- **Case 2.** If $x \in \partial\Omega \cap \partial\Omega^P_{L,i}$, then

$$(E_{L,i}v)(x) = 0,$$

  where $\partial\Omega^P_{L,i}$ is the set of vertices of the triangulation $\mathcal{T}_{L,i}$ that are on $\partial\Omega_i$.

For the operator $E_{L,i}$, we have:

**Lemma 5.2.** *For any $v \in V_{L,i}$, it holds that*

$$\begin{aligned} |E_{L,i}v|_{H^1(\Omega_i)} &\le C\|v\|_{L,i}, &(5.3)\\ \|E_{L,i}v - v\|_{L^2(\Omega_i)} &\le Ch_L\|v\|_{L,i}, &(5.4)\\ \|E_{L,i}v - v\|_{L^2(\gamma_m)} &\le Ch_L^{1/2}\|v\|_{L,i}, &(5.5)\end{aligned}$$

*where $\gamma_m$ is an edge of $\Omega_i$.*

*Proof.* The proof of (5.3) and (5.4) can be found in [24,25,35]. For the proof of (5.5), we refer to Lemma 3.3 in [17]. □

Based on the operator $E_{L,i}$, we define an intergrid transfer operator $E_L : \tilde{V}_L \to \tilde{W}_L$ as follows: for any $v = (v_1, \ldots, v_N) \in \tilde{V}_L$,

$$E_L v = (E_{L,1}v_1, \ldots, E_{L,N}v_N) \in \tilde{W}_L.$$

We then define the transfer operator $J_L$ as follows:

$$J_L v = E_L v + \sum_{m=1}^{M} \tilde{\Xi}_{L,\delta_{m(j)}}(E_L v), \tag{5.6}$$

where $\tilde{\Xi}_{L,\delta_{m(j)}} : \tilde{W}_L \to \tilde{W}_L$ is given by

$$(\tilde{\Xi}_{L,\delta_{m(j)}}(v))(x) = \begin{cases} (\Pi_{L,\delta_{m(j)}}(v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}))(x) & x \in \delta^P_{L,m(j)}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Here $\delta^P_{L,m(j)}$ denotes the set of the vertices belonging to $\delta_{m(j)}$. It is easy to check that $J_L v \in W_L$.

**Theorem 5.3.** *For any $v \in V_L$, it holds that*

$$\begin{aligned} \|v - J_L v\|_{L^2(\Omega)} &\le Ch_L \|v\|_L, && (5.8)\\ \|J_L v\|_L &\le C\|v\|_L. && (5.9) \end{aligned}$$

*Proof.* We prove (5.8) first. Using Lemma 5.2, we get

$$\begin{aligned} \|v - J_L v\|^2_{L^2(\Omega)} &\le C\left( \|v - E_L v\|^2_{L^2(\Omega)} + \sum_{m=1}^{M} \left\| \tilde{\Xi}_{L,\delta_{m(j)}} (E_L v) \right\|^2_{L^2(\Omega)} \right) \\ &\le C\left( h_L^2 \|v\|_L^2 + \sum_{m=1}^{M} \left\| \tilde{\Xi}_{L,\delta_{m(j)}} (E_L v) \right\|^2_{L^2(\Omega)} \right). \end{aligned} \quad (5.10)$$

For each nonmortar edge $\delta_{m(j)}$,

$$\begin{aligned} \|\tilde{\Xi}_{L,\delta_{m(j)}} (E_L v)\|^2_{L^2(\Omega)} &\le Ch_L^2 \sum_{x \in \delta^P_{L,m(j)}} \left( \tilde{\Xi}_{L,\delta_{m(j)}} (E_L v)(x) \right)^2 \\ &= Ch_L^2 \sum_{x \in \delta^P_{L,m(j)}} \left( \Pi_{L,\delta_{m(j)}} \left( (E_L v)_{|\gamma_{m(i)}} - (E_L v)_{|\delta_{m(j)}} \right) \right)^2 (x) \\ &\le Ch_L \left\| \Pi_{L,\delta_{m(j)}} \left( (E_L v)_{|\gamma_{m(i)}} - (E_L v)_{|\delta_{m(j)}} \right) \right\|^2_{L^2(\delta_{m(j)})} \\ &\le Ch_L \left\| (E_L v)_{|\gamma_{m(i)}} - (E_L v)_{|\delta_{m(j)}} \right\|^2_{L^2(\gamma_m)} \\ &\le Ch_L \Bigg( \left\| (E_L v)_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}} \right\|^2_{L^2(\gamma_m)} \\ &\quad + \left\| v_{|\delta_{m(j)}} - (E_L v)_{|\delta_{m(j)}} \right\|^2_{L^2(\delta_{m(j)})} \Bigg) \\ &:= Ch_L (K_1 + K_2). \end{aligned} \quad (5.11)$$

It follows immediately from Lemma 5.2 that

$$K_2 \le Ch_L \|v\|^2_{L,j}. \quad (5.12)$$

In the following, we estimate the term $K_1$. Since $v \in V_L$, we get

$$\begin{aligned} \left\| (E_L v)_{|\gamma_{m(i)}} - v_{|\delta_{m(j)}} \right\|^2_{L^2(\gamma_m)} &\le 2 \left\| (E_L v)_{|\gamma_{m(i)}} - Q_{L,\delta_{m(j)}} \left( v_{|\gamma_{m(i)}} \right) \right\|^2_{L^2(\gamma_m)} \\ &\quad + 2 \left\| Q_{L,\delta_{m(j)}} \left( v_{|\delta_{m(j)}} \right) - v_{|\delta_{m(j)}} \right\|^2_{L^2(\delta_{m(j)})}. \end{aligned} \quad (5.13)$$

For the second term in the above inequality, *cf.* [17,33], we have

$$\left\| Q_{L,\delta_{m(j)}} \left( v_{|\delta_{m(j)}} \right) - v_{|\delta_{m(j)}} \right\|_{L^2(\delta_{m(j)})} \le Ch_L^{1/2} \|v\|_{L,j}. \quad (5.14)$$

For the first term on the right hand side of (5.13), we deduce that

$$\begin{aligned}\left\|(E_L v)_{|\gamma_{m(i)}} - Q_{L,\delta_{m(j)}}\left(v_{|\gamma_{m(i)}}\right)\right\|^2_{L^2(\gamma_m)} &\leq 2\left\|(E_{L,i}v)_{|\gamma_{m(i)}} - Q_{L,\delta_{m(j)}}\left((E_{L,i}v)_{|\gamma_{m(i)}}\right)\right\|^2_{L^2(\gamma_m)} \\ &\quad + 2\left\|Q_{L,\delta_{m(j)}}\left((E_{L,i}v)_{|\gamma_{m(i)}}\right) - v_{|\gamma_{m(i)}}\right)\right\|^2_{L^2\left(\delta_{m(j)}\right)} \\ &:= F_1 + F_2. \end{aligned} \quad (5.15)$$

Applying a similar argument as in the proof of (5.14), and Lemma 5.2 we get

$$F_1 \leq Ch_L |E_{L,i}v|^2_{H^1(\Omega_i)} \leq Ch_L \|v\|^2_{L,i}. \quad (5.16)$$

For $F_2$, using Lemma 5.2 and the stability results of $Q_{L,\delta_{m(j)}}$ [17], we have

$$F_2 \leq C\left\|(E_{L,i}v)_{|\gamma_{m(i)}} - v_{|\gamma_{m(i)}}\right\|^2_{L^2(\gamma_{m(i)})} \leq Ch_L\|v\|^2_{L,i}, \quad (5.17)$$

which, together with (5.10)–(5.13), and (5.14)–(5.16), gives (5.8).

We now prove (5.9). In fact, by the definition of $E_L$, and Lemma 5.2, we deduce

$$\begin{aligned}\|v - J_L v\|^2_L &\leq \|v - E_L v\|^2_L + \sum_{m=1}^{M} \|\tilde{\Xi}_{L,\delta_{m(j)}}(E_L v)\|^2_L \\ &\leq C(\|v\|^2_L + \sum_{m=1}^{M} \|\tilde{\Xi}_{L,\delta_{m(j)}}(E_L v)\|^2_L).\end{aligned} \quad (5.18)$$

Using a similar argument as in the proof of (5.8), we can derive

$$\|\tilde{\Xi}_{L,\delta_{m(j)}}(E_L v)\|^2_L \leq C\left(\|v\|^2_{L,i} + \|v\|^2_{L,j}\right). \quad (5.19)$$

Combining (5.18) with (5.19) yields (5.9). □

Based on Theorem 5.3, we know that (5.1) is true. On the other hand, by Lemma 4.1 and Theorem 5.3, we have

$$\begin{aligned}\|v - I_L J_L v\|_{L^2(\Omega)} &\leq \|v - J_L v\|_{L^2(\Omega)} + \|(I - I_L)J_L v\|_{L^2(\Omega)} \\ &\leq Ch_L\|v\|_L + Ch_L\|J_L v\|_L \\ &\leq Ch_L\|v\|_L,\end{aligned}$$

which is (5.2). The assumptions of Theorem 5.1 are thereby satisfied.

**Remark 5.4.** Using the same technique developed in this paper, we can construct an effective multilevel preconditioner for the mortar Wilson element proposed in [26].

**Remark 5.5.** Normally, when we consider a multigrid or a multilevel method for the mortar method, we always have to require that the mesh sizes $h_i^l$, for all $i$, are comparable (*cf.* [8,9,15] for instance). It is a big challenge to design a multilevel or a multigrid algorithm that is independent of the mesh size ratio $h_i^l/h_j^l$ across interfaces, or even mildly dependent. This topic will be further investigated.

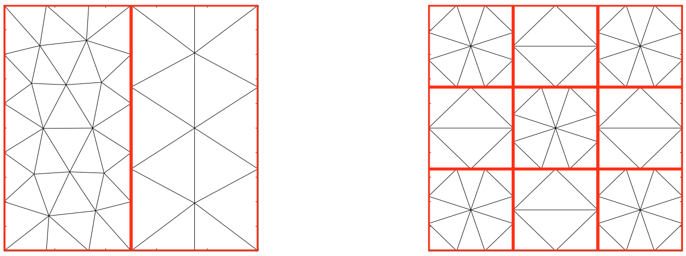

FIGURE 1. Initial nonmatching grids as a result of calling Matlab's triangulation routine `initmesh` separately for each subdomain, with the input parameter `hmax` $= 0.2$ and $0.35$ (on the left) and `hmax` alternatingly equal to $0.15$ and $0.30$ (on the right).

**Remark 5.6.** Although, by using the domain decomposition framework, we have already succeeded to design a number of powerful preconditioners for the nonconforming $P_1$ mortar element on problems with discontinuous coefficients, it is still a difficult task to design a multilevel preconditioner for the nonconforming $P_1$ mortar element for the problem. It is a topic of further investigation.

## 6. Numerical results

We present in this section the numerical results from our experiments. The model problem is defined on a unit square domain, $\Omega = (0,1)^2$, with the forcing function $f$ equal to $2\pi^2 \sin(\pi x)\sin(\pi y)$, and a homogeneous Dirichlet boundary condition resulting in the exact solution $u$ equals to $\sin(\pi x)\sin(\pi y)$. At first, the domain $\Omega$ is partitioned into a $d_x \times d_y$ rectangular subdomains (subregions). Initially, each subdomain $\Omega_i$ is triangulated using Matlab's discretization routine `initmesh` with the mesh size parameter `hmax` as an input parameter. The parameter `hmax` is one of two real numbers defining two different mesh sizes distributed among the subdomains in a checkerboard fashion. Starting from the initial triangulation ($l = 1$), we decompose each triangle into four subtriangles in each refinement step. The resulting grid is nonmatching across all interfaces, and we use the CR mortar finite element [17] for the discretization of the model problem. The resulting discrete problem is solved using the Preconditioned Conjugate Gradients (PCG) method with the multilevel preconditioner for the CR mortar finite element proposed in this paper. For the smoothing operator we consider only the Richardson type here as it is simple, and turned out to be very effective for our case.

For our first experiment, we consider two different partitions of the domain, one with two subdomains without resulting in any internal crosspoint, *cf.* Figure 1 (left picture), and one with nine subdomains resulting in four internal crosspoints, *cf.* Figure 1 (right picture). In the first case there are no corner triangles in the entire triangulation, and we choose the mortar side with `hmax` $= 0.35$. In the second case there are corner triangles in the triangulation of each subdomain, and we choose mortar sides with the smallest `hmax` $= 0.15$. Clearly, the assumption made in the beginning of Section 4 is supported in both cases.

The numerical results are presented in Tables 1 and 2, showing, for each test case, the following quantities: number of levels ('levels'), number of interior degrees of freedom ('dofs'), condition number estimate ('$\kappa$') with the number of iterations ('iter', inside parentheses) required to reduce the residual norm by the factor $10^{-6}$, and the $L^2$ norm of the error ('error$_{L^2}$') in the computed solution. For the comparison, we also include the corresponding numerical results from applying, to our model problem, the original multilevel preconditioner for the conforming $P_1$ mortar finite element as proposed in [13]. Mortar sides are chosen so the best convergence results are observed, *i.e.*, mortar sides with `hmax` $= 0.35$ and `hmax` $= 0.30$ in the first and second partitions, respectively.

TABLE 1. The multilevel Schwarz for the conforming $P_1$ mortar element and the proposed multilevel Schwarz for the CR mortar element on $d_x \times d_y = 2 \times 1$ nonmatching grids.

| | Multilevel for $P_1$ mortar | | | Multilevel for CR mortar | | |
|---|---|---|---|---|---|---|
| Levels | Dofs | $\text{Error}_{L^2}$ | $\kappa$ (iter) | Dofs | $\text{Error}_{L^2}$ | $\kappa$ (iter) |
| 3 | 377 | 4.269 e–3 | 15.00 (21) | 1208 | 1.131 e–3 | 19.21 (26) |
| 4 | 1585 | 1.069 e–3 | 17.58 (24) | 4912 | 2.839 e–4 | 20.51 (30) |
| 5 | 6497 | 2.672 e–4 | 19.46 (27) | 19 808 | 7.103 e–5 | 21.72 (32) |
| 6 | 26 305 | 6.679 e–5 | 20.95 (30) | 79 552 | 1.776 e–5 | 22.90 (34) |

TABLE 2. The multilevel Schwarz for the conforming $P_1$ mortar element and the proposed multilevel Schwarz for the CR mortar element on $d_x \times d_y = 3 \times 3$ nonmatching grids.

| | Multilevel for $P_1$ mortar | | | Multilevel for CR mortar | | |
|---|---|---|---|---|---|---|
| Levels | Dofs | $\text{Error}_{L^2}$ | $\kappa$ (iter) | Dofs | $\text{Error}_{L^2}$ | $\kappa$ (iter) |
| 3 | 597 | 2.742 e–3 | 50.17 (25) | 1976 | 1.109 e–3 | 59.17 (34) |
| 4 | 2525 | 6.903 e–4 | 68.36 (32) | 7984 | 2.737 e–4 | 77.90 (40) |
| 5 | 10 413 | 1.729 e–4 | 89.69 (39) | 32 096 | 6.772 e–5 | 103.79 (43) |
| 6 | 42 317 | 4.326 e–5 | 114.49 (44) | 128 320 | 1.692 e–5 | 134.37 (50) |

Generally speaking, the condition number estimates and the iteration counts as seen from the tables, support our theory presented in this paper. The numerical results simply reflect the fact that the convergence behavior of the multilevel preconditioner for the CR mortar finite element is similar to that of the original multilevel preconditioner for the conforming $P_1$ mortar finite element. Table 1 corresponds to the initial discretization shown in the left picture of Figure 1, representing the case without any crosspoint. As seen from the table, the condition number estimates depend very mildly on the number of refinement levels. Table 2 corresponds to the initial discretization shown in the right picture of Figure 1, where there are four internal crosspoints. We observe asymptotically a quadratic and a linear dependence of the condition number estimates and the iteration counts, respectively, on the number of grid refinement levels. This is in agreement with our theory.

In our next experiment, we show how the multilevel Schwarz methods depend on the subdomain size using two different subdomain sizes $H$ and keeping the level $L$ fixed. We note that the level $L$ is proportional to $\log_2(H/h_L)$, and hence fixing $L$ implies that the ratio $H/h_L$ should be kept fixed. We start with the partition $d_x \times d_y = 3 \times 3$ with `hmax` $= 0.15$ and $0.30$ as in Figure 1, and halve the subdomain size by doubling the number of subdomains in both directions and using `hmax` $= 0.075$ and $0.15$. The mortar sides are chosen in the same way as before, *i.e.*, in case of the nonconforming $P_1$ mortar element, on the side of smallest `hmax`, and in case of the conforming $P_1$ mortar element, on the side of largest `hmax`. The numerical results are presented in Table 3. For a fixed level, as we halve the subdomain size, the condition number estimate quadruples, showing a quadratic dependence of the condition number on the mesh size $H$. This is in accordance with the theory.

TABLE 3. Illustrating the $H$-dependence of the multilevel Schwarz for the conforming $P_1$ mortar element and the proposed multilevel Schwarz for the CR mortar element.

| | Multilevel for $P_1$ mortar | | | | Multilevel for CR mortar | | | |
|---|---|---|---|---|---|---|---|---|
| | $d_x \times d_y = 3 \times 3$ | | $d_x \times d_y = 6 \times 6$ | | $d_x \times d_y = 3 \times 3$ | | $d_x \times d_y = 6 \times 6$ | |
| Levels | Dofs | $\kappa$ (iter) | Dofs | $\kappa$ (iter) | Dofs | $\kappa$ (iter) | Dofs | $\kappa$ (iter) |
| 2 | 137 | 39.29 (18) | 604 | 140.14 (37) | 484 | 43.03 (29) | 1944 | 160.99 (44) |
| 3 | 597 | 50.17 (25) | 2428 | 181.86 (43) | 1976 | 59.17 (34) | 7776 | 217.24 (53) |
| 4 | 2525 | 68.36 (32) | 9964 | 242.20 (50) | 7984 | 77.90 (40) | 31 104 | 284.75 (61) |

*Acknowledgements.* We thank the anonymous referees who meticulously read through the paper, and made many helpful suggestions which lead to an improved presentation of this paper.

## REFERENCES

[1] Y. Achdou and Yu.A. Kuznetsov, Subtructuring preconditioners for finite element methods on nonmatching grids. *J. Numer. Math.* **3** (1995) 1–28.
[2] Y. Achdou, Yu.A. Kuznetsov and O. Pironneau, Substructuring preconditioner for the $Q_1$ mortar element method. *Numer. Math.* **71** (1995) 419–449.
[3] Y. Achdou, Y. Maday and O.B. Widlund, Iterative substructing preconditioners for mortar element methods in two dimensions. *SIAM J. Numer. Anal.* **36** (1999) 551–580.
[4] F. Ben Belgacem, The mortar finite element method with Lagrange multipliers. *Numer. Math.* **84** (1999) 173–198.
[5] F. Ben Belgacem and Y. Maday, The mortar element method for three dimensional finite elements. *RAIRO Modél. Math. Anal. Numér.* **31** (1997) 289–302.
[6] C. Bernardi, Y. Maday and A.T. Patera, A new nonconforming approach to domain decomposition: the mortar element method, in *Nonlinear partial differential equations and their applications*, Vol. XI, Collège de France Seminar, H. Brezis and J.L. Lions Eds., *Pitman Research Notes in Mathematics Series* **299**, Longman Scientific & Technical, Harlow (1994) 13–51.
[7] D. Braess and W. Dahmen, Stability estimates of the mortar finite element method for 3-dimensional problems. *J. Numer. Math.* **6** (1998) 249–264.
[8] D. Braess, W. Dahmen and C. Wieners, A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.* **37** (2000) 48–69.
[9] D. Braess, P. Deuflhard and K. Lipnikov, A subspace cascadic multigrid method for the mortar elements. *Computing* **69** (2002) 202–225.
[10] S. Brenner, Preconditioning complicated finite elements by simple finite elements. *SIAM J. Sci. Comput.* **17** (1996) 1269–1274.
[11] P.G. Ciarlet, *The Finite Element Method for Elliptic Problem.* North-Holland, Amsterdam (1978).
[12] M. Dryja, An iterative substructuring method for elliptic mortar finite element problems with discontinous coefficients, in *Domain Decomposition Methods* **10**, J. Mandel, C. Farhat and X.C. Cai Eds., *Contemp. Math.* **218** (1998) 94–103.
[13] M. Dryja, A. Gantner, O. Widlund and B. Wohlmuth, Multilevel additive Schwarz preconditioner for nonconforming mortar finite element methods. *J. Numer. Math.* **12** (2004) 23–38.
[14] J. Gopalakrishnan and J.P. Pasciak, Multigrid for the mortar finite element method. *SIAM J. Numer. Anal.* **37** (2000) 1029–1052.
[15] R.H.W. Hoppe and B. Wohlmuth, Adaptive multilevel iterative techniques for nonconforming finite element discretizations. *J. Numer. Math.* **3** (1995) 179–198.
[16] C. Kim, R. Lazarov, J. Pasciak and P. Vassilevski, Multiplier spaces for the mortar finite element method in three dimensions. *SIAM J. Numer. Anal.* **39** (2001) 519–538.
[17] L. Marcinkowski, The mortar element method with locally nonconforming elements. *BIT Numer. Math.* **39** (1999) 716–739.
[18] L. Marcinkowski, Additive Schwarz method for mortar discretization of elliptic problems with $P_1$ nonconforming finite element. *BIT Numer. Math.* **45** (2005) 375–394.
[19] L. Marcinkowski and T. Rahman, Neumann – Neumann algorithms for a mortar Crouzeix-Raviart element for 2nd order elliptic problems. *BIT Numer. Math.* **48** (2008) 607–626.
[20] S.V. Nepomnyaschikh, Fictitious components and subdomain alternating methods. *Sov. J. Numer. Anal. Math. Modelling* **5** (1990) 53–68.
[21] P. Oswald, Preconditioners for nonconforming elements. *Math. Comp.* **65** (1996) 923–941.

[22] T. Rahman, X. Xu and R. Hoppe, Additive Schwarz method for the Crouzeix-Raviart mortar finite element for elliptic problems with discontinuous coefficients. *Numer. Math.* **101** (2005) 551–572.

[23] T. Rahman, P.E. Bjørstad and X. Xu, Crouzeix-Raviart FE on nonmatching grids with an approximate mortar condition. *SIAM J. Numer. Anal.* **46** (2008) 496–516.

[24] M. Sarkis, *Schwarz Preconditioners for Elliptic Problems with Discontinuous Coefficients Using Conforming and Nonconforming Elements*. Tech. Report **671**, Ph.D. Thesis, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, USA (1994).

[25] M. Sarkis, Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using nonconforming elements. *Numer. Math.* **77** (1997) 383–406.

[26] Z.C. Shi and X. Xu, Multigrid for the Wilson mortar element method. *Comput. Methods Appl. Math.* **1** (2001) 99–112.

[27] P. Vassilevski and J. Wang, An application of the abstract multilevel theory to nonconforming finite element methods. *SIAM J. Numer. Anal.* **32** (1995) 235–248.

[28] B. Wohlmuth, A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.* **38** (2000) 989–1012.

[29] B. Wohlmuth, A multigrid method for saddlepoint problems arising from mortar finite element discretizations. *Electron. Trans. Numer. Anal.* **11** (2000) 43–54.

[30] J. Xu, *Theory of Multilevel Methods*. Ph.D. Thesis, Cornell University, USA (1989).

[31] J. Xu, Iterative methods by space decomposition and subspace correction. *SIAM Rev.* **34** (1992) 581–613.

[32] J. Xu, The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grid. *Computing* **56** (1996) 215–235.

[33] X. Xu and J. Chen, Multigrid for the mortar element method for $P_1$ nonconforming element. *Numer. Math.* **88** (2001) 381–398.

[34] H. Yserentant, Old and new convergence proofs for multigrid methods. *Acta Numer.* (1993) 285–326.

[35] S. Zhang and Z. Zhang, Treatments of discontinuity and bubble functions in the multigrid method. *Math. Comp.* **66** (1997) 1055–1072.