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**A new definition of a fluid. I. The Stokesian fluid**

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*A new definition of a fluid.*

*I. The Stokesian Fluid;*

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*Dedicated to the Memory of P. DUHEM.*

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**Contents** <sup>(1)</sup>.

1. Preliminary discourse.
2. Phenomenological methods.
3. Heuristic discussion of fluidity. Stokes's Principle.
4. Heuristic discussion of viscosity.
5. Definition of the Stokesian fluid.
6. Isotropic fluids.
7. The form of the coefficients  $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ .
8. Dynamical similarity.
9. Mean pressure and thermodynamic pressure.
10. Contrast with elasticity.
11. The Reiner-Rivlin fluid.
12. An example : shearing flow.

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<sup>(1)</sup> I am obliged to the Office of Naval Research for supporting my preliminary studies under Contracts nos 45-47 and 53-47, and to the Applied Mathematics Branch, Mechanics Division, Naval Research Laboratory, for supporting the completion of this and the succeeding Memoir; to Dr P. Neményi for presenting part of the material in Part II at the VII International Congress of Applied Mechanics; to Dr A. Van Tuyl and M. R. N. Schwartz for checking some of the calculations in Part II; to Miss Charlotte Brudno for her great assistance in preparing the paper. The methods used here were first announced in [1947, 1].

13. Heuristic discussion of heat flow.
  14. Postulates for the heat flux vector.
  15. Thermally isotropic fluids.
  16. The form of the coefficient  $\mathcal{G}$ .
  17. Provisional conclusions.
- APPENDIX. — Properties of continuous media in general.

1. PRELIMINARY DISCOURSE. — For the description of aerodynamic phenomena at high altitudes the classical Navier-Stokes equations are no longer sufficient. It is generally believed, however, that up to an altitude of 100<sup>km</sup> the air may be regarded as a continuous medium, so that what is required is a formula for the stress tensor more general than that forming the basis of the classical theory of viscous compressible fluids but reducing to it when the pressure is not too low. Such an expression has been sought in the kinetic theory of gases<sup>(2)</sup>. But for gross phenomena in a continuous medium a gross assumption is preferable. History teaches us that the conjectures of natural philosophers, though often positively proclaimed as “physical laws”, are subject to unforeseen revisions. Molecular hypotheses have come and gone, but the phenomenological equations of d’Alembert, Euler and Cauchy remain exact as at the day of their discovery, exempt from fashion. In adopting a phenomenological and at the same time rigorously mathematical approach to the general theory of fluids I follow the path opened fifty years ago by the profound researches of Duhem.

2. PHENOMENOLOGICAL METHODS. — To search out the true foundation of the theory of fluids, it is not to experiments of the laboratory but rather to ordinary experience, whence comes our intuitive concept of

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Much of the material in Part I, of which [1949, 7] is a preliminary version, I presented in lectures at the University of Illinois, december 16, 1948; at the University of Toronto, february 4, 1949; and at the Institut de Mécanique de la Faculté des Sciences de Paris, june 9, 1949. Part II, of which [1948, 1] is a preliminary version, is partially summarized in [1948, 2], and was presented to the Americal Physical Society, january 30, 1948, New-York City; paragraphe 11 was presented to that Society, january 28, 1949, New-York City.

<sup>(2)</sup> See § 18 of the succeeding Memoir.

fluidity, that our first questions should be directed. For the mathematical realization of an elementary concept we are in no way more able than our forebears, those giants upon whose shoulders we stand like dwarfs who would leap over mountains. On the contrary, we have somewhat dissipated the wealth which they bequeathed us. But today we have a full knowledge of the rules of invariance, which permit us to attain in the mathematical formulæ the generality, simplicity, and elegance of the intuitive concepts we design them to represent, and from formal simplicity comes what is called “understanding”.

The basic method of this paper, which may be recommended for establishing an phenomenological theory upon a sound foundation, consists of two parts. First, simple and immediate experience is summarized in a *formal definition of an ideal medium*, which is neither more nor less than a *postulate of the existence of certain dimensional quantities* and a *specification of the functional dependence of some of the quantities upon others*. It need hardly be mentioned that in such a definition no special type of function, such as a linear or quadratic form, is singled out for preference. Second, the forms of the defining functions are rendered definite by a *full use of the principles of invariance, both dimensional and tensorial*. From expansions of these functions in power series, any number of terms of which may be retained at will, complete dynamical equations of any order of approximation are an immediate consequence.

**5. HEURISTIC DISCUSSION OF FLUIDITY. STOKES'S PRINCIPLE.** — What then is a fluid? The most evident mechanical property of an isolated volume of water is that its shape depends largely upon the form of the containing vessel, exhibiting no tendency whatever to re-assume any previous form it may have had; such a body is thus possessed of no finite “Memory”. It would be quite wrong, however, to conclude that a fluid offers no resistance to deformation. Indeed, if we drag our hand rapidly through a body of water, even at constant speed, we experience marked resistance, while the same amount of deformation is easily effected more slowly. Even though offering no resistance to a permanent deformation, once effected, water does

indeed resist a change of volume, and no ordinary pressure can produce a sensible compression. If we attempt to compress a mass of air, however, we find that a slight pressure produces a considerable change of volume, but that when this pressure is removed the air quickly regains its original volume, provided the surroundings be otherwise kept constant. Euler <sup>(3)</sup> in effect defined a fluid as a body possibly endowed with elasticity of volume, but without elasticity of shape, and it was perhaps in this sense that Cauchy <sup>(4)</sup> in first giving the complete expressions for the stresses in a viscous compressible fluid spoke of “*corps solides entièrement dépourvus d'élasticité*”. The elasticity of shape of a compressible fluid is attendant upon the thermodynamic static pressure  $\pi = \pi(\varrho, \theta)$ , where  $\varrho$  is the density and  $\theta$  is the temperature. In an incompressible fluid there is no such functional dependence for the static pressure <sup>(5)</sup>. In all types of fluids the remaining portion  $v^i_j$  of the total stress  $t^i_j$ ,

$$(3.1) \quad v^i_j \equiv t^i_j - (-p \delta^i_j) = t^i_j + p \delta^i_j,$$

is that which arises in resistance to instantaneous change of shape, and thus evidently depends upon it. This concept of fluidity is embodied in Stokes's *principle* <sup>(6)</sup>: “That the difference between the

<sup>(3)</sup> Euler [1769, 1, § 1, 13, 24] defined a fluid in terms of three phenomena : 1. “ Si fluidum a vi quacunq̄ue pressum in æquilibrio versetur, tum pressio per totam fluidi massam ita æqualiter diffunditur, ut omnes ejus particulæ parem vim sustineant. ” 2. “ Alia fluida ita comparata deprehenduntur, ut quantumvis magna vi premantur, idem semper volumen retineant : alia vero hujus sunt indolis, ut quo majori vi premantur, in eo minus spatium redigantur, antequam ad æquilibrium perveniant : in utroque autem genere proprietas fluiditatis ante memorata æque locum habet. ” 3. “ Omnis generis fluida a calore in majus spatium expandi, a frigore autem in minus spatium contrahi experientia declarat, quatenus quidem ob vires sollicitantes hoc fieri licet. ” His discussion [§ 2-12] of 1 reveals that he regarded a fluid as a body without elasticity of shape. An earlier formulation may be found in [1757, 1, § VI-XIX].

<sup>(4)</sup> [1823, 1], [1828, 1, § III]. See, § 18, however.

<sup>(5)</sup> For a discussion of the various pressures, and a hasty exposition of these properties of continuous media in general which are employed in the present paper, see the Appendix below.

<sup>(6)</sup> [1845, 1, § 1]. For the history of the fundamental equations of the classical theory, see § 18 of the succeeding Memoir.

pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned. " Now the measure of the *rate of deformation* of a medium was shown by Euler (<sup>7</sup>) to be the tensor

$$(3.2) \quad d^i_j \equiv \frac{1}{2}(\dot{x}^i_{,j} + \dot{x}^j_{,i}),$$

where  $\dot{x}^i$  is the velocity vector (<sup>8</sup>). Since the tensor

$$(3.3) \quad \omega^i_j \equiv \frac{1}{2}(\dot{x}^i_{,j} - \dot{x}^j_{,i})$$

was shown by Cauchy (<sup>9</sup>) and Stokes (<sup>10</sup>) to be a measure of the local and instantaneous rate of rotation (vorticity) of the medium,  $d^i_j = \dot{x}^i_{,j} - \omega^i_j$  is indeed the measure of " relative motion " after the rotation has been eliminated. Thus in modern terms Stokes's principle in its full generality (<sup>11</sup>) is simply

$$(3.4) \quad v^i_j = f(d^k_l), \quad f(0) = 0.$$

These equations were used as the defining property of fluids by Reiner (<sup>12</sup>) and Rivlin (<sup>13</sup>). Since in general the rate at which the stress does work (per unit volume) is  $v^i_j d^j_i$ , from (3.4) we have the scalar equation

$$(3.5) \quad v^i_j d^j_i = g(d^k_l).$$

(<sup>7</sup>) [1770, 1, § 9-12]. Equivalent analyses were given by Cauchy [1827, 1, p. 88-93] (for infinitesimal strain) and Stokes [1845, 1, § 2].

(<sup>8</sup>) We employ the ordinary notations of the absolute differential calculus, as presented e. g. in [1927, 1].

(<sup>9</sup>) [1841, 1, th. IV].

(<sup>10</sup>) [1845, 1, § 2].

(<sup>11</sup>) After stating his principle, Stokes himself [1845, 1, § 3] by an argument referring to the ultimate molecules concluded that the functions (3.4) should be linear.

(<sup>12</sup>) [1945, 1, § 2].

(<sup>13</sup>) [1947, 2], [1948, 4, § 9].

Thus an equivalent formulation of Stokes's principle is : *the stress power is a function of the rate of deformation only.*

**4. HEURISTIC DISCUSSION OF VISCOSITY.** — The phenomenon embodied by (3.4) is called *viscosity*, and the stresses  $\sigma^i_j$  are called *viscous stresses*. In dimensional form (3.4<sub>1</sub>) becomes

$$(4.1) \quad \frac{\mathbf{M}}{\mathbf{L}\mathbf{T}^2} = f\left(\frac{1}{\mathbf{T}}\right),$$

a relation manifestly impossible without the insertion of a further dimensional quantity, say, P. That is, in place of (3.4<sub>1</sub>) we must have

$$(4.2) \quad \sigma^i_j = f(d^k_l, P),$$

or, dimensionally,

$$(4.3) \quad f\left(\frac{\mathbf{M}}{\mathbf{L}\mathbf{T}^2}, \frac{1}{\mathbf{T}}, \dim P\right) = 0,$$

where  $\dim P$  is such that a dimensionless ratio

$$\frac{\left(\frac{\mathbf{M}}{\mathbf{L}\mathbf{T}^2}\right)^\alpha \mathbf{T}^\beta}{\dim P},$$

can be formed. Consequently

$$(4.4) \quad \dim P = \mathbf{M}^\alpha \mathbf{L}^{-\alpha} \mathbf{T}^{\beta-\alpha}.$$

Choice of one of the exponents is equivalent to raising P to an arbitrary power, and hence one exponent may be given any convenient value. Taking  $\alpha = 1$ , we have

$$(4.5) \quad \dim P = \frac{\mathbf{M}}{\mathbf{L}\mathbf{T}^{\beta-1}}.$$

The number  $\beta$  remains arbitrary. The phenomenon of resistance, therefore, requires a nearer analysis.

According to the celebrated hypothesis of Newton (<sup>14</sup>), as reformulated by Maxwell (<sup>15</sup>), when a viscous fluid is confined between

(<sup>14</sup>) [1726, 1, lib. II, sect. IX].

(<sup>15</sup>) [1866, 1, p. 7-8], [1871, 1, p. 277-278].

infinite parallel plane boundaries which are in constant relative motion at a velocity  $\dot{x}$ , the walls experience a retarding force according to the formula

$$(4.6) \quad \frac{\text{force}}{\text{area}} = \mu \frac{\dot{x}}{\text{distance between walls}},$$

where  $\mu$  is a quantity called the *coefficient of viscosity*, which is independent, of the other magnitudes in the formula, but dependent upon the particular fluid. The dimensions of  $\mu$  are  $\mathbf{ML}^{-1}\mathbf{T}^{-1}$ , corresponding to the choice  $\beta = 2$  in (4.5). We are not concerned here with the exactness of (4.6); suffice that a crude actual experiment indicates it to be a rough approximation, for our only purpose at the moment is to indicate that a quantity  $\mu$  of dimensions  $\mathbf{ML}^{-1}\mathbf{T}^{-1}$  should enter the definition of a fluid.

Now this viscosity  $\mu$  is a manifest function of the temperature  $\theta$ , and more generally may be supposed to be a function of the thermodynamic state

$$(4.7) \quad \mu = f(p, \theta).$$

But this relation connects three quantities composed of the three independent dimensions  $\mathbf{ML}^{-1}\mathbf{T}^{-1}$ ,  $\mathbf{T}$ , and  $\Theta$ , and hence cannot subsist unless some of the variables be struck out or else other quantities composed of the same dimensions be inserted. The only alternative to  $\mu = \text{const.}$  is then to introduce new quantities. At first sight it would seem reasonable to propose

$$(4.8) \quad \frac{\mu}{\mu_0} = f\left(\frac{p}{p_0}, \frac{\theta}{\theta_0}\right),$$

where  $\mu_0$ ,  $p_0$ , and  $\theta_0$  are certain reference values of  $\mu$ ,  $p$  and  $\theta$ , characteristic of the particular fluid. Now  $\dim p_0 = \mathbf{ML}^{-1}\mathbf{T}^{-2}$ , so that  $p_0$  is an elasticity. But if there be both a viscosity  $\mu_0$  and an elasticity  $p_0$  representing properties of the fluid, then the quantity  $\frac{\mu_0}{p_0}$  must also represent a property of the fluid. Now  $\frac{\mu_0}{p_0}$  is of the dimension  $\mathbf{T}$ : *any substance with a natural viscosity and natural elasticity possesses also a natural time*, and must necessarily be susceptible of time dependent phenomena, such as relaxation effects. These effects are expressly



excluded by our initial statement that it is contrary to experience for a fluid (as we simply conceive it) to exhibit any response to a configuration assumed a finite time in the past<sup>(16)</sup>. Hence we may either adopt the usual expedient of excluding  $p$  altogether from (4.7), as is done implicitly at the outset in the ordinary kinetic theory of gases, or embracing a more liberal view we may seek another scalar quantity of the dimensions of  $p$ , but not a constant of the fluid. Such a quantity is readily found in the mean pressure  $p_m \equiv -\frac{t_i^i}{3}$ . In place of (4.8) we have then

$$(4.9) \quad \mu = \mu_0 f\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right).$$

Recall that from the basic definitions usually employed in the kinetic theory of gases follows at the outset the confluence of these two pressures. It is not surprising that one of the results of that theory is that  $\mu$  is independent of the pressure.

All the remarks of this and the preceding section are heuristic, serving only to motivate the formal definition of a fluid which follows now.

**5. DEFINITION OF THE STOKESIAN FLUID.** — A Stokesian fluid is a continuous medium such that :

I. *There exist material constants  $\mu_n$  and  $\theta_0$ , called respectively the natural viscosity and the reference temperature, whose dimensions are*

$$(5.1) \quad \dim \mu_n = \frac{\mathbf{M}}{\mathbf{LT}},$$

$$(5.2) \quad \dim \theta_0 = \Theta.$$

II. *The stress power is of the functional form*

$$(5.3) \quad t_j^i d_i = f(\mu_n, \theta_0, p_m, p, \theta, d_{kl}),$$

*and is an analytic function of the components  $d_{kl}$ .*

III. *If  $d_{kl} = 0$  then*

$$(5.4) \quad t_j^i = -p \delta^i_j,$$

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<sup>(16)</sup> As Cauchy [1828, 1, § III] expressed it, "l'élasticité disparaît entièrement".

where for incompressible fluids  $p$  is an unspecified scalar, while for compressible fluids  $p$  is the thermodynamic pressure  $p = p(\rho, \theta)$ .

Note that  $\mu_n$  is a constant, not a function of the thermodynamic state, and thus must not be confused with the ordinary coefficient of viscosity  $\mu$ . Rather, it is to be regarded as the dimensional part of  $\mu$ , and we shall see later [cf. equat. (7.8)] that in fact  $\mu = \mu_n f\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right)$ , where the function  $f$  is dimensionless.

**6. ISOTROPIC FLUIDS.** — Since  $d_{ij}$  is symmetric, from (5.3), and the assumed analyticity of  $t^i_j$  we have

$$(6.1) \quad t^i_j = k^i_j + k^{ijk}_l d^l_k + k^{ijkl}_m d^m_n d^l_k + \dots,$$

where  $k^i_j, k^{ijk}_l, \dots$  are certain tensors independent of  $d^i_j$ . By (5.4),  $k^i_j = -p \delta^i_j$ . A fluid will be said to be *isotropic* if the matrix  $\mathbf{t}$  of  $t^i_j$  be a function of the matrix  $\mathbf{d}$  of  $d^i_j$ :

$$(6.2) \quad \mathbf{t} = -p\mathbf{I} + k_1 \mathbf{d} + k_2 \mathbf{d}^2 + k_3 \mathbf{d}^3 + k_4 \mathbf{d}^4 + \dots,$$

where the  $k_i$  are scalars independent of  $\mathbf{d}$ . From this definition it follows that in an isotropic fluid there are no preferred directions of response:

$t^1_1$  must depend upon  $d^1_1$  in exactly the same way that  $t^2_2$  depends upon  $d^2_2, \dots$ , in every co-ordinate system, and the principal axes of  $\mathbf{t}$  coincide with those of  $\mathbf{d}$ . Reiner<sup>(17)</sup> has given an elegant reduction of the series (6.2) to a simpler form by observing that the Cayley-Hamilton equation yields

$$(6.3) \quad \mathbf{d}^3 = \text{I}_a \mathbf{d}^2 - \text{II}_a \mathbf{d} + \text{III}_a \mathbf{I},$$

where  $\text{I}_a, \text{II}_a, \text{III}_a$  are the principal invariants of  $\mathbf{d}$ , so that the third and all higher powers of  $\mathbf{d}$  may be eliminated from (6.2). Thus

$$(6.4) \quad \mathbf{t} = \mathcal{F}_0 \mathbf{I} + \mathcal{F}_1 \mathbf{d} + \mathcal{F}_2 \mathbf{d}^2,$$

where  $\mathcal{F}_0, \mathcal{F}_1$ , and  $\mathcal{F}_2$  are functions of the principal invariants. In component form,

$$(6.5) \quad t^i_j = \mathcal{F}_0 \delta^i_j + \mathcal{F}_1 d^i_j + \mathcal{F}_2 d^i_k d^k_j.$$

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<sup>(17)</sup> [1945, 1, § 4].

The definition of paragraph 5 neither mentions nor requires isotropy, but in the present paper we shall confine our attention to this special case in the interest of simplicity. In the succeeding paper, our method is applied more generally.

7. THE FORM OF THE COEFFICIENTS  $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ . — From (5.3) we have now

$$(7.1) \quad \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 = f(\mu_n, \theta_0, p_m, p, \theta, \mathbf{I}_d, \mathbf{II}_d, \mathbf{III}_d).$$

But the quantities  $\frac{\mathcal{F}_0}{p}, \frac{\mathcal{F}_1}{\mu_n}, \frac{\mathcal{F}_2 p}{\mu_n^2}$  are dimensionless. It follows that the relations

$$(7.2) \quad \frac{\mathcal{F}_0}{p}, \frac{\mathcal{F}_1}{\mu_n}, \frac{\mathcal{F}_2 p}{\mu_n^2} = g(\mu_n, \theta_0, p_m, p, \theta, \mathbf{I}_d, \mathbf{II}_d, \mathbf{III}_d),$$

wich connect eight quantities composed of the three fundamental dimensions  $\frac{\mathbf{M}}{\mathbf{L}\mathbf{T}}, \mathbf{T}$ , and  $\Theta$ , must reduce to relations connecting  $8 - 3 = 5$  dimensionless ratios of these quantities. The desired ratios rise beneath our hands :

$$(7.3) \quad \frac{\mu_n}{p} \mathbf{I}_d, \frac{\mu_n^2}{p^2} \mathbf{II}_d, \frac{\mu_n^3}{p^3} \mathbf{III}_d, \frac{p}{p_m}, \frac{\theta}{\theta_0}.$$

If then we introduce the dimensionless coefficients  $F_\Gamma$  :

$$(7.4) \quad F_0 \equiv \frac{\mathcal{F}_0}{p}, \quad F_1 \equiv \frac{\mathcal{F}_1}{\mu_n}, \quad F_2 \equiv \frac{\mathcal{F}_2 p}{\mu_n^2},$$

we have

$$(7.5) \quad F_0, F_1, F_2 = f\left(\frac{\mu_n}{p} \mathbf{I}_d, \frac{\mu_n^2}{p^2} \mathbf{II}_d, \frac{\mu_n^3}{p^3} \mathbf{III}_d, \frac{p}{p_m}, \frac{\theta}{\theta_0}\right),$$

where the functions  $F_\Gamma$  require only dimensionless constants for their full expression. Putting (7.4) into (6.5), we thus reduce the theory of the Stokesian fluid to the definite form

$$(7.6) \quad t_j = p F_0 \delta^i_j + \mu_n F_1 d^i_j + \frac{\mu_n^2}{p} F_2 d^i_k d^k_j,$$

where  $F_0\left(0, 0, 0, \frac{p}{p_m}, \frac{\theta}{\theta_0}\right) = -1$ .

So as to fix our ideas upon this result let us write out explicitly the terms of first and second order <sup>(18)</sup> in the viscosity  $\mu_n$  :

$$(7.7) \quad \begin{aligned} \dot{t}_j = & - p \delta^j_j + \mu_n [A I_a \delta^j_j + B d^j_j] \\ & + \frac{\mu_n^2}{p} [C I_a^2 \delta^j_j + D I_a d^j_j + E I_a \delta^j_j + F d^k_k d^k_j] + \dots, \end{aligned}$$

where A, B, . . . , F are dimensionless functions of  $\frac{\theta}{\theta_0}$  and  $\frac{p}{p_m}$  only. For conformity with the classical notations, we may define the *linear coefficients* of viscosity  $\lambda$  and  $\mu$  :

$$(7.8) \quad \lambda \equiv \mu_n A, \quad 2\mu \equiv \mu_n B.$$

The terms of degree 0 and 1 in  $\mu_n$ , in the series (7.1) now become the Newton-Cauchy-Poisson law, which forms the basis of the classical theory of viscous fluids. C, D, E and F are the four dimensionless coefficients of viscosity of second order.

At the commencement of this Memoir I spoke of the need for an expression for the stresses in a fluid medium at low pressures, but since then I have not mentioned the matter again. Instead, I have only given a precise form to our ordinary concept of fluidity, and here is the result : an expansion in reciprocal powers of the pressure, to which we have been driven simply by the irrefutable force of the principles of invariance. The linear terms are exactly those of the classical theory. The terms of higher order can become significant only at sufficiently low pressures.

More generally, let us write the power series for the dimensionless coefficients  $F_\Upsilon$  in the form <sup>(19)</sup>

$$(7.9) \quad \begin{aligned} F_\Upsilon &= F_{\Upsilon \mathcal{J} \mathcal{J} \mathcal{K}} \left( \frac{\mu_n}{p} I_a \right)^{\mathcal{J}} \left( \frac{\mu_n^2}{p^2} II_a \right)^{\mathcal{J}} \left( \frac{\mu_n^3}{p^3} III_a \right)^{\mathcal{K}}, \\ &= F_{\Upsilon \mathcal{J} \mathcal{J} \mathcal{K}} \left( \frac{\mu_n}{p} \right)^{\mathcal{J}+2\mathcal{J}+3\mathcal{K}} I_a^{\mathcal{J}} II_a^{\mathcal{J}} III_a^{\mathcal{K}}, \end{aligned}$$

<sup>(18)</sup>These second order terms, but with coefficients unspecified in form, are included in a more general result [equat (18.3) of the succeeding memoir] given by Boussinesq (1868, 1, Note I). Five of the six terms of third order, again with coefficients unspecified in form, are derived by Girault [1931, 1, Chap. III], whose work is discussed in § 12, <sup>(27)</sup>.

<sup>(19)</sup>In this paper a diagonally repeated index, whether tensorial or not, is always taken as summed over its full range of variability. Thus e. g. in (7.9) all three indices  $\mathcal{J}$ ,  $\mathcal{J}$ ,  $\mathcal{K}$  are summed from 0 to  $\infty$ .

where the *dimensionless coefficients of viscosity*  $F_{\gamma\beta\gamma\kappa}$  are functions of  $\frac{p}{p_m}$  and  $\frac{\theta}{\theta_0}$  only

$$(7.10) \quad F_{\gamma\beta\gamma\kappa} = f\left(\frac{p_m}{p}, \frac{\theta}{\theta_0}\right).$$

The coefficients of (7.7) are related to those of this full expansion as follows

$$F_{0000} = -1, \quad \lambda \equiv \mu_n, \quad \Lambda = \mu_n F_{0100}, \quad 2\mu \equiv \mu_n, \quad B = \mu_n F_{1000}, \\ C = F_{0200}, \quad D = F_{1100}, \quad E = F_{0010}, \quad F = F_{2000}.$$

If  $F_{1000} \neq 0$  the fluid is *viscous*, as we shall henceforth assume.

**8. DYNAMICAL SIMILARITY.** — By the ordinary rigorous method of affine transformations it is easy to deduce criteria for dynamical similarity for the Stokesian fluid. In addition to the Mach number, the Reynolds number, and the other dimensionless characteristic numbers of the classical theory, we obtain also the number  $\mathfrak{J}$ :

$$(8.1) \quad \mathfrak{J} \equiv \frac{\mu d}{p},$$

where  $d$  is a rate of deformations, frequency, expansion, vorticity, or reciprocal of a time which is characteristic for the class of motions being compared. Now in the classical linear theory no such quantity as  $d$  occurs in the complete set of parameters for local dynamic similarity (although for certain classes of motions such a quantity may occur in parameters governing similarity at boundaries). In a general Stokesian fluid however, a time, frequency, expansion, or vorticity must always be taken into account in considerations of similarity, and furthermore this quantity is the only new dimensional parameter which need be employed no matter how many higher order terms in (7.7) are retained, for the only new characteristic numbers besides  $\mathfrak{J}$  to which these give rise are ratios of the several dimensionless coefficients, viz.

$$(8.2) \quad \frac{F_{\gamma\beta\gamma\kappa}}{F_{1000}}.$$

Thus the classical linear theory altogether neglects one effect of

primary physical significance : although it is to represent *dynamical* phenomena in fluids, all the variables which occur in the consequent numbers governing dynamic similarity are *statical* quantities (in the usual Eulerian description), so that it would seem that time rates are of secondary importance in the structure even though they alone are considered in its basic dynamical laws. This strange quality results only from omitting the higher order terms in (7.7), and thus is not proper to the Stokesian concept, but a mere accident of the classical linearization. From a physical point of view we may say that in the general theory here presented time rates play the dominant part we should expect from the concept of fluidity, and that moreover this part is essentially the same no matter how many terms in the series for the stresses are retained, so long as they go past the linear terms.

To grasp the significance of this new effect, consider a case in which the higher order terms in (7.7) are not negligible, but yet act only as small perturbations. Then if motions of two different Stokesian fluids be compared, it is not unreasonable to suppose that the effect of different values of the higher order coefficients  $A, B, \dots$  for the two fluids may be neglected, so that  $\mathfrak{J}$  becomes the only characteristic number which need be added to govern the effects of the perturbing terms. Suppose now we have two identical vessels containing specimens of the same fluid, which we place in a vacuum at the same altitude above sea level. Let cocks be opened so as to permit the fluids to escape, and at suitable corresponding points in each issuing jet let measurements be taken. Suppose the velocity, density, and temperature at these corresponding points be found identical in each. Then all the criteria for dynamical similarity in the classical theory are satisfied, and while in the absence of a uniqueness theorem we cannot actually prove, yet we may with some confidence expect, that the two flows will be altogether identical. In the case of the Stokesian fluid in order to gain the same expectation we must first verify that, say, the corresponding vorticities are equal, or that the total times of efflux are equal, etc. In other words, two samples of a Stokesian fluid issuing from identical vessels under identical *statical* conditions at a typical point may require quite

different times of efflux. This greater freedom arises mathematically from the fact that the dynamical equations for the Stokesian fluid are of higher order than the Navier-Stokes equations, and it is possible for the two motions to be started in different manners.

Now in general in order to consider dynamical similarity one must know the following properties of the *fluid*

$$(8.3) \quad \mu_n, \theta_0, F_{\Upsilon\mathfrak{J}\mathfrak{K}},$$

and the following properties of the *particular flow* at certain corresponding points

$$(8.4) \quad p, \rho, l, \dot{x}, d.$$

*Note that there are but five of these latter, no matter how high an order of approximation for the stresses be selected, and that in the classical linear theory one of these ( $d$ ) may be neglected; while the higher the order of approximation in the stress formulae, the greater the number of properties of the *fluid* which must be ascertained.*

If we may suppose that all the dimensionless coefficients  $F_{\Upsilon\mathfrak{J}\mathfrak{K}}$  are of the order of 1 or less in absolute value, then a criterion for the validity of the ordinary linear theory is simply

$$(8.5) \quad \mathfrak{J} \equiv \frac{\mu d}{p} \ll 1.$$

That is to say, the higher order terms are appropriate to a combination of high viscosity, high rate of deformation, or low pressure. The first two effects are in some measure contradictory, though in any case we may notice that since the rate of deformation often becomes infinite at a singularity, the results of the classical theory cannot be regarded as meaningful in its vicinity; but to find the third, one need only ascend the atmosphere. A numerical example is instructive. For water at 1<sup>mm</sup>, in order for  $\mathfrak{J}$  to reach the order of 1 the rate of deformation  $d$  must be of the order of  $7 \cdot 10^5 \text{ sec}^{-1}$ ; our higher order terms, then, can be significant only for very high frequency oscillations. But on the contrary, in the upper atmosphere there are very low pressures, which can render our higher order terms of predominant importance even for ordinary rates of deformation.

**9. MEAN PRESSURE AND THERMODYNAMIC PRESSURE.** — Most treatments of fluid dynamics employ the assumption  $p = p_m$ , although it is supported neither by sound theory nor by experiment and effects but slight simplification of the resulting equations. By (7.8), an immediate consequence is that all the coefficients of viscosity  $F_{\gamma\beta\alpha}$  become independent of pressure. It is sometimes claimed that in the kinetic theory of gases it is proved that  $\mu$  is independent of  $p$ . Since the basic definitions of that theory (<sup>20</sup>) as usually employed imply  $p = p'_m$  at the outset, the result is not surprising.

In the general case, by (7.6) and (7.9) we have

$$(9.1) \quad -3p_m = 3pF_0 + \mu_n \mathbf{I}_d F_1 + \frac{\mu_n^2}{\rho} (\mathbf{I}_d^2 - 2\mathbf{II}_d) F_2,$$

or

$$(9.2) \quad \left\{ \begin{aligned} 3(p - p_m) &= (3\lambda + 2\mu)\mathbf{I}_d + \frac{\mu_n^2}{\rho} [(3C + D + F)\mathbf{I}_d^2 + (3E - 2F)\mathbf{II}_d] + \dots, \\ \frac{3(p - p_m)}{\rho} &= [3F_{0\beta\gamma\alpha} + F_{1,\beta-1,\gamma,\alpha} + F_{2,\beta-2,\gamma,\alpha} - 2F_{2,\beta,\gamma-1,\alpha}] \cdot \\ &\quad \left(\frac{\mu_n}{\rho}\right)^{\beta+2\gamma+3\alpha} \mathbf{I}_d^\beta \mathbf{II}_d^\gamma \mathbf{III}_d^\alpha, \end{aligned} \right.$$

where the term containing  $F_{0000}$  in the sum on the right is to be omitted. In the classical linear theory we have  $p \geq p_m$  in an expanding motion and  $p \leq p_m$  in a contracting motion, but (9.2) shows that if the number  $\mathfrak{J}$ , as given by (8.1), be sufficiently large, the sign of  $p - p_m$  may be that of  $3C + D + F$ , irrespective of whether the motion be expanding or contracting.

A necessary and sufficient condition that  $p = p'_m$  in all motions is

$$(9.3) \quad 3F_{0\beta\gamma\alpha} + F_{1,\beta-1,\gamma,\alpha} + F_{2,\beta-2,\gamma,\alpha} - 2F_{2,\beta,\gamma-1,\alpha} = 0.$$

The first few cases special are

$$(9.4) \quad \left\{ \begin{aligned} 3\lambda + 2\mu &= 0, \\ 3C + D + F &= 0, \\ 3E - 2F &= 0, \end{aligned} \right.$$

of which the first is the classical Stokes relation. Now in the classical theory the Stokes relation is also a necessary and sufficient

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(<sup>20</sup>) Cf. § 20 of the succeeding Memoir.



condition that a motion of uniform dilatation  $d^i_j = d \delta^i_j$  can take place without dissipation of energy. In the general theory it is easy to work out a condition for this most improbable phenomenon, similar in form but not identical with (9.3).

**10. CONTRAST WITH ELASTICITY.** — From the formal analogy between Cauchy's generalization of Hooke's law and the classical linear terms of (7.7) hasty persons sometimes erroneously conclude that a proper generalization of the equations of elasticity simultaneously yields a proper generalization of the Navier-Stokés equations. Since fluids and elastic solids, being the very extremes of possible types of deformable continua, embody entirely different mechanical concepts, such an analogy would be surprising; that it exists in the classical theories is in fact an accident attendant upon linearization. Comparison of (7.7) and the corresponding expansion for the stress in the general theory of elasticity <sup>(21)</sup> shows their entirely different form and character as soon as the first terms of higher order are considered, a difference reflecting the fact that strain is dimensionless <sup>(22)</sup> while rate of deformation is of dimension  $\mathbf{T}^{-1}$ . In the classical linear theories, the elastic stress is the gradient of the elastic energy with respect to the strain, while the viscous stress is the gradient of Rayleigh's dissipation function with respect to the rate of deformation. Neither property carries over in a rational generalization. In the theory of elasticity the Eulerian stress  $t^i_j$  remains expressible in terms of derivatives of the elastic energy with respect to the strain,

<sup>(21)</sup> E. g. [1937, 1, p. 251].

<sup>(22)</sup> Thus the criterion for the validity of the classical linear stress-strain relations in elasticity is simply that the *strain be small*, in contrast to the more complicated criterion (8.5) for the validity of the classical linear expressions for the stress as a function of the rate of deformation in fluid dynamics. Several authors [1874, 1, p. 109], [1888, 1, § 466], [1931, 1, Chap. I], [1933, 1, p. 359], [1945, 1, p. 355] in pursuing a false analogy between elasticity and fluid dynamics have spoken of "small velocities", "small relative velocities", and "small rates of strain", etc. in the present connection. Since none of these quantities is dimensionless, the statements are meaningless, except perhaps as loose references to a limit process.

but no longer as a simple gradient. In the present theory of fluids the viscous stresses are no longer derivable from any scalar by differentiation, and no Rayleigh function exists. To prove this last statement we need only observe that a scalar function of  $\mathbf{d}$  alone can contain (in the isotropic case) at most three terms of third order in the components  $d_{ij}$ , since  $I_a^3$ ,  $I_a II_a$ , and  $III_a$  form a complete set of third order invariants of  $\mathbf{d}$ . An expression derived from a Rayleigh function by differentiation can then contain at most three third order coefficients of viscosity, while in (7.7) there are four. As more terms, are considered, the discrepancy in number increases, and the difference in form becomes more striking, a difference to be expected because of the different physical situations considered. The non-linear theories are thus of greater conceptual clarity than the classical linear theories, whose simplicity is misleading.

**11. THE REINER-RIVLIN FLUID.** — Reiner<sup>(23)</sup> bases his general theory of fluids upon Stokes's principle, and hence for an isotropic fluid obtains an expression of type (6.5) for the viscous stresses

$$(11.1) \quad t_j = -p \delta^i_j + \mathcal{F}_0 \delta^i_j + \mathcal{F}_1 d^i_j + \mathcal{F}_2 d^i_k d^k_j,$$

where  $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 = f(I_a, II_a, III_a)$ . Rivlin<sup>(24)</sup> gives a similar analysis for incompressible fluids, and hence obtains

$$(11.2) \quad t_j = -p \delta^i_j + \mathcal{F}_1 d^i_j + \mathcal{F}_2 d^i_k d^k_j,$$

where  $\mathcal{F}_1, \mathcal{F}_2 = f(II_a, III_a)$ ; here the term  $\mathcal{F}_0 \delta^i_j$  has been incorporated into the arbitrary pressure  $-p \delta^i_j$ . According to Reiner, the coefficients "while generally be functions of the hydrostatic pressure or of the density of the material," while according to Rivlin the coefficients in the power series expansions of the  $\mathcal{F}_r$  are "constants characterizing the fluid considered." Both authors imply also that these coefficients depend upon the temperature.

Now

$$(11.3) \quad \dim \frac{\mathcal{F}_2}{\mathcal{F}_1} = \mathbf{T}.$$

<sup>(23)</sup> [1945, 1].

<sup>(24)</sup> [1947, 2], [1948, 4].

Hence if the leading coefficients  $\mathfrak{F}_2(o, o, o)$  and  $\mathfrak{F}_1(o, o, o)$  in the power series for  $\mathfrak{F}_2$  and  $\mathfrak{F}_1$  be material constants, (11.3) demonstrates *the existence of a natural time  $t_n$*  in the Reiner-Rivlin theory. Similar reasoning does not apply to the Stokesian fluid, since the coefficients  $\mathfrak{F}_{\gamma\beta\gamma\alpha}$  in the series for the  $\mathfrak{F}_\gamma$  are generally *not* material constants. Neither Reiner nor Rivlin comments upon the dimensions of any of the terms, but I believe it a fair interpretation of the quotations above to conclude that both imply the existence of material constants having the same dimensions as the coefficients  $\mathfrak{F}_{\gamma\beta\gamma\alpha}$  in the power series for the  $\mathfrak{F}_\gamma$ . For a fluid with a natural time  $t_n$ , the ratio  $\frac{\mu_n}{t_n}$  is also a material constant, and since  $\dim \frac{\mu_n}{t_n} = \text{ML}^{-1}\text{T}^{-2}$ , we may say that the Reiner-Rivlin theory concerns fluids having a *natural elasticity* (cf. § 4).

To give a complete formal definition of the Reiner-Rivlin fluid, we simply make two additions in the definition of the Stokesian fluid in paragraph 5: First, *add a third material constant  $t_n$  such that*

$$(11.4) \quad \dim t_n = \mathbf{T};$$

second, *in place of (5.3) suppose*

$$(11.5) \quad t_j d^j_i = f(\mu_n, t_n, \theta_0, \rho_m, p, \theta, d_{kl}).$$

Then for the coefficients  $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2$  of an isotropic fluid in place of (7.1) we obtain

$$(11.6) \quad \mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2 = f(\mu_n, t_n, \theta_0, \rho_m, p, \theta, \text{I}_a, \text{II}_a, \text{III}_a).$$

The relations of the type (7.2) now connect nine quantities composed of three fundamental dimensions, and therefore must reduce to relations connecting  $9 - 3 = 6$  dimensionless ratios of these quantities. A simple set of six such ratios is

$$(11.7) \quad t_n \text{I}_a, \quad t_n^2 \text{II}_a, \quad t_n^3 \text{III}_a, \quad \frac{p t_n}{\mu_n}, \quad \frac{p}{\rho_m}, \quad \frac{\theta}{\theta_0},$$

rather than (7.3). In place of the dimensionless functions  $F_0, F_1, F_2$  given by (7.4) it is now preferable to use those resulting from multiplying them by the dimensionless quantities  $\frac{p t_n}{\mu_n}$ ,  $\mathbf{1}$ , and  $\frac{t_n}{p t_n}$ , respectively.

In place of (7.6) and (7.9) we now obtain

$$(11.8) \quad t_j = -p \delta^i_j + \mu_n \left[ \frac{1}{t_n} F_0 \delta^i_j + F_1 \mathbf{d}^i_j + t_n F_2 \mathbf{d}^i_k \mathbf{d}^k_j \right],$$

where the expressions of the coefficients  $F_\gamma$  are

$$(11.9) \quad F_\gamma = F_{\gamma j_1 j_2 \dots j_n} t_n^{j_1 + 2j_2 + \dots + nj_n} \mathbf{I}_d^{j_1} \mathbf{I}_d^{j_2} \dots \mathbf{I}_d^{j_n},$$

the coefficients  $F_{\gamma j_1 j_2 \dots j_n}$  being dimensionless functions of  $\frac{\rho t_n}{\mu_n}$ ,  $\frac{\rho}{\rho_m}$ ,  $\frac{\theta}{\theta_0}$ . By (5.4),  $F_{0000} = 0$ . Here too the classical Newton-Cauchy-Poisson law results if only the linear terms be retained. Notice also that *the classical viscous compressible fluid is the limiting case  $t_n \rightarrow 0$  of the Reiner-Rivlin fluid.*

For the Reiner-Rivlin fluid it is not the number  $\mathfrak{J}$  defined by (8.1), but rather the number  $\mathfrak{R}_r$  :

$$(11.10) \quad \mathfrak{R}_r = t_n d,$$

which is of primary importance in similarity considerations. The criterion for the validity of the classical theory becomes  $\mathfrak{R}_r \ll 1$ , or  $t_n^{-1} \ll d$  : *the classical theory becomes inadequate as the vorticity or the rate of déformation approaches the natural frequency  $t_n^{-1}$  of the fluid.* Thus the higher order terms in the Reiner-Rivlin theory represent “memory” effects, while those in the theory of the Stokesian fluid are appropriate for a fluid without “memory” subjected to low pressures.

**12. AN EXAMPLE : SHEARING FLOW.** — To illustrate the difference between the Stokesian and Reiner-Rivlin fluids, we now consider the case of a rectilinear shearing flow

$$(12.1) \quad \dot{x} = f(y), \quad \dot{y} = 0, \quad \dot{z} = 0;$$

following the analysis of Rivlin <sup>(25)</sup>. All the components of  $\mathbf{d}$  except  $d^x_y = d_y^x$  vanish

$$(12.2) \quad \mathbf{d} = \begin{vmatrix} 0 & \frac{1}{2} f' & 0 \\ \frac{1}{2} f' & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

<sup>(25)</sup> [1948, 4, §15].

and we have

$$(12.3) \quad \text{I}_a = 0, \quad \text{II}_a = -\frac{1}{4}f'^2, \quad \text{III}_a = 0.$$

By (11.2), then

$$(12.4) \quad \begin{cases} v^x_x = v^y_y = \bar{\mathfrak{F}}_2 \left( -\frac{f'^2}{4}, 0 \right), \\ v^x_y = \bar{\mathfrak{F}}_1 \left( -\frac{f'^2}{4}, 0 \right) \frac{f'}{2}. \end{cases}$$

Notice that  $v^x_y$  is an *odd* function<sup>(26)</sup> of  $d^x_y$ , while  $v^x_x$  and  $v^y_y$  are *even*<sup>(27)</sup> functions of  $d^x_x$ . Since  $\ddot{x}^i = 0$ , when  $f^i = 0$  the dynamical equation become simply  $t^{ij}_{,j} = 0$ . Hence  $v^x_y = C_1$ , where  $C_1$  is a constant specifying the uniform resistance per unit area. Thus (12.4<sub>3</sub>) yields

$$(12.5) \quad f' \bar{\mathfrak{F}}_1 \left( -\frac{f'^2}{4}, 0 \right) = 2C_1,$$

or, in the notation (11.9),

$$(12.6) \quad \mu_n f'_{10,30} \left( \frac{t_n f'^2}{4} \right)^{\bar{3}} = 2C_1.$$

Thus such a flow is possible only when the ratios  $\frac{\mu_n}{\rho t_n}$ ,  $\frac{\theta}{\theta_0}$ , and  $\frac{p}{P_m}$  are functions of  $y$  alone. For simplicity, suppose indeed that the coefficients  $F_{10,30}$  may be taken as constant. Then for any finite approximation, there are an odd number of possible velocity profiles  $f'$

<sup>(26)</sup> Cf. § 18, <sup>(12)</sup> of the succeeding Memoir.

<sup>(27)</sup> Here we may take note of the theory of Girault [1931, 1, Chap. III], who attempted to generalize the Navier-Stokes equations by means of Stokes's principle, but at once added the assumption that when a flow is reversed all the stresses change sign. Thus he concluded that no terms of even degree in the  $d^i_j$  can occur in the general expression for  $t^i_j$ . Premise and conclusion are alike false. In the shearing flow under consideration, for example, if the flow is reversed the shearing stress  $t^x_y$  should indeed change sign, but if the cross stress  $t^y_x$  should change it would imply the absurd conclusion that if pressure upon the  $y = \text{const.}$  planes be required to maintain such a flow from left to right, then tension is required for a similar flow from right to left. Girault's theory fails altogether to reveal the basically important phenomenon of cross-viscosity, and its application by Viguier [1947, 3] to boundary layer problems thus cannot yield correct results.

satisfying (12.6) and yielding the *same value of the resistance*  $C_1$  : all are linear. In the case of a second order theory only the classical  $f = \frac{2C_1\gamma}{\mu_n F_{1000}} + C_2$  is possible, <sup>(28)</sup> but in a third order theory there may be one or three profiles, depending upon the values of  $C_1, F_{1000}, F_{1010}, \mu_n$ , and  $t_n$ . It is not unlikely that of all the possibilities for a given  $C_1$  only one will satisfy the thermodynamic requirement that the dissipation of energy be positive <sup>(29)</sup>. In any case the pressure  $p$  is determined from the equation

$$(12.7) \quad v_y = -p + \mathfrak{F}_2\left(-\frac{f'^2}{4}, 0\right) \frac{f'^2}{4} = C_3,$$

where  $C_3$  is a constant. Since  $f'$  is constant,  $p$  may assume an arbitrary constant value. Just as in the classical theory, then, we may select the pressure and the specific resistance at will. The general case is distinguished by : (A) *Cross-viscosity* : a normal stress  $v_y$  must be exerted on the shearing planes in order to produce the flow, and there is a normal stress  $v_x$  in the direction of flow, and (B) *Multiplicity of solution* : more than one linear velocity profile yielding the same resistance, but different cross-stresses, may exist. Notice that the *resistance* as a function of the profile  $f'$ , which is the classical subject of measurement, is *independant of the second coefficient*  $\mathfrak{F}_2$ , and its *departure from the classical value is an effect of third order in the number*  $\mathfrak{A}_r \equiv t_n d$ ; the *second order effect of cross-viscosity depends only on*  $\mathfrak{F}_2$  and has no influence upon the result of the classical measurement. So much for the Reiner-Rivlin fluid.

For a similar flow in a Stokesian compressible fluid, again we deduce (12.4), but by (7.6) and (7.9) these conditions now take the form.

$$(12.8) \quad \left\{ \begin{array}{l} \frac{\mu_n}{2} f' F_{10} \mathfrak{J}_0 \left(-\frac{\mu_n^2 f'^2}{4p^2}\right) \mathfrak{J} = C_1, \\ \left(p F_{00} \mathfrak{J}_0 + \frac{\mu_n^2 f'^2}{4p} F_{20} \mathfrak{J}_0\right) \left(-\frac{\mu_n^2 f'^2}{4p^2}\right) \mathfrak{J} = C_3. \end{array} \right.$$

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<sup>(28)</sup> Boussinesq [1868, 1, Note I] observed that in a second order theory the resistance is the same as in the classical theory.

<sup>(29)</sup> Cf. § 29 of the following Memoir.

The possible velocity profiles  $f'$  now depend upon the pressure  $p$  as well as the resistance  $C_1$ ; or, otherwise expressed, *for a given velocity profile  $f'$  the resistance depends upon the pressure*. As the pressure is reduced, the resistance is diminished, as should be expected in a gas. Both in the classical theory and in the second approximation we have  $\mu_n f' F_{1000} = 2C_1$ , but in the third approximation *the resistance is diminished by  $\frac{\mu_n^2 f'^2 F_{1010}}{8\rho^2}$  from the value predicted by the classical theory for the same velocity profile  $f'$* .

**15. HEURISTIC DISCUSSION OF HEAT FLOW.** — In order to solve nearly any sort of boundary problem concerning a compressible fluid it is necessary to employ not only the dynamical equation but also the energy equation. Since the energy equation contains the heat flux vector  $q_i$ , some sort of definite form for this vector in terms of the other variables is required.

Now the flow of heat is most evidently associated with temperature differences. According to the celebrated hypothesis of Fourier<sup>(30)</sup>, when a conducting material is confined between infinite parallel plane boundaries, which are maintained at a constant temperature difference  $\Delta\theta$ , heat energy must be supplied at a rate given by

$$(13.1) \quad \frac{\text{heat flow}}{\text{area}} = -\alpha \frac{\Delta\theta}{\text{distance between walls}},$$

where  $\alpha$  is a quantity called the *coefficient of thermal conductivity*, which is independent of the other magnitudes in the formula, but dependent upon the particular substances. The analogy between (4.6) and (13.1) is immediate; heat flow being the analogue of force (or heat energy the analogue of momentum) and temperature the analogue of velocity. The dimensions of  $\alpha$  are  $\mathbf{MLT}^{-3}\Theta^{-1}$ . The differential form of (13.1) for thermally isotropic media is<sup>(31)</sup>

$$(13.2) \quad q_i = -\alpha\theta_{,i}.$$

We are not concerned here with the exactitude of Fourier's law

<sup>(30)</sup> [1822, 1, § 65].

<sup>(31)</sup> [1822, 1, § 138].

(13.2); suffice that a crude experiment indicates it to be a rough approximation, for our only purpose at the moment is to indicate that in general the heat flux  $q_i$  should depend upon the temperature gradient  $\theta_{,i}$  and a quantity  $\alpha$  of dimensions  $\mathbf{MLT}^{-3}\Theta^{-1}$ . The thermal conductivity  $\alpha$  is observed to be a function of temperature, and more generally we may suppose that in a fluid it depends upon the thermodynamic state.

**14. POSTULATE FOR THE HEAT FLUX VECTOR.** — The foregoing remarks suggest that our definition of the Stokesian fluid of paragraphe 5 should be amplified by the following three further postulates :

IV. *There exists a material constant  $\alpha_n$ , called the natural conductivity, whose dimensions are*

$$(14.1) \quad \dim \alpha_n = \frac{\mathbf{ML}}{\mathbf{T}^3\Theta}.$$

V. *The heat flux  $q_i$  is of the functional form*

$$(14.2) \quad q_i = f(\alpha_n, \theta_0, p_m, p, \theta, \theta_{,j}),$$

*and is analytic an function of  $\theta_{,j}$*

VI. *If  $\theta_{,j} = 0$  then  $q_i = 0$ .*

The natural conductivity  $\alpha_n$  is a constant, not a function of temperature, and thus is not to be identified with the ordinary conductivity  $\alpha$ . Indeed, it will appear [cf. (16.3)] that  $\alpha_n$  is the dimensional part of  $\alpha$ , just as  $\mu_n$  is the dimensional par of  $\mu$ .

**15. THERMALLY ISOTROPIC FLUIDS.** — From (14.2) and the assumed analyticity of  $q_i$  we have

$$(15.1) \quad q_i = k_i + k_i^j \theta_{,j} + k_i^{jk} \theta_{,j} \theta_{,k} + \dots,$$

where  $k_i, k_i^j, \dots$ , are certain tensors independent of  $\theta_{,i}$ . By VI of paragraphe 14,  $k_i = 0$ . A fluid is *thermally isotropic* if the coefficients in the expansion (15.1) reduce to numerical tensors. Now the only numerical tensors at our disposal are multiples and products of  $\delta_{ij}^i, \delta_{ij}^k, \dots$ , and  $\varepsilon^{ijk}$ . Hence (15.1) reduces to

$$(15.2) \quad q_i = \mathcal{G} \theta_{,i}$$



where the function  $\mathcal{G}$  is of the form

$$(15.3) \quad \mathcal{G} = f(I_0, x_n, \theta_0, p_m, p, \theta),$$

where  $I_0 \equiv \theta_i \theta^i$ .

**16. THE FORM OF THE COEFFICIENT  $\mathcal{G}$ .** — Now the quantity  $\frac{\mathcal{G}}{x_n}$  is dimensionless, and hence the relation

$$(16.1) \quad \frac{\mathcal{G}}{x_n} = f(I_0, x_n, \theta_0, p_m, p, \theta)$$

connects six quantities composed of four independent dimensions, and so must reduce to a relation connecting  $6 - 4 = 2$  dimensionless ratios of these quantities. For the desired ratios we may select  $\frac{p}{p_m}, \frac{\theta}{\theta_0}$ . Thus (15.2) becomes simply

$$(16.2) \quad q_i = -x \theta_{,i}$$

where

$$(16.3) \quad \frac{x}{x_n} = f\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right);$$

*the Stokesian fluid obeys Fourier's law of heat conduction.*

**17. PROVISIONAL CONCLUSIONS.** — It may at first sight seem somewhat surprising that in the theory of the Stokesian fluid while the classical Newton-Cauchy-Poisson law for the stress turns out to be only a first approximation, nevertheless Fourier's law for the heat flux remains unchanged. The underlying reason for the difference is the separation of stress and heat flux as associated each separately with only one macroscopic phenomena-velocity gradients give rise to stresses and thermal gradients give rise to heat flow. This distinction, which we have embodied in our defining equations (5.3) and (14.2), from a physical point of view is an artificial one, which in reality should appear only as a first approximation to a broader concept of fluidity. The formulation and development of this broader concept is the subject of the succeeding Memoir.

APPENDIX.

PROPERTIES OF CONTINUOUS MEDIA IN GENERAL.

We summarize here these properties of continuous media with which the reader of the foregoing memoir has been presumed to be familiar. If  $ds^2$  be the squared element of arc length and  $\dot{\phantom{x}}$  the symbol of material differentiation, we have

$$(A.1) \quad \dot{\overline{ds^2}} = 2 d_{ij} dx^i dx^j,$$

where the *rate of deformation tensor*  $d_{ij}$  is given by

$$(A.2) \quad d_{ij} \equiv \frac{1}{2} (\dot{x}_{i,j} + \dot{x}_{j,i}),$$

$\dot{x}_i$  being the velocity vector. Thus the components  $d_{ij}$  are measures of the local and instantaneous rates at which the medium is suffering deformation. Their physical components are of dimension  $\mathbf{T}^{-1}$ , where  $\mathbf{T}$  is a unit of time. Similarly the components  $\omega_{ij}$  given by

$$(A.3) \quad \omega_{ij} \equiv \frac{1}{2} (\dot{x}_{i,j} - \dot{x}_{j,i})$$

are measures of the *rate of rotation* (vorticity). The principle of *conservation of mass* (or *continuity of motion*) is expressed by Euler's equation

$$(A.4) \quad \dot{\log \rho} + \dot{x}^i{}_{,i} = 0,$$

where  $\rho$  is the *density*, whose dimensions are  $\mathbf{ML}^{-3}$ ,  $\mathbf{M}$  being a unit of mass and,  $\mathbf{L}$  a unit of length. The principle of *conservation of momentum* is expressed by Cauchy's equations

$$(A.5) \quad t^{i,j}{}_{,j} + \rho f^i = \rho \ddot{x}^i, \quad t^{ij} = t^{ji},$$

where  $t^{ij}$  is the *stress tensor* and  $f^i$  is the *extraneous force vector*. The dimensions of the physical components of  $t^i_j$  are  $\mathbf{ML}^{-1}\mathbf{T}^{-2}$ . The *mean pressure*  $p_m$  is given by the definition

$$(A.6) \quad p_m \equiv -\frac{1}{3} t^i{}_i.$$

Let  $\varepsilon$  be the *internal energy* (per unit mass). Then the principle of *conservation of energy* implies that the rate of change of the total energy (internal plus kinetic) of any material volume must equal the sum of the rate at which the extraneous force does work upon it, the rate at which the stresses upon its boundary do work, and the rate at which thermal energy flows into it from the exterior, in terms of the *heat flux vector*  $q^i$ . The resulting equation may be simplified by (A.5), yielding finally the Fourier-Kirchhoff-C. Neumann equation

$$(A.7) \quad \rho \dot{\varepsilon} = t_j d^j_i - q^{i,i}.$$

Thus  $t_j d^j_i$ , the *stress power*, is the rate at which the stress does work, per unit volume. Note that the stress power vanishes in any rigid motion ( $d^j_i = 0$ ). The equation (A.7) may be taken as a definition of the internal energy if we please, for from it the value of  $\varepsilon$  for each particle may be calculated up to an arbitrary constant, if the other variables be known. The dimensions of  $\varepsilon$  are  $\mathbf{L}^2 \mathbf{T}^{-2}$ ; those of  $q^i$  are  $\mathbf{M} \mathbf{T}^{-3}$ . Usually it is assumed further that each particle of the continuum constitutes a *thermodynamic system*; that is, regardless of the state of motion each particle has an *equation of state*

$$(A.8) \quad \varepsilon = \varepsilon(\nu; \eta),$$

where  $\varepsilon$  is the *internal energy* (per unit mass),  $\nu$  is the *specific volume* ( $\nu \equiv \frac{1}{\rho}$ ), and  $\eta$  is the *entropy* (per unit mass). The form of (A.8) may vary from one particle to another, but it is to remain constant in time; that is, for the present simple analysis the medium may be heterogeneous, but if changes of phase, chemical reactions, etc., are to be included, further elaborations are required. Then if the *pressure*  $\pi$  and *temperature*  $\theta$  be defined by

$$(A.9) \quad \pi \equiv - \left( \frac{\partial \varepsilon}{\partial \nu} \right)_{\eta}, \quad \theta \equiv \left( \frac{\partial \varepsilon}{\partial \eta} \right)_{\nu},$$

we have

$$(A.10) \quad \dot{\varepsilon} = \theta \dot{\eta} - \pi \dot{\nu}.$$

The dimensions of  $\pi$  so defined are  $\mathbf{M} \mathbf{L}^{-1} \mathbf{T}^{-2}$ , the same as the dimensions of  $t_j$  and of  $p_m$ . While we are here considering the

entropy as a primitive variable, in accord with common physical practice we shall take a unit of temperature  $\Theta$  as the fundamental thermal dimension; then the dimensions of  $\eta$  are  $\mathbf{L}^2 \mathbf{T}^{-2} \Theta^{-1}$ . Let an *extra stress*  $v^i_j$  be defined by

$$(A.11) \quad v^i_j \equiv p \delta^i_j + t^i_j$$

where  $p$  is an arbitrary scalar. Cauchy's laws (A.5) now become

$$(A.12) \quad v^i_{j,i} - p_{,i} + \rho f_i = \rho \dot{x}_i, \quad v_{ij} = v_{ji}.$$

Combination of (A.7), (A.10), and (A.11) yields

$$(A.13) \quad \rho \theta \dot{\eta} = (\pi - p) \overline{\log v} + \Phi - q^i_{,i}$$

where  $\Phi$  is given by

$$(A.14) \quad \Phi \equiv v^i_j \dot{d}i_i.$$

We have

$$(A.15) \quad t^i_j \dot{d}i_i = \Phi + (-p \overline{\log v}).$$

While it is tempting to regard this equation as a decomposition of the stress power into a portion associated with change of shape plus a portion associated with change of volume, since  $p$  remains up to this point an arbitrary scalar, no specific interpretation can yet be justified. Here we observe a fundamental distinction. For an incompressible medium (A.13) reduces simply to

$$(A.16) \quad \rho \theta \dot{\eta} = \Phi - q^i_{,i}.$$

Thus, *for an incompressible substance, whatever the choice of the scalar  $p$  in the decomposition (A.11),  $\Phi$  is the rate at which mechanical work is being transformed into thermal energy (per unit volume).* For a compressible substance, however, in order to obtain the simple equation (A.15) and the consequent unique interpretation of  $\Phi$ , it is necessary and sufficient to add the definition

$$(A.17) \quad p \equiv \pi,$$

which reduces the number of unknowns by one. The total number of unknowns, however, is the same for compressible and for incom-

pressible media, since for the latter  $\rho$  is a given function or constant, while for the former it is an unknown. With this distinction, then, the same formal structure may be employed for describing the thermodynamics of both compressible and incompressible media, although the meanings of some of the variables are not quite the same in the two cases. When (A.17) is adopted for compressible media, the decomposition (A.11) acquires primary physical significance, for it resolves the stress into two portions, the first,  $-p\delta^i_j$ , being that whose work (if any) is always mechanically recoverable, and the second,  $v^i_j$ , being that whose work<sup>(32)</sup> in deforming the medium is irretrievably lost as heat, which may be stored in terms of entropy or conducted away. The decomposition (A.15) similarly resolves the stress power into dissipated and recoverable portions; part of the work done in producing a change of volume is generally recoverable, while part is not. It cannot be too strongly emphasized that: (1) no resolution of the stress other than that consequent upon (A.17) can yield the simple energy equation (A.16) for compressible media; (2) both  $\pi$  and  $p_m$  are *defined* variables whose equations of definition (A.6) and (A.9) show them to represent quite different physical concepts, and there is no reason whatever *a priori* to suppose them equal or even related to each other, and (3) the power loss  $\Phi$  is a *defined* quantity which exists in all continua, requiring no special hypotheses<sup>(33)</sup>.

Finally, the irreversibility of many processes is expressed by the Clausius-Duhem inequality

$$(A.18) \quad \int_{\mathcal{V}} \rho \dot{\eta} dV + \oint_{\mathcal{S}} \frac{q^i dS_i}{\theta} \geq 0,$$

where  $V$  is an arbitrary volume and  $\mathcal{S}$  its bounding surface.

<sup>(32)</sup> It would not be correct to conclude that from  $v^i_j \neq 0$  it necessarily follows that dissipation takes place, for it is possible that  $v^i_j$  does no work, i. e.  $v^i_j dl_i = 0$ .

<sup>(33)</sup>  $\Phi$  is not to be confused with Rayleigh's dissipation function, which for continuous media could better be called a "stress potential", and exists only in very special circumstances.

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