EXPONENTIAL CONVERGENCE TO EQUILIBRIUM *VIA* LYAPOUNOV FUNCTIONALS FOR REACTION-DIFFUSION EQUATIONS ARISING FROM NON REVERSIBLE CHEMICAL KINETICS

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Abstract. We show that the entropy method, that has been used successfully in order to prove exponential convergence towards equilibrium with explicit constants in many contexts, among which reaction-diffusion systems coming out of reversible chemistry, can also be used when one considers a reaction-diffusion system corresponding to an irreversible mechanism of dissociation/recombination, for which no natural entropy is available.

1. Introduction

1.1. Entropy methods

Entropy methods have recently been used in order to prove exponential convergence towards the equilibrium with *explicit* constants in many situations (*e.g.* integral equations, *cf.* [20], fourth order equations, *cf.* [5], nonlinear diffusion equations, *cf.* [6,7,12]). A nice survey of these methods may be found in the review paper [1]. In particular, reaction-diffusion equations in the context of reversible chemistry have been systematically studied in [9–11].

We recall that the principle of this method is to find a Lyapounov functional $E(f)$ and its dissipation $D(f)$ such that

$$\partial_t E(f) = -D(f) \leq 0,$$

when f is a solution of the equation, and such that the following (sometimes called entropy/entropy dissipation) functional inequality holds:

$$D(f) \geq C\left(E(f) - E(f_{\mathrm{eq}})\right),$$

where f_{eq} is the unique minimum of E (once conservations have been taken into account). In this situation, it is usually possible to prove that

$$\|f - f_{\mathrm{eq}}\| \leq C_1\, \mathrm{e}^{-C_2\, t},$$

where C_1 and C_2 are explicit (note that a linearization usually leads to exponential decay, but with a constant C_1 which is not explicit).

For reaction-diffusion equations appearing in reversible chemistry, it is in general possible to take for E the natural physical entropy of the problem (such an entropy exists because the equations can be obtained as the limit of kinetic equations of Boltzmann type describing the microscopic processes, *cf.* [2]).

This paper is devoted to showing that in a typical example of irreversible chemistry in which only one equilibrium appears, it is also possible to find a Lyapounov functional E which satisfies the requirements of the entropy method. The model that we intend to study is related to a set of dissociation/recombination chemical reactions.

1.2. **A model of dissociation/recombination**

We consider a diatomic gas with dissociation/recombination reactions, made up by atoms A with mass m_1 and molecules A_2 with mass $m_2 = 2\,m_1$. The two species in the binary mixture are labeled by an index $i = 1, 2$. Generally speaking, the reaction-diffusion system is expected in the form

$$\partial_t n_i - d_i \, \Delta_{\mathbf{x}} n_i = \mathcal{Q}_i(n_1, n_2), \qquad i = 1, 2, \tag{1.1}$$

where n_i denotes number density, d_i diffusion coefficient, and $\mathcal{Q}_i$ the chemical source term. Since all chemical encounters preserve the global mass of participating species (or, equivalently, preserve the total number of atoms), any reasonable dissociation/recombination model must fulfil "*a priori*" the consistency constraint

$$\mathcal{Q}_1(n_1, n_2) + 2\,\mathcal{Q}_2(n_1, n_2) = 0. \tag{1.2}$$

If the mixture is embedded in a fixed background, to be labeled by an additional index $i = 0$, dissociation reactions may occur by binary encounter of a molecule A_2 with any of the possible collision partners (field particle A_0, single atom A_1, or other molecule A_2), whereas a recombination reaction is due solely to a binary interaction of two atoms between themselves. In the first process, one molecule is lost and two atoms are gained in the collision balance. In the second process, two atoms coalesce in one molecule, and the balance is just reversed. Therefore, the simplest heuristic model one could think of in order to describe the reaction effects on the whole evolution problem is represented by

$$\mathcal{Q}_2^h(n_1, n_2) = \alpha_{11}^r (n_1)^2 - \left(\alpha_{20}^d\, n_0 + \alpha_{21}^d\, n_1 + \alpha_{22}^d\, n_2 \right) n_2, \tag{1.3}$$

where α_{11}^r and α_{2j}^d $(j = 0, 1, 2)$ are suitable averaged rate constants quantifying the probability of a recombination collision A_1–A_1 or of a dissociation collision A_2–A_j, respectively. Of course, $\mathcal{Q}_1^h$ follows from (1.2), and superscripts are used to denote the type of reaction. The constant background density n_0 is given.

The same problem can be tackled at the kinetic level [18], in the frame of a recently developed literature (see for instance [13]). According to a common kinetic model, the gas is described as a mixture of three species, with an additional component, labeled by $i = 3$, representing unstable molecules $A_3 \equiv A_2^*$ (with mass $m_3 = m_2$) and playing the role of a transition state [21]. The mixture is then taken to diffuse in a much denser medium [3], whose evolution is not affected by the collisions going on, assumed in local thermodynamical equilibrium, namely with distribution function $f_0 = n_0 M_0$, where M_0 stands for the normalized Maxwellian

$$M_0 = \left(\frac{m_0}{2\pi\, T_0} \right)^{\frac{3}{2}} \exp\left(-\frac{m_0}{2\, T_0} |\mathbf{v}|^2 \right) \tag{1.4}$$

and T_0 is also constant. According to the model, both atoms A_1 and stable molecules A_2 may undergo elastic collisions with other atoms, stable molecules and background particles. Moreover, atoms A_1 may form a stable molecule A_2 passing through the transition state A_2^*, while, on the other hand, both stable and unstable diatomic

molecules may dissociate into two atoms. More precisely, the recombination process occurs in two steps:

$$
\begin{array}{llll}
(R) & A_1 + A_1 & \rightarrow & A_2^* \\
(I) & A_2^* + P & \rightarrow & A_2 + P,
\end{array}
$$
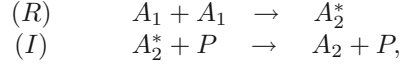
where $P = A$, A_2, B, while dissociation occurs *via* two possible reactions:

$$
\begin{array}{llll}
(D1) & A_2 + P & \rightarrow & 2\,A_1 + P \\
(D2) & A_2^* + P & \rightarrow & 2\,A_1 + P.
\end{array}
$$
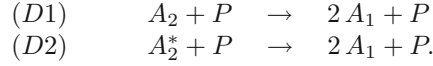
It must be stressed that all above interactions, modelling the chemical reactions at the kinetic level, have to be understood as irreversible processes. Chemical operators are given in terms of total microscopic collision frequencies ν_{ij}^{k} (constant for Maxwellian molecules), where the superscript k may take the values s, r, i, d, corresponding to elastic scattering, recombination R, inelastic scattering I, dissociations $D1$, $D2$, respectively, and of suitable transition probabilities, accounting for the correct exchange rates for mass, momentum, and various forms of energy [14]. Then, kinetic integrodifferential equations have been scaled in terms of the typical relaxation times, a small parameter defining the dominant process(es) has been introduced, and the formal asymptotic limit when this parameter vanishes has been consistently investigated [3]. This leads to the derivation of hydrodynamic limiting equations, whose nature varies considerably according to the relative importance of the various processes and to the corresponding pertinent scaling, but which are typically of reaction-diffusion type as long as the scattering with the background plays an important role. Some non-exhaustive examples were given in [3] itself, and also in [4]. However, we shall deal here with one of the asymptotic limits which seems more realistic in practice [3], and leads to (1.1), (1.2) with specialization

$$
\mathcal{Q}_2(n_1, n_2) = \frac{\nu_{30}^i n_0 + \nu_{31}^i n_1 + \nu_{32}^i n_2}{\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2}\, \nu_{11}^r\, (n_1)^2 - \left(\nu_{20}^d n_0 + \nu_{21}^d n_1 + \nu_{22}^d n_2\right) n_2 \tag{1.5}
$$

with $\nu_{3j}^t = \nu_{3j}^i + \nu_{3j}^d$, $j = 0, 1, 2$. This rather simple expression was derived under the simplifying assumption of Maxwell-type interactions, in which reactive collision frequencies are constant. However, with respect to (1.3), this expression shows a more complicated and realistic dependence on the participating species densities (a rational function rather than a quadratic polynomial), with rates well defined in terms of microscopic parameters. The same is true for the (positive) diffusion coefficients, which take the form

$$
d_1 = \frac{m_1 + m_0}{2\,m_1\,m_0}\,\frac{T_0}{\bar{\nu}_{10}^s\,n_0}, \qquad d_2 = \frac{2\,m_1 + m_0}{4\,m_1\,m_0}\,\frac{T_0}{\bar{\nu}_{20}^s\,n_0}, \tag{1.6}
$$

where the constants $\bar{\nu}_{j0}^s$ are suitable angle averaged scattering collision frequencies [3]. The fraction in (1.5) accounts for the important physical fact that recombination occurs *via* a transition state, and it is remarkable that the microscopic parameters of this metastable species, ν_{3j}^i and ν_{3j}^d, do influence the evolution of the two stable species, though the third species is not present at the hydrodynamic level (its density has collapsed to zero in the asymptotic limit). This is a so-called ghost-effect, not unfrequent in fluid dynamics [19], namely a trace in the evolution of something that does not exist. This is due in our case to the clearance of an indeterminate form, coming from the simultaneous vanishing of the species density and of its relaxation times. It can be noticed that the simple heuristic model (1.3) may be considered as a special case of the more physical model (1.5) under the simplifying assumption $\nu_{3j}^d = 0$, $j = 0, 1, 2$, with all rate constants α_{ij}^{k} provided exactly by the corresponding microscopic collision frequencies ν_{ij}^{k}. Another special case of the same type as (1.3) is achieved by assuming instead $\nu_{3j}^i = \eta\,\nu_{3j}^t$, $j = 0, 1, 2$, with $0 < \eta < 1$; in this case the rate constant for recombination α_{11}^r would be given by the microscopic recombination collision frequency ν_{11}^r reduced by the factor η.

1.3. **Exponential convergence towards equilibrium**

The main goal of this paper is to show that the solution of system (1.1)–(1.2)–(1.5), together with the Neumann boundary conditions

$$\hat{\mathbf{n}}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} n_i = 0, \qquad \mathbf{x} \in \partial\Omega, \tag{1.7}$$

(where $\hat{\mathbf{n}}(\mathbf{x})$ is the outward normal unit vector to the spatial domain Ω at point $\mathbf{x}$) and the nonnegative initial conditions

$$n_i(0, \mathbf{x}) = n_i^0(\mathbf{x}) \geq 0, \tag{1.8}$$

converges exponentially fast with explicit constants towards the unique equilibrium of the system. Since total number of atoms is preserved, we have that

$$\forall t \geq 0, \qquad \int_\Omega \Big(n_1(t, \mathbf{x}) + 2\, n_2(t, \mathbf{x})\Big) \mathrm{d}\mathbf{x} = \int_\Omega \Big(n_1^0(\mathbf{x}) + 2\, n_2^0(\mathbf{x})\Big) \mathrm{d}\mathbf{x} = \bar{n}^0,$$

or, equivalently, setting $\bar{n}_i(t) = \int_\Omega n_i(t, \mathbf{x})\, \mathrm{d}\mathbf{x}$,

$$\forall t \geq 0, \qquad \bar{n}_1(t) + 2\, \bar{n}_2(t) = \bar{n}^0. \tag{1.9}$$

We are able to prove the following theorem:

Theorem 1.1. *Let Ω be a bounded regular (C^2) open set of $\mathbb{R}^N$, let $n_i^0 > 0$ ($i = 1, 2$) be initial data in $C^2(\bar{\Omega})$ compatible with Neumann boundary conditions. Finally, let ν_{3j}^i, ν_{3j}^d, ν_{2j}^d ($j = 0, 1, 2$), ν_{11}^r, d_1, d_2 be strictly positive constants.*

Then, there exists a unique strong ($C^2(\mathbb{R}^+ \times \bar{\Omega})$) solution n_1, n_2 to system (1.1)–(1.2)–(1.5) with boundary conditions (1.7) and initial data n_i^0 ($i = 1, 2$). This solution is bounded from above and from below:

$$k_1 \leq n_1(t, \mathbf{x}) \leq \mathcal{K}_1, \qquad k_2 \leq n_2(t, \mathbf{x}) \leq \mathcal{K}_2,$$

where k_1, k_2, $\mathcal{K}_1$ and $\mathcal{K}_2$ are strictly positive constants depending only on corresponding bounds for initial data, and moreover it satisfies

$$\sum_{i=1}^2 \|n_i(t, \cdot) - n_i^*\|_{L^2(\Omega)}^2 \leq \mathsf{C}_1\, \mathrm{e}^{-\,\mathsf{C}_2\, t},$$

where C_1 and C_2 are explicitly computable. Here n_i^* is the unique positive (independent of $\mathbf{x}$) solution of

$$\mathcal{Q}_i(n_1^*, n_2^*) = 0, \quad i = 1, 2,$$

satisfying the conservation (1.9).

As announced in Section 1.1, the method of proof will consist in finding a suitable Lyapounov functional $E(n_1, n_2)$ and its dissipation $D(n_1, n_2)$ such that

$$\partial_t E(n_1, n_2) = -\, D(n_1, n_2) \leq 0,$$

when n_1, n_2 is a solution of system (1.1)–(1.2)–(1.5), and such that

$$D(n_1, n_2) \geq C\, \Big(E(n_1, n_2) - E(n_1^*, n_2^*)\Big), \tag{1.10}$$

for any n_1, n_2 (functions of $\mathbf{x}$ only) satisfying the same *a priori* assumptions as those of system (1.1)–(1.2)–(1.5), *i.e.* conservation of mass, minimum and maximum principle. A suitable scaling of the space variable $\mathbf{x}$ allows us to carry out the proof under the assumption $|\Omega| = 1$, without loss of generality.

In Section 2, we begin by treating a particular case in which the computations are quite simple and make the method easily understandable. In that section, we do not prove in detail the existence and uniqueness of the solution of the system, for the sake of conciseness.

Then, in Section 3, we treat the general case. Section 3.1 is devoted to a first study of the reaction terms $\mathcal{Q}_i$. Then, in Section 3.2 we present the minimum/maximum principle and existence/uniqueness of solutions to system (1.1)–(1.2)–(1.5). Finally, Section 3.3 is devoted to the establishment of estimate (1.10), which enables to recover the result of exponential convergence with explicit rates.

Remark 1.2. The theorem in the present version will be proved in Section 3. The assumption on strict positivity for the collision frequencies ν_{ij}^k can be easily weakened, and several of them can be allowed to vanish, making proofs easier, as shown indeed by the simpler case dealt with preliminarily in Section 2. From a mathematical point of view, an interesting question consists in asking if it is really necessary that both diffusivities are strictly positive. We shall show in Remark 3.8 that if one of them vanishes, the results remain true (though with different constants).

Remark 1.3. Concerning computability of the constants in the decay estimate, explicit formulas are too involved to be given in the text of the theorem. However, they can be estimated in terms of the initial distributions n_i^0 (specifically, of their upper and lower bounds), of the domain Ω (specifically, of its Poincaré constant), in addition to the physical parameters ν_{3j}^i, ν_{3j}^d, ν_{2j}^d ($j = 0, 1, 2$), ν_{11}^r, and the diffusivity constants (more precisely, a lower bound of one of them) d_1, d_2. The explicit formulas can be put together anyhow by following the various steps of the proofs and the relevant estimates, which are all given in full detail.

2. Particular case

For readers' convenience, we introduce at first a self-contained particular case, in which the main steps of our procedure may be summarized without tedious technical complications. The generalization to the collision contributions (1.5) will be presented in next section.

If we assume $\nu_{3j}^d = 0$, $j = 0, 1, 2$, and $\nu_{20}^d = 0$ in (1.5), system (1.1) takes the form

$$\begin{cases} \partial_t n_1 - d_1 \, \Delta_{\mathbf{x}} n_1 = -2 \, \nu_{11}^r \, (n_1)^2 + 2 \Big(\nu_{21}^d \, n_1 + \nu_{22}^d \, n_2 \Big) n_2, \\ \partial_t n_2 - d_2 \, \Delta_{\mathbf{x}} n_2 = \nu_{11}^r \, (n_1)^2 - \Big(\nu_{21}^d \, n_1 + \nu_{22}^d \, n_2 \Big) n_2, \end{cases} \tag{2.1}$$

so that, after a suitable rescaling, we have to deal with the dimensionless equations

$$\begin{cases} \partial_t n_1 - d_1 \, \Delta_{\mathbf{x}} n_1 = -2 \, (n_1)^2 + 4 \, \alpha \, n_1 \, n_2 + 2 \, \beta \, (n_2)^2 := Q_1^s(n_1, n_2), \\ \partial_t n_2 - d_2 \, \Delta_{\mathbf{x}} n_2 = (n_1)^2 - 2 \, \alpha \, n_1 \, n_2 - \beta \, (n_2)^2 := Q_2^s(n_1, n_2), \end{cases} \tag{2.2}$$

where

$$\alpha = \frac{\nu_{21}^d}{2 \, \nu_{11}^r} > 0, \qquad \beta = \frac{\nu_{22}^d}{\nu_{11}^r} > 0.$$

2.1. Study of the reaction terms

Collision equilibria for the set (2.2) are given by the nonnegative solutions of the equation $\mathcal{Q}_i^s(n_1, n_2) = 0$. It can be trivially checked that

$$\mathcal{Q}_2^s(n_1, n_2) = \Big(n_1 - \gamma \, n_2 \Big) \Big(n_1 + \delta \, n_2 \Big), \tag{2.3}$$

where

$$\gamma = \alpha + \sqrt{\alpha^2 + \beta} > 0, \qquad \delta = -\alpha + \sqrt{\alpha^2 + \beta} > 0, \tag{2.4}$$

hence the physical equilibrium states are characterized by $n_1 = \gamma\, n_2$. Taking into account the conservation property (1.9), since the initial data are known we get that the unique global (homogeneous in space) equilibrium is determined by the positive constants

$$n_1^* = \frac{\gamma}{2+\gamma}\,\bar{n}^0, \qquad\qquad n_2^* = \frac{1}{2+\gamma}\,\bar{n}^0. \tag{2.5}$$

Notice that $\mathcal{Q}_2^s \geq 0 \Leftrightarrow n_1 \geq \gamma\, n_2$, and conversely for $\mathcal{Q}_1^s$.

This suggests that a suitable entropy could be given by

$$E(n_1, n_2) = \int_\Omega \left(\frac{1}{4}\,(n_1)^2 + \frac{\gamma}{2}\,(n_2)^2 \right) \mathrm{d}\mathbf{x}, \tag{2.6}$$

since, in space homogeneous conditions,

$$\partial_t E(n_1, n_2) = -\int_\Omega \left(n_1 - \gamma\, n_2 \right)^2 \left(n_1 + \delta\, n_2 \right) \mathrm{d}\mathbf{x} \leq 0,$$

with $\partial_t E = 0$ only at the local equilibrium $n_1 = \gamma\, n_2$.

2.2. Minimum principle

We now turn to a result of "minimum principle" type for equation (2.2).

Proposition 2.1. *Let $d_1, d_2 > 0$, $\alpha, \beta > 0$, and Ω be a bounded regular (C^2) open set of $\mathbb{R}^N$. Let $(n_1(t, \mathbf{x}), n_2(t, \mathbf{x}))$ be a strong solution (that is, in $C^2(\mathbb{R}^+ \times \bar{\Omega})$) to system (2.2) with Neumann boundary conditions (1.7) and with initial conditions such that*

$$n_1(0, \mathbf{x}) = n_1^0(\mathbf{x}) > c_1 > 0, \qquad n_2(0, \mathbf{x}) = n_2^0(\mathbf{x}) > c_2 > 0. \tag{2.7}$$

This solution $(n_1(t, \mathbf{x}), n_2(t, \mathbf{x}))$ is strictly positive for $(t, \mathbf{x}) \in [0, \infty) \times \Omega$, and satisfies the following lower bounds:

$$n_1(t, \mathbf{x}) \geq k_1, \qquad n_2(t, \mathbf{x}) \geq k_2, \tag{2.8}$$

where

$$k_1 = \min\left\{ c_1, \gamma\, c_2 \right\}, \qquad k_2 = \gamma^{-1}\, k_1, \tag{2.9}$$

with γ given by (2.4).

Proof of Proposition 2.1. The proof shall be carried out following the same lines as in reference [15]. For any $\varepsilon > 0$, let us consider the functions

$$n_1^\varepsilon(t, \mathbf{x}) = n_1(t, \mathbf{x})\, \mathrm{e}^{\varepsilon\, t}, \qquad n_2^\varepsilon(t, \mathbf{x}) = n_2(t, \mathbf{x})\, \mathrm{e}^{\varepsilon\, t}, \tag{2.10}$$

and let us prove that

$$n_1^\varepsilon(t, \mathbf{x}) > k_1, \qquad n_2^\varepsilon(t, \mathbf{x}) > k_2. \tag{2.11}$$

From equations (2.2), it follows that the evolution of n_1^ε, n_2^ε is governed by the system

$$\begin{cases} \partial_t n_1^\varepsilon - d_1\, \Delta_\mathbf{x} n_1^\varepsilon = \left[-2\,(n_1^\varepsilon)^2 + 4\,\alpha\, n_1^\varepsilon\, n_2^\varepsilon + 2\,\beta\,(n_2^\varepsilon)^2 \right] \mathrm{e}^{-\varepsilon\, t} + \varepsilon\, n_1^\varepsilon, \\[2mm] \partial_t n_2^\varepsilon - d_2\, \Delta_\mathbf{x} n_2^\varepsilon = \left[(n_1^\varepsilon)^2 - 2\,\alpha\, n_1^\varepsilon\, n_2^\varepsilon - \beta\,(n_2^\varepsilon)^2 \right] \mathrm{e}^{-\varepsilon\, t} + \varepsilon\, n_2^\varepsilon. \end{cases} \tag{2.12}$$

Suppose that the inequalities (2.11) do not hold for all $(t, \mathbf{x}) \in [0, \infty) \times \Omega$, and define the set B^ε as

$$B^\varepsilon = \left\{ \tau > 0 \,:\, n_1^\varepsilon(t, \mathbf{x}) > k_1, \; n_2^\varepsilon(t, \mathbf{x}) > k_2, \qquad \forall \, (t, \mathbf{x}) \in [0, \tau) \times \Omega \right\}. \tag{2.13}$$

If we denote $\tilde{t} = \sup B^\varepsilon$, there must exist $\tilde{\mathbf{x}} \in \bar{\Omega}$ such that at least one of the following equalities holds:

$$n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = k_1 \qquad \text{or} \qquad n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = k_2.$$

Case 1. $n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = k_1$.

By definitions of $\tilde{t}$ and $\tilde{\mathbf{x}}$, we have $n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq n_1^\varepsilon(\tilde{t}, \mathbf{x})$ $\forall \mathbf{x} \in \Omega$, namely the function $n_1^\varepsilon(\tilde{t}, \mathbf{x})$ takes minimum for $\mathbf{x} = \tilde{\mathbf{x}}$. So, if $\tilde{\mathbf{x}} \in \Omega$, then $d_1 \, \Delta_\mathbf{x} n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq 0$. If $\tilde{\mathbf{x}} \in \partial\Omega$, we can claim again that $d_1 \, \Delta_\mathbf{x} n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq 0$; in fact, if it were $d_1 \, \Delta_\mathbf{x} n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) < 0$ with $n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}})$ minimum, it would follow (see references [15,17]) $\nabla_\mathbf{x} n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \cdot \hat{\mathbf{n}} < 0$, that would contradict Neumann boundary conditions (1.7).

Moreover, by evaluating the chemical contributions in the first line of (2.12) at $(\tilde{t}, \tilde{\mathbf{x}})$, we get

$$
\begin{aligned}
-2 \, (n_1^\varepsilon)^2(\tilde{t}, \tilde{\mathbf{x}}) + 4 \, \alpha \, n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \, n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + 2 \, \beta \, (n_2^\varepsilon)^2(\tilde{t}, \tilde{\mathbf{x}}) &= -2 \, k_1^2 + 4 \, \alpha \, k_1 \, n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + 2 \, \beta \, (n_2^\varepsilon)^2(\tilde{t}, \tilde{\mathbf{x}}) \\
&\geq -2 \, k_1^2 + 4 \, \alpha \, k_1 \, k_2 + 2 \, \beta \, k_2^2 = 0
\end{aligned}
$$

(the inequality holds since $n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq k_2$, and the last line vanishes bearing in mind that $k_1 = \left(\alpha + \sqrt{\alpha^2 + \beta}\right) k_2$).

Consequently, the equation (2.12) for n_1^ε implies that $\partial_t n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq \varepsilon \, n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) > 0$, hence $n_1^\varepsilon(t, \tilde{\mathbf{x}}) < n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = k_1$ for some $t < \tilde{t}$, contradicting the definition of $\tilde{t}$.

Case 2. $n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = k_2$.

In this case, we have $n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq n_2^\varepsilon(\tilde{t}, \mathbf{x})$ $\forall \mathbf{x} \in \Omega$, namely $n_2^\varepsilon(\tilde{t}, \mathbf{x})$ takes its minimum for $\mathbf{x} = \tilde{\mathbf{x}}$. Therefore we get, as above, $d_2 \, \Delta_\mathbf{x} n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq 0$.

As concerns the first term on the right hand side of the second line of (2.12), we obtain

$$
\begin{aligned}
(n_1^\varepsilon)^2(\tilde{t}, \tilde{\mathbf{x}}) - 2 \, \alpha \, n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \, n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) - \beta \, (n_2^\varepsilon)^2(\tilde{t}, \tilde{\mathbf{x}}) &= (n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) - \gamma \, k_2) \, (n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + \delta \, k_2) \\
&\geq (k_1 - \gamma \, k_2) \, (n_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + \delta \, k_2) = 0.
\end{aligned}
$$

It follows $\partial_t n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq \varepsilon \, n_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) > 0$, which leads to a contradiction as in Case 1.

Consequently, the set B^ε is unbounded, hence $n_1^\varepsilon(t, \mathbf{x}) > k_1$ and $n_2^\varepsilon(t, \mathbf{x}) > k_2$ for all $\mathbf{x} \in \Omega$ and for all $t \geq 0$. This means that $n_1(t, \mathbf{x}) > k_1 \, \mathrm{e}^{-\varepsilon t}$ and $n_2(t, \mathbf{x}) > k_2 \, \mathrm{e}^{-\varepsilon t}$, thus, passing to the limit $\varepsilon \to 0$, we have $n_1(t, \mathbf{x}) \geq k_1$ and $n_2(t, \mathbf{x}) \geq k_2$. $\qquad \square$

2.3. Convergence to equilibrium

In this subsection we shall derive an explicit rate of convergence towards the equilibrium state (n_1^*, n_2^*) given in (2.5). Precisely, we shall prove:

Theorem 2.2. *Let $d_1, d_2 > 0$, $\alpha, \beta > 0$ and Ω be a bounded regular (C^2) open set of $\mathbb{R}^N$. Let $(n_1(t, \mathbf{x}), n_2(t, \mathbf{x}))$ be a strong solution (that is, in $C^2(\mathbb{R}^+ \times \bar{\Omega})$) to system (2.2) with Neumann boundary conditions (1.7) and with initial conditions (2.7). Then, this solution satisfies the following property of exponential decay towards equilibrium with explicit constants:*

$$\frac{1}{4} \, \|n_1 - n_1^*\|_2^2 + \frac{\gamma}{2} \, \|n_2 - n_2^*\|_2^2 \leq \left(E(n_1^0, n_2^0) - E(n_1^*, n_2^*) \right) \mathrm{e}^{-\mathcal{C} t}, \tag{2.14}$$

with

$$\mathcal{C} = \min \left\{ \min \left\{ 1, \frac{2 + \gamma}{6} \right\} \frac{d_1}{2 \, P(\Omega)}, \; \min \left\{ \frac{1}{2}, \frac{2 + \gamma}{6 \, \gamma} \right\} \frac{d_2}{P(\Omega)}, \; \frac{2 + \gamma}{6} \, d_3 \right\}, \tag{2.15}$$

where $P(\Omega)$ is the Poincaré constant of Ω, and $d_3 = k_1 + \delta\, k_2 = 2\sqrt{\alpha^2 + \beta}\, k_2$ is the lower bound for $n_1 + \delta\, n_2$ (remember that γ and δ are defined by (2.4)*, and that E is defined by* (2.6)*).*

The proof of this theorem is based on the following functional inequality:

Lemma 2.3 (entropy dissipation). *Let $n_1 := n_1(\mathbf{x})$ and $n_2 := n_2(\mathbf{x})$ be two nonnegative functions of $L^1(\Omega)$ such that*

$$\int_\Omega (n_1(\mathbf{x}) + 2\, n_2(\mathbf{x}))\, \mathrm{d}\mathbf{x} = \bar{n}_0, \qquad and \qquad \forall \mathbf{x} \in \Omega, \quad n_1(\mathbf{x}) + \delta\, n_2(\mathbf{x}) \geq d_3.$$

Then, the entropy dissipation

$$
\begin{aligned}
D(n_1, n_2) \;=\; & \frac{d_1}{2} \int_\Omega |\nabla_{\mathbf{x}} n_1|^2 \mathrm{d}\mathbf{x} + d_2\, \gamma \int_\Omega |\nabla_{\mathbf{x}} n_2|^2 \mathrm{d}\mathbf{x} \\
& + \int_\Omega (n_1 - \gamma\, n_2)\Big((n_1)^2 - 2\,\alpha\, n_1\, n_2 - \beta\, (n_2)^2\Big) \mathrm{d}\mathbf{x}
\end{aligned}
\tag{2.16}
$$

fulfils the inequality

$$D(n_1, n_2) \geq \mathcal{C}\Big[E(n_1, n_2) - E(n_1^*, n_2^*)\Big],\tag{2.17}$$

where the constant $\mathcal{C}$ is defined in (2.15)*.*

Proof of Lemma 2.3*.* For convenience, the proof will be divided into five steps.

Step 1. A direct computation shows that the relative entropy with respect to the equilibrium state (n_1^*, n_2^*) is related to the L^2-distance from the equilibrium itself:

$$E(n_1, n_2) - E(n_1^*, n_2^*) = \frac{1}{4}\, \|n_1 - n_1^*\|_2^2 + \frac{\gamma}{2}\, \|n_2 - n_2^*\|_2^2.\tag{2.18}$$

This ensures that the entropy $E(n_1, n_2)$ takes its minimum for $(n_1, n_2) = (n_1^*, n_2^*)$.

Step 2. By resorting to Poincaré inequality, we have

$$\int_\Omega |\nabla_{\mathbf{x}} n_i|^2 \mathrm{d}\mathbf{x} \geq \frac{1}{P(\Omega)}\, \|n_i - \bar{n}_i\|_2^2 \qquad \text{where} \qquad \bar{n}_i = \int_\Omega n_i\, \mathrm{d}\mathbf{x}.$$

In addition, as concerns the last integral in (2.16) we note that

$$(n_1 - \gamma\, n_2)\Big((n_1)^2 - 2\,\alpha\, n_1\, n_2 - \beta\, (n_2)^2\Big) = (n_1 - \gamma\, n_2)^2(n_1 + \delta\, n_2) \geq d_3(n_1 - \gamma\, n_2)^2.$$

Hence, the following estimate holds for the entropy dissipation:

$$D(n_1, n_2) \geq \frac{d_1}{2\, P(\Omega)}\|n_1 - \bar{n}_1\|_2^2 + \frac{d_2\, \gamma}{P(\Omega)}\|n_2 - \bar{n}_2\|_2^2 + d_3\|n_1 - \gamma\, n_2\|_2^2.\tag{2.19}$$

Step 3. Owing to estimate (2.19) and to the first step, in order to prove Lemma 2.3 it suffices to show that

$$
\begin{aligned}
I \;:=\; & \int_\Omega \left[\frac{d_1}{2\, P(\Omega)}|n_1 - \bar{n}_1|^2 + \frac{d_2\, \gamma}{P(\Omega)}|n_2 - \bar{n}_2|^2 + d_3|n_1 - \gamma\, n_2|^2\right] \mathrm{d}\mathbf{x} \\
\geq \;& \mathcal{C} \int_\Omega \left[\frac{1}{4}\, |n_1 - n_1^*|^2 + \frac{\gamma}{2}\, |n_2 - n_2^*|^2\right] \mathrm{d}\mathbf{x}.
\end{aligned}
\tag{2.20}
$$

Step 4. Using the inequality

$$|n_i - n_i^*|^2 \le 2\Big[|n_i - \bar{n}_i|^2 + |\bar{n}_i - n_i^*|^2\Big],$$

we see that in order to get estimate (2.20), it is enough to prove that

$$I \ge C \int_\Omega \left[\frac{1}{2}|n_1 - \bar{n}_1|^2 + \gamma|n_2 - \bar{n}_2|^2 + \frac{1}{2}|\bar{n}_1 - n_1^*|^2 + \gamma|\bar{n}_2 - n_2^*|^2\right] d\mathbf{x}. \tag{2.21}$$

Since it obviously holds

$$\frac{1}{2}I \ge C_1 \int_\Omega \left[\frac{1}{2}|n_1 - \bar{n}_1|^2 + \gamma|n_2 - \bar{n}_2|^2\right] d\mathbf{x},$$

with

$$C_1 = \min\left\{\frac{d_1}{2\,P(\Omega)}, \frac{d_2}{2\,P(\Omega)}\right\}, \tag{2.22}$$

it remains to prove that

$$\frac{1}{2}I \ge C_2 \left[\frac{1}{2}|\bar{n}_1 - n_1^*|^2 + \gamma|\bar{n}_2 - n_2^*|^2\right], \tag{2.23}$$

and to take $C = \min\{C_1, C_2\}$.

Step 5. Since

$$|\bar{n}_1 - \gamma\,\bar{n}_2|^2 \le 3\Big[|\bar{n}_1 - n_1|^2 + |n_1 - \gamma\,n_2|^2 + \gamma^2|n_2 - \bar{n}_2|^2\Big],$$

we have

$$\frac{1}{3}\,\min\left\{\frac{d_1}{2\,P(\Omega)}, \frac{d_2}{\gamma\,P(\Omega)}, d_3\right\}|\bar{n}_1 - \gamma\,\bar{n}_2|^2 \le I,$$

therefore in order to prove (2.23) it suffices to show that

$$|\bar{n}_1 - \gamma\,\bar{n}_2|^2 \ge \frac{6}{\min\left\{\frac{d_1}{2\,P(\Omega)}, \frac{d_2}{\gamma\,P(\Omega)}, d_3\right\}}\,C_2\left[\frac{1}{2}|\bar{n}_1 - n_1^*|^2 + \gamma|\bar{n}_2 - n_2^*|^2\right]. \tag{2.24}$$

At this point, bearing in mind the expressions of (n_1^*, n_2^*) given in (2.5), together with the fact that $n_1^* + 2\,n_2^* = \bar{n}_1 + 2\,\bar{n}_2 = \bar{n}^0$, we get

$$\bar{n}_2 - n_2^* = -\frac{1}{2}(\bar{n}_1 - n_1^*) \qquad \text{and} \qquad \bar{n}_1 - \gamma\,\bar{n}_2 = \frac{2 + \gamma}{2}(\bar{n}_1 - n_1^*),$$

so that (2.24) becomes

$$|\bar{n}_1 - n_1^*|^2 \ge \frac{6}{2 + \gamma}\,\frac{1}{\min\left\{\frac{d_1}{2\,P(\Omega)}, \frac{d_2}{\gamma\,P(\Omega)}, d_3\right\}}\,C_2\,|\bar{n}_1 - n_1^*|^2,$$

that is true once we put

$$C_2 = \frac{2 + \gamma}{6}\,\min\left\{\frac{d_1}{2\,P(\Omega)}, \frac{d_2}{\gamma\,P(\Omega)}, d_3\right\}. \tag{2.25}$$

Taking $C = \min\{C_1, C_2\}$ concludes the proof of Lemma 2.3. $\qquad\square$

We are now in condition to conclude the

Proof of Theorem 2.2. Let's evaluate the entropy dissipation along the solution of system (2.2):

$$- \partial_t E(n_1, n_2) = - \int_\Omega \left(\frac{n_1}{2} \, \partial_t n_1 + \gamma \, n_2 \, \partial_t n_2 \right) d\mathbf{x}.$$

Taking into account that for Neumann boundary conditions

$$\int_\Omega n_i \, \Delta_\mathbf{x} n_i \, d\mathbf{x} = - \int_\Omega |\nabla_\mathbf{x} n_i|^2 d\mathbf{x},$$

we have

$$- \partial_t E(n_1, n_2) = D(n_1, n_2). \tag{2.26}$$

Owing to Lemma 2.3, we end up with

$$\partial_t \Big[E(n_1, n_2) - E(n_1^*, n_2^*) \Big] \leq - \mathcal{C} \Big[E(n_1, n_2) - E(n_1^*, n_2^*) \Big],$$

thus, by Gronwall's inequality,

$$E(n_1, n_2) - E(n_1^*, n_2^*) \leq \Big(E(n_1^0, n_2^0) - E(n_1^*, n_2^*) \Big) e^{- \mathcal{C} \, t}.$$

Finally, the first step of Lemma 2.3 provides the sought estimate (2.14). □

Remark 2.4. The same procedure could be applied to whatever bi-species reaction-diffusion system in which chemical reaction contributions take the form

$$\mathcal{Q}_2 = \mathsf{C} \, (n_1 - \gamma \, n_2) \, \mathcal{W}(n_1, n_2),$$

with C and γ positive constants and $\mathcal{W}(n_1, n_2)$ (smooth) function satisfying the estimate

$$\mathcal{W}(n_1, n_2) \geq \mathcal{A} \, n_1 + \mathcal{B} \, n_2$$

(where $\mathcal{A}$ and $\mathcal{B}$ are positive constants).

3. MATHEMATICAL STUDY IN THE GENERAL CASE

In this section, we prove Theorem 1.1. In next subsection, we begin by giving a few results about the reaction terms $\mathcal{Q}_i$.

3.1. **Study of the reaction terms**

System (1.1)–(1.2)–(1.5) writes

$$\partial_t n_i - d_i \, \Delta_\mathbf{x} n_i = \mathcal{Q}_i(n_1, n_2) \qquad i = 1, 2, \tag{3.1}$$

where the collision contributions may be rearranged as

$$\mathcal{Q}_1 = - 2 \, \mathcal{Q}_2, \qquad \mathcal{Q}_2 = \frac{1}{\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2} \, \mathcal{F}(n_1, n_2), \tag{3.2}$$

with

$$\mathcal{F}(n_1, n_2) = \nu_{11}^r (n_1)^2 \left(\nu_{30}^i n_0 + \nu_{31}^i n_1 + \nu_{32}^i n_2 \right)$$
$$- n_2 \left(\nu_{20}^d n_0 + \nu_{21}^d n_1 + \nu_{22}^d n_2 \right) \left(\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2 \right). \tag{3.3}$$

This function is homogeneous (of third order) in the variables n_1, n_2, n_0, and it can be seen as a cubic function of the variable n_1:

$$\mathcal{F}(n_1, n_2) = A(n_1)^3 + B(n_1)^2 - C\, n_1 - D,$$

where

$$A = \nu_{11}^r \nu_{31}^i > 0,$$
$$B = \left(\nu_{11}^r \nu_{32}^i - \nu_{31}^t \nu_{21}^d \right) n_2 + \nu_{11}^r \nu_{30}^i n_0,$$
$$C = \left(\nu_{22}^d \nu_{31}^t + \nu_{32}^t \nu_{21}^d \right)(n_2)^2 + \left(\nu_{21}^d \nu_{30}^t + \nu_{31}^t \nu_{20}^d \right) n_2\, n_0 > 0,$$
$$D = \nu_{22}^d \nu_{32}^t (n_2)^3 + \left(\nu_{20}^d \nu_{32}^t + \nu_{22}^d \nu_{30}^t \right)(n_2)^2 n_0 + \nu_{20}^d \nu_{30}^t n_2 (n_0)^2 > 0.$$

So, using the sign of A, B, D, it can be checked that there exists only one admissible (*i.e.* strictly positive) root $n_1 = \mathcal{G}(n_2, n_0)$ of $\mathcal{F}(n_1, n_2) = 0$, with $\mathcal{G}$ homogeneous of order 1 in the variables n_2, n_0, and that

$$\frac{\partial \mathcal{F}}{\partial n_1}(\mathcal{G}(n_2, n_0), n_2) > 0.$$

Therefore, the collision contributions may be rewritten as $\mathcal{Q}_1 = -2\,\mathcal{Q}_2$ and

$$\mathcal{Q}_2 = \frac{1}{\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2} \left[n_1 - \mathcal{G}(n_2, n_0) \right] \mathcal{P}(n_1, n_2, n_0), \tag{3.4}$$

where $\mathcal{P}(n_1, n_2, n_0)$ is an homogeneous function of order 2 that is strictly positive (for $n_1 > 0$, $n_2 > 0$, $n_0 > 0$).

Lemma 3.1. *With the notations above, $\frac{\partial \mathcal{G}}{\partial n_2}(n_2) > 0$ for each $n_2 > 0$ (we skip here the dependence of $\mathcal{G}$ on the background fixed density n_0 to keep reasonable notations).*

Proof of Lemma 3.1. Differentiating the equality

$$\mathcal{F}(\mathcal{G}(n_2), n_2) = 0, \tag{3.5}$$

we get

$$\frac{\partial \mathcal{G}}{\partial n_2}(n_2) = -\frac{\dfrac{\partial \mathcal{F}}{\partial n_2}(\mathcal{G}(n_2), n_2)}{\dfrac{\partial \mathcal{F}}{\partial n_1}(\mathcal{G}(n_2), n_2)}. \tag{3.6}$$

By resorting to (3.5), identity (3.6) is equivalent to

$$\frac{\partial \mathcal{G}}{\partial n_2}(n_2) = -\frac{\dfrac{\partial \mathcal{F}}{\partial n_2}(\mathcal{G}(n_2), n_2) - \dfrac{1}{n_2} \mathcal{F}(\mathcal{G}(n_2), n_2)}{\dfrac{\partial \mathcal{F}}{\partial n_1}(\mathcal{G}(n_2), n_2) - \dfrac{2}{\mathcal{G}(n_2)} \mathcal{F}(\mathcal{G}(n_2), n_2)} =: -\frac{\mathcal{N}}{\mathcal{D}}. \tag{3.7}$$

The numerator of this fraction turns out to be

$$\mathcal{N} = -2\,\nu_{22}^d \nu_{32}^t (n_2)^2 - \nu_{11}^r \nu_{31}^i \frac{\mathcal{G}^3(n_2)}{n_2} - \left(\nu_{22}^d \nu_{31}^t + \nu_{32}^t \nu_{21}^d \right) \mathcal{G}(n_2)\, n_2$$
$$- \left(\nu_{20}^d \nu_{32}^t + \nu_{22}^d \nu_{30}^t \right) n_0\, n_2 - \nu_{11}^r \nu_{30}^i n_0 \frac{\mathcal{G}^2(n_2)}{n_2} < 0. \tag{3.8}$$

As concerns the denominator of fraction (3.7), we obtain analogously

$$
\begin{aligned}
\mathcal{D} = {} & \nu_{11}^r\,\nu_{31}^i\,\mathcal{G}^2(n_2) + 2\,\nu_{22}^d\,\nu_{32}^t\,\frac{(n_2)^3}{\mathcal{G}(n_2)} + \big(\nu_{22}^d\,\nu_{31}^t + \nu_{32}^t\,\nu_{21}^d\big)(n_2)^2 \\
& + \big(\nu_{21}^d\,\nu_{30}^t + \nu_{20}^d\,\nu_{31}^t\big)n_0\,n_2 + 2\big(\nu_{20}^d\,\nu_{32}^t + \nu_{22}^d\,\nu_{30}^t\big)n_0\frac{(n_2)^2}{\mathcal{G}(n_2)} + 2\,\nu_{20}^d\,\nu_{30}^t(n_0)^2\,\frac{n_2}{\mathcal{G}(n_2)} > 0.
\end{aligned}
\tag{3.9}
$$

By inserting results (3.8) and (3.9) into equality (3.7), we get

$$
\frac{\partial \mathcal{G}}{\partial n_2}(n_2) > 0.
\tag{3.10}
$$

$\square$

Lemma 3.2. *If n_1 and n_2 are bounded from above and from below (that is,*

$$
k_1 \le n_1 \le \mathcal{K}_1, \qquad k_2 \le n_2 \le \mathcal{K}_2,
\tag{3.11}
$$

with k_1, k_2, $\mathcal{K}_1$, $\mathcal{K}_2 > 0$), then there exist positive constants g, G, $\tilde{g}$, $\tilde{G}$, p, P such that

$$
g \le \mathcal{G}(n_2, n_0) \le G,
\tag{3.12}
$$

$$
\tilde{g} \le \frac{\partial \mathcal{G}}{\partial n_2}(n_2, n_0) \le \tilde{G},
\tag{3.13}
$$

$$
p \le \mathcal{P}(n_1, n_2, n_0) \le \mathsf{P}.
\tag{3.14}
$$

Proof of (3.12) *of Lemma* 3.2. Since $k_2 \le n_2 \le \mathcal{K}_2$ and $\mathcal{G}$ is increasing with respect to n_2 (we have proved that $\frac{\partial \mathcal{G}}{\partial n_2}(n_2, n_0) > 0$), we immediately get

$$
g = \mathcal{G}(k_2, n_0) \le \mathcal{G}(n_2, n_0) \le \mathcal{G}(\mathcal{K}_2, n_0) = G. \qquad\qquad \square
$$

Proof of (3.13) *of Lemma* 3.2. Since n_2 and $\mathcal{G}(n_2, n_0)$ are bounded from above and from below, we can obtain lower and upper bounds for expressions (3.8) and (3.9), thus, coming back to equality (3.7), also for $\dfrac{\partial \mathcal{G}}{\partial n_2}(n_2, n_0)$. More precisely, $c_N < -\mathcal{N} < C_N$ where

$$
\begin{aligned}
c_N = {} & 2\,\nu_{22}^d\,\nu_{32}^t(k_2)^2 + \nu_{11}^r\,\nu_{31}^i\,\frac{\mathcal{G}^3(k_2)}{\mathcal{K}_2} + \big(\nu_{22}^d\,\nu_{31}^t + \nu_{32}^t\,\nu_{21}^d\big)\mathcal{G}(k_2)\,k_2 \\
& + \big(\nu_{20}^d\,\nu_{32}^t + \nu_{22}^d\,\nu_{30}^t\big)n_0\,k_2 + \nu_{11}^r\,\nu_{30}^i\,n_0\frac{\mathcal{G}^2(k_2)}{\mathcal{K}_2},
\end{aligned}
$$

$$
\begin{aligned}
C_N = {} & 2\,\nu_{22}^d\,\nu_{32}^t(\mathcal{K}_2)^2 + \nu_{11}^r\,\nu_{31}^i\,\frac{\mathcal{G}^3(\mathcal{K}_2)}{k_2} + \big(\nu_{22}^d\,\nu_{31}^t + \nu_{32}^t\,\nu_{21}^d\big)\mathcal{G}(\mathcal{K}_2)\,\mathcal{K}_2 \\
& + \big(\nu_{20}^d\,\nu_{32}^t + \nu_{22}^d\,\nu_{30}^t\big)n_0\,\mathcal{K}_2 + \nu_{11}^r\,\nu_{30}^i\,n_0\frac{\mathcal{G}^2(\mathcal{K}_2)}{k_2},
\end{aligned}
$$

and $c_D < \mathcal{D} < C_D$ where

$$
\begin{aligned}
c_D &= \nu_{11}^r \, \nu_{31}^i \, \mathcal{G}^2(k_2) + 2\,\nu_{22}^d \, \nu_{32}^t \frac{(k_2)^3}{\mathcal{G}(\mathcal{K}_2)} + \big(\nu_{22}^d \, \nu_{31}^t + \nu_{32}^t \, \nu_{21}^d\big)(k_2)^2 \\
&\quad + \big(\nu_{21}^d \, \nu_{30}^t + \nu_{20}^d \, \nu_{31}^t\big)n_0 \, k_2 + 2\big(\nu_{20}^d \, \nu_{32}^t + \nu_{22}^d \, \nu_{30}^t\big)n_0\frac{(k_2)^2}{\mathcal{G}(\mathcal{K}_2)} + 2\,\nu_{20}^d \, \nu_{30}^t (n_0)^2 \frac{k_2}{\mathcal{G}(\mathcal{K}_2)},
\end{aligned}
$$

$$
\begin{aligned}
C_D &= \nu_{11}^r \, \nu_{31}^i \, \mathcal{G}^2(\mathcal{K}_2) + 2\,\nu_{22}^d \, \nu_{32}^t \frac{(\mathcal{K}_2)^3}{\mathcal{G}(k_2)} + \big(\nu_{22}^d \, \nu_{31}^t + \nu_{32}^t \, \nu_{21}^d\big)(\mathcal{K}_2)^2 \\
&\quad + \big(\nu_{21}^d \, \nu_{30}^t + \nu_{20}^d \, \nu_{31}^t\big)n_0 \, \mathcal{K}_2 + 2\big(\nu_{20}^d \, \nu_{32}^t + \nu_{22}^d \, \nu_{30}^t\big)n_0\frac{(\mathcal{K}_2)^2}{\mathcal{G}(k_2)} + 2\,\nu_{20}^d \, \nu_{30}^t (n_0)^2 \frac{\mathcal{K}_2}{\mathcal{G}(k_2)},
\end{aligned}
$$

so that

$$
\tilde{g} = \frac{c_N}{C_D}, \qquad \tilde{G} = \frac{C_N}{c_D}. \qquad\qquad \square
$$

Proof of (3.14) *of Lemma* 3.2. Let us recall that

$$
\Big[n_1 - \mathcal{G}(n_2, n_0)\Big]\mathcal{P}(n_1, n_2, n_0) = \mathcal{F}(n_1, n_2, n_0)
$$

given in (3.5), hence $\mathcal{P}$ may be written in the form

$$
\mathcal{P}(n_1, n_2, n_0) = \Lambda(n_1)^2 + \Upsilon(n_2, n_0)\, n_1 + \Theta(n_2, n_0), \tag{3.15}
$$

with Υ homogeneous of order 1 and Θ homogeneous of order 2. Let us compare the following "polynomial function"

$$
\begin{aligned}
\Big[n_1 - \mathcal{G}(n_2, n_0)\Big]\mathcal{P}(n_1, n_2, n_0) &= \Lambda(n_1)^3 + \Big[\Upsilon(n_2, n_0) - \Lambda\,\mathcal{G}(n_2, n_0)\Big](n_1)^2 \\
&\quad + \Big[\Theta(n_2, n_0) - \Upsilon(n_2, n_0)\,\mathcal{G}(n_2, n_0)\Big]n_1 - \Theta(n_2, n_0)\,\mathcal{G}(n_2, n_0)
\end{aligned}
$$

with the function $\mathcal{F}(n_1, n_2, n_0)$ (see (3.5)). Looking at the coefficient of $(n_1)^3$, we immediately get $\Lambda = \nu_{11}^r \, \nu_{31}^i$. Then, by considering the terms of order 0 in n_1, we have

$$
\Theta(n_2, n_0)\,\mathcal{G}(n_2, n_0) = \nu_{22}^d \, \nu_{32}^t \, (n_2)^3 + \big(\nu_{20}^d \, \nu_{32}^t + \nu_{22}^d \, \nu_{30}^t\big)\,(n_2)^2 \, n_0 + \nu_{20}^d \, \nu_{30}^t \, n_2 \, (n_0)^2,
$$

so there exist positive constants θ_1, θ_2 such that $\theta_1 \leq \Theta(n_2, n_0) \leq \theta_2$; precisely

$$
\begin{aligned}
\theta_1 &= \frac{1}{\mathcal{G}(\mathcal{K}_2, n_0)}\Big[\nu_{22}^d \, \nu_{32}^t \, (k_2)^3 + \big(\nu_{20}^d \, \nu_{32}^t + \nu_{22}^d \, \nu_{30}^t\big)\,(k_2)^2 \, n_0 + \nu_{20}^d \, \nu_{30}^t \, k_2 \, (n_0)^2\Big], \\
\theta_2 &= \frac{1}{\mathcal{G}(k_2, n_0)}\Big[\nu_{22}^d \, \nu_{32}^t \, (\mathcal{K}_2)^3 + \big(\nu_{20}^d \, \nu_{32}^t + \nu_{22}^d \, \nu_{30}^t\big)\,(\mathcal{K}_2)^2 \, n_0 + \nu_{20}^d \, \nu_{30}^t \, \mathcal{K}_2 \, (n_0)^2\Big].
\end{aligned}
$$

Finally, by comparing the coefficients of n_1, we get

$$
\Upsilon(n_2, n_0)\,\mathcal{G}(n_2, n_0) = \Theta(n_2, n_0) + \big(\nu_{22}^d \, \nu_{31}^t + \nu_{32}^t \, \nu_{21}^d\big)(n_2)^2 + \big(\nu_{21}^d \, \nu_{30}^t + \nu_{20}^d \, \nu_{31}^t\big)n_2 \, n_0,
$$

thus there exist positive constants y_1, y_2 such that $y_1 \leq \Upsilon(n_2, n_0) \leq y_2$:

$$y_1 = \frac{1}{\mathcal{G}(\mathcal{K}_2, n_0)} \left[\theta_1 + \left(\nu_{22}^d \, \nu_{31}^t + \nu_{32}^t \, \nu_{21}^d \right)(k_2)^2 + \left(\nu_{21}^d \, \nu_{30}^t + \nu_{20}^d \, \nu_{31}^t \right) k_2 \, n_0 \right],$$

$$y_2 = \frac{1}{\mathcal{G}(k_2, n_0)} \left[\theta_2 + \left(\nu_{22}^d \, \nu_{31}^t + \nu_{32}^t \, \nu_{21}^d \right)(\mathcal{K}_2)^2 + \left(\nu_{21}^d \, \nu_{30}^t + \nu_{20}^d \, \nu_{31}^t \right) \mathcal{K}_2 \, n_0 \right].$$

By inserting these results, together with the bounds of n_1, into (3.15), we get the sought lower and upper bounds for $\mathcal{P}(n_1, n_2, n_0)$:

$$p = \Lambda(k_1)^2 + y_1 \, k_1 + \theta_1, \qquad \mathsf{P} = \Lambda(\mathcal{K}_1)^2 + y_2 \, \mathcal{K}_1 + \theta_2. \qquad \square$$

3.2. **Existence and uniqueness of a strong solution**

Proposition 3.3. *Let $d_1, d_2 > 0$ and Ω be a bounded regular (C^2) open set of $\mathbb{R}^N$. We consider initial data in $C^2(\bar{\Omega})$, compatible with Neumann boundary conditions, and satisfying the bounds (for some strictly positive constants c_1, c_2, $\mathcal{C}_1$ and $\mathcal{C}_2$):*

$$0 < c_1 < n_1(0, \mathbf{x}) < \mathcal{C}_1, \qquad 0 < c_2 < n_2(0, \mathbf{x}) < \mathcal{C}_2. \tag{3.16}$$

Then, for each $T > 0$, there exists a unique (strong) solution $n_1(t, \mathbf{x}), n_2(t, \mathbf{x})$ in $C^2([0, T] \times \bar{\Omega})$ to system (3.1)–(1.7) such that, for $(t, \mathbf{x}) \in [0, T] \times \Omega$,

$$k_1 \leq n_1(t, \mathbf{x}) \leq \mathcal{K}_1, \qquad k_2 \leq n_2(t, \mathbf{x}) \leq \mathcal{K}_2, \tag{3.17}$$

where

$$k_1 = \min \left\{ c_1, \mathcal{G}(c_2, n_0) \right\}, \qquad k_2 = \mathcal{G}^{-1}(k_1, n_0) = \min \left\{ \mathcal{G}^{-1}(c_1, n_0), c_2 \right\}, \tag{3.18}$$

$$\mathcal{K}_1 = \max \left\{ \mathcal{C}_1, \mathcal{G}(\mathcal{C}_2, n_0) \right\}, \qquad \mathcal{K}_2 = \mathcal{G}^{-1}(\mathcal{K}_2, n_0) = \max \left\{ \mathcal{G}^{-1}(\mathcal{C}_1, n_0), \mathcal{C}_2 \right\}. \tag{3.19}$$

Remark 3.4. Given a fixed positive value n_0, we have proved that the function $n_2 \mapsto \mathcal{G}(n_2, n_0)$ is strictly increasing, and moreover it can be checked that $\mathcal{G}(0, n_0) = 0$ and $\lim_{n_2 \to +\infty} \mathcal{G}(n_2, n_0) = +\infty$; consequently, the function $\mathcal{G}^{-1}$ is well defined.

Proof of Proposition 3.3. Initial bounds (3.16) imply of course

$$k_1 < n_1(0, \mathbf{x}) < \mathcal{K}_1, \qquad k_2 < n_2(0, \mathbf{x}) < \mathcal{K}_2. \tag{3.20}$$

At first let us prove that the "maximum principle" holds for $(t, \mathbf{x}) \in [0, T] \times \Omega$.

Lemma 3.5. *Let $\varepsilon > 0$ and Ω be a bounded regular (C^2) open set of $\mathbb{R}^N$. For any $T > 0$, there exists a strong solution $(N_1^\varepsilon(t, \mathbf{x}), N_2^\varepsilon(t, \mathbf{x}))$ (in $C^2([0, T] \times \bar{\Omega})$) to system*

$$\begin{cases} \partial_t N_1^\varepsilon - d_1 \, \Delta_{\mathbf{x}} N_1^\varepsilon = -\dfrac{2 \, \mathcal{P}(N_1^\varepsilon, N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon \, t})}{\nu_{30}^t n_0 \, \mathrm{e}^{\varepsilon \, t} + \nu_{31}^t N_1^\varepsilon + \nu_{32}^t N_2^\varepsilon} \left[N_1^\varepsilon - \mathcal{G}(N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon \, t}) \right] \\[4mm] \partial_t N_2^\varepsilon - d_2 \, \Delta_{\mathbf{x}} N_2^\varepsilon = \dfrac{\mathcal{P}(N_1^\varepsilon, N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon \, t})}{\nu_{30}^t n_0 \, \mathrm{e}^{\varepsilon \, t} + \nu_{31}^t N_1^\varepsilon + \nu_{32}^t N_2^\varepsilon} \left[N_1^\varepsilon - \mathcal{G}(N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon \, t}) \right] \end{cases} \tag{3.21}$$

with Neumann boundary conditions (1.7) and initial conditions

$$N_1^\varepsilon(0, \mathbf{x}) = n_1(0, \mathbf{x}), \qquad N_2^\varepsilon(0, \mathbf{x}) = n_2(0, \mathbf{x}), \tag{3.22}$$

where $n_1(0, \mathbf{x})$, $n_2(0, \mathbf{x})$ satisfy the bounds (3.16). *This solution $(N_1^\varepsilon(t, \mathbf{x}), N_2^\varepsilon(t, \mathbf{x}))$ satisfies the upper bounds:*

$$N_1^\varepsilon(t, \mathbf{x}) < \mathcal{K}_1 \, e^{\varepsilon t}, \qquad N_2^\varepsilon(t, \mathbf{x}) < \mathcal{K}_2 \, e^{\varepsilon t}, \tag{3.23}$$

where $\mathcal{K}_1$, $\mathcal{K}_2$ are the constants defined in (3.19).

Proof of Lemma 3.5. We first suppose to have a strong solution $(N_1^\varepsilon, N_2^\varepsilon)$ (in $C^2([0, T] \times \bar{\Omega})$) to system (3.21) with Neumann boundary conditions (1.7) and initial conditions (3.22). Let us consider the functions

$$\tilde{N}_1^\varepsilon(t, \mathbf{x}) = N_1^\varepsilon(t, \mathbf{x}) \, e^{-\varepsilon t}, \qquad \tilde{N}_2^\varepsilon(t, \mathbf{x}) = N_2^\varepsilon(t, \mathbf{x}) \, e^{-\varepsilon t}, \tag{3.24}$$

and let us prove that

$$\tilde{N}_1^\varepsilon(t, \mathbf{x}) < \mathcal{K}_1, \qquad \tilde{N}_2^\varepsilon(t, \mathbf{x}) < \mathcal{K}_2. \tag{3.25}$$

From equations (3.1), it follows that the evolution of $\tilde{N}_1^\varepsilon$, $\tilde{N}_2^\varepsilon$ is governed by

$$\begin{cases} \partial_t \tilde{N}_1^\varepsilon - d_1 \, \Delta_\mathbf{x} \tilde{N}_1^\varepsilon = -\dfrac{2 \, \mathcal{P}(\tilde{N}_1^\varepsilon, \tilde{N}_2^\varepsilon, n_0) \, e^{\varepsilon t}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon + \nu_{32}^t \tilde{N}_2^\varepsilon} \Big[\tilde{N}_1^\varepsilon - \mathcal{G}(\tilde{N}_2^\varepsilon, n_0) \Big] - \varepsilon \, \tilde{N}_1^\varepsilon \\[4mm] \partial_t \tilde{N}_2^\varepsilon - d_2 \, \Delta_\mathbf{x} \tilde{N}_2^\varepsilon = \dfrac{\mathcal{P}(\tilde{N}_1^\varepsilon, \tilde{N}_2^\varepsilon, n_0) \, e^{\varepsilon t}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon + \nu_{32}^t \tilde{N}_2^\varepsilon} \Big[\tilde{N}_1^\varepsilon - \mathcal{G}(\tilde{N}_2^\varepsilon, n_0) \Big] - \varepsilon \, \tilde{N}_2^\varepsilon. \end{cases} \tag{3.26}$$

Suppose that the inequalities (3.25) do not hold for all $(t, \mathbf{x}) \in [0, T] \times \Omega$, and define

$$B^\varepsilon = \Big\{ \tau > 0 : \; \tilde{N}_1^\varepsilon(t, \mathbf{x}) < \mathcal{K}_1, \quad \tilde{N}_2^\varepsilon(t, \mathbf{x}) < \mathcal{K}_2 \qquad \forall \, (t, \mathbf{x}) \in [0, \tau) \times \Omega \Big\}.$$

If we denote $\tilde{t} = \sup B^\varepsilon$, there must exist $\tilde{\mathbf{x}} \in \bar{\Omega}$ such that at least one of the following equalities holds:

$$\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = \mathcal{K}_1 \qquad \text{or} \qquad \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = \mathcal{K}_2.$$

Case 1. $\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = \mathcal{K}_1$.

By definitions of $\tilde{t}$ and $\tilde{\mathbf{x}}$, we have $\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq \tilde{N}_1^\varepsilon(\tilde{t}, \mathbf{x}) \; \forall \mathbf{x} \in \Omega$, namely the function $\tilde{N}_1^\varepsilon(\tilde{t}, \mathbf{x})$ takes maximum for $\mathbf{x} = \tilde{\mathbf{x}} \in \bar{\Omega}$, and consequently $d_1 \, \Delta_\mathbf{x} \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq 0$ (for more details, see the proof of Prop. 2.1).

Moreover, by evaluating the chemical contribution of equation (3.26) for $\tilde{N}_1^\varepsilon$ at $(\tilde{t}, \tilde{\mathbf{x}})$, we get

$$-\dfrac{2 \, \mathcal{P}(\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \, e^{-\varepsilon \tilde{t}}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + \nu_{32}^t \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}})} \Big[\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) - \mathcal{G}(\tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \Big] =$$

$$-\dfrac{2 \, \mathcal{P}(\mathcal{K}_1, \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \, e^{-\varepsilon \tilde{t}}}{\nu_{30}^t n_0 + \nu_{31}^t \mathcal{K}_1 + \nu_{32}^t \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}})} \Big[\mathcal{K}_1 - \mathcal{G}(\tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \Big] \leq$$

$$-\dfrac{2 \, \mathcal{P}(\mathcal{K}_1, \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \, e^{-\varepsilon \tilde{t}}}{\nu_{30}^t n_0 + \nu_{31}^t \mathcal{K}_1 + \nu_{32}^t \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}})} \Big[\mathcal{K}_1 - \mathcal{G}(\mathcal{K}_2, n_0) \Big] = 0,$$

where the inequality holds since $\tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq \mathcal{K}_2$ and $n_2 \mapsto \mathcal{G}(n_2, n_0)$ is increasing. Therefore, from equation (3.21) for $\tilde{N}_1^\varepsilon$, we get that $\partial_t \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq -\varepsilon \, \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = -\varepsilon \, \mathcal{K}_1 < 0$, hence $\tilde{N}_1^\varepsilon(t, \tilde{\mathbf{x}}) > \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = \mathcal{K}_1$ for some $t < \tilde{t}$, contradicting the definition of $\tilde{t}$.

Case 2. $\tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = \mathcal{K}_2$.

In this case we have $\tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \geq \tilde{N}_2^\varepsilon(\tilde{t}, \mathbf{x})$ $\forall \mathbf{x} \in \Omega$, namely $\tilde{N}_2^\varepsilon(\tilde{t}, \mathbf{x})$ takes its maximum for $\mathbf{x} = \tilde{\mathbf{x}} \in \bar{\Omega}$, therefore $d_2 \Delta_{\mathbf{x}} \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq 0$. For the chemical contributions at $(\tilde{t}, \tilde{\mathbf{x}})$ appearing in the equation for $\tilde{N}_2^\varepsilon$, we get

$$\frac{\mathcal{P}(\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \, \mathrm{e}^{-\varepsilon \tilde{t}}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + \nu_{32}^t \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}})} \left[\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) - \mathcal{G}(\tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), n_0) \right] =$$

$$\frac{\mathcal{P}(\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), \mathcal{K}_2, n_0) \, \mathrm{e}^{-\varepsilon \tilde{t}}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + \nu_{32}^t \mathcal{K}_2} \left[\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) - \mathcal{G}(\mathcal{K}_2, n_0) \right] \leq$$

$$\frac{\mathcal{P}(\tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}), \mathcal{K}_2, n_0) \, \mathrm{e}^{-\varepsilon \tilde{t}}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) + \nu_{32}^t \mathcal{K}_2} \left[\mathcal{K}_1 - \mathcal{G}(\mathcal{K}_2, n_0) \right] = 0,$$

hence we obtain $\partial_t \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) \leq -\varepsilon \, \tilde{N}_2^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = -\varepsilon \, \mathcal{K}_2 < 0$, which leads to a contradiction as in Case 1.

Consequently, the set B^ε is unbounded, hence $\tilde{N}_1^\varepsilon(t, \mathbf{x}) < \mathcal{K}_1$ and $\tilde{N}_2^\varepsilon(t, \mathbf{x}) < \mathcal{K}_2$ for all $\mathbf{x} \in \Omega$ and for all $t \in [0, T]$. This means that $N_1^\varepsilon(t, \mathbf{x}) < \mathcal{K}_1 \, \mathrm{e}^{-\varepsilon t}$ and $N_2^\varepsilon(t, \mathbf{x}) < \mathcal{K}_2 \, \mathrm{e}^{-\varepsilon t}$.

In order to prove the existence of a solution (in $C^2([0, T] \times \bar{\Omega})$) to system (3.21) with Neumann boundary conditions (1.7) and initial conditions (3.22), let us consider a function $\chi := \chi(N_1^\varepsilon, N_2^\varepsilon)$ in C^2 such that $\chi(N_1^\varepsilon, N_2^\varepsilon) = 1$ when $|N_1^\varepsilon|$ and $|N_2^\varepsilon| \leq \mathrm{e}^{\varepsilon T} \max \{\mathcal{K}_1, \mathcal{K}_2\}$, and $\chi(N_1^\varepsilon, N_2^\varepsilon) = 0$ when $|N_1^\varepsilon|$ or $|N_2^\varepsilon| \geq 2 \, \mathrm{e}^{\varepsilon T} \max \{\mathcal{K}_1, \mathcal{K}_2\}$. Denoting by $\mathcal{R}_i$ the right hand side of the i-th equation (3.21), the function

$$\mathbf{f}(N_1^\varepsilon, N_2^\varepsilon) = \left(-2 \, \mathcal{R}_2(N_1^\varepsilon, N_2^\varepsilon) \, \chi(N_1^\varepsilon, N_2^\varepsilon), \; \mathcal{R}_2(N_1^\varepsilon, N_2^\varepsilon) \, \chi(N_1^\varepsilon, N_2^\varepsilon) \right)$$

lies in $C^2 \cap W^{1,\infty}$. Thus, we can resort to a simple fixed point argument (see for instance [8]) that guarantees that, under these assumptions on the function $\mathbf{f}$, there exists a unique strong solution on $[0, T]$ to the problem

$$\begin{cases} \partial_t \mathbf{U} - \mathbf{D} \cdot \Delta_{\mathbf{x}} \mathbf{U} = \mathbf{f}(\mathbf{U}) \\ \hat{\mathbf{n}} \cdot \nabla_{\mathbf{x}} \mathbf{U} = \mathbf{0} \qquad \text{for } \mathbf{x} \in \partial\Omega \\ \mathbf{U}(0, \mathbf{x}) = \mathbf{U_0}(\mathbf{x}), \end{cases} \tag{3.27}$$

where in our case the vector $\mathbf{U} \equiv (N_1^\varepsilon, N_2^\varepsilon)$, and $\mathbf{D}$ stands for the diagonal matrix containing the diffusion coefficients d_1, d_2 as diagonal entries.

As long as $\forall \mathbf{x} \in \Omega$, $|N_1^\varepsilon(t, \mathbf{x})|$ and $|N_2^\varepsilon(t, \mathbf{x})| \leq \mathrm{e}^{\varepsilon T} \max \{\mathcal{K}_1, \mathcal{K}_2\}$, the solution $\mathbf{U}(t, \mathbf{x}) = (N_1^\varepsilon(t, \mathbf{x}), N_2^\varepsilon(t, \mathbf{x}))$ is also a (strong) solution of system (3.21) (in this range $\chi \equiv 1$, hence $\mathbf{f} \equiv (\mathcal{R}_1, \mathcal{R}_2)$), and satisfies therefore the maximum principle (3.23), so that at the end, $|N_1^\varepsilon(t, \mathbf{x})|$ and $|N_2^\varepsilon(t, \mathbf{x})| \leq \mathrm{e}^{\varepsilon T} \max \{\mathcal{K}_1, \mathcal{K}_2\}$ holds up to time T. $\qquad \square$

By using Lemma 3.5, we get a sequence of solutions $\{N_1^\varepsilon\}_{\varepsilon > 0}$ of system (3.21), which is uniformly bounded in $L^\infty([0, T] \times \bar{\Omega})$. Therefore, it converges (up to a subsequence) weakly $*$ in L^∞ to some function n_1. The same holds for $\{N_2^\varepsilon\}_{\varepsilon > 0}$.

Boundedness from above of N_1^ε and N_2^ε implies boundedness (in $[0, T]$) of the polynomial function $\Big[N_1^\varepsilon - \mathcal{G}(N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon t}) \Big] \mathcal{P}(N_1^\varepsilon, N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon t}) = \mathcal{F}(N_1^\varepsilon, N_2^\varepsilon, n_0 \, \mathrm{e}^{\varepsilon t})$ (see (3.3)–(3.4)) appearing in the right hand sides of (3.21). Therefore, using the smoothing properties of the heat equation, we can pass to the limit $\varepsilon \to 0$ in the weak form of equations (3.21), obtaining that the limit functions (n_1, n_2) are weak solutions of the system (3.1). Passing to the limit in (3.23), we get the upper bounds in (3.17).

Since (3.1) is a strict parabolic system, it is clear that (n_1, n_2) lies in $C^2([0, T] \times \bar{\Omega})$ (*cf.* [16]), and is therefore the (unique) solution of system (3.1) satisfying the upper bounds in (3.17).

We now turn to the minimum principle, that is the lower bounds in estimate (3.17). We begin by establishing the following result:

Lemma 3.6. *Let $\varepsilon > 0$ and Ω be a bounded regular (C^2) open set of $\mathbb{R}^N$. Let $(N_1^\varepsilon(t, \mathbf{x}), N_2^\varepsilon(t, \mathbf{x}))$ be a strong solution (that is, in $C^2([0, T] \times \bar{\Omega})$) to system*

$$
\begin{cases}
\partial_t N_1^\varepsilon - d_1 \, \Delta_\mathbf{x} N_1^\varepsilon &= -\dfrac{2 \, \mathcal{P}(N_1^\varepsilon, N_2^\varepsilon, n_0 \, \mathrm{e}^{-\varepsilon t})}{\nu_{30}^t n_0 \, \mathrm{e}^{-\varepsilon t} + \nu_{31}^t N_1^\varepsilon + \nu_{32}^t N_2^\varepsilon} \Big[N_1^\varepsilon - \mathcal{G}(N_2^\varepsilon, n_0 \, \mathrm{e}^{-\varepsilon t}) \Big] \\[4mm]
\partial_t N_2^\varepsilon - d_2 \, \Delta_\mathbf{x} N_2^\varepsilon &= \dfrac{\mathcal{P}(N_1^\varepsilon, N_2^\varepsilon, n_0 \, \mathrm{e}^{-\varepsilon t})}{\nu_{30}^t n_0 \, \mathrm{e}^{-\varepsilon t} + \nu_{31}^t N_1^\varepsilon + \nu_{32}^t N_2^\varepsilon} \Big[N_1^\varepsilon - \mathcal{G}(N_2^\varepsilon, n_0 \, \mathrm{e}^{-\varepsilon t}) \Big]
\end{cases}
\tag{3.28}
$$

with Neumann boundary conditions (1.7) and with initial conditions satisfying the bounds (3.16). This solution $(N_1^\varepsilon(t, \mathbf{x}), N_2^\varepsilon(t, \mathbf{x}))$ satisfies the lower bounds:

$$
k_1 \, \mathrm{e}^{-\varepsilon t} < N_1^\varepsilon(t, \mathbf{x}), \qquad k_2 \, \mathrm{e}^{-\varepsilon t} < N_2^\varepsilon(t, \mathbf{x}),
\tag{3.29}
$$

where k_1, k_2 are the constants defined in (3.18).

Proof of Lemma 3.6. The proof is very similar to the one performed for the maximum principle. We now have to resort to the auxiliary functions

$$
\tilde{N}_1^\varepsilon(t, \mathbf{x}) = N_1^\varepsilon(t, \mathbf{x}) \, \mathrm{e}^{\varepsilon t}, \qquad \tilde{N}_2^\varepsilon(t, \mathbf{x}) = N_2^\varepsilon(t, \mathbf{x}) \, \mathrm{e}^{\varepsilon t},
\tag{3.30}
$$

and to prove that

$$
k_1 < \tilde{N}_1^\varepsilon(t, \mathbf{x}), \qquad k_2 < \tilde{N}_2^\varepsilon(t, \mathbf{x}).
\tag{3.31}
$$

The evolution of $\tilde{N}_1^\varepsilon$, $\tilde{N}_2^\varepsilon$ is now governed by

$$
\begin{cases}
\partial_t \tilde{N}_1^\varepsilon - d_1 \, \Delta_\mathbf{x} \tilde{N}_1^\varepsilon &= -\dfrac{2 \, \mathcal{P}(\tilde{N}_1^\varepsilon, \tilde{N}_2^\varepsilon, n_0) \, \mathrm{e}^{-\varepsilon t}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon + \nu_{32}^t \tilde{N}_2^\varepsilon} \Big[\tilde{N}_1^\varepsilon - \mathcal{G}(\tilde{N}_2^\varepsilon, n_0) \Big] + \varepsilon \, \tilde{N}_1^\varepsilon \\[4mm]
\partial_t \tilde{N}_2^\varepsilon - d_2 \, \Delta_\mathbf{x} \tilde{N}_2^\varepsilon &= \dfrac{\mathcal{P}(\tilde{N}_1^\varepsilon, \tilde{N}_2^\varepsilon, n_0) \, \mathrm{e}^{-\varepsilon t}}{\nu_{30}^t n_0 + \nu_{31}^t \tilde{N}_1^\varepsilon + \nu_{32}^t \tilde{N}_2^\varepsilon} \Big[\tilde{N}_1^\varepsilon - \mathcal{G}(\tilde{N}_2^\varepsilon, n_0) \Big] + \varepsilon \, \tilde{N}_2^\varepsilon,
\end{cases}
\tag{3.32}
$$

and we conclude like in Lemma 3.5. □

Let us now take initial data such that (3.16), and consequently (3.20), hold. Then, there exists a time $T^* > 0$ depending only on $\mathcal{K}_1$, $\mathcal{K}_2$ (and not on ε nor n_1^0, n_2^0) such that (3.28) (with the initial datum $N_1^\varepsilon(0, \mathbf{x}) = n_1(0, \mathbf{x})$, $N_2^\varepsilon(0, \mathbf{x}) = n_2(0, \mathbf{x})$, and the Neumann boundary conditions) admits a strong solution on $[0, T^*]$ satisfying (3.29) and such that

$$
N_1^\varepsilon(t, \mathbf{x}) \leq C, \qquad N_2^\varepsilon(t, \mathbf{x}) \leq C \qquad \text{for } t \in [0, T^*], \quad \mathbf{x} \in \Omega,
$$

where C does not depend on ε (this can be obtained by a standard fixed point argument).

As a consequence, passing to the limit in (3.29), we obtain that $n_1(t, \mathbf{x})$ and $n_2(t, \mathbf{x})$ satisfy the lower bounds in (3.17) on $[0, T^*]$. Since we can apply this argument at time T^* (because the data at this time are still bounded from above by $\mathcal{K}_1$, $\mathcal{K}_2$), we see that the lower bounds in (3.17) hold on $[T^*, 2T^*]$. By induction, they will hold on $[0, T]$, and this concludes the proof of Proposition 3.3. □

By sticking together the solutions on $[0, T]$ (for $T \in \mathbb{R}^+$), we obtain a (unique) solution in $C^2(\mathbb{R}^+ \times \bar{\Omega})$ to system (3.1)–(1.7). This concludes the proof of existence and uniqueness of a solution stated in Theorem 1.1, since the assumption of strictly positive initial data in $C^2(\bar{\Omega})$ compatible with the Neumann boundary conditions (introduced in Thm. 1.1) coincides with assumption (3.16).

3.3. **Entropy functionals and convergence to equilibrium**

A crucial feature of the reaction-diffusion system (3.1) is that it admits a unique collision equilibrium (n_1^*, n_2^*), characterized by the relation $n_1^* = \mathcal{G}(n_2^*, n_0)$, plus the conservation of number of atoms $n_1^* + 2\, n_2^* = \bar{n}^0$. This allows to build up several different Lyapounov functionals.

Let $\varphi(\cdot)$ be a strictly increasing function (regular enough). It can be proved that the following functional is suitable for our problem:

$$E_\varphi = \int_\Omega \left[\frac{1}{2}\, \Psi(n_1) + \Phi(n_2) \right] \mathrm{d}\mathbf{x}, \tag{3.33}$$

with Ψ and Φ such that

$$\Psi'(n_1) = \varphi(n_1), \qquad\qquad \frac{\partial \Phi}{\partial n_2}(n_2, n_0) = \varphi\big(\mathcal{G}(n_2, n_0)\big).$$

Let us explain now why E_φ is indeed suitable:

Dissipation of the functional

$$
\begin{aligned}
D(n_1, n_2) &= -\partial_t E_\varphi(n_1, n_2) = -\int_\Omega \left(\frac{1}{2}\, \Psi'(n_1)\, \partial_t n_1 + \frac{\partial \Phi}{\partial n_2}(n_2, n_0)\, \partial_t n_2 \right) \mathrm{d}\mathbf{x} \\
&= -\frac{d_1}{2} \int_\Omega \varphi(n_1)\, \Delta_\mathbf{x} n_1\, \mathrm{d}\mathbf{x} - d_2 \int_\Omega \varphi\big(\mathcal{G}(n_2, n_0)\big)\, \Delta_\mathbf{x} n_2\, \mathrm{d}\mathbf{x} \\
&\quad - \int_\Omega \left(\frac{1}{2}\, \varphi(n_1)\, \mathcal{Q}_1 + \varphi\big(\mathcal{G}(n_2, n_0)\big)\, \mathcal{Q}_2 \right) \mathrm{d}\mathbf{x} \\
&= \frac{d_1}{2} \int_\Omega \varphi'(n_1) |\nabla_\mathbf{x} n_1|^2 \mathrm{d}\mathbf{x} + d_2 \int_\Omega \varphi'\big(\mathcal{G}(n_2, n_0)\big)\, \frac{\partial \mathcal{G}}{\partial n_2}(n_2, n_0)\, |\nabla_\mathbf{x} n_2|^2\, \mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \frac{\mathcal{P}(n_1, n_2, n_0)}{\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2} \Big[n_1 - \mathcal{G}(n_2, n_0) \Big] \Big[\varphi(n_1) - \varphi\big(\mathcal{G}(n_2, n_0)\big) \Big] \mathrm{d}\mathbf{x} \geq 0,
\end{aligned}
$$

since $\varphi(\cdot)$ is strictly increasing.

Strict coercivity of the Lyapounov functional

$$
\begin{aligned}
E_\varphi(n_1, n_2) - E_\varphi(n_1^*, n_2^*) &= \int_\Omega \left[\frac{1}{2}\, \Psi(n_1) - \frac{1}{2}\, \Psi(n_1^*) + \Phi(n_2) - \Phi(n_2^*) \right] \mathrm{d}\mathbf{x} \\
&= \int_\Omega \left\{ \frac{1}{2} \left[\Psi'(n_1)\big|_{n_1 = n_1^*} (n_1 - n_1^*) + \frac{1}{2}\, \Psi''(\xi)(n_1 - n_1^*)^2 \right] \right. \\
&\quad \left. + \left[\frac{\partial \Phi}{\partial n_2}(n_2, n_0)\bigg|_{n_2 = n_2^*} (n_2 - n_2^*) + \frac{1}{2}\, \frac{\partial^2 \Phi}{\partial n_2^2}(\xi, n_0)(n_2 - n_2^*)^2 \right] \right\} \mathrm{d}\mathbf{x} \\
&= \int_\Omega \left\{ \frac{1}{2}\, \varphi(n_1^*)(n_1 - n_1^*) + \varphi\big(\mathcal{G}(n_2^*, n_0)\big)(n_2 - n_2^*) \right. \\
&\quad \left. + \frac{1}{4}\, \varphi'(\xi)(n_1 - n_1^*)^2 + \frac{1}{2}\, \varphi'\big(\mathcal{G}(\xi, n_0)\big)\, \frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)(n_2 - n_2^*)^2 \right\} \mathrm{d}\mathbf{x} \\
&= \int_\Omega \left\{ \frac{1}{4}\, \varphi'(\xi)(n_1 - n_1^*)^2 + \frac{1}{2}\, \varphi'\big(\mathcal{G}(\xi, n_0)\big)\, \frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)(n_2 - n_2^*)^2 \right\} \mathrm{d}\mathbf{x},
\end{aligned}
$$

because of conservation of total number of atoms

$$\int_\Omega (n_1 + 2\,n_2)\mathrm{d}\mathbf{x} = \bar{n}^0 = \int_\Omega (n_1^* + 2\,n_2^*)\mathrm{d}\mathbf{x}$$

and of the relation $n_1^* = \mathcal{G}(n_2^*, n_0)$. Since $\varphi(\cdot)$ and $n_2 \mapsto \mathcal{G}(n_2, n_0)$ are strictly increasing functions, the relative entropy is strictly positive for each $(n_1, n_2) \neq (n_1^*, n_2^*)$.

From now on we shall stick to the choice $\varphi(s) = s$, already adopted in the simplified case dealt with in Section 2, hence to the corresponding "quadratic" entropy functional:

$$E(n_1, n_2) = \int_\Omega \left[\frac{1}{4}\,(n_1)^2 + \Phi(n_2, n_0) \right] \mathrm{d}\mathbf{x}, \tag{3.34}$$

with

$$\frac{\partial \Phi}{\partial n_2}(n_2, n_0) = \mathcal{G}(n_2, n_0).$$

Lemma 3.7 (entropy dissipation inequality)**.** *Let $\mathcal{P}$ and $\mathcal{G}$ be defined as in Subsection* 3.1 *from $\mathcal{Q}_1$, $\mathcal{Q}_2$ defined by* (1.5)*. We suppose that n_0, $n_1 := n_1(\mathbf{x})$, $n_2 := n_2(\mathbf{x}) \geq 0$ are such that (for some $d_3 > 0$)*

$$d_3 \leq \frac{\mathcal{P}(n_1, n_2, n_0)}{\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2}.$$

We also suppose that estimate (3.13) *holds.*

Then, the entropy dissipation relevant to the quadratic entropy (3.34)*:*

$$\begin{aligned} D(n_1, n_2) &= \frac{d_1}{2} \int_\Omega |\nabla_\mathbf{x} n_1|^2 \mathrm{d}\mathbf{x} + d_2 \int_\Omega \frac{\partial \mathcal{G}}{\partial n_2}(n_2, n_0)\,|\nabla_\mathbf{x} n_2|^2\,\mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \frac{\mathcal{P}(n_1, n_2, n_0)}{\nu_{30}^t n_0 + \nu_{31}^t n_1 + \nu_{32}^t n_2} \left[n_1 - \mathcal{G}(n_2, n_0) \right]^2 \mathrm{d}\mathbf{x} \end{aligned} \tag{3.35}$$

fulfils the inequality:

$$D(n_1, n_2) \geq \mathcal{C}\left[E(n_1, n_2) - E(n_1^*, n_2^*) \right], \tag{3.36}$$

with

$$\mathcal{C} = \min \left\{ \min \left\{ 1, \frac{(2 + \tilde{g})^2}{6(2 + \tilde{G})} \right\} \frac{d_1}{2\,P(\Omega)}, \ \min \left\{ \frac{1}{2}, \frac{(2 + \tilde{g})^2}{6\,\tilde{G}(2 + \tilde{G})} \right\} \frac{d_2\,\tilde{g}}{P(\Omega)\,\tilde{G}}, \ \frac{(2 + \tilde{g})^2}{6(2 + \tilde{G})}\,d_3 \right\}, \tag{3.37}$$

where $P(\Omega)$ is the Poincaré constant.

Proof of Lemma 3.7. For convenience we divide the proof into four steps.

Step 1. Taking into account the bounds (3.13),

$$D(n_1, n_2) \geq \frac{d_1}{2} \int_\Omega |\nabla_\mathbf{x} n_1|^2 \mathrm{d}\mathbf{x} + d_2\,\tilde{g} \int_\Omega |\nabla_\mathbf{x} n_2|^2\,\mathrm{d}\mathbf{x} + d_3 \int_\Omega \left[n_1 - \mathcal{G}(n_2, n_0) \right]^2 \mathrm{d}\mathbf{x}.$$

Then, by resorting to Poincaré inequality,

$$D(n_1, n_2) \geq \frac{d_1}{2\,P(\Omega)}\|n_1 - \bar{n}_1\|_2^2 + \frac{d_2\,\tilde{g}}{P(\Omega)}\|n_2 - \bar{n}_2\|_2^2 + d_3\|n_1 - \mathcal{G}(n_2, n_0)\|_2^2, \tag{3.38}$$

where

$$\bar{n}_i(t) = \int_\Omega n_i(t, \mathbf{x}) \, d\mathbf{x}.$$

Step 2. Considering the relative entropy, we have

$$
\begin{aligned}
E(n_1, n_2) - E(n_1^*, n_2^*) &= \int_\Omega \left\{ \frac{1}{4}(n_1 - n_1^*)^2 + \frac{1}{2}\frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)(n_2 - n_2^*)^2 \right\} d\mathbf{x} \\
&\leq \frac{1}{4}\|n_1 - n_1^*\|_2^2 + \frac{\tilde{G}}{2}\|n_2 - n_2^*\|_2^2.
\end{aligned}
$$

Since

$$|n_i - n_i^*|^2 \leq 2\left[|n_i - \bar{n}_i|^2 + |\bar{n}_i - n_i^*|^2 \right],$$

we see that in order to prove Lemma 3.7, it is enough to show that

$$
\begin{aligned}
J &:= \int_\Omega \left[\frac{d_1}{2\,P(\Omega)}|n_1 - \bar{n}_1|^2 + \frac{d_2\,\tilde{g}}{P(\Omega)}|n_2 - \bar{n}_2|^2 + d_3|n_1 - \mathcal{G}(n_2, n_0)|^2 \right] d\mathbf{x} \\
&\geq \mathcal{C} \int_\Omega \left[\frac{1}{2}|n_1 - \bar{n}_1|^2 + \tilde{G}\,|n_2 - \bar{n}_2|^2 + \frac{1}{2}\,|\bar{n}_1 - n_1^*|^2 + \tilde{G}\,|\bar{n}_2 - n_2^*|^2 \right] d\mathbf{x}.
\end{aligned}
$$

Since it obviously holds

$$\frac{1}{2}\,J \geq \mathcal{C}_1 \int_\Omega \left[\frac{1}{2}\,|n_1 - \bar{n}_1|^2 + \tilde{G}\,|n_2 - \bar{n}_2|^2 \right] d\mathbf{x},$$

with

$$\mathcal{C}_1 = \min\left\{ \frac{d_1}{2\,P(\Omega)}, \frac{d_2\,\tilde{g}}{2\,P(\Omega)\,\tilde{G}} \right\}, \tag{3.39}$$

it remains to prove that

$$\frac{1}{2}\,J \geq \mathcal{C}_2 \left[\frac{1}{2}\,|\bar{n}_1 - n_1^*|^2 + \tilde{G}\,|\bar{n}_2 - n_2^*|^2 \right], \tag{3.40}$$

and to take $\mathcal{C} = \min\{\mathcal{C}_1, \mathcal{C}_2\}$.

Step 3. Notice that

$$
\begin{aligned}
|\bar{n}_1 - \mathcal{G}(\bar{n}_2, n_0)|^2 &\leq 3\left[|\bar{n}_1 - n_1|^2 + |n_1 - \mathcal{G}(n_2, n_0)|^2 + |\mathcal{G}(n_2, n_0) - \mathcal{G}(\bar{n}_2, n_0)|^2 \right] \\
&= 3\left[|\bar{n}_1 - n_1|^2 + |n_1 - \mathcal{G}(n_2, n_0)|^2 + \left| \frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)\,(n_2 - \bar{n}_2) \right|^2 \right] \\
&\leq 3\left[|\bar{n}_1 - n_1|^2 + |n_1 - \mathcal{G}(n_2, n_0)|^2 + \tilde{G}^2|n_2 - \bar{n}_2|^2 \right].
\end{aligned}
$$

Therefore we have

$$\frac{1}{6}\min\left\{ \frac{d_1}{2\,P(\Omega)}, \frac{d_2\,\tilde{g}}{P(\Omega)\,\tilde{G}^2}, d_3 \right\} |\bar{n}_1 - \mathcal{G}(\bar{n}_2, n_0)|^2 \leq \frac{1}{2}\,J.$$

Thus, in order to prove (3.40), it suffices to show that

$$|\bar{n}_1 - \mathcal{G}(\bar{n}_2, n_0)|^2 \geq \frac{6}{\min\left\{ \frac{d_1}{2\,P(\Omega)}, \frac{d_2\,\tilde{g}}{P(\Omega)\,\tilde{G}^2}, d_3 \right\}}\,\mathcal{C}_2 \left[\frac{1}{2}\,|\bar{n}_1 - n_1^*|^2 + \tilde{G}\,|\bar{n}_2 - n_2^*|^2 \right]. \tag{3.41}$$

Step 4. Bearing in mind the preservation of total number of atoms: $\bar{n}_1 + 2\,\bar{n}_2 = \bar{n}^0 = n_1^* + 2\,n_2^*$, we have

$$\bar{n}_1 - n_1^* = -2\,(\bar{n}_2 - n_2^*).$$

Moreover,

$$
\begin{aligned}
|\bar{n}_1 - \mathcal{G}(\bar{n}_2, n_0)|^2 &= \left|\bar{n}_1 - n_1^* + \mathcal{G}(n_2^*, n_0) - \mathcal{G}(\bar{n}_2, n_0)\right|^2 = \left|\bar{n}_1 - n_1^* + \frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)\,(n_2^* - \bar{n}_2)\right|^2 \\
&= \left|2 + \frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)\right|^2 |n_2^* - \bar{n}_2|^2 \geq \left[2 + \tilde{g}\right]^2 |n_2^* - \bar{n}_2|^2.
\end{aligned}
$$

Consequently it suffices to prove that

$$\left[2 + \tilde{g}\right]^2 |n_2^* - \bar{n}_2|^2 \geq \frac{6}{\min\left\{\frac{d_1}{2\,P(\Omega)},\ \frac{d_2\,\tilde{g}}{P(\Omega)\,\tilde{G}^2},\ d_3\right\}}\,\mathcal{C}_2 \left[2 + \tilde{G}\right] |\bar{n}_2 - n_2^*|^2. \tag{3.42}$$

This is true once we put

$$\mathcal{C}_2 = \frac{1}{6} \min\left\{\frac{d_1}{2\,P(\Omega)},\ \frac{d_2\,\tilde{g}}{P(\Omega)\,\tilde{G}^2},\ d_3\right\} \frac{(2 + \tilde{g})^2}{2 + \tilde{G}}.$$

Taking $\mathcal{C} = \min\{\mathcal{C}_1, \mathcal{C}_2\}$ concludes the proof. □

Notice that if we had $\mathcal{G}(n_2, n_0) = \gamma\,n_2$, we would obtain $\tilde{g} = \tilde{G} = \gamma$, hence we would correctly reproduce the constant $\mathcal{C}$ found in the particular case treated in Section 2.

End of the Proof of Theorem 1.1. Owing to Proposition 3.3, we consider the unique solution $n_1 = n_1(t, \mathbf{x})$, $n_2 = n_2(t, \mathbf{x})$ of equations (1.1)–(1.2)–(1.5) together with (1.7). It satisfies the bound (3.17), so that estimates (3.13) and (3.14) are also satisfied (for a suitable $d_3 > 0$). As a consequence, we can use Lemma 3.7 for $n_1(t, \cdot)$ and $n_2(t, \cdot)$. Bearing in mind that $\partial_t E(n_1, n_2) = -D(n_1, n_2)$, we have

$$\partial_t \left[E(n_1, n_2) - E(n_1^*, n_2^*)\right] \leq -\mathcal{C}\left[E(n_1, n_2) - E(n_1^*, n_2^*)\right],$$

thus, by Gronwall inequality,

$$E(n_1, n_2) - E(n_1^*, n_2^*) \leq \left(E(n_1^0, n_2^0) - E(n_1^*, n_2^*)\right) e^{-\mathcal{C} t}.$$

Finally, proceeding like in (3.39), we see that

$$
\begin{aligned}
E(n_1, n_2) - E(n_1^*, n_2^*) &= \int_\Omega \left\{\frac{1}{4}(n_1 - n_1^*)^2 + \frac{1}{2}\frac{\partial \mathcal{G}}{\partial n_2}(\xi, n_0)(n_2 - n_2^*)^2\right\} d\mathbf{x} \\
&\geq \frac{1}{4}\|n_1 - n_1^*\|_2^2 + \frac{\tilde{g}}{2}\|n_2 - n_2^*\|_2^2.
\end{aligned}
$$

This concludes the proof of Theorem 1.1, and defines properly the constants C_1 and C_2. □

Remark 3.8. We end this section by a remark about the case when one diffusivity constant is 0. In this case, the entropy-entropy dissipation estimate (3.36) still holds (and so does the theorem of exponentially fast decay), though with different constants.

This comes out (in the case when $d_1 = 0$) from the following inequality

$$
\begin{aligned}
\int_\Omega |n_1 - \bar{n}_1|^2 \mathrm{d}\mathbf{x} &\leq 3\int_\Omega |n_1 - \mathcal{G}(n_2, n_0)|^2 \mathrm{d}\mathbf{x} + 3\int_\Omega |\mathcal{G}(n_2, n_0) - \overline{\mathcal{G}(n_2, n_0)}|^2 \mathrm{d}\mathbf{x} + 3\int_\Omega |\overline{\mathcal{G}(n_2, n_0)} - \bar{n}_1|^2 \mathrm{d}\mathbf{x} \\
&\leq 6\int_\Omega |n_1 - \mathcal{G}(n_2, n_0)|^2 \mathrm{d}\mathbf{x} + 3\tilde{G}^2 P(\Omega) \int_\Omega |\nabla_\mathbf{x} n_2|^2 \mathrm{d}\mathbf{x}.
\end{aligned}
$$

The case $d_2 = 0$ can be treated in the same way.

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