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THE TREATMENT OF "PINCHING LOCKING" IN 3*D*-SHELL ELEMENTS

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Abstract. We consider a family of shell finite elements with quadratic displacements across the thickness. These elements are very attractive, but compared to standard general shell elements they face another source of numerical locking in addition to shear and membrane locking. This additional locking phenomenon – that we call "pinching locking" – is the subject of this paper and we analyse a numerical strategy designed to overcome this difficulty. Using a model problem in which only this specific source of locking is present, we are able to obtain error estimates independent of the thickness parameter, which shows that pinching locking is effectively treated. This is also confirmed by some numerical experiments of which we give an account.

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Introduction

As defined and discussed in [10] (see also [6]), "3D-shell" elements are finite elements – for shell structures – which belong to the category of general shell elements (see [1,8]) and are based on a complete quadratic expansion of the displacements across the thickness (as opposed to, *e.g.*, the Reissner–Mindlin kinematical assumption which corresponds to an incomplete linear expansion). These shell elements feature three major advantages compared to standard general shell elements:

1. The complete quadratic expansion allows the use of nodes on the outer (*i.e.* upper and lower) surfaces of the shell, with degrees of freedom that correspond to displacements only (rotational degrees of freedom are not required). Hence these elements have the same external characteristics (geometry, shape functions) as 3D isoparametric elements. This – in particular – makes the coupling with other elements (fluids, solids, ...) sharing the same outer surfaces straightforward, see [10].
2. These shell elements can be used with a 3D variational formulation without modifying the constitutive equation, namely without resorting to a plane stress assumption which is rather cumbersome to handle when considering large strains and stresses.
3. We can expect that structures that undergo large strains are much better accounted for with this approach, since the quadratic kinematical assumption allows for complex transverse strains.

One of the main challenges in the design of shell finite elements is their ability to resist numerical locking phenomena, see in particular [1,7,9]. This is – to a large extent – still an open problem since we do not know of

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any shell finite element procedure that would have been proven to give uniform error estimates (with respect to the thickness parameter) for the main two categories of asymptotic behaviour encountered, namely the bending-dominated and membrane-dominated behaviours [7, 9]. Nevertheless, the same treatments already applied on standard general shell elements (in particular those substantiated by thorough numerical assessments, see [2]) can be applied on 3D-shell elements also. In addition to membrane and shear locking encountered with classical elements, however, it can be shown that 3D-shell elements face another source of locking, that we call "pinching locking" in this paper. Although many shell finite element procedures using kinematical assumptions of higher order (in the transverse direction) than the Reissner–Mindlin assumptions have already been proposed and discussed, see *e.g.* [4, 12, 14], a mathematical analysis of the pinching locking phenomenon leading to efficient mathematically-substantiated treatments is missing, and this is the objective of the present paper.

In order to explain the concept of pinching locking, we consider the quadratic kinematical assumption written in the form

$$\vec{U}(\xi^1,\xi^2,\xi^3) = \vec{u}(\xi^1,\xi^2) + \xi^3\vec{\theta}(\xi^1,\xi^2) + (\xi^3)^2\vec{\tau}(\xi^1,\xi^2), \quad (\xi^1,\xi^2)\in\omega,\ \xi^3\in\left[-\frac{t}{2},\frac{t}{2}\right], \tag{1}$$

where (ξ^1,ξ^2) denote the curvilinear coordinates (lying in ω, a domain of $\mathbb{R}^2$) that parametrize the midsurface of the shell while ξ^3 corresponds to the transverse direction and t denotes the thickness (assumed to be constant in this paper). With this notation, the transverse linearized strain e_{33} reads

$$e_{33} = \theta_3 + 2\xi^3\tau_3, \tag{2}$$

where the index "3" denotes the transverse component, namely, defining $\vec{a}_3$ as the unit normal vector to the midsurface,

$$\theta_3 = \vec{\theta}\cdot\vec{a}_3, \quad \tau_3 = \vec{\tau}\cdot\vec{a}_3. \tag{3}$$

We call θ_3, the first term in the expansion of e_{33}, the "pinching strain". Then, when the geometry and the loading are such that pure bending is not inhibited (see *e.g.* [7, 9]), it can be shown that the pinching strains "tend to vanish" when the thickness parameter tends to zero, just as the membrane and shear strains (the first terms in the expansions of the twice-tangential and tangential-transverse strains, respectively) also tend to vanish [10]. Namely, there exists a well-defined limit solution and this solution satisfies the constraint

$$\vec{\theta}\cdot\vec{a}_3\ (=\theta_3) = 0. \tag{4}$$

Clearly, this induces an additional source of locking since the corresponding constraint enforced on a finite element solution is that

$$\vec{\theta}_h\cdot\vec{a}_3 = 0 \tag{5}$$

over ω, denoting by $\vec{\theta}_h$ the first term in the expansion of the discrete solution. Indeed, as $\vec{\theta}_h$ is typically a piecewise polynomial while $\vec{a}_3$ is an arbitrary function, this constraint is frequently fulfilled for $\vec{\theta}_h = \vec{0}$ only, which means that locking occurs. This is what we call "pinching locking". This phenomenon is also called "curvature thickness locking" in [3], because it does not arise when $\vec{a}_3$ is constant (zero curvature). It should not be confused with the phenomenon sometimes referred to as "Poisson thickness locking" which is – in fact – not a numerical locking phenomenon [3].

In this paper we analyse a technique designed to circumvent pinching locking. The basic idea to that purpose consists in using, in the discrete variational formulation, the interpolation of the pinching strain – with the interpolation operator corresponding to the finite element shape functions – instead of the pinching strain directly computed from the displacements. Hence, instead of (5) we impose the relaxed constraint

$$\mathcal{I}\left(\vec{\theta}_h \cdot \vec{a}_3\right) = 0, \tag{6}$$

where $\mathcal{I}$ denotes the interpolation operator, which means that we typically impose (5) *at the nodes only.* This natural idea has already been proposed by several authors, see [3] and references therein, and we emphasize that the originality of our approach lies in the mathematical analysis performed.

The outline of the paper is as follows. In Section 1 we define the model problem that we consider throughout the paper, namely, a problem in which pinching locking occurs without any other specific numerical difficulty (such as membrane and shear locking in actual shell models). In Section 2 we introduce the modified discrete problem based on the above interpolation strategy, and we obtain *uniform a priori* estimates for this problem. Then, in Section 3 we present some numerical results and these results confirm the previous theoretical discussion. Finally, we give some concluding remarks in Section 4.

1. The model problem

In order to analyse and overcome the pinching locking phenomenon, we introduce a model problem which features the same specific source of locking as $3D$-shell models without the additional difficulties and technicalities present in shell formulations.

$(\mathcal{P})$: Find $\vec{\theta}^{\,\varepsilon} \in H^1_0(\omega)^3$ such that

$$a\left(\vec{\theta}^{\,\varepsilon}, \vec{\eta}\right) + \frac{1}{\varepsilon^2}\int_\omega \left(\vec{\theta}^{\,\varepsilon} \cdot \vec{a}_3\right)(\vec{\eta} \cdot \vec{a}_3)\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2 = f(\vec{\eta}), \qquad \forall \vec{\eta} \in H^1_0(\omega)^3, \tag{7}$$

where

$$a\left(\vec{\theta}, \vec{\eta}\right) = \int_\omega \left(\frac{\partial\vec{\theta}}{\partial\xi^1}\cdot\frac{\partial\vec{\eta}}{\partial\xi^1} + \frac{\partial\vec{\theta}}{\partial\xi^2}\cdot\frac{\partial\vec{\eta}}{\partial\xi^2}\right)\mathrm{d}\xi^1\,\mathrm{d}\xi^2, \tag{8}$$

$$f(\vec{\eta}) = \int_\omega \vec{f}\cdot\vec{\eta}\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2, \qquad \vec{f} \in L^2(\omega)^3. \tag{9}$$

Here, ε – which plays the role of the thickness in shell theories – is a parameter meant to tend to zero, and we assume that $\vec{a}_3$ is a smooth field of unit-length vectors defined over $\bar{\omega}$. Of course, the bilinear form in the left-hand side of (7) is continuous and H^1_0-coercive and f is a L^2-continuous linear form. Problem $(\mathcal{P})$ is then well-posed because we satisfy the assumptions of the Lax–Milgram lemma. Note that this is a penalty problem, since as $\varepsilon \to 0$ we tend to impose on the solution of the limit problem that the pinching strain vanish.

By introducing the auxiliary unknown

$$p^{\varepsilon} = \frac{1}{\varepsilon^2}\vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3, \tag{10}$$

problem $(\mathcal{P})$ can be written as the equivalent mixed formulation

$(\mathcal{P}_\varepsilon)$: Find $\vec{\theta}^{\,\varepsilon} \in H^1_0(\omega)^3$ and $p^{\varepsilon} \in L^2(\omega)$ such that

$$\begin{cases} a\left(\vec{\theta}^{\,\varepsilon}, \vec{\eta}\right) + b(\vec{\eta}, p^{\varepsilon}) = f(\vec{\eta}) & \forall \vec{\eta} \in H^1_0(\omega)^3, \\ b\left(\vec{\theta}^{\,\varepsilon}, q\right) - \varepsilon^2\langle p^{\varepsilon}, q\rangle_{L^2} = 0 & \forall q \in L^2(\omega), \end{cases}$$

with

$$b(\vec{\eta}, q) = \int_\omega q\,(\vec{\eta}\cdot\vec{a}_3)\,\mathrm{d}\xi^1\mathrm{d}\xi^2 = \langle q\,,\,\vec{\eta}\cdot\vec{a}_3\rangle_{H^{-1}\times H_0^1}. \tag{11}$$

The solution depends on the small parameter ε and the tentative limit problem, as $\varepsilon \to 0$, reads

$(\mathcal{P}_0)$: Find $\vec{\theta} \in H_0^1(\omega)^3$ and $p \in H^{-1}(\omega)$ such that

$$\begin{cases} a\left(\vec{\theta}, \vec{\eta}\right) + b(\vec{\eta}, p) = f(\vec{\eta}) & \forall \vec{\eta} \in H_0^1(\omega)^3, \\ b\left(\vec{\theta}, q\right) = 0 & \forall q \in H^{-1}(\omega). \end{cases}$$

Observing that $(\mathcal{P}_\varepsilon)$ is a regularized form of the saddle-point problem $(\mathcal{P}_0)$, the well-posedness of these problems crucially relies on two (classical) conditions, namely, the coercivity of the bilinear form $a(\cdot,\cdot)$ and the inf-sup condition for the bilinear form b (see in particular [5]). The coercivity of a holds over the whole space $H_0^1(\omega)^3$, hence we only need to check the inf-sup condition for the bilinear form b.

In the sequel, we use the symbol C to denote a generic positive constant – independent of ε and of the mesh parameter h – that may take different values at successive occurrences (including possibly in a single equation). Likewise γ will denote a generic *strictly positive* constant (also independent of ε).

Proposition 1.1. *There exists a strictly positive constant δ, such that*

$$\sup_{\vec{\eta}\in H_0^1(\omega)^3} \frac{b(\vec{\eta}, q)}{\|\,\vec{\eta}\,\|_1} \geq \delta \|\, q \,\|_{-1}, \qquad \forall q \in H^{-1}(\omega). \tag{12}$$

Proof. Take an arbitrary $q \in H^{-1}(\omega)$. By the Riesz representation theorem, there exists a unique $\phi \in H_0^1(\omega)$ such that

$$\int_\omega \nabla\phi\cdot\nabla\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2 = \langle q, \psi\rangle_{H^{-1}\times H_0^1}, \qquad \forall \psi \in H_0^1(\omega) \tag{13}$$

and

$$|\,\phi\,|_1 = \|\,q\,\|_{-1}. \tag{14}$$

Hence, by using (13) and considering the particular field $\vec{\eta} = \phi\vec{a}_3 \in H_0^1(\omega)^3$ we have

$$\sup_{\vec{\eta}\in H_0^1(\omega)^3} \frac{\langle\, q\,,\,\vec{\eta}\cdot\vec{a}_3\,\rangle_{H^{-1}\times H_0^1}}{\|\,\vec{\eta}\,\|_1} = \sup_{\vec{\eta}\in H_0^1(\omega)^3} \frac{\int_\omega \nabla\phi\,\cdot\nabla(\,\vec{\eta}\cdot\vec{a}_3\,)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\,\vec{\eta}\,\|_1}$$
$$\geq \frac{\int_\omega \nabla\phi\,\cdot\nabla(\,\phi\vec{a}_3\cdot\vec{a}_3\,)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\,\phi\vec{a}_3\,\|_1} = \frac{\int_\omega (\,\nabla\phi\,)^2\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\,\phi\vec{a}_3\,\|_1}\cdot \tag{15}$$

Then, since

$$\|\,\phi\vec{a}_3\,\|_1 \leq \gamma\|\,\phi\,\|_1 \leq \gamma|\,\phi\,|_1, \tag{16}$$

from (14), (15) and (16) we infer

$$\sup_{\vec{\eta}\in H_0^1(\omega)^3} \frac{\langle q\,,\,\vec{\eta}\cdot\vec{a}_3\,\rangle_{H^{-1}\times H_0^1}}{\|\,\vec{\eta}\,\|_1} \geq \gamma|\,\phi\,|_1 = \gamma\|\,q\,\|_{-1}, \tag{17}$$

and (12) follows. □

As a consequence, problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{P}_0)$ are well-posed and $(\mathcal{P}_0)$ is the limit problem for $(\mathcal{P}_\varepsilon)$ as stated in the following proposition, see [5] (Rem. 1.13, Prop. 4.2).

Proposition 1.2. *Problem $(\mathcal{P}_\varepsilon)$ has a unique solution $\left(\vec{\theta}^{\,\varepsilon}, p^{\,\varepsilon}\right) \in H_0^1(\omega)^3 \times L^2(\omega)$ and this solution satisfies*

$$\left\|\,\vec{\theta}^{\,\varepsilon}\,\right\|_1 + \|\,p^{\,\varepsilon}\,\|_{-1} + \varepsilon\|\,p^{\,\varepsilon}\,\|_0 \leq C\|\,f\,\|_0. \tag{18}$$

In addition, Problem $(\mathcal{P}_0)$ also has a unique solution $\left(\vec{\theta}, p\right)$ that is the limit of $\left(\vec{\theta}^{\,\varepsilon}, p^{\,\varepsilon}\right)$, namely,

$$\left\|\,\vec{\theta}^{\,\varepsilon} - \vec{\theta}\,\right\|_1 + \|\,p^{\,\varepsilon} - p\,\|_{-1} \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{19}$$

Remark 1.3. We observe that – as is the case in, *e.g.*, plate and shell formulations [5,9] – we “lose control” on the L^2-norm of the “Lagrange multiplier” (namely, $p^{\,\varepsilon}$ here) in the estimate (18). This is – of course – because the inf-sup condition only holds in H^{-1}, which is also why the limit problem is posed in this space.

Remark 1.4. We note that the above asymptotic behaviour “degenerates” when the “loading” considered does not activate the fields that satisfy the constraint (4), namely when $\forall\vec{\eta}\in H_0^1(\omega)^3$

$$\vec{\eta}\cdot\vec{a}_3 = 0 \Longrightarrow f(\vec{\eta}) = 0. \tag{20}$$

This may occur *e.g.* when $\vec{f}$ is everywhere directed along $\vec{a}_3$. In such a case we indeed have the zero solution for the limit problem $(\mathcal{P}_0)$, which may indicate that the asymptotic assumptions – in particular the scaling of the right-hand side in $(\mathcal{P})$ – is not appropriate. However this specific situation would not occur as such in a shell problem for which $\vec{\theta}$ is coupled to the other displacement unknowns through the energy. Hence we need not be concerned by this particular case (for which all the results given are also valid) in the sequel.

2. The discrete problem

2.1. Variational formulation

We consider the following displacement-based discrete formulation for $(\mathcal{P})$

$(\mathcal{P}_h)$: Find $\vec{\theta}_h^{\varepsilon} \in \mathcal{V}_h$ such that

$$a\left(\vec{\theta}_h^{\varepsilon}, \vec{\eta}\right) + \frac{1}{\varepsilon^2}\int_\omega \mathcal{I}\left(\vec{\theta}_h^{\varepsilon}\cdot\vec{a}_3\right)\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2 = f(\vec{\eta}), \qquad \forall\vec{\eta}\in\mathcal{V}_h. \tag{21}$$

We consider

$$\mathcal{V}_h = \{\,\vec{\eta}\in\mathcal{C}_0(\omega)^3 \text{ such that } \vec{\eta}\,|_T \in P_k(T)^3, \forall T\in\mathcal{T}_h,\ \vec{\eta} = \vec{0} \text{ on } \partial\omega\}, \tag{22}$$

where $\mathcal{T}_h$ represents a triangulation given over the domain ω, $P_k(T)$ the (finite-dimensional) space consisting of the restrictions of polynomials of degree k to the finite element $T\in\mathcal{T}_h$, and h denotes the maximum of all element diameters. We also recall that $\mathcal{I}$ denotes the interpolation operator associated with $\mathcal{V}_h$.

By introducing

$$p_h^\varepsilon = \frac{1}{\varepsilon^2}\mathcal{I}\left(\vec{\theta}_h^\varepsilon \cdot \vec{a}_3\right), \tag{23}$$

problem $(\mathcal{P}_h)$ can be written as the equivalent mixed approximation procedure

$(\mathcal{P}_{\varepsilon h})$: Find $\vec{\theta}_h^\varepsilon \in \mathcal{V}_h$ and $p_h^\varepsilon \in \mathcal{Q}_h$ such that

$$\begin{cases} a\left(\vec{\theta}_h^\varepsilon, \vec{\eta}\right) + b_h(\vec{\eta}\,, p_h^\varepsilon) = f(\vec{\eta}) & \forall \vec{\eta} \in \mathcal{V}_h, \\ b_h\left(\vec{\theta}_h^\varepsilon, q\right) - \varepsilon^2 \langle p_h^\varepsilon, q \rangle_{L^2} = 0 & \forall q \in \mathcal{Q}_h, \end{cases}$$

where $\mathcal{Q}_h \subset L^2(\omega)$ is the finite element space defined by

$$\mathcal{Q}_h = \{\, q \in \mathcal{C}_0(\omega) \text{ such that } q\,|_T \in P_k(T), \forall\, T \in \mathcal{T}_h,\; q\, = 0 \text{ on } \partial\omega\}, \tag{24}$$

and b_h denotes the modified bilinear form

$$b_h(\vec{\eta}, q) = \int_\omega q\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\mathrm{d}\xi^1\mathrm{d}\xi^2 = \langle\, q\,,\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\rangle_{H^{-1}\times H_0^1}. \tag{25}$$

Since $\mathcal{V}_h = (\mathcal{Q}_h)^3$ we will use the same notation for the operators associated to both spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$, without any risk of confusion.

By using the modified bilinear form b_h we relax the constraint that we tend to impose as $\varepsilon \to 0$ in the form

$$\mathcal{I}\left(\vec{\theta}_h \cdot \vec{a}_3\right) = 0. \tag{26}$$

We claim that this relaxed constraint is the discrete analogue of (4), hence we expect this modified formulation to be free of locking, which we proceed to demonstrate.

Recalling that a is coercive over the whole space $H_0^1(\omega)^3$, the well-posedness of $(\mathcal{P}_{\varepsilon h})$ crucially relies on a discrete inf-sup condition [5]. We first establish this inf-sup condition before using it to obtain the stability in a stability-consistency argument. We henceforth assume that the family of triangulations $\mathcal{T}_h$ is *quasi-uniform*, *i.e.* $\exists\,\nu > 0$ such that

$$\nu h \le h_T,\;\; \forall T \in \mathcal{T}_h. \tag{27}$$

Proposition 2.1. *There exists a strictly positive constant δ, such that*

$$\sup_{\vec{\eta}\in\mathcal{V}_h} \frac{b_h(\vec{\eta}\,, q)}{\|\,\vec{\eta}\,\|_1} \ge \delta \|\, q\,\|_{-1}, \qquad \forall q \in \mathcal{Q}_h. \tag{28}$$

Proof. We take an arbitrary $q \in \mathcal{Q}_h$ and we define $\phi_h \in \mathcal{Q}_h$ as the solution of the finite element problem

$$\int_\omega q\,\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2 = \int_\omega \nabla\phi_h\cdot\nabla\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2, \qquad \forall \psi \in \mathcal{Q}_h. \tag{29}$$

Note that $\mathcal{Q}_h \subset H_0^1(\omega)^3$, hence this is a well-posed problem and we have

$$|\,\phi_h\,|_1 \le C\|\, q\,\|_{-1}. \tag{30}$$

Then,

$$\sup_{\vec{\eta}\in\mathcal{V}_h} \frac{\int_\omega q\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\vec{\eta}\|_1} = \sup_{\vec{\eta}\in\mathcal{V}_h} \frac{\int_\omega \nabla\phi_h\cdot\nabla\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\vec{\eta}\|_1}$$

$$\geq \frac{\int_\omega \nabla\phi_h\cdot\nabla\mathcal{I}\,(\mathcal{I}(\phi_h\vec{a}_3)\cdot\vec{a}_3)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\mathcal{I}(\phi_h\vec{a}_3)\|_1} = \frac{\int_\omega \nabla\phi_h\cdot\nabla\phi_h\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\mathcal{I}(\phi_h\vec{a}_3)\|_1}\cdot \tag{31}$$

The key point in the above inequality is that the particular vector $\vec{\eta} = \mathcal{I}(\phi_h\vec{a}_3)$ belongs to $\mathcal{V}_h$. We also used the relation

$$\mathcal{I}\,(\mathcal{I}(\phi_h\vec{a}_3\,)\cdot\vec{a}_3) = \mathcal{I}(\phi_h) = \phi_h. \tag{32}$$

We also have, by a standard scaling argument,

$$\|\mathcal{I}(\phi_h\vec{a}_3)\|_1 \leq \gamma\|\phi_h\|_1. \tag{33}$$

Hence, from (31) and (33) we obtain that

$$\sup_{\vec{\eta}\in\mathcal{V}_h} \frac{\int_\omega q\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\vec{\eta}\|_1} \geq \gamma\frac{\int_\omega \nabla\phi_h\cdot\nabla\phi_h\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\phi_h\|_1} \geq \gamma\|\phi_h\|_1. \tag{34}$$

We still need to prove that

$$\|q\|_{-1} \leq \gamma\|\phi_h\|_1. \tag{35}$$

Defining Π_h as the L^2-projection onto the subspace $\mathcal{Q}_h$, we have

$$\|q\|_{-1} = \sup_{\psi\in H_0^1} \frac{\int_\omega q\,\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\psi\|_1} = \sup_{\psi\in H_0^1} \frac{\int_\omega q\,\Pi_h\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\psi\|_1} \tag{36}$$

$$= \sup_{\psi\in H_0^1} \frac{\int_\omega \nabla\phi_h\nabla(\Pi_h\psi)\,\mathrm{d}\xi^1\mathrm{d}\xi^2}{\|\psi\|_1} \leq |\,\phi_h\,|_1 \sup_{\psi\in H_0^1}\frac{|\,\Pi_h\psi\,|_1}{\|\psi\|_1}, \tag{37}$$

where we used

$$\int_\omega q\,\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2 = \int_\omega q\,\Pi_h\psi\,\mathrm{d}\xi^1\mathrm{d}\xi^2,\ \forall\psi\in H_0^1(\omega). \tag{38}$$

Under the quasi-uniformity assumption (27) the following estimate holds for any $\psi \in H_0^1(\omega)$

$$\|\Pi_h\psi - \psi\|_1 \leq C\|\psi\|_1. \tag{39}$$

Indeed, denoting by Λ the Clément interpolation operator, we have

$$\begin{aligned}\|\Pi_h\psi-\psi\|_1 &\leq \|\Pi_h\psi-\Lambda\psi\|_1 + \|\Lambda\psi-\psi\|_1\\ &\leq C\left(h^{-1}\|\Pi_h\psi-\Lambda\psi\|_0 + \|\psi\|_1\right)\\ &\leq C\big[h^{-1}(\|\Pi_h\psi-\psi\|_0 + \|\psi-\Lambda\psi\|_0) + \|\psi\|_1\big]\\ &\leq C\big[h^{-1}\times h\|\psi\|_1 + \|\psi\|_1\big] \leq C\|\psi\|_1,\end{aligned} \tag{40}$$

where we used an inverse inequality – under the assumption that the mesh is quasi-uniform – and the following classical estimates, see *e.g.* [13]

$$\| \Pi_h\psi - \psi \|_0 \le Ch\| \psi \|_1, \tag{41}$$

$$\| \Lambda\psi - \psi \|_0 \le Ch\| \psi \|_1. \tag{42}$$

Then, from (39) we have

$$\sup_{\psi\in H_0^1} \frac{| \Pi_h\psi |_1}{\| \psi \|_1} \le C, \tag{43}$$

hence from (37),

$$\| q \|_{-1} \le C| \phi_h |_1, \tag{44}$$

and the conclusion follows. □

2.2. *A priori* estimates

We introduce the mixed bilinear forms associated to problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{P}_{\varepsilon h})$, namely, respectively,

$$M\left(\vec{\theta}, p; \vec{\eta}, q\right) = a\left(\vec{\theta}, \vec{\eta}\right) + b\left(\vec{\theta}, q\right) + b(\vec{\eta}, p) - \varepsilon^2\langle p, q\rangle_{L^2}, \qquad \forall\left(\vec{\theta}, p\right), (\vec{\eta}, q) \in H_0^1(\omega)^3 \times L^2(\omega), \tag{45}$$

$$M_h\left(\vec{\theta}, p; \vec{\eta}, q\right) = a\left(\vec{\theta}, \vec{\eta}\right) + b_h\left(\vec{\theta}, q\right) + b_h(\vec{\eta}, p) - \varepsilon^2\langle p, q\rangle_{L^2}, \qquad \forall\left(\vec{\theta}, p\right), (\vec{\eta}, q) \in \mathcal{V}_h \times \mathcal{Q}_h. \tag{46}$$

Then, $(\mathcal{P}_\varepsilon)$ is equivalent to finding $\left(\vec{\theta}^{\,\varepsilon}, p^\varepsilon\right) \in H_0^1(\omega)^3 \times L^2(\omega)$ such that

$$M\left(\vec{\theta}^{\,\varepsilon}, p^\varepsilon; \vec{\eta}, q\right) = f(\vec{\eta}), \qquad \forall(\vec{\eta}, q) \in H_0^1(\omega)^3 \times L^2(\omega), \tag{47}$$

and $(\mathcal{P}_{\varepsilon h})$ is equivalent to finding $\left(\vec{\theta}_h^\varepsilon, p_h^\varepsilon\right) \in \mathcal{V}_h \times \mathcal{Q}_h$ such that

$$M_h\left(\vec{\theta}_h^\varepsilon, p_h^\varepsilon; \vec{\eta}, q\right) = f(\vec{\eta}), \qquad \forall(\vec{\eta}, q) \in \mathcal{V}_h \times \mathcal{Q}_h. \tag{48}$$

Defining the norm

$$\| \vec{\eta}, q \|_\varepsilon = \left(\| \vec{\eta} \|_1^2 + \| q \|_{-1}^2 + \varepsilon^2\| q \|_0^2 \right)^{\frac{1}{2}}, \tag{49}$$

we now establish the stability of M_h for this norm.

Lemma 2.2. *The bilinear form M_h defined above is stable with respect to the norm $\|\cdot,\cdot\|_\varepsilon$, i.e.*

$$\forall(\vec{\eta}, q) \in \mathcal{V}_h \times \mathcal{Q}_h, \quad \exists\,(\vec{\eta}^{\#}, q^{\#}) \in \mathcal{V}_h \times \mathcal{Q}_h \text{ such that}$$

$$\| \vec{\eta}^{\#}, q^{\#} \|_\varepsilon \le C\| \vec{\eta}, q \|_\varepsilon, \tag{50}$$

$$M_h(\vec{\eta}, q; \vec{\eta}^{\#}, q^{\#}) \ge \gamma\| \vec{\eta}, q \|_\varepsilon^2. \tag{51}$$

Proof. *(i) Stability in $\| \vec{\eta} \|_1$ and $\varepsilon\| q \|_0$.*

Taking $(\vec{\eta}_1, q_1) = (\vec{\eta}, -q)$ we have

$$\| \vec{\eta}_1, q_1 \|_\varepsilon = \| \vec{\eta}, q \|_\varepsilon \tag{52}$$

and

$$M_h(\vec{\eta}, q; \vec{\eta}_1, q_1) = a(\vec{\eta}; \vec{\eta}) + \varepsilon^2 \| q \|_0^2. \tag{53}$$

Hence the coercivity of $a(\cdot, \cdot)$ implies

$$M_h(\vec{\eta}, q; \vec{\eta}_1, q_1) \geq \gamma \Big(\| \vec{\eta} \|_1^2 + \varepsilon^2 \| q \|_0^2 \Big). \tag{54}$$

(ii) Stability in $\| q \|_{-1}$ and conclusion.

The bilinear form b_h satisfies the inf-sup condition (28). Hence, we can find $\vec{\eta}_2$ in $\mathcal{V}_h$ such that

$$\| \vec{\eta}_2 \|_1 = \| q \|_{-1}, \qquad b_h(\vec{\eta}_2, q) \geq \frac{\delta}{2} \| q \|_{-1}^2. \tag{55}$$

We now take $q_2 = 0$ and we obtain

$$\| \vec{\eta}_2, q_2 \|_\varepsilon = \| \vec{\eta}_2 \|_1 = \| q \|_{-1} \leq \| \vec{\eta}, q \|_\varepsilon \tag{56}$$

and

$$\begin{aligned} M_h(\vec{\eta}, q; \vec{\eta}_2, q_2) &= a(\vec{\eta}, \vec{\eta}_2) + b_h(\vec{\eta}_2, q) \\ &\geq -C \| \vec{\eta} \|_1 \| \vec{\eta}_2 \|_1 + \frac{\delta}{2} \| q \|_{-1}^2 \\ &= \frac{\delta}{2} \| q \|_{-1}^2 - C \| \vec{\eta} \|_1 \| q \|_{-1} \\ &\geq \gamma \| q \|_{-1}^2 - C \| \vec{\eta} \|_1^2, \end{aligned} \tag{57}$$

using the Cauchy-Schwarz inequality. Finally, a well-chosen linear combination of $(\vec{\eta}_1, q_1)$ and $(\vec{\eta}_2, q_2)$ provides the desired test function $(\vec{\eta}^{\#}, q^{\#})$. □

We proceed to obtain a “best approximation” estimate, the proof of which is based on a classical stability-consistency argument provided here for completeness.

Proposition 2.3. *Problem $(\mathcal{P}_{\varepsilon h})$ has a unique solution $\left(\vec{\theta}_h^\varepsilon, p_h^\varepsilon\right)$ and this solution satisfies*

$$\begin{aligned} \left\| \vec{\theta}^\varepsilon - \vec{\theta}_h^\varepsilon \right\|_1 + \| p^\varepsilon - p_h^\varepsilon \|_{-1} + \varepsilon \| p^\varepsilon - p_h^\varepsilon \|_0 \leq C \inf_{(\vec{\tau}, r) \in \mathcal{V}_h \times \mathcal{Q}_h} \Bigg(\left\| \vec{\theta}^\varepsilon - \vec{\tau} \right\|_1 + \| p^\varepsilon - r \|_{-1} + \varepsilon \| p^\varepsilon - r \|_0 \\ + \sup_{\vec{\eta} \in \mathcal{V}_h} \frac{|(b - b_h)(\vec{\eta}, r)|}{\| \vec{\eta} \|_1} + \sup_{q \in \mathcal{Q}_h} \frac{|(b - b_h)(\vec{\tau}, q)|}{\| q \|_{-1}} \Bigg). \end{aligned} \tag{58}$$

Proof. Let $(\vec{\tau}, r)$ be an arbitrary element of $\mathcal{V}_h \times \mathcal{Q}_h$. By Lemma 2.2 we can find $\left(\vec{\theta}^{\#}, p^{\#}\right)$ in $\mathcal{V}_h \times \mathcal{Q}_h$ such that

$$M_h\left(\vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r; \vec{\theta}^{\#}, p^{\#}\right) \geq \gamma \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon}^2, \tag{59}$$

$$\left\| \vec{\theta}^{\#}, p^{\#} \right\|_{\varepsilon} \leq C \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon}. \tag{60}$$

Furthermore,

$$\begin{aligned} M_h\left(\vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r; \vec{\theta}^{\#}, p^{\#}\right) &= M_h\left(\vec{\theta}_h^{\varepsilon}, p_h^{\varepsilon}; \vec{\theta}^{\#}, p^{\#}\right) - M_h\left(\vec{\tau}, r; \vec{\theta}^{\#}, p^{\#}\right) \\ &= M\left(\vec{\theta}^{\varepsilon}, p^{\varepsilon}; \vec{\theta}^{\#}, p^{\#}\right) - M_h\left(\vec{\tau}, r; \vec{\theta}^{\#}, p^{\#}\right) \\ &= M\left(\vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r; \vec{\theta}^{\#}, p^{\#}\right) + (M - M_h)\left(\vec{\tau}, r; \vec{\theta}^{\#}, p^{\#}\right). \end{aligned} \tag{61}$$

By using the continuity of M with (59) and (60) we obtain

$$\gamma \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon}^2 \leq \left| (M - M_h)\left(\vec{\tau}, r; \vec{\theta}^{\#}, p^{\#}\right) \right| + C \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon} \left\| \vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r \right\|_{\varepsilon}. \tag{62}$$

Then,

$$\gamma \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon} \leq \frac{\left| (M - M_h)\left(\vec{\tau}, r; \vec{\theta}^{\#}, p^{\#}\right) \right|}{\left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon}} + C \left\| \vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r \right\|_{\varepsilon}. \tag{63}$$

From the definitions of M and M_h we have that

$$(M - M_h)\left(\vec{\tau}, r; \vec{\theta}^{\#}, p^{\#}\right) = (b - b_h)\left(\vec{\theta}^{\#}, r\right) + (b - b_h)\left(\vec{\tau}, p^{\#}\right). \tag{64}$$

Therefore the inequality (63) becomes

$$\gamma \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon} \leq C \left\| \vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r \right\|_{\varepsilon} + \frac{\left| (b - b_h)\left(\vec{\theta}^{\#}, r\right) \right|}{\left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon}} + \frac{\left| (b - b_h)\left(\vec{\tau}, p^{\#}\right) \right|}{\left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon}}. \tag{65}$$

Combining (60) and (65) we get

$$\begin{aligned} \left\| \vec{\theta}_h^{\varepsilon} - \vec{\tau}, p_h^{\varepsilon} - r \right\|_{\varepsilon} &\leq C \left(\left\| \vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r \right\|_{\varepsilon} + \frac{\left| (b - b_h)\left(\vec{\theta}^{\#}, r\right) \right|}{\left\| \vec{\theta}^{\#}, p^{\#} \right\|_{\varepsilon}} + \frac{\left| (b - b_h)\left(\vec{\tau}, p^{\#}\right) \right|}{\left\| \vec{\theta}^{\#}, p^{\#} \right\|_{\varepsilon}} \right) \\ &\leq C \left(\left\| \vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r \right\|_{\varepsilon} + \sup_{\vec{\eta} \in \mathcal{V}_h} \frac{|(b - b_h)(\vec{\eta}, r)|}{\| \vec{\eta} \|_1} + \sup_{q \in \mathcal{Q}_h} \frac{|(b - b_h)(\vec{\tau}, q)|}{\| q \|_{-1}} \right). \end{aligned} \tag{66}$$

Then, by using a triangular inequality, we infer

$$\left\| \vec{\theta}_h^{\varepsilon} - \vec{\theta}^{\varepsilon}, p_h^{\varepsilon} - p^{\varepsilon} \right\|_{\varepsilon} \leq C \left(\left\| \vec{\theta}^{\varepsilon} - \vec{\tau}, p^{\varepsilon} - r \right\|_{\varepsilon} + \sup_{\vec{\eta} \in \mathcal{V}_h} \frac{|(b - b_h)(\vec{\eta}, r)|}{\| \vec{\eta} \|_1} + \sup_{q \in \mathcal{Q}_h} \frac{|(b - b_h)(\vec{\tau}, q)|}{\| q \|_{-1}} \right), \tag{67}$$

and since this holds for any $(\vec{\tau}, r) \in \mathcal{V}_h \times \mathcal{Q}_h$ the conclusion directly follows. □

2.3. Consistency and final estimates

Note that the first three terms in the right-hand side of (58) represent the interpolation errors, whereas the last two terms are the consistency errors. Our goal in this section is to determine the orders of convergence for the consistency terms, in order to know whether the numerical procedure is optimal.

We will use the following estimates, already proved in [8].

Lemma 2.4. *For any $\vec{\eta}_{\|} \in \mathcal{C}_0(\omega)^3$ tangent to the midsurface at all points (i.e. $\vec{\eta}_{\|} \cdot \vec{a}_3 = 0$),*

$$\| \mathcal{I}(\vec{\eta}_{\|}) \cdot \vec{a}_3 \|_1 \leq C\, h \,\| \mathcal{I}(\vec{\eta}_{\|}) \|_1, \tag{68}$$

$$\| \mathcal{I}(\vec{\eta}_{\|}) \cdot \vec{a}_3 \|_0 \leq C\, h^2 \,\| \mathcal{I}(\vec{\eta}_{\|}) \|_1. \tag{69}$$

Note that this result also holds for $\vec{\eta}_{\|}$ tangent to the midsurface only at nodes.

We also need the following lemma.

Lemma 2.5. *For any $q \in \mathcal{Q}_h$,*

$$\| \mathcal{I}(q\, \vec{a}_3) \cdot \vec{a}_3 - q \|_0 \leq C\, h^2 \,\| \, q \,\|_1. \tag{70}$$

Proof. Let q be an arbitrary element of $\mathcal{Q}_h$. Introducing the notation

$$\delta(q) = \mathcal{I}(q\, \vec{a}_3) \cdot \vec{a}_3 - q, \tag{71}$$

we start by bounding $|\,\delta(q)\,|$. In every element T of the mesh we denote by λ_i the Lagrange shape function associated to the node (i) and by $q^{(i)}$ the value of q at this node. We have

$$\begin{aligned} |\,\delta(q)\,| &= \left| \sum_{i \in T} \lambda_i\, q^{(i)}\, \vec{a}_3 \cdot \left(\vec{a}_3^{(i)} - \vec{a}_3 \right) \right| \\ &\leq C \left(\sup_{i \in T} \| \lambda_i \|_{L^\infty} \sup_{i \in T} \left| q^{(i)} \right| \sup_{i \in T} \left\| \vec{a}_3 \cdot \left(\vec{a}_3^{(i)} - \vec{a}_3 \right) \right\|_{L^\infty} \right) \\ &\leq C \left(1 \times h_T^2 \right) \sup_{i \in T} \left| q^{(i)} \right| \leq C\, h_T^2 \sup_T |\, q\,|, \end{aligned} \tag{72}$$

where h_T denotes the diameter of the element T, using the fact that

$$\vec{a}_3 \cdot \left(\vec{a}_3^{(i)} - \vec{a}_3 \right) = \mathcal{O}\left(h_T^2 \right),$$

since $\vec{a}_3$ is a smooth field of unit-length vectors. Using standard scaling arguments (see *e.g.* [11]) we have

$$\begin{aligned} \sup_T |\, q\,| = \|\, q\,\|_{L^\infty(T)} &\leq C(\mathrm{meas}\,(T))^{-1/2} (\,\|\, q\,\|_{0,T} + h_T |\, q\,|_{1,T}\,) \\ &\leq C h_T^{-1} \,\|\, q\,\|_{1,T}. \end{aligned} \tag{73}$$

Hence,

$$|\,\delta(q)\,| \leq C\, h_T \|\, q\,\|_{1,T}, \tag{74}$$

so that

$$\begin{aligned}\| \delta(q) \|_0^2 &\leq \sum_{T\in\mathcal{T}_h} \| \delta(q) \|_{0,T}^2 = \sum_{T\in\mathcal{T}_h} \int_T | \delta(q) |^2 \,\mathrm{d}\xi^1\mathrm{d}\xi^2 \\ &\leq C \sum_{T\in\mathcal{T}_h} h_T^2 \times \operatorname{meas}(T) \times \| q \|_{1,T}^2 \\ &\leq C \left(\sup_{T\in\mathcal{T}_h} h_T^4 \right) \sum_{T\in\mathcal{T}_h} \| q \|_{1,T}^2 \leq C\, h^4 \| q \|_1^2, \end{aligned} \tag{75}$$

and (70) follows. □

In order to assess the impact of consistency errors, we assume for the solution the regularity required for optimal interpolation estimates, namely, $\vec{\theta}^{\,\varepsilon} \in H^{k+1}(\omega)^3$ and $p^\varepsilon \in H^k(\omega)$, with k as used in the definitions (22) and (24). The interpolation errors are then given by

$$\inf_{\vec{\tau}\in\mathcal{V}_h} \left\| \vec{\theta}^{\,\varepsilon} - \vec{\tau} \right\|_1 \leq \left\| \vec{\theta}^{\,\varepsilon} - \mathcal{I}(\vec{\theta}^{\,\varepsilon}) \right\|_1 \leq C\, h^k \left\| \vec{\theta}^{\,\varepsilon} \right\|_{k+1}, \tag{76}$$

$$\inf_{r\in\mathcal{Q}_h} \| p^\varepsilon - r \|_0 \leq \| p^\varepsilon - \Pi_h(p^\varepsilon) \|_0 \leq C\, h^{k+1} \| p^\varepsilon \|_{k+1}, \tag{77}$$

recalling that Π_h denotes the L^2 projection operator onto $\mathcal{Q}_h$.

Theorem 2.6. *We have*

$$\left\| \vec{\theta}^{\,\varepsilon} - \vec{\theta}_h^{\,\varepsilon} \right\|_1 + \| p^\varepsilon - p_h^\varepsilon \|_{-1} + \varepsilon \| p^\varepsilon - p_h^\varepsilon \|_0 \leq C \left[h^k \left(\left\| \vec{\theta}^{\,\varepsilon} \right\|_{k+1} + \| p^\varepsilon \|_k \right) + h^2 \| p^\varepsilon \|_0 \right]. \tag{78}$$

Proof. We take in (58) $\vec{\tau} = \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right)$ and $r = \Pi_h(p^\varepsilon)$. Then

$$\begin{aligned}\sup_{q\in\mathcal{Q}_h} \frac{\left| (b-b_h)\left(\mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right), q\right) \right|}{\| q \|_{-1}} &= \sup_{q\in\mathcal{Q}_h} \frac{\left| \int_\omega q \left[\mathcal{I}\left(\mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right)\cdot\vec{a}_3\right) - \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right)\cdot\vec{a}_3 \right] \mathrm{d}\xi^1\mathrm{d}\xi^2 \right|}{\| q \|_{-1}} \\ &\leq \sup_{q\in\mathcal{Q}_h} \frac{\left\| \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3\right) - \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right)\cdot\vec{a}_3 \right\|_1 \| q \|_{-1}}{\| q \|_{-1}} \\ &= \left\| \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3\right) - \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right)\cdot\vec{a}_3 \right\|_1 \\ &\leq \left\| \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3\right) - \vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3 \right\|_1 + \left\| \vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3 - \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right)\cdot\vec{a}_3 \right\|_1 \\ &\leq C \left(\left\| \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3\right) - \vec{\theta}^{\,\varepsilon}\cdot\vec{a}_3 \right\|_1 + \left\| \vec{\theta}^{\,\varepsilon} - \mathcal{I}\left(\vec{\theta}^{\,\varepsilon}\right) \right\|_1 \right) \\ &\leq C\, h^k \left\| \vec{\theta}^{\,\varepsilon} \right\|_{k+1}. \end{aligned} \tag{79}$$

We also have

$$\sup_{\vec{\eta}\in\mathcal{V}_h} \frac{|\,(b-b_h)(\vec{\eta},\Pi_h(p_\varepsilon))\,|}{\|\,\vec{\eta}\,\|_1} = \sup_{\vec{\eta}\in\mathcal{V}_h} \frac{\left|\displaystyle\int_\omega \left[\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) - \vec{\eta}\cdot\vec{a}_3\right]\Pi_h(p_\varepsilon)\,\mathrm{d}\xi^1\mathrm{d}\xi^2\right|}{|\,\vec{\eta}\,\|_1}$$

$$\le C\|\,\Pi_h(p^\varepsilon)\,\|_0 \sup_{\vec{\eta}\in\mathcal{V}_h} \frac{\|\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) - \vec{\eta}\cdot\vec{a}_3\,\|_0}{\|\,\vec{\eta}\,\|_1}$$

$$\le C\|\,p^\varepsilon\,\|_0 \sup_{\vec{\eta}\in\mathcal{V}_h} \frac{\|\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) - \vec{\eta}\cdot\vec{a}_3\,\|_0}{\|\,\vec{\eta}\,\|_1}. \tag{80}$$

Considering an arbitrary $\vec{\eta}$ in $\mathcal{V}_h$, inside any element $T\in\mathcal{T}_h$ we have

$$\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) = \sum_{i\in T}\lambda_i\,\vec{\eta}^{\;(i)}\cdot\vec{a}_3^{(i)}. \tag{81}$$

At every node (i), $\vec{\eta}^{(i)}$ can be decomposed as the sum of a vector $\vec{\eta}_{||}^{(i)}$ tangent to the midsurface $\left(\vec{\eta}_{||}^{(i)}\cdot\vec{a}_3^{(i)} = 0\right)$ and of a vector normal to the midsurface, namely,

$$\vec{\eta}^{\;(i)} = \vec{\eta}_{\|}^{(i)} + \eta_3^{(i)}\,\vec{a}_3^{(i)}. \tag{82}$$

Then,

$$\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) = \sum_{i\in T}\lambda_i\,\eta_3^{(i)}\,\vec{a}_3^{(i)}\cdot\vec{a}_3^{(i)} = \sum_{i\in T}\lambda_i\,\eta_3^{(i)} = \eta_3, \tag{83}$$

where η_3 is the polynomial of degree k which takes the value $\eta_3^{(i)}$ at each node (i). Introducing also $\vec{\eta}_{\|}$ the polynomial which takes at each node (i) the value $\vec{\eta}_{\|}^{(i)}$, we have

$$\vec{\eta} = \vec{\eta}_{\|} + \mathcal{I}(\eta_3\vec{a}_3), \tag{84}$$

hence

$$\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) - \vec{\eta}\cdot\vec{a}_3 = \eta_3 - \mathcal{I}(\eta_3\vec{a}_3)\cdot\vec{a}_3 - \vec{\eta}_{\|}\cdot\vec{a}_3. \tag{85}$$

Using Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}\|\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3) - \vec{\eta}\cdot\vec{a}_3\,\|_0 &\le \|\,\eta_3 - \mathcal{I}(\eta_3\vec{a}_3)\cdot\vec{a}_3\,\|_0 + \|\,\mathcal{I}(\vec{\eta}_{\|})\cdot\vec{a}_3\,\|_0\\ &\le C\,(\,h^2\,\|\,\eta_3\,\|_1 + h^2\,\|\,\mathcal{I}(\vec{\eta}_{\|})\,\|_1\,)\\ &\le C\,h^2\,\|\,\vec{\eta}\,\|_1,\end{aligned} \tag{86}$$

noting that

$$\|\,\mathcal{I}(\vec{\eta}_{||})\,\|_1 \le C\|\,\vec{\eta}\,\|_1, \tag{87}$$

$$\|\,\mathcal{I}(\vec{\eta}\cdot\vec{a}_3)\,\|_1 \le C\|\,\vec{\eta}\,\|_1. \tag{88}$$

Therefore, from (80) it follows that

$$\sup_{\vec{\eta}_h\in\mathcal{V}_h} \frac{|\,(b-b_h)(\vec{\eta}_h,\Pi_h(p^\varepsilon))\,|}{\|\,\vec{\eta}_h\,\|_1} \le C\,h^2\,\|\,p^\varepsilon\,\|_0. \tag{89}$$

Gathering (58, 76, 77, 79) and (89) we obtain the desired estimate, taking into account that

$$\| p^\varepsilon - \Pi_h(p^\varepsilon) \|_{-1} \leq \| p^\varepsilon - \Pi_h(p^\varepsilon) \|_0 \leq h^k \| p^\varepsilon \|_k. \tag{90}$$

□

Remark 2.7. Due to the consistency errors, the proposed numerical procedure is sub-optimal if the finite element method uses polynomials of degree $k > 2$.

Remark 2.8. The uniform convergence obtained in Theorem 2.6 (recalling that the constant C is independent of ε and h) shows that the numerical procedure proposed in this chapter is locking-free.

3. Numerical experiments

We implemented the above numerical strategy and obtained the corresponding numerical solutions in the case of a one-dimensional problem, namely, considering geometric and mechanical data for which the solution of problem $(\mathcal{P})$ is invariant by translation in one direction. More specifically, we considered the one-dimensional domain corresponding to $\xi^1 \in [0, \pi/2]$ with the "normal vector" given by

$$\vec{a}_3 = \begin{pmatrix} \cos \xi^1 \\ \sin \xi^1 \end{pmatrix}, \tag{91}$$

and the loading chosen so that the exact solution is, for any value of ε,

$$\vec{\theta}^\varepsilon = \begin{pmatrix} -\sin 4\xi^1 \sin \xi^1 \\ \sin 4\xi^1 \cos \xi^1 \end{pmatrix}. \tag{92}$$

Note that this choice satisfies

$$\vec{\theta}^\varepsilon \cdot \vec{a}_3 = 0 \tag{93}$$

exactly, which ensures that the corresponding loading does not depend on ε.

Figure 1 shows the numerical results obtained for a direct P_1 discretization of this 1D problem, for varying values of t (we henceforth do not distinguish between t and ε). In this figure, N denotes the number of elements along the total length. We observe a deterioration of the convergence behaviour when t decreases, compared to the curve obtained for $t = 0.1$. This deterioration is still limited for $t = 0.01$, but dramatically worsens for smaller values. This behaviour is characteristic of locking.

In contrast with this behaviour, Figure 2 displays the numerical results obtained when applying the above-described treatment of pinching strain interpolation, and we can see that the four convergence curves are practically superimposed. This clearly confirms that locking is not present.

4. Conclusions

We presented a numerical procedure designed to circumvent the pinching locking for $3D$-shell finite elements. Namely, we use the interpolated value of the pinching strains (with the same interpolation as for displacements) in the variational formulation, instead of the value directly computed from the displacements. The analysis of this modified finite element formulation as a mixed formulation allowed us to obtain an error estimate independent of the "thickness" parameter, hence locking is effectively remedied.

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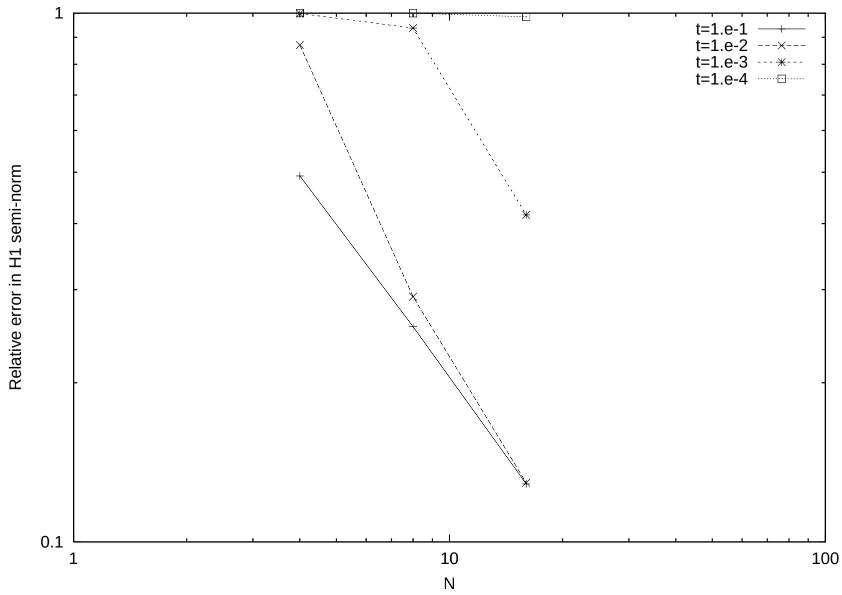

FIGURE 1. Convergence with direct discretization.

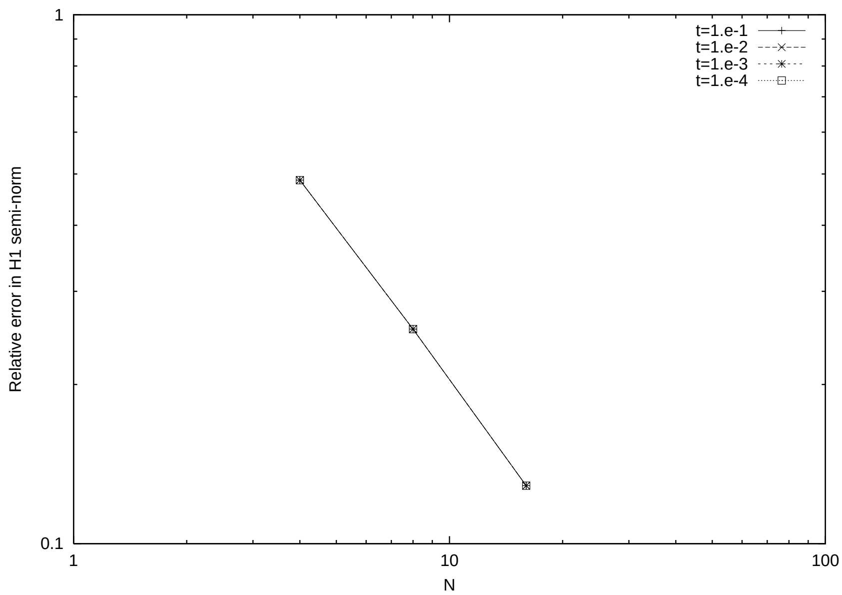

FIGURE 2. Convergence with pinching strain interpolation.

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