

APPROXIMATION OF A NONLINEAR ELLIPTIC PROBLEM ARISING
IN A NON-NEWTONIAN FLUID FLOW MODEL IN GLACIOLOGYROLAND GLOWINSKI¹ AND JACQUES RAPPAZ²

Abstract. The main goal of this article is to establish *a priori* and *a posteriori* error estimates for the numerical approximation of some non linear elliptic problems arising in glaciology. The stationary motion of a glacier is given by a non-Newtonian fluid flow model which becomes, in a first two-dimensional approximation, the so-called infinite parallel sided slab model. The approximation of this model is made by a finite element method with piecewise polynomial functions of degree 1. Numerical results show that the theoretical results we have obtained are almost optimal.

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1. INTRODUCTION

In this paper, we consider the stationary motion of an idealized glacier made with an infinite ice mass between two parallel planes. We suppose that the inclination of the glacier is slight and the ice is assumed as being an incompressible viscous fluid. In order to conduct theoretical studies, glaciologists consider often this simplified model, so-called *infinite parallel sided slab*, and described in Blatter [3] for instance. Let us remark that glacier ice is treated as a non-Newtonian fluid with a nonlinear relationship between the rate deformation tensor and the deviatoric stress tensor like in the Bingham model (see [7] and its references). In Blatter's model, the viscosity is an implicit function of the deformation tensor. The mathematical analysis of this problem is done in Colinge–Rappaz [5] and we will adopt in the following, the formulation of this last paper.

If x_1, x_2 are the two Cartesian coordinates in the plane of the glacier occupying the Lipschitzian domain Ω , we will denote by $u(x_1, x_2)$ the horizontal velocity component of the ice at the point $(x_1, x_2) \in \Omega$. After a rescaling of the physical velocity of the ice, u will be satisfying the following equation:

$$-\operatorname{div}(f(|\nabla u|)\nabla u) = p \quad \text{in } \Omega, \quad (1)$$

where p is a hydrostatic pressure force acting on the glacier and f is a function resulting from a constitutive law for the ice. Clearly speaking we have to add some boundary conditions that we take as homogeneous Dirichlet

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conditions for the sake of simplicity:

$$u = 0 \quad \text{on the boundary } \partial\Omega. \tag{2}$$

In Blatter’s model (see [3] for instance), the function $f : (0, +\infty) \rightarrow (0, +\infty)$ is a smooth function implicitly given by the following relationship:

$$f(s)^{-1} = (sf(s))^{\frac{\alpha}{1-\alpha}} + T_0^{\frac{\alpha}{1-\alpha}}, \quad \forall s \in (0, +\infty), \tag{3}$$

where α is a parameter belonging to the open interval $(0, 1)$ and T_0 is a positive number. This ice behavior law is used in several computer codes (see [9] for instance). It is a little different from the one given in Colinge–Rappaz [5] but doesn’t change the conclusions obtained in this last paper. We will begin in Section 2 by proving that f defined in (3) satisfies the following properties:

- (H1) $f \in C^1(0, +\infty)$ and $f'(s) \stackrel{\text{def}}{=} \frac{df}{ds}(s) < 0$ for all $s \in (0, +\infty)$;
- (H2) $\exists \beta, \gamma > 0$ such that $\beta(1+s)^{-\alpha} \leq f(s) \leq \gamma(1+s)^{-\alpha}$, $\forall s \in (0, +\infty)$;
- (H3) the function $g(s) \stackrel{\text{def}}{=} sf(s)$ is such that there exist $\omega, \rho > 0$ satisfying

$$\omega(1+s+r)^{-\alpha}(s-r) \leq g(s) - g(r) \leq \rho(1+s+r)^{-\alpha}(s-r), \quad \forall s \geq r \geq 0.$$

Let us notice that property (H3) is exactly the assumption (A) of Barrett and Liu [2] in which the number $p-2$ is set to $-\alpha$ and the value α taken by Barrett and Liu in [2] is vanishing. A direct consequence is that we will be able to obtain *a priori* error estimates directly from [2].

It is proven in Colinge–Rappaz [5] that with the properties (H1)–(H3) and if $p \in W^{-1, \frac{2-\alpha}{1-\alpha}}(\Omega)$, where Ω is a Lipschitzian domain, problem (1, 2) has a unique solution u in the usual Sobolev space $W_0^{1,2-\alpha}(\Omega)$. Actually this solution minimizes the functional

$$v \rightarrow J(v) = \int_{\Omega} [F(|\nabla v|) - p \cdot v] \, dx, \tag{4}$$

where $F(s)$ is a primitive of $sf(s)$ given by $F(s) = \int_0^s g(\tau) \, d\tau$. Observe that property (H3) implies that $F''(s) \geq \omega(1+2s)^{-\alpha}$ for all s in $(0, \infty)$ so that the functional J is strictly convex. Due to this property, it is also shown that if X_h is a finite dimensional subspace of $W_0^{1,2-\alpha}(\Omega)$ satisfying

$$\lim_{h \rightarrow 0} \inf_{v_h \in X_h} \|v - v_h\|_{W_0^{1,2-\alpha}(\Omega)} = 0 \quad \text{for all } v \in W_0^{1,2-\alpha}(\Omega), \tag{5}$$

then there exists a unique $u_h \in X_h$ which minimizes J on X_h and $\|u - u_h\|_{W_0^{1,2-\alpha}(\Omega)}$ converges to zero when h tends to zero. It is the case when we choose for X_h the classical finite element subspace of piecewise polynomial functions of degree 1 on a regular triangulation Υ_h of $\bar{\Omega}$, vanishing on $\partial\Omega$.

The main goal of this paper is to establish some results concerning *a priori* and *a posteriori* error estimates for $\|u - u_h\|$ in the $W_0^{1,2-\alpha}(\Omega)$ – norm or in other norms.

For the approximations and error estimates of degenerate or non-degenerate quasilinear elliptic problems, we refer to the Barrett–Liu’s paper [2] and all its references. Let us still mention that the error estimates given in [2] improve the ones for the p -Laplacian established originally by Glowinski–Marrocco [6]. We also establish *a posteriori* error estimates in a same way as in Baranger–El Amri [1].

Finally remark that Liu–Yan [8] have recently improved, in some particular cases, *a posteriori* error estimates for p -Laplacian. To do this, they extend a quasi-norm technique which is not trivial and not applied in this paper for obtaining *a posteriori* error estimates. However we numerically compare our estimator with the Liu–Yan’s one.

An outline of this paper is as follows. In Section 2 we show that the function f , implicitly given by (3), satisfies properties (H1), (H2) and (H3) and consequently error estimates for $\|u - u_h\|$ in different norms are obtained from [2]. Sections 3 and 4 are devoted to *a priori* and *a posteriori* error estimates respectively. In Section 5 we give some numerical results that confirm these results.

2. ABOUT THE FUNCTION f

In this paragraph we show that the function f is well defined by (3) and that properties (H1), (H2) and (H3) are fulfilled when $\alpha \in (0, 1)$ in the definition of f .

Lemma 1. *Function f is well defined by (3) and (H1), (H2) hold.*

Proof. Let s be fixed in the open interval $(0, \infty)$ and consider the two following functions:

$$T(y) \stackrel{\text{def}}{=} (sy)^{\frac{\alpha}{1-\alpha}} + T_0^{\frac{\alpha}{1-\alpha}}, \quad R(y) \stackrel{\text{def}}{=} 1/y \quad \text{where } y \in (0, +\infty).$$

Clearly speaking, since $\alpha \in (0, 1)$, then T is an increasing continuous function on $(0, +\infty)$ and its graph possesses exactly one intersection point with the graph of R . We call $y \stackrel{\text{def}}{=} f(s)$ the abscissa of this intersection point and in this way, $f(s)$ satisfies relationship (3). We have $f(s) > 0$ when $s \in (0, +\infty)$.

By differentiating (3) with respect to s , we obtain:

$$f'(s) \left[\frac{\alpha}{1-\alpha} s (sf(s))^{\frac{2\alpha-1}{1-\alpha}} + \frac{1}{f(s)^2} \right] = -\frac{\alpha}{1-\alpha} f(s) (sf(s))^{\frac{2\alpha-1}{1-\alpha}} \tag{6}$$

and consequently $f'(s) < 0, \forall s \in (0, +\infty)$ which proves that (H1) holds.

Equality (3) shows immediately that $\frac{1}{f(s)} \geq (sf(s))^{\frac{\alpha}{1-\alpha}}$ and $\frac{1}{f(s)} \geq T_0^{\frac{\alpha}{1-\alpha}}$. It follows that $f(s) \leq s^{-\alpha}$ and $f(s) \leq T_0^{-\frac{\alpha}{1-\alpha}}$ and consequently the upper bound of (H2) is proven. By using this inequality in (3) we obtain:

$$\frac{1}{f(s)} \leq T_0^{\frac{\alpha}{1-\alpha}} + s^\alpha \tag{7}$$

which implies the lower bound of (H2). □

Lemma 2. *If $g(s) = sf(s)$, then g satisfies (H3).*

Proof. We have from (3):

$$\frac{s}{g(s)} = g(s)^{\frac{\alpha}{1-\alpha}} + T_0^{\frac{\alpha}{1-\alpha}}. \tag{8}$$

By differentiating (8) with respect to s we obtain:

$$g'(s) = \left[\frac{\alpha}{1-\alpha} g(s)^{\frac{\alpha}{1-\alpha}} + \frac{s}{g(s)} \right]^{-1}. \tag{9}$$

Since $g'(s) \geq 0$, then $g(s)^{\frac{\alpha}{1-\alpha}}$ is an increasing function. Moreover $sg(s)^{-1} = f(s)^{-1}$ is also an increasing function because $f'(s) < 0$. It suffices to consider (9) to see that g' is decreasing. Now we use (9) together with (H2) in order to obtain a lower bound on g' :

$$g'(s) \geq \left[\frac{\alpha}{1-\alpha} \gamma^{\frac{\alpha}{1-\alpha}} + \beta^{-1} \right]^{-1} (1+s)^{-\alpha}. \tag{10}$$

Now we define the function

$$\Psi(r, s) = (g(s) - g(r)) (s - r)^{-1} (1 + r + s)^\alpha.$$

If we prove there exist two positive constants λ_1 and λ_2 satisfying

$$\lambda_1 \leq \Psi(r, s) \leq \lambda_2 \quad \text{for all } r, s \in (0, \infty), \quad (11)$$

then property (H3) will be satisfied. Moreover, since $\Psi(r, s) = \Psi(s, r)$, we can consider in the following only the case $s > r$.

In order to obtain the lower bound λ_1 , it suffices to write:

$$g(s) - g(r) = g'(\xi) (s - r)$$

where ξ belongs to the interval with extremities r and s . From (10) and because g' is decreasing, we obtain

$$\begin{aligned} g(s) - g(r) &\geq \left[\frac{\alpha}{1-\alpha} \gamma^{\frac{\alpha}{1-\alpha}} + \beta^{-1} \right]^{-1} (1+s)^{-\alpha} (s-r) \\ &\geq \left[\frac{\alpha}{1-\alpha} \gamma^{\frac{\alpha}{1-\alpha}} + \beta^{-1} \right]^{-1} (1+r+s)^{-\alpha} (s-r) \end{aligned} \quad (12)$$

which implies the lower bound $\lambda_1 = \left[\frac{\alpha}{1-\alpha} \gamma^{\frac{\alpha}{1-\alpha}} + \beta^{-1} \right]^{-1}$.

In order to prove the upper bound, we use properties (H1, H2) and we verify the following relationships:

$$\begin{aligned} \Psi(r, s) &= (sf(s) - rf(r)) (s - r)^{-1} (1 + r + s)^\alpha \\ &= \left[f(s) + r(f(s) - f(r)) (s - r)^{-1} \right] (1 + r + s)^\alpha \\ &\leq f(s) (1 + r + s)^\alpha \leq f(s) (1 + 2s)^\alpha \leq 2^\alpha \gamma. \end{aligned} \quad \square$$

3. APPROXIMATIONS AND A PRIORI ERROR ESTIMATES

In this section we set $X = W_0^{1,2-\alpha}(\Omega)$ and the natural Sobolev norm in X will be denoted by $\|\cdot\|_{W^{1,2-\alpha}(\Omega)}$ which is equivalent to $\|\cdot\| = (\int_\Omega |\nabla \cdot|^{2-\alpha} dx)^{1/(2-\alpha)}$. The weak formulation of problem (1, 2) corresponding to the Euler equation of the minimization of J on X is the following one: we are looking for $u \in X$ satisfying

$$\int_\Omega f(|\nabla u|) \nabla u \cdot \nabla v \, dx = \int_\Omega p v \, dx \quad \text{for all } v \in X. \quad (13)$$

In order to establish a Galerkin approximation of (13) we choose a family of finite dimensional subspaces X_h of X satisfying

$$\lim_{h \rightarrow 0} \min_{v_h \in X_h} \|v - v_h\| = 0, \quad \forall v \in X, \quad (14)$$

and we are looking for $u_h \in X_h$ such that:

$$\int_\Omega f(|\nabla u_h|) \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega p v_h \, dx \quad \text{for all } v_h \in X_h. \quad (15)$$

By subtracting (15) to (13), we obtain:

$$\int_{\Omega} (f(|\nabla u|) \nabla u - f(|\nabla u_h|) \nabla u_h) \cdot \nabla v_h \, dx = 0 \quad \text{for all } v_h \in X_h. \tag{16}$$

As a consequence we will have the following relationship:

$$\begin{aligned} \int_{\Omega} (f(|\nabla u|) \nabla u - f(|\nabla u_h|) \nabla u_h) \cdot (\nabla u - \nabla u_h) \, dx = \\ \int_{\Omega} (f(|\nabla u|) \nabla u - f(|\nabla u_h|) \nabla u_h) \cdot (\nabla u - \nabla v_h) \, dx \quad \text{for all } v_h \in X_h. \end{aligned} \tag{17}$$

Now let us introduce a quasi-norm $|\cdot|_{\alpha}$ as in [2] by

$$|v|_{\alpha} = \left\| (1 + |\nabla u| + |\nabla v|)^{-\alpha/2} |\nabla v| \right\|_{L^2(\Omega)}, \tag{18}$$

where $\|\cdot\|_{L^2(\Omega)}$ is the quadratic norm and u is the solution of (13). $|v|_{\alpha}$ is not a norm because the homogeneity property is missing. However the other properties of the norm are true. From this definition together with properties (17) and (H3) we obtain:

Theorem 1. *We assume that the function f satisfies (3) and let u and u_h be the solutions of (13) and (15) respectively. Then there exists a constant C such that the following error estimate holds:*

$$|u - u_h|_{\alpha} \leq C |u - v_h|_{\alpha}, \quad \forall v_h \in X_h. \tag{19}$$

Proof. We can find the proof of this result in [2], Theorem 1 and we don't completely repeat it here. Let us just mention a sketch of the proof that it is essentially a consequence of property (H3).

Since (H3) is satisfied (see Lem. 2), then the following relationships are true (see for instance [2]):

$$\langle f(|\xi|) \xi - f(|\eta|) \eta, \xi - \eta \rangle \geq \chi (1 + |\xi| + |\eta|)^{-\alpha} \cdot |\xi - \eta|^2 \tag{20}$$

and

$$|f(|\xi|) \xi - f(|\eta|) \eta| \leq C (1 + |\xi| + |\eta|)^{-\alpha} |\xi - \eta|, \tag{21}$$

for all $\xi, \eta \in R^2$, where $\langle \cdot, \cdot \rangle$ is the scalar product and $|\cdot|$ is the Euclidian norm in R^2 . Since, for $\xi, \eta \in R^2$ we have the obvious inequalities $\frac{1}{2} (|\xi| + |\eta|) \leq |\xi| + |\xi - \eta| \leq 2 (|\xi| + |\eta|)$, we obtain by using (18) with $v = u - u_h$ and successively (20, 17, 21):

$$\begin{aligned} |u - u_h|_{\alpha}^2 &= \int_{\Omega} (1 + |\nabla u| + |\nabla u - \nabla u_h|)^{-\alpha} |\nabla u - \nabla u_h|^2 \, dx \\ &\leq C \int_{\Omega} (f(|\nabla u|) \nabla u - f(|\nabla u_h|) \nabla u_h) \cdot (\nabla u - \nabla u_h) \, dx \\ &\leq C \int_{\Omega} (f(|\nabla u|) \nabla u - f(|\nabla u_h|) \nabla u_h) \cdot (\nabla u - \nabla v_h) \, dx \\ &\leq C \int_{\Omega} (1 + |\nabla u| + |\nabla u_h|)^{-\alpha} |\nabla u - \nabla u_h| |\nabla u - \nabla v_h| \, dx \\ &\leq C \int_{\Omega} (1 + |\nabla u| + |\nabla(u - u_h)|)^{\alpha} |\nabla(u - u_h)| \cdot |\nabla(u - v_h)| \, dx. \end{aligned} \tag{22}$$

By using inequalities $(1+a+r)^{-\alpha} rs \leq \varepsilon(1+a+r)^{-\alpha} r^2 + \frac{1}{\varepsilon}(1+a+s)^{-\alpha} s^2$ for all $a, r, s \geq 0$ and $\varepsilon \in]0, 1[$, by setting $a = |\nabla u|$, $r = |\nabla(u - u_h)|$, $s = |\nabla(u - v_h)|$ and by choosing ε small enough, we obtain

$$|u - u_h|_{\alpha}^2 \leq C |u - v_h|_{\alpha}^2$$

where C denotes a generic constant independent of the parameter h . \square

In order to obtain error estimates in classical Sobolev spaces, we can use Holder inequalities applying to the quasi-norm $|\cdot|_{\alpha}$ as in [2]. If $\sigma \in [2 - \alpha, 2]$ we have:

$$M \|\nabla v\|_{L^{2-\alpha}(\Omega)}^2 \leq |v|_{\alpha}^2 \leq C \|\nabla v\|_{L^{\sigma}(\Omega)}^{\sigma}, \quad (23)$$

for all $v \in W_0^{1,\sigma}(\Omega)$ satisfying $\|\nabla v\|_{L^{2-\alpha}(\Omega)} \leq D$, where D, C, M are positive constants (M is depending on D). From (23), together with the fact that $\lim_{h \rightarrow 0} \|u - u_h\|_{W_0^{1,2-\alpha}(\Omega)} = 0$ and Theorem 1, we can easily prove the following result:

Theorem 2. *We assume that the function f satisfies (3) and let u and u_h be the solutions of (13) and (15) respectively. If $\sigma \in [2 - \alpha, 2]$, if $u \in W_0^{1,\sigma}(\Omega)$ and if $X_h \subset W_0^{1,\sigma}(\Omega)$ then there exists a constant C such that the following error estimate holds:*

$$\|\nabla(u - u_h)\|_{L^{2-\alpha}(\Omega)} \leq C \|\nabla(u - v_h)\|_{L^{\sigma}(\Omega)}^{\sigma/2}, \quad \forall v_h \in X_h. \quad (24)$$

Remark 1. Assume that X_h is a finite element subspace of $H_0^1(\Omega)$ made of piecewise polynomial functions of degree 1 on each triangle of a regular triangulation Υ_h of $\bar{\Omega}$. It is well known (see Ciarlet [4]) that we have the following error estimates when $\sigma \in [2 - \alpha, 2]$:

$$\min_{v_h \in X_h} \|\nabla u - \nabla v_h\|_{L^{\sigma}(\Omega)} \leq Ch \quad \text{if } u \in W^{2,\sigma}(\Omega)$$

where h is the maximum of diameters of triangles contained in Υ_h .

Following Theorem 2, we will obtain in this case:

$$\|u - u_h\|_{W^{1,2-\alpha}(\Omega)} \leq Ch^{1-\alpha/2} \quad \text{if } u \in W^{2,2-\alpha}(\Omega), \quad (25)$$

and

$$\|u - u_h\|_{W^{1,2-\alpha}(\Omega)} \leq Ch \quad \text{if } u \in H^2(\Omega). \quad (26)$$

Let us remark that with the same techniques used in Glowinski–Marrocco [6] for the p -Laplacian problem with $p = 2 - \alpha$, we can prove that function f given by (3) satisfies the two following inequalities:

$$\langle f(|\xi|)\xi - f(|\eta|)\eta, \xi - \eta \rangle \geq \chi \min(|\xi|^{-\alpha}, |\eta|^{-\alpha}, 1) \cdot |\xi - \eta|^2 \quad (27)$$

and

$$|f(|\xi|)\xi - f(|\eta|)\eta| \leq C |\xi - \eta|^{1-\alpha}, \quad (28)$$

for all $\xi, \eta \in R^2$, where $\langle \cdot, \cdot \rangle$ is the scalar product and $|\cdot|$ is the Euclidian norm in R^2 .

From (17), (27) and (28) we easily prove by using a Holder inequality that the following estimate holds:

$$\|\nabla(u - u_h)\|_{L^{2-\alpha}(\Omega)} \leq C \|\nabla(u - v_h)\|_{L^{2-\alpha}(\Omega)}^{1/(1+\alpha)}, \quad \forall v_h \in X_h. \quad (29)$$

As a consequence, we obtain with this argument an error estimate given by $\|\nabla(u - u_h)\|_{L^{2-\alpha}(\Omega)} \leq h^{1/(1+\alpha)}$ if $u \in W^{2,2-\alpha}(\Omega)$, which is lesser order than the one in (25) since $1/(1 + \alpha) \leq 1 - \alpha/2$.

4. APPROXIMATIONS AND A POSTERIORI ERROR ESTIMATES

In this paragraph, we assume that X_h is a finite element subspace of $H_0^1(\Omega)$ made of piecewise polynomial functions of degree 1 on each triangle K of a regular triangulation Υ_h of Ω . The parameter h represents the maximum of diameters of triangles contained in Υ_h . In order to establish a posteriori error estimates, we follow the paper by Baranger and El Amri [1], in which we can find residuals error estimates for the p -Laplacian model.

Let us define the residual quantity $R \in W^{-1, \frac{2-\alpha}{1-\alpha}}(\Omega)$ by

$$\langle R, v \rangle_{X'X} = \int_{\Omega} f(|\nabla u_h|) \nabla u_h \cdot \nabla v \, dx - \int_{\Omega} p v \, dx \quad \forall v \in X,$$

where here, $\langle \cdot, \cdot \rangle_{X'X}$ is the duality pairing between $X' = W^{-1, \frac{2-\alpha}{1-\alpha}}(\Omega)$ and $X = W_0^{1,2-\alpha}(\Omega)$.

As in Baranger–El Amri [1], we obtain by integrating by parts:

$$\begin{aligned} \langle R, v \rangle_{X'X} &= \sum_{K \in \Upsilon_h} \left(\int_K f(|\nabla u_h|) \nabla u_h \cdot \nabla v \, dx - \int_K p v \, dx \right) \\ &= \sum_{K \in \Upsilon_h} \left(- \int_K \operatorname{div} (f(|\nabla u_h|) \nabla u_h) v \, dx - \int_K p v \, dx + \int_{\partial K} f(|\nabla u_h|) \frac{\partial u_h}{\partial n} v \, ds \right). \end{aligned}$$

Since $\langle R, v_h \rangle_{X'X} = 0$ for all $v_h \in X_h$, we write:

$$\begin{aligned} \langle R, v \rangle_{X'X} &= \sum_{K \in \Upsilon_h} \left(- \int_K \operatorname{div} (f(|\nabla u_h|) \nabla u_h) (v - \pi_h v) \, dx \right. \\ &\quad \left. - \int_K p (v - \pi_h v) \, dx + \int_{\partial K} f(|\nabla u_h|) \frac{\partial u_h}{\partial n} (v - \pi_h v) \, ds \right), \end{aligned}$$

where π_h is the Clement’s interpolation operator.

By using the approximation properties of π_h we prove by following the same arguments given by Baranger–El Amri [1] that:

$$\|R\|_{X'} = \sup_{\|v\|=1} \langle R, v \rangle_{X'X} \leq C \left(\sum_{K \in \Upsilon_h} \eta^m(K) \right)^{1/m} \quad \text{with } m = \frac{2-\alpha}{1-\alpha};$$

where the estimator $\eta(K)$ is given by

$$\eta(K) = \left\{ h_K^m \|\operatorname{div} (f(|\nabla u_h|) \nabla u_h) + p\|_{L^m(K)}^m + \sum_{t \in \partial K} h_t \left\| \left[f(|\nabla u_h|) \frac{\partial u_h}{\partial n} \right]_t \right\|_{L^m(\partial K)}^m \right\}^{1/m}.$$

In the above expression, h_K is the diameter of K , h_t is the length of the side $t \in \partial K$ and $[\cdot]_t$ denotes the jump through the side t of the triangle K . If the measure of $\partial K \cap \partial\Omega$ is not vanishing, the jump $[\cdot]_t$ is only defined by the internal value in Ω . At this point, let us remark that the power of h_K in the estimator $\eta(K)$ of [1] contains a misprint.

Clearly speaking in our estimator $\eta(K)$, since ∇u_h is constant on each triangle K , the term $\operatorname{div} (f (|\nabla u_h|) \nabla u_h)$ is vanishing and we obtain:

$$\eta(K) = \left\{ h_K^m \|p\|_{L^m(K)}^m + \sum_{t \in \partial K} h_t \left\| \left[f (|\nabla u_h|) \frac{\partial u_h}{\partial n} \right]_t \right\|_{L^m(\partial K)}^m \right\}^{1/m}.$$

Now we are able to prove the following *a posteriori* error estimate:

Theorem 3. *We assume that the function f satisfies (3) and let u and u_h be the solutions of (13) and (15) respectively. Then there exists a constant C such that:*

$$\|\nabla (u - u_h)\|_{L^{2-\alpha}(\Omega)} \leq C \left(\sum_{K \in \mathcal{T}_h} \eta^m(K) \right)^{1/m}, \quad \text{with } m = \frac{2-\alpha}{1-\alpha}.$$

Proof. We have to link the error $\|u - u_h\|$ to $\|R\|_{X'}$. From the definition of R and since u is the solution of problem (1, 2), we have:

$$\langle R, v \rangle_{X'X} = \int_{\Omega} (f (|\nabla u_h|) \nabla u_h - f (|\nabla u|) \nabla u) \cdot \nabla v \, dx$$

and consequently

$$\|R\|_{X'} = \|f (|\nabla u_h|) \nabla u_h - f (|\nabla u|) \nabla u\|_{L^m(\Omega)}.$$

From (22) and (23) we obtain:

$$\begin{aligned} \|\nabla (u - u_h)\|_{L^{2-\alpha}(\Omega)}^2 &\leq C \int_{\Omega} (f (|\nabla u|) \nabla u - f (|\nabla u_h|) \nabla u_h) \cdot (\nabla u - \nabla u_h) \, dx \\ &\leq C \|f (|\nabla u_h|) \nabla u_h - f (|\nabla u|) \nabla u\|_{L^m(\Omega)} \cdot \|\nabla (u - u_h)\|_{L^{2-\alpha}(\Omega)}. \end{aligned}$$

It follows that $\|\nabla (u - u_h)\|_{L^{2-\alpha}(\Omega)} \leq C \|R\|_{X'}$. □

5. NUMERICAL RESULTS

In this paragraph, we give some numerical results in order to illustrate the *a priori* and *a posteriori* error estimates results obtained in Sections 3 and 4.

A priori error estimates

We start by showing that the error estimate (26) we obtained in Section 3 is optimal if the solution is regular (say $u \in H^2(\Omega)$). However we will see that the error estimate (25) is not optimal in the case of non-regular solutions. To do this, we have chosen $\alpha = 1/2$ which leads to the explicit function:

$$f(s) = 2 \left[T_0 + \sqrt{T_0^2 + 4s} \right]^{-1}.$$

Starting from a given function $u(x_1, x_2)$, we can compute

$$p = -\operatorname{div} (f (|\nabla u|) \nabla u).$$

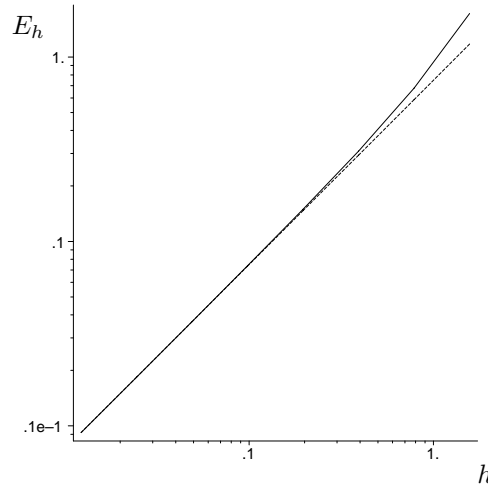


FIGURE 1. *A priori* error estimate E_h in function of h when $u(x_1, x_2) = \sin(x_1) \sin(x_2)$. (The dotted line is a straightline with slope 1.)

After solving approximate problem (15) with that function p , we can plot in a log-log diagram the *a priori* error E_h defined by

$$E_h = \|\nabla u - \nabla u_h\|_{L^{3/2}(\Omega)}.$$

The slope of the graph of E_h as a function of h in log-log scale gives the rate of convergence.

As an example we chose the domain $\Omega = (0, \pi) \times (0, \pi)$ and the function $u(x_1, x_2) = (\sin(x_1) \sin(x_2))$ which is in the space $H^2(\Omega)$. The square Ω is divided into $N \times N$ equal squares and each square is splitted into two triangles by its diagonal with direction (1,1). In Figure 1, we represent the *a priori* error in a log-log scale when the approximation space X_h is the finite element space of degree 1 on this triangulation. We observe a rate of convergence of order h like predicted by (26).

If we choose the function $u(x_1, x_2) = (\sin(x_1) \sin(x_2))^{1.34}$ which is in the space $W_0^{2, \frac{3}{2}}(\Omega)$ but not in $W_0^{2, 1.6}(\Omega)$, we observe in Figure 2 a rate of convergence of order $h^{0.95}$ which shows that the error estimate (25) is not optimal. In fact error estimate (25) gives in this case a rate of convergence of order $h^{\frac{3}{4}} = h^{0.75}$ which is less accurate than $h^{0.95}$.

A posteriori error estimates

As above, we consider the case $\alpha = 1/2$ which allows us to obtain an explicit expression for $f(s)$. We *a priori* fix the solution u and we compute the right-hand side p which corresponds to u . On the boundary $\partial\Omega$ of the unit square $\Omega = (0, 1) \times (0, 1)$ we prescribe a Dirichlet condition (not necessary homogeneous) for u which doesn't change our theoretical results.

In Table 1 we can see the true error $E_h = \|\nabla u - \nabla u_h\|_{L^{3/2}(\Omega)}$, the estimated error $E_s = (\sum_{K \in \mathcal{T}_h} \eta^3(K))^{1/3}$ and the effectivity index E_s/E_h in the case where $u = x_1^2 + x_2^2$. As above, the square Ω has been divided into $N \times N$ equal squares and each square has been splitted into two triangles by its diagonal with direction (1,1). We set $h = 1/N$.

As we observe in Table 1, the effectivity index is very close to 2 and doesn't depend on h . Except the factor 2, our estimated error E_s seems to be a good estimation of the error E_h .

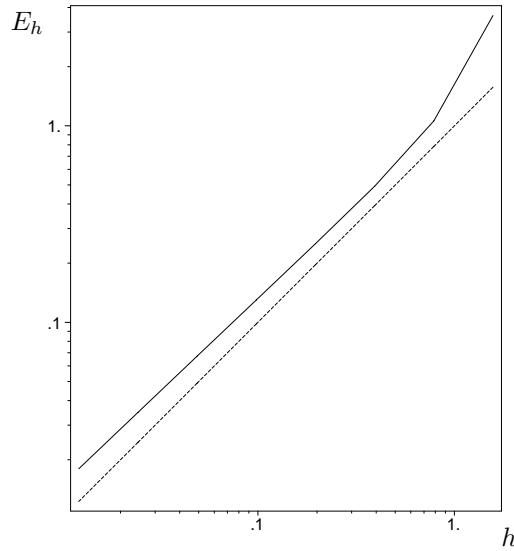


FIGURE 2. *A priori* error estimate E_h in function of h when $u(x_1, x_2) = (\sin(x_1) \sin(x_2))^{1.34}$. (The dotted line is a straightline with slope 1.)

TABLE 1. True error $E_h = \|\nabla(u - u_h)\|_{L^{3/2}(\Omega)}$, estimated error $E_s = (\sum_{K \in \mathcal{T}_h} \eta^m(K))^{1/m}$ and effectivity index E_s/E_h .

h	vertices	E_h	E_s	E_s/E_h
1/5	36	0.07634	0.1414	1.853
1/10	121	0.03816	0.07391	1.937
1/20	441	0.01908	0.03775	1.979
1/40	1681	0.009540	0.01907	1.999
1/80	6561	0.004771	0.009586	2.009
1/160	25921	0.002389	0.004805	2.011
1/320	103041	0.001220	0.002405	1.972

In Table 2 we have computed the estimated error \tilde{E}_s given by Liu and Yan [8] for the p -Laplacian. In this example where $\alpha = 1/2$, we have

$$\tilde{E}_s = \left(\sum_{K \in \mathcal{T}_h} \tilde{\eta}^2(K) \right)^{1/2}$$

where

$$\begin{aligned} \tilde{\eta}^2(K) &= \int_K (|\nabla u_h|^{1/2} + h_K |p|) h_K^2 |p|^2 \, dx \\ &+ \sum_{t \in \partial K} \int_{K_t} \left(|\nabla u_h|^{1/2} + \left[|\nabla u_h|^{-1/2} \frac{\partial u_h}{\partial n} \right] \right) \left[|\nabla u_h|^{-1/2} \frac{\partial u_h}{\partial n} \right]^2 \, dx. \end{aligned}$$

TABLE 2. True error $E_h = \|\nabla(u - u_h)\|_{L^{3/2}(\Omega)}$, estimated error $\tilde{E}_s = (\sum_{K \in \Upsilon_h} \tilde{\eta}^2(K))^{1/2}$ and effectivity index \tilde{E}_s/E_h .

h	vertices	E_h	\tilde{E}_s	\tilde{E}_s/E_h
1/5	36	0.07634	0.2152	2.819
1/10	121	0.03816	0.1330	3.485
1/20	441	0.01908	0.07271	3.811
1/40	1681	0.009540	0.03792	3.975
1/80	6561	0.004771	0.01936	4.058
1/160	25921	0.002389	0.009778	4.093
1/320	103041	0.001220	0.004913	4.028

Here, we denote by K_t the triangle satisfying

$$|\nabla u_h|_{K_t} = \min_{i=1,2} (|\nabla u_h|_{K_t^i})$$

where K_t^1, K_t^2 are the two elements sharing the common side t .

If we compare Table 1 and Table 2 we can conclude that E_s and \tilde{E}_h are two estimators with some accuracy order (in h) but the effectivity index is better for E_s (close to 2) than for \tilde{E}_s (close to 4).

In the following, we present some numerical results related to an adaptive finite element method based on the *a posteriori* error estimate of Theorem 3. Our goal is now to build a mesh such that the estimated relative error is close to a preset tolerance Tol, namely

$$0.75\text{Tol} \leq E_s / \|\nabla u_h\|_{L^{2-\alpha}(\Omega)} \leq 1.25\text{Tol}.$$

A sufficient condition to build such a mesh is to check that, for all triangle $K \in \Upsilon_h$ we have

$$\nu_1 \frac{\|\nabla u_h\|_{L^{2-\alpha}(\Omega)}}{(N_T)^{1/m}} \leq \eta(K) \leq \nu_2 \frac{\|\nabla u_h\|_{L^{2-\alpha}(\Omega)}}{(N_T)^{1/m}} \tag{30}$$

where $\nu_1 = 0.75\text{Tol}$, $\nu_2 = 1.25\text{Tol}$ and N_T is the number of triangles in the mesh Υ_h . The adaptive algorithm we have used is an iterative method which adds or suppresses some vertices in the triangulation in order to generate a new Delaunay–Voronoi triangulation satisfying (30) for the best.

To illustrate our purpose, we still choose $\alpha = 1/2$ and for solution of (1), we choose a function u , the graph of which is very sharp in a neighborhood of the circle centered at the middle of the square Ω and with radius 0.2. More precisely we choose $u = \exp((r - a)^2 / ((r - a)^2 - \varepsilon^2))$ if $a < r < a + \varepsilon$, with $r^2 = x_1^2 + x_2^2$, $\varepsilon = 0.02$ and $a = 0.2$. If $r < a$ we set $u = 1$ and if $r > a + \varepsilon$ we set $u = 0$. Figures 3 to 6 show the initial mesh and the meshes obtained after 1, 5 and 10 iterations respectively. We conclude that the adaptive finite element method combined with the *a posteriori* error estimate given in Theorem 3 is efficient.

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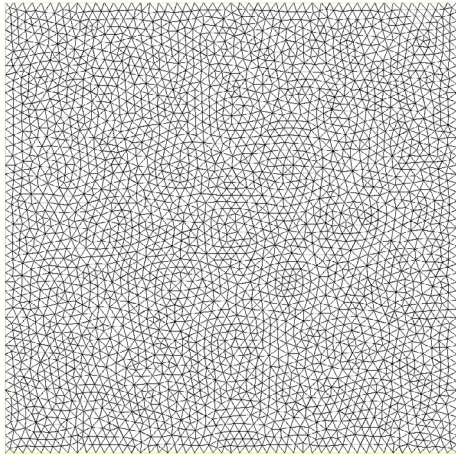


FIGURE 3. Initial mesh.

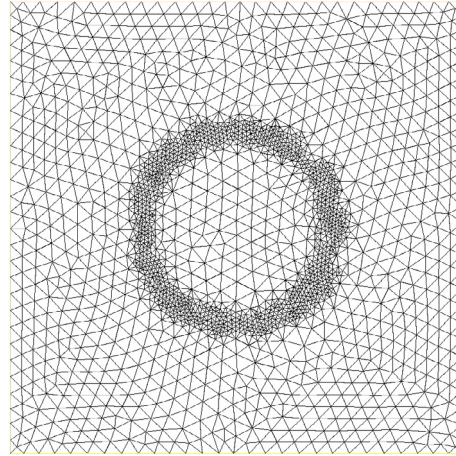


FIGURE 4. Mesh after one iteration.

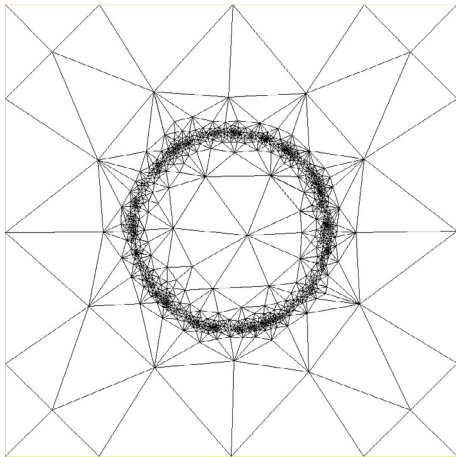


FIGURE 5. Mesh after five iterations.

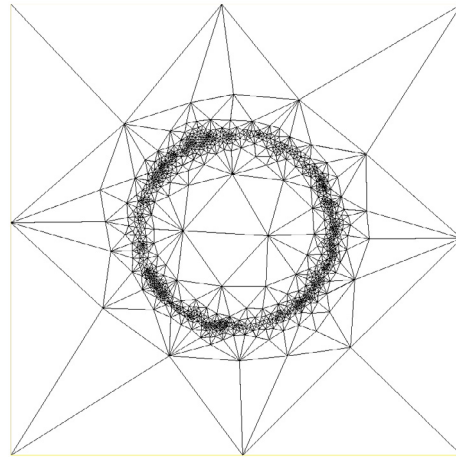


FIGURE 6. Mesh after ten iterations.

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