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Boundedness in a topological space;

BY SZE-TSEN HU (¹).

In 1939, J. W. Alexander [1] (²) introduced the notion of boundedness into the realm of general topology. It is a contribution of real importance, for it opens a vas region of investigation which is so far foreign to topology. However, this promising Note of J. W. Alexander has been completely neglected during the last seven years since its publication. A possible reason of its being neglected might be that boundedness is not a topological invariant. But, not all properties studied in topology are invariants under homeomorphisms; uniformity is an outstanding example.

The object of the present work is to give a detailed axiomatic approach of boundedness in general topology and its consequences. The original definition of J. W. Alexander yields an unsatisfactory result that every non-bounded point of the space must be an isolated point a fact which does not agree with the usual notion of geometry, as for example, the points at infinity of a projective plane form a line at infinity but not a set of isolated points. Instead of combining boundedness within the definition of a topological space as J. W. Alexander did, we consider a topological space given *a priori* and introduce a boundedness by picking up a family of subsets, called bounded sets. Thus the topology of a space is independent

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(²) Numbers in brackets denote references in the bibliography at the end of the paper.

of the boundedness introduced, and the latter is rather a superstructure built upon the given space.

With regard to topological spaces, no separation axiom is assumed unless explicitly staded. Following N. Bourbaki, we shall denote by $\emptyset$ the empty set, by $\overline{M}$ and $\mathring{M}$ respectively the closure and the interior of the set M. Following S. Lefschetz, continuous transformations will be called *mappings*.

1 Boundedness and universes. — Definition 1.1. — *A boundedness in a topological space* X *is a non-void family of subsets* $\{B\}$ *of* X, *called the bounded sets of* X, *satisfying :*

(1.11) *every subset of a bounded set is bounded;*

(1.12) *the union of a finite number of bounded sets is bounded.*

From (1.11) it follows immediately that

(1.13) $\emptyset$ *is bounded*;

(1.14) *the intersection of a non-void collection of bounded sets is bounded.*

Definition 1.2. — *A universe is a topological space with a given boundedness.*

Definition 1.3. — *A universe* X *is said to be bounded, if the whole space* X *is a bounded set and hence every set is bounded.*

The following theorem is trivial.

Theorem 1.4. — *If a universe* X *with a boundedness* $\{B\}$ *is not bounded, then the family* $\{X - B\}$ *of the complementary sets* $X - B$, $B \in \{B\}$, *has the following properties :*

(1.41) *none of the sets* $X - B$ *is bounded;*

(1.42) $\{X - B\}$ *form a filter* [3, p. 20], *called the filter at infinity of the universe* X.

2. Calculus of boundedness. — Since a boundedness in a space X is but a family of subsets, the calculus of boundedness in a given space X can be naturally introduced by the analogue of the calculus of sets. Throughout this paper, German capitals are only used to denote families of subsets, e. g. boundedness, coverings, filters, etc...

Definition 2.1. — *Given two boundedness $\mathfrak{A}$ and $\mathfrak{B}$ in a given topological space* X, *we say that $\mathfrak{A}$ is stronger than $\mathfrak{B}$ and $\mathfrak{B}$ is weaker than $\mathfrak{A}$, if $\mathfrak{A} \supset \mathfrak{B}$.*

It is clear that the system of all the boundedness in X is properly ordered by $\supset$, [7, p. 3]. The weakest boundedness is that which consists only one member, i. e. the only bounded set is $\emptyset$; the strongest is the one in which X itself is a bounded set and hence every set is bounded.

Theorem 2.2. — *Given an arbitrary family $\mathfrak{A} = \{A\}$ of subsets of a topological space* X, *there exists a weakest boundedness $\mathfrak{B} = \{B\}$ in* X *containing $\mathfrak{A}$, which will be called the boundedness generated by $\mathfrak{A}$.*

Proof. — Let $\mathfrak{B} = \{B\}$ denote the family of subsets of X which consists of the totality of the subsets of the finite unions of the familily $\mathfrak{A}$. It is easily seen that $\mathfrak{B}$ contains $\mathfrak{A}$ and is a boundedness.

On the other hand, let $\mathfrak{C}$ be an arbitrary boundedness which contains the family $\mathfrak{A}$. By (**1**.12), $\mathfrak{C}$ contains every finite union of $\mathfrak{A}$; then by (**1**.11), $\mathfrak{C}$ contains $\mathfrak{B}$. Hence $\mathfrak{C}$ is stronger than $\mathfrak{B}$, and our theorem is proved Q. E. D.

Definition 2.3. — *Given a system $\mathfrak{B}_i$ of boundedness in a topological space* X. *indexed by a set* I, [5, p. 3], *the two boundedness generated by the union $\bigcup_{i \in I} \mathfrak{B}_i$ and by the intersection $\bigcap_{i \in I} \mathfrak{B}_i$ are respectively called the join $\bigvee_{i \in I} \mathfrak{B}_i$ and the meet $\bigwedge_{i \in I} \mathfrak{B}_i$ of the given system of boundedness $\mathfrak{B}_i$, $i \in I$.*

Theorem 2.4. — $\bigwedge_{i \in I} \mathfrak{B}_i = \bigcap_{i \in I} \mathfrak{B}_i$.

Proof. — It is enough to prove that the intersection $\bigcap_{i\in I} \mathcal{B}_i$ is a boundedness. Let $B \in \bigcap_{i\in I} \mathcal{B}_i$, then $B \in \mathcal{B}_i$ for each $i \in I$; hence by (1.11), every subset of B belongs to each $\mathcal{B}_i$, $i \in I$. Therefore, $\bigcap_{i\in I} \mathcal{B}_i$ satisfies (1.11). Similarly, $\bigcap_{i\in I} \mathcal{B}_i$ also statisfies (1.12).

Theorem 2.5. — *The join $\bigvee_{i\in I} \mathcal{B}_i$ consists of all the subsets of* X *which are of the form $\bigcup_{i\in I} B_i$, where $B_i \in \mathcal{B}_i$ for each $i \in I$ and at most a finite number of the B_i are different from the empty set $\emptyset$.*

Proof. — Let B be a set of the form described in the theorem, then B is a finite union of the family $\bigcup_{i\in I} \mathcal{B}_i$. Hence we have $B \in \bigvee_{i\in I} \mathcal{B}_i$.

Conversely, suppose A be an arbitrary set of the boundedness $\bigvee_{i\in I} \mathcal{B}_i$, then A is by definition a subset of a finite union of $\bigcup_{i\in I} \mathcal{B}_i$. Since each B_i satisfies (1.11), A itself is a finite union of $\bigcup_{i\in I} \mathcal{B}_i$. Suppose $A = A_1 \cup A_2 \cup \ldots \cup A_q$, where $A_1, A_2, \ldots, A_q$ are members of $\bigcup_{i\in I} \mathcal{B}_i$. For each $A_p (p = 1, 2, \ldots, q)$, we choose an $i_p \in I$ such that $A_p \in \mathcal{B}_{i_p}$. Let $B_i = \bigcup_{i_p = i} A_p$, then $A = \bigcup_{i\in I} B_i$ is of the described form.

Theorem 2.6. — *The boundedness $\mathcal{B}_1 \vee \mathcal{B}_2 \vee \ldots \vee \mathcal{B}_g$ consists of all the subsets of* X *which are of the form $B_1 \cup B_2 \cup \ldots \cup B_q$, where $B_p \in \mathcal{B}_p (p = 1, 2, \ldots, q)$. The boundedness $\mathcal{B}_1 \wedge \mathcal{B}_2 \wedge \ldots \wedge \mathcal{B}_g$ consists of all the subsets of* X *which are of the form $B_1 \cap B_2 \cap \ldots \cap B_q$, where $B_p \in \mathcal{B}_p (p = 1, 2, \ldots, q)$.*

Proof. — This theorem is an immediate consequence of (2.5) and (2.4).

THEOREM 2.7. — *The system of all boundedness in a topological space* X *is directed both by* $\supset$ *and by* $\subset$ [7, p. 10].

PROOF. — For an arbitrary pair of boundedness $\mathcal{A}$ and $\mathcal{B}$ in X, the join $\mathcal{A} \vee \mathcal{B}$ is stronger than both of them and the meet $\mathcal{A} \wedge \mathcal{B}$ is weaker than both of them.

3. THE CLOSURE AND THE INTERIOR OF A BOUNDEDNESS. DEFINITION 3.1. — *Given a boundedness* $\mathcal{B} = \{B\}$ *in a topological space* X, *the boundedness* $\overline{\mathcal{B}}$ *generated by the family* $\{\overline{B}\}$ *is called the closure of* $\mathcal{B}$ *and the boundedness* $\mathring{\mathcal{B}}$ *generated by the family* $\{\mathring{B}\}$ *is called the interior of* $\mathcal{B}$.

The following two theorems are trivial.

THEOREM 3.2. — *For an arbitrary boundedness* $\mathcal{B}$ *in a topological space* X, *we always have* $\mathring{\mathcal{B}} \subset \mathcal{B} \subset \overline{\mathcal{B}}$.

THEOREM 3.3. — *For any two boundedness* $\mathcal{A}$ *and* $\mathcal{B}$ *in a topological space* X, $\mathcal{A} \supset \mathcal{B}$ *implies* $\overline{\mathcal{A}} \supset \overline{\mathcal{B}}$ *and* $\mathring{\mathcal{A}} \supset \mathring{\mathcal{B}}$.

DEFINITION 3.4. — *A boundedness* $\mathcal{B}$ *is said to by closed if* $\overline{\mathcal{B}} = \mathcal{B}$, *open if* $\mathring{\mathcal{B}} = \mathcal{B}$, *and proper if it is both closed and open. A universe* X *is said to be closed, open, or proper, according as its boundedness is closed, open, or proper respectively.*

THEOREM 3.5. — *For a given boundedness* $\mathcal{B}$ *in a topological space* X, *the following conditions are equivalent:*

(3.51) $\mathcal{B}$ *is closed*;
(3.52) $\mathcal{B}$ *is generated by its subfamily of the bounded closed sets*;
(3.53) *the closure of every bounded set is a bounded set.*

PROOF. — (3.51) $\rightarrow$ (3.52). Since $\mathcal{B}$ is closed, $\overline{\mathcal{B}} = \mathcal{B}$; therefore, $\{\overline{B}\} \subset \mathcal{B}$. On the other hand, if B is a bounded closed set, then $B = \overline{B} \in \{\overline{B}\}$. Hence $\{\overline{B}\}$ is the subfamily of the bounded closed sets, and $\mathcal{B} = \overline{\mathcal{B}}$ is generated by $\{\overline{B}\}$.

(3.52) $\rightarrow$ (3.53). Let B be an arbitrary bounded set of $\mathcal{B}$.

Since $\mathcal{B}$ is generated by its subfamily $\{F\}$ of the bounded closed sets, B is a subset of a finite union of bounded closed sets. Hence there exists a bounded closed set F such that $B \subset F$. It follows that the closure $\overline{B} \subset F$ is a bounded set.

(5.53) $\rightarrow$ (5.51). Since $\overline{B} \in \mathcal{B}$ for each $B \in \mathcal{B}$, we obtain $\overline{\mathcal{B}} \subset \mathcal{B}$. By (5.2), $\overline{\mathcal{B}} \supset \mathcal{B}$; hence $\overline{\mathcal{B}} = \mathcal{B}$. Q. E. D.

Remark. — It is easy to see that the boundedness originally introduced by J. W. Alexander is a special case of closed boundedness.

THEOREM 5.6. — *For a given boundedness $\mathcal{B}$ in a topological space* X, *the following conditions are equivalent :*

(5.61) *$\mathcal{B}$ is open;*

(5.62) *$\mathcal{B}$ is generated by its subfamily of the bounded open sets;*

(5.63) *every bounded set is contained in the interior of some bounded set.*

PROOF. — (5.61) $\rightarrow$ (5.62). It is clear that $\{\mathring{B}\}$ is the subfamily of the bounded open sets of $\mathcal{B}$. Since $\mathcal{B}$ is open, $\mathring{\mathcal{B}} = \mathcal{B}$; hence $\mathcal{B}$ is generated by $\{\mathring{B}\}$.

(5.62) $\rightarrow$ (5.63). Let B be an arbitrary bounded set of $\mathcal{B}$. Since $\mathcal{B}$ is generated by its subfamily $\{G\}$ of the bounded open sets, B is a subset of a finite union of bounded open sets. Hence there exists a bounded open set G such that $B \subset G$. Hence (5.63).

(5.63) $\rightarrow$ (5.61). Since every bounded set is contained in the interior of some bounded set, then $\mathring{\mathcal{B}} \subset \mathcal{B}$. By (5.2), $\mathring{\mathcal{B}} \subset \mathcal{B}$; hence $\mathring{\mathcal{B}} = \mathcal{B}$ and $\mathcal{B}$ is open.

Combining (5.5) and (5.6), we obtain the following theorem concerning proper boundedness.

THEOREM 5.7. — *For a given boundedness $\mathcal{B}$ in a topological space* X, *the following conditions are equivalent :*

(5.71) *$\mathcal{B}$ is proper;*

(5.72) *$\mathcal{B}$ is generated both by its subfamily of the bounded closed sets and by its subfamily of the bounded open sets;*

(**3**.73) *the closure of every bounded set is contained in the interior of some bounded set.*

THEOREM **3**.8. — *Given a non-void family* $\{A\}$ *of closed (open) sets of a topological space* X, *satisfying :*

(**3**.81) *every closed (open) subset of a set* A *of* $\{A\}$ *is a set of* $\{A\}$;

(**3**.82) *every finite union of* $\{A\}$ *belongs to* $\{A\}$,

then there exists a unique closed (open) boundedness in X *with* $\{A\}$ *as the family of the bounded closed (open) sets.*

PROOF. — Let us prove the theorem for closed sets $\{A\}$. We define a boundedness in X by calling a subset B of X to be a bounded set, if the closure $\overline{B} \in \{A\}$. Then it is easily verified that $\mathcal{B} = \{B\}$ is the closed boundedness generated by the family $\{A\}$. Now let B be an arbitrary bounded closed set of $\mathcal{B}$, then $B = \overline{B} \in \{A\}$. Conversely, every set $A \in \{A\}$ is a bounded closed set of $\mathcal{B}$. Hence $\{A\}$ is the family of the bounded closed sets of $\mathcal{B}$. The uniqueness of $\mathcal{B}$ follows from the fact that every closed bouudedness is generated by its family of the bounded closed sets. Q. E. D.

THFOREM **3**.9. — *Given a family* $\{F\}$ *of closed sets and a family* $\{G\}$ *of open sets of a topological space* X, *statisfying :*

(**3**.91) $\{F\}$ *and* $\{G\}$ *both satisfy* (**3**.81) *and* (**3**.82);

(**3**.92) *for each* $F \in \{F\}$, *there exists a set* $G_F \in \{G\}$ *with* $G_F \supset F$,

(**3**.93) *for each* $G \in \{G\}$, *there exists a set* $F_G \in \{F\}$ *with* $F_G \supset G$,

then there exists a unique proper boundedness in X *with* $\{F\}$ *and* $\{G\}$ *as its families of bounded closed sets and of bounded open sets respectively. Conversely, the families of bounded closed sets and of bounded open sets of an arbitrary proper boundedness in* X *satisfy* (**3**.91) $\rightarrow$ (**3**.93).

PROOF. — The second part of the theorem follows from (**1**.11), (**1**.12), and (**3**.73). It remains to prove the first part of the theorem.

Let $\mathcal{A}$ be the closed boundedness generated by $\{F\}$, and $\mathcal{B}$ be the open boundedness generated by $\{G\}$, as are described in (**3**.8). For an arbitrary $A \in \mathcal{A}$, there is a set $F \in \{F\}$ such that $A \subset F$. By (**3**.92), there exists a set $G_F \in \{G\}$ such that $G_F \supset F \supset A$; hence $A \in \mathcal{B}$, and $\mathcal{A} \subset \mathcal{B}$. Similarly, we can prove that $\mathcal{A} \supset \mathcal{B}$. Therefore, $\mathcal{A} = \mathcal{B}$ is a proper boundedness in X; and $\{F\}$, $\{G\}$ are the families of the bounded closed sets and of the bounded open sets by (**5**.8).

Q. E. D.

It will be seen in § **15** that the proper boundedness is the most important one. However we shall give two examples to show the existence of non-proper boundedness.

Example (**1**). — *A boundedness which is open but not closed.* — Let X be a Hausdorff space, and x be a non-isolated point of X. Define a boundedness in X by calling a subset B of X to be bounded if B does not contain x. The conditions (**1**.11) and (**1**.12) are trivial. The boundedness $\mathcal{B} = \{B\}$ is open, for each bounded set B is contained in the bounded open set $X - x$. Since x is not isolated, the closure $\overline{X - x}$ of the bounded set $X - x$ is the space X which is not bounded by our definition. Hence $\mathcal{B}$ is not closed.

Example (1) (**2**). — *A boundedness which is closed but not open.* — In the Hilbert space R^ω, let $X_n (n = 1, 2, \ldots)$ denote the closed interval on the x_n-axis defined by $0 \leqq x_n \leqq 1$ and $x_i = 0\ (i \neq n)$. Let X denote the union $\bigcup_{n=1}^{\infty} X_n$. Define a boundedness in X by calling a subset B of X to be bounded if the closure $\overline{B}$ of B in X is compact. The conditions (**1**.11) and (**1**.12) are trivial. The boundedness $\mathcal{B} = \{B\}$ is closed, for the closure $\overline{B}$ of a bounded set B is by definition bounded. That $\mathcal{B}$ is not open will be proved as follows. Since X_1 is compact, it is bounded. It $\mathcal{B}$ is open, then there exists a bounded open set G which contains X_1. Since X_1 is compact, G may be assumed to be an ε-neighbourhood of X_1 in X for a sufficiently small $\varepsilon > 0$. From our definition of the boundedness it follows that $\overline{G}$ should be compact. Let y_n denote the point of X_n with $x_n = \varepsilon$; then $y_n \in G$ for each $n = 1, 2, \ldots$. The sequence $\{y_n\}$ has no cluster point, which is a contradiction to the compactness of $\overline{G}$.

(1) The author is grateful to Dr. A. J. Ward for this example.

4. THE BASES OF A BOUNDEDNESS. — DEFINITION 4.1. — *A subfamily $\mathcal{A} = \{A\}$ of bounded sets of a given boundedness $\mathcal{B} = \{B\}$ in a topological space X is called a basis of $\mathcal{B}$, if every bounded set $B \in \mathcal{B}$ is a subset of some $A \in \mathcal{A}$. $\mathcal{A}$ is called a closed (open) basis of $\mathcal{B}$, if each set $A \in \mathcal{A}$ is closed (open).*

It follows from (3.6) and (3.7) that the family of bounded closed (open) sets of a closed (open) boundedness $\mathcal{B}$ forms a closed (open) basis of $\mathcal{B}$.

THEOREM 4.2. — *A family $\mathcal{A} = \{A\}$ of subsets of a topological space X is a basis of some boundedness in X, if and only if the union of any two sets of $\mathcal{A}$ is contained in a set of $\mathcal{A}$.*

PROOF. — *Necessity.* — Suppose $\mathcal{C}$ to by a basis of a boundedness $\mathcal{B}$ in X, then $\mathcal{A} \subset \mathcal{B}$. Let A_1, A_2 be two arbitrary sets of $\mathcal{A}$. By (1.12), $A_1 \cup A_2 \in \mathcal{B}$. Since $\mathcal{A}$ is a basis of $\mathcal{B}$, $A_1 \cup A_2$ is a subset of a set $A \in \mathcal{A}$.

Sufficiency. — Suppose the condition be satisfied, and let $\mathcal{B}$ be the boundedness generated by $\mathcal{A}$. Since the union of any two sets of $\mathcal{A}$ is contained in a set of $\mathcal{A}$, then the union of any finite number of sets of $\mathcal{A}$ is contained in a set of $\mathcal{A}$. Therefore, every set of $\mathcal{B}$ is a subset of some set of $\mathcal{A}$; and hence $\mathcal{A}$ is a basis of $\mathcal{B}$. Q. E. D.

THEOREM 4.3. — *If the boundedness $\mathcal{B}$ in X is generated by $\mathcal{A}$, then the finite unions of $\mathcal{A}$ form a basis of $\mathcal{B}$.*

PROOF. — Let $\mathcal{A}^\star$ be the family of finite unions of $\mathcal{A}$, then $\mathcal{A}^\star$ generates $\mathcal{B}$. Since $\mathcal{A}^\star$ satisfies the condition in (4.2), $\mathcal{A}^\star$ is a basis of $\mathcal{B}$. Q. E. D.

THEOREM 4.4. — *In a given topological space X, a necessary and sufficient condition for the boundedness $\mathcal{B}^\star$ with a basis $\mathcal{A}^\star$ to be stronger than the boundedness $\mathcal{B}$ with a basis $\mathcal{A}$ is that each set $A \in \mathcal{A}$ is contained in a set $A^\star \in \mathcal{A}^\star$.*

PROOF. — *Necessity.* — Suppose $\mathcal{B}^\star \supset \mathcal{B}$, and let A be an arbytrary set of $\mathcal{A}$. Then we have $A \in \mathcal{B}^\star$. Since $\mathcal{A}^\star$ is a basis of $\mathcal{B}^\star$, A is contained in a set $A^\star \in \mathcal{A}^\star$.

Sufficiency. — Let B be an arbitrary set of $\mathcal{B}$. Since $\mathcal{A}$ is a basis of $\mathcal{B}$, B is contained in a set $A \in \mathcal{A}$. By our condition, A is contained in a set $A^\star \in \mathcal{A}^\star$. Hence $\mathcal{B}B \in \mathcal{B}^\star$ and $\mathcal{B}^\star \supset \mathcal{B}$. Q. E. D.

DEFINITION **4**.5. — *A family $\mathcal{A} = \{A\}$ of subsets of a topological space* X *is called a basis of boundedness, if it is a basis of some boundedness in* X. *Two bases of boundedness are said to be equivalent, if they are the bases of the same boundedness.*

The following theorem is an immediate consequence of (**4**.4).

THEOREM **4**.6. — *Two bases $\mathcal{A}$ and $\mathcal{A}^\star$ of boundedness in* X *are equivalent, if and only if each set of $\mathcal{A}$ is contained in a set of $\mathcal{A}^\star$ and each set of $\mathcal{A}^\star$ is contained in a set of $\mathcal{A}$.*

THEOREM **4**.7. — *A boundedness $\mathcal{B}$ in* X *is closed (open) if and only if it admits a closed (open) basis.*

PROOF. — Let us prove the theorem for closed boundedness. If $\mathcal{B}$ is closed, then the family of the bounded closed sets of $\mathcal{B}$ is a closed basis of $\mathcal{B}$. Conversely, if $\mathcal{B}$ admits a closed basis $\mathcal{A}$, then each bounded set $B \in \mathcal{B}$ is contained in a bounded closed set $A \in \mathcal{A}$. Hence $\overline{B}$ is bounded and $\mathcal{B}$ is closed by (**3**.5). Q. E. D.

5. BOUNDEDNESS WITH A COUNTABLE BASIS. — DEFINITION **5**.1. — *A bounded set B_* of a boundedness $\mathcal{B}$ in a topological space* X *is said to be maximal, if every bounded set of $\mathcal{B}$ is a subset of B_*.*

In example (1) of § **3**, the set X-x is the maximal bounded set of the boundedness $\mathcal{B}$ defined there.

THEOREM **5**.2. — *For a given boundedness $\mathcal{B}$ in a topological space* X, *the following conditions are equivalent:*

(**5**.21) *$\mathcal{B}$ has a maximal bounded set B_*;*

(**5**.22) *$\mathcal{B}$ admits a finite basis;*

(**5**.23) *the union of any number of bounded sets is bounded.*

Proof. — (5.21) → (5.22). The maximal bounded set B_* itself form a basis of $\mathcal{B}$.

(5.22) → (5.23). Suppose $\mathcal{B}$ admits a finite basis $\mathcal{A} = \{A_1, \ldots, A_q\}$, then the union $A = A_1 \cup \ldots \cup A_q$ is a bounded set and forms a basis of $\mathcal{B}$. Since every bounded set is a subset of A, the union of any number of bounded sets is still a subset of A; hence (5.23).

(5.23) → (5.21). Let B_* be the union of all bounded sets of $\mathcal{B}$. B_* is bounded by (5.23), and hence it is a maximal bounded set.

Q. E. D.

The following theorem is trivial.

Theorem 5.3. — *Suppose the boundedness $\mathcal{B}$ in X has a maximal bounded set B_*, then :*

(5.31) *$\mathcal{B}$ is closed if and only if B_* is closed;*
(5.32) *$\mathcal{B}$ is open if and only if B_* is open;*
(5.33) *$\mathcal{B}$ is proper if and only if B_* is both closed and open.*

The following theorem is an immediate consequence of (5.33).

Theorem 5.4. — *A connected proper universe whose boundedness admits a finite basis is bounded.*

Theorem 5.5. — *If $\mathcal{A}$ is an arbitrary basis of a boundedness $\mathcal{B}$ with a countable basis but without maximal bounded set, then there exists a countable basis $\mathcal{C}$ of $\mathcal{B}$ which consists of a strictly increasing sequence of sets of $\mathcal{A}$.*

Proof. — Let $\mathcal{D} = \{D_1, D_2, \ldots\}$ be an arbitrary countable basis of $\mathcal{B}$. Define a subfamily $\mathcal{C} = \{C_1, C_2, \ldots\}$ of $\mathcal{A}$ as follows. Choose $C_1 \in \mathcal{A}$ wich containe D_1. Suppose that $C_1, C_2, \ldots, C_n$ have been so chosen from $\mathcal{A}$ that C_i contains the union $C_{i-1} \cup D_i$ and is different from C_{i-1} for $i = 2, \ldots, n$. Now there exists at least one set $B_0 \in \mathcal{B}$ which is not a subset of C_n, for otherwise C_n would be a maximal bounded set of $\mathcal{B}$. Since $\mathcal{A}$ is a basis of $\mathcal{B}$, there exists a set $C_{n+1} \in \mathcal{A}$ which contains the union $C_n \cup D_{n+1} \cup B_0$. Thus a strictly increasing sequence of sets $\mathcal{C} = \{C_1, C_2, \ldots\}$ has been chosen from $\mathfrak{A}$ such that $D_n \subset C_n$ for $n = 1, 2, \ldots$. $\mathcal{C}$ is a basis of $\mathcal{B}$, for every bounded set $B \in \mathcal{B}$ is a subset of some D_n and therefore a subset of C_n. This completes the proof. Q. E. D.

THEOREM 5.6. — *A proper boundedness $\mathcal{B}$ whith a countable basis but without maximal bounded set admits an open basis $\mathcal{G}$ which consists of a strictly increasing sequence of bounden open sets* $G_1, G_2, \ldots, G_n, \ldots$, *such that* $\overline{G}_n \subset G_{n+1}$ *for each* $n = 1, 2, \ldots$.

PROOF. — By (5.5), $\mathcal{B}$ admits a basis $\mathcal{C} = \{C_1, C_2, \ldots\}$ which consists of a strictly increasing sequence of bounded sets. Let G_1 be an arbitrary bounded open set. Suppose that bounded open sets G_1, $G_2, \ldots, G_n$ have been so chosen that G_i contains the union $\overline{G}_{i-1} \cup C_{i-1}$ and is different from G_{i-1} for each $i = 2, \ldots, n$. Now there exists at least one bounded set B_0 which is not contained in G_n, for otherwise G_n would be a maximal bounded set of $\mathcal{B}$. Since $\mathcal{B}$ is closed, $\overline{G}_n$ is a bounded set. Since $\mathcal{B}$ is open, there exists at least one bounded open set which contains $\overline{G}_n \cup C_n \cup B_0$. Choose such a set for our G_{n+1}, then G_{n+1} contains $\overline{G}_n \cup C_n$ and is different from G_n. Let $\mathcal{G} = \{G_1, G_2, \ldots\}$ be the sequence of bounded open sets thus defined. It remains to prove that $\mathcal{G}$ is a basis of $\mathcal{B}$. Let B be an arbitrary bounded set of $\mathcal{B}$. Since $\mathcal{C}$ is a basis of $\mathcal{B}$, B is a subset of some set $C_n \in \mathcal{C}$ and hence a subset of G_{n+1}. Q. E. D.

REMARK. — We have also shown in the above proof that the leader G_1 of the sequence $\mathcal{G}$ can be any given bounded open set and that the sequence of the closed sets $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_n, \ldots$ form a closed basis of $\mathcal{B}$.

6. LOCAL BOUNDEDNESS, POINTS AT INFINITY. — DEFINITION 6.1. — *A point x of universe X is said to be bounded, if the set $\{x\}$ which consists of the single point x is bounded; otherwise, it is said to be non-bounded.*

The set of all bounded points of X and that of all non-bounded points of X will be denoted by $L = L(X)$ and $W = W(X)$ respectively. The following theorem is trivial.

THEOREM **6**.2. — $L(X)$ *is the union of all bounded sets of* X *and* $W(X) = X - L$; *therefore,*

(**6**.21) $L(X)$ *is an open set and* $W(X)$ *is a closed set, if* X *is an open universe;*

(**6**.22) $L(X)$ *is an* F_σ*-set and* $W(X)$ *is a* G_δ*-set, if* X *is a closed universe with a countable basis.*

DEFINITION **6**.3. — *A point x of a universe* X *is said to be finite, if it is an interior point of some bounded set; otherwise, it is called a point at infinity of* X. *The set of all finite points is called the kernel of* X, *denoted by* $\Lambda = \Lambda(X)$; *the set of all points at infinity is called the set at infinity of* X, *denoted by* $\Omega = \Omega(X)$.

The following two theorems are obvious.

THEOREM **6**.4. — $\Lambda(X)$ *is the union of all bounded open sets of the universe* X; *therefore,* $\Lambda(X)$ *is open and* $\Omega(X)$ *is closed.*

THEOREM **6**.5. — *For an open universe* X, *we have*

$$L(X) = \Lambda(X), \qquad W(X) = \Omega(X).$$

For a boundedness which is not open, a bounded point need not be finite. In the example (2) of paragraph **5**, the point $\xi_0 = (0, 0, \ldots)$ is bounded by definition. ξ_0 is a point at infinity, for the closure of every ε-neighbourhood of ξ_0 in X is not compact and hence there is no bounded set containing ξ_0 in its interior.

DEFINITION **6**.6. — *A universe* X *is said to be locally bounded at x, if x is a finite point of* X. X *is said to be locally bounded, if each point of* X *is finite.*

THEOREM **6**.7. — *Suppose* $\mathcal{V} = \{V\}$ *be a base of the open sets of a locally bounded universe* X, *then there exists a subfamily* $\mathcal{V}_*$ *of* $\mathcal{V}$ *which consists of only bounded open sets and which forms already a base of the open sets of* X.

Proof. — Since X is locally bounded, for each point $x \in X$ there is a bounded open set U_x containing x. Then we obtain a covering $\{U_x\}$ of X with bounded open sets. Let $\mathcal{V}_*$ denote the totality of the open sets $V \in \mathcal{V}$ such that V is contained in some U_x. It remains to prove that $\mathcal{V}_*$ is still a base of the open sets of X.

Let G be an arbitrary open set of X and x be an arbitrary point of G. Let $G_x = U_x \cap G$. Since $\mathcal{V}$ is a base of X, there is a set $V_x \in \mathcal{V}$ such that $x \in V_x \subset G_x$. Then $V_x \in \mathcal{V}_*$ and $G = \bigcup_{x \in G} V_x$.

Q. E. D.

Theorem 6.8. — *A compact subset of a locally bounded universe is bounded.*

Proof. — Let X be a locally bounded universe and X_0 a compact subset of X. Since X is locally bounded, for each point $x \in X_0$ there is a bounded open set U_x containing x. The family $\{U_x \mid x \in X_0\}$ forms a covering of X_0 with bounded open sets of X. Since X_0 is compact, there is a finite subfamily of $\{U_x\}$ which covers X_0. Hence X_0 is bounded. Q. E. D.

7. Trivial boundedness and compact boundedness. — The boundedness of a space is, in general, not a topological property. However, if we define a particular boundedness topologically, it forms a topological invariant. The most important of these topologically invariant boundedness are the trivial and the compact boundedness which were used to construct the examples given in paragraph 5.

Definition 7.1. — *The trivial boundedness of a topological space* X *is that which makes* X *a bounded universe. The trivial boundedness of a topological space* X *relative to a subset* X_0 *is that which makes* $X - X_0$ *a maximal bounded set.*

Definition 7.2. — *The compact boundedness of a topological space* X *is that which consists of the totality of the subsets* $B \subset X$ *such that the closure* $\overline{B}$ *of* B *is compact. The compact boundedness of a topological space* X *relative to a subset* X_0 *is that which consists of the totality of* *the subsets* $B \subset X$ *such that the closure* $\overline{B}$ *of* B *is compact and does not meet* X_0.

The following theorem is obvious.

THEOREM 7.3. — *The compact boundedness of a topological space* X *relative to an arbitrary subset* X_0 (*which might by empty*) *is closed.*

THEOREM 7.4. — *For an arbitrary topological space* X, *the following conditions are equivalent :*

(7.41) X *is locally compact:*

(7.42) *the compact boundedness makes* X *a locally bounded universe.*

If, in addition, X *be a Hausdorff space, then each of the conditions* (7.41) *and* (7.42) *is equivalent to the following :*

(7.43) *the compact boundedness is open.*

PROOF. — The equivalence of (7.41) and (7.42) is trivial. Suppose X to be a Hausdorff space.

(7.43) → (7.42). Let $x \in X$ be an arbitrary point. Since the closure $\overline{x} = x$ is compact, x is a bounded set of the compact boundedness. It follows from (7.43) that there exists a bounded open set containing x; hence (7.42).

(7.42) → (7.43). Let B be an arbitrary bounded set of the compact boundedness. By definition, $\overline{B}$ is compact. Since X is locally bounded, for each $x \in \overline{B}$ there is a bounded open set G_x containing x. Since $\overline{B}$ is compact, there are a finite number of these bounded sets, say G_{x_1}, G_{x_2}, ..., G_{x_q}, whose union G contains $\overline{B}$. G is a bounded open set containing B; hence we have (7.43).

Q. E. D.

THEOREM 7.5. — *The compact boundedness of a Hausdorff space* X *relative to a subset* X_0 *is open if and only if* X_0 *is closed and* $X - X_0$ *is locally compact.*

PROOF. — *Necessity.* — Suppose the compact boundedness $\mathcal{B}$ of X relative to X_0 be open. Let x be an arbitrary point $X - X_0$.

Since $\bar{x}=x\in X-X_0$, x is bounded. Since $\mathcal{B}$ is open, there is a bounded open set G containing x. By the definition, the closure $\overline{G}$ of G is compact and contained in $X-X_0$; hence $X-X_0$ is locally compact. Further, we have proved that every point of $X-X_0$ is a finite point. It is trivial that every point of X_0 is a point at infinity. Hence X_0, as the set at infinity of an open universe, must be a closed set.

Sufficiency. — Suppose X_0 be closed and $X-X_0$ be locally compact, and denote by $\mathcal{B}$ the compact boundedness of X relative to X_0. Let B be an arbitrary bounded set of $\mathcal{B}$, then $\overline{B}$ is compact and contained in $X-X_0$. Since $X-X_0$ is locally compact, for each $x\in\overline{B}$ there exists an open set G_x containing x such that the closure $\overline{G}_x$ of G_x in $X-X_0$ is compact. Since X is a Hausdorff space, $\overline{G}_x$ is a closed set of X and hence $\overline{G}_x$ is the closure of G_x in X. Since $\overline{B}$ is compact, there are a finite number of these open sets, say $G_{x_1}, G_{x_2}, \ldots, G_{x_q}$, whose union $G=\bigcup_{i=1}^{q} G_{x_i}$ contains $\overline{B}$. Now $\overline{G}=\bigcup_{i=1}^{q} \overline{G}_{x_i}\subset X-X_0$ is compact, G is a bounded open set containing B. Hence $\mathcal{B}$ is open. Q. E. D.

8. Boundedly compact universes. — The notion of bounded compactness, studied in the present paragraph, was introduced by J. W. Alexander [1].

Definition **8**.1. — *A universe X is said to be boundedly compact, if every bounded closed set of X is compact.*

Definition **8**.2. — *A collection of sets is said to have the finite intersection property, if every finite subcollection has a non-empty intersection.*

Theorem **8**.3. — *A universe X is boundedly compact if and only if every collection of bounded closed sets with the finite intersection property has at least one common point.*

PROOF. — *Necessity.* — Suppose X by boundedly compact, and let $\{F\}$ be a collection of bounded closed sets with the finite intersection property. Let F_0 be a particular bounded closed set $\{F\}$, then F_0 is compact. Let $\{F^*\}$ by the collection of closed sets of F_0 which consists of the totality of the sets $F^* = F \cap F_0$, $F \in \{F\}$. Let $F_1^*, \ldots, F_n^*$ be an arbitrary finite subcollection of $\{F^*\}$. Since $\{F\}$ has the finite intersection property, $F_0, F_1, \ldots, F_n$ have a common point, say x; hence $F_1^*, \ldots, F_n^*$ have x as a common point and $\{F^*\}$ has the finite intersection property. Since F_0 is compact, $\{F^*\}$ has a common point x_0 which is clearly a common point of the collection $\{F\}$.

Sufficiency. — Suppose the condition be satisfied, and let F_0 be an arbitrary bounded closed set. Let $\{F\}$ be an arbitrary collection of closed sets of F_0 which has the finite intersection property. Since F_0 is a bounded closed set, $\{F\}$ is a collection of bounded closed sets of X. Hence it follows from the condition that $\{F\}$ has at least one common point and F_0 is compact. Q. E. D.

DEFINITION **8**.4. — *A bounded covering of a subset* X_0 *of a universe* X *is a collection of bounded sets of* X *whose union contains* X_0.

THEOREM **8**.5. — *A locally bounded universe* X *boundedly compact, if and only if every bounded open covering* $\{U\}$ *of a bounded closed set* $F \subset X$ *has a finite subcovering of* F.

PROOF. — *Necessity.* — Suppose X be boundedly compact, and $\{U\}$ be an arbitrary bounded open covering of an arbitrary bounded closed set $F \subset X$. By (**8**.1), F is compact and, therefore, $\{U\}$ has a finite subcollection which covere F.

Sufficiency. — Suppose F be an arbitrary bounded closed set of X and $\{V\}$ be an arbitrary open covering of F in X. Since X is locally bounded, for each point $x \in F$ there exists a bounded open set W_x containing x. Since $\{V\}$ is an open covering of F in X, for each point $x \in F$ there exists an open set $V_x \in \{V\}$ containing x. Let $U_x = V_x \cap W_x$, then the collection $\{U_x\}$ forms a bounded open covering of F. It follows from the condition that $\{U_x\}$ has a finite subcollection, say $U_{x_1}, U_{x_2}, \ldots, U_{x_q}$, which covers F. Since $U_{x_i} = V_{x_i} \cap W_{x_i}$, the finite subcollection $V_{x_1}, V_{x_2}, \ldots, V_{x_q}$ covers F. Hence F is compact. Q. E. D.

THEOREM 8.6. — *If a closed universe X is locally bounded and boundedly compact, then X is locally compact.*

PROOF. — Let $x \in X$. Since X is locally bounded, there is a bounded open set G which contains x. Since X is a closed universe, $\overline{G}$ is a bounded closed set of X. Since X is boundedly compact, $\overline{G}$ is compact. Hence X is locally compact. Q. E. D.

The following theorem is evident.

THEOREM 8.7. — *Every universe X with the compact boundedness relative to a subset X_0 (which might by empty) is boundedly compact.*

Therefore, bounded compactness is a property of the given boundedness instead of the given space X. However, if the space X is compact, then every universe X with an arbitrary boundedness is boundedly compact.

9. SIMPLICIAL UNIVERSES. — Throughout the present paragraph, let X be a polytope in the sense of S. Lefschetz [6, p. 9] with $\{\sigma\}$ as its open simplexes.

DEFINITION 9.1. — *The simplicial boundedness of a polytope X is the one which is generated by the family $\{\sigma\}$ of open simplexes. A polytope X with the simplicial boundedness is called a simplicial universe X.*

THEOREM 9.2. — *The simplicial boundedness of a polytope X is closed.*

PROOF. — Since the closure $\overline{\sigma}$ of an open simplex σ consists of a finite number of open simplexes, $\overline{\sigma}$ is a bounded closed set. Let B be an arbitrary bounded set, then there exists a finite number of open simplexes $\sigma_1, \sigma_2, \ldots, \sigma_8$ such that $B \subset \bigcup_{i=1}^{q} \sigma_i$. Hence $\overline{B} \subset \bigcup_{i=1}^{q} \overline{\sigma}_i$ is a bounded closed set. Q. E. D.

Theorem 9.3. — *For a given polytope X, the following conditions are equivalent :*

(**9**.31) *The polytope X is locally finite.*
(**9**.32) *The simplicial universe X is locally bounded.*
(**9**.33) *The simplicial boundedness of X is open.*

Proof. — (**9**.31) → (**9**.32). Let $x \in X$ by an arbitrary point, then there is a unique open simplex σ which contains x. Let $S(\sigma)$ denote the open star of σ, i. e. the totality of the open simplexes each of which contains σ in its closure. Since X is locally finite, $S(\sigma)$ consists of a finite number of open simplexes. Hence $S(\sigma)$ is a bounded open set containing x, and X is locally bounded.

(**9**.32) → (**9**.33). Let B be an arbitrary bounded set, then there exist a finite number of open simplexes $\sigma_1, \sigma_2, \ldots, \sigma_8$ such that $B \subset \bigcup_{i=1}^{q} \sigma_i$. Hence $\overline{B} \subset \bigcup_{i=1}^{q} \overline{\sigma}_i$. As a closed subset of a compact set $\bigcup_{i=1}^{q} \overline{\sigma}_i$, $\overline{B}$ is compact. Since X is locally bounded, for each $x \in \overline{B}$ there is a bounded open set G_x containing x. It follows from the compactness of $\overline{B}$ that there exist a finite number of these bounded open sets, say $G_{x_1}, G_{x_2}, \ldots, G_{x_q}$, whose union contains $\overline{B}$. Hence $\bigcup_{i=1}^{q} G_{x_i}$ is a bounded open set containing B, and the simplicial boundedness of X is open.

(**9**.33) → (**9**.31). Let σ be an arbitrary open simplex of X. Since the simplicial boundedness of X is open, there exists a bounded open set G containing σ. By definition, there are a finite number of open simplexes, say $\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_8$, such that $G \subset \bigcup_{i=1}^{q} \sigma_i$. Let σ_* by an open simplex of X different from each of the simplexes $\sigma_1, \sigma_2, \ldots, \sigma_s$. Then $\sigma_* \subset X - G$, and the closure $\bar{\sigma}_*$ does not meet σ. Hence the open star $S(\sigma)$ consists of only a finite number of open simplexes; and, therefore, X is locally finite

Q. E. D.

THEOREM **9.4**. — *For a locally finite polytope* X, *the simplicial boundedness coincides with the compact boundedness.*

PROOF. — It is trivial that the closure of a bounded set in the simplicial boundedness is compact. Conversely, let F be an arbitrary compact subset of X. For each point $x \in F$, let σ denote the open simplex containing x and $G_x = S(\sigma)$ the open star of σ. Since X is locally finite, G_x consists of only a finite number of simplexes. From the compactness of F, there exist a finite number of these open stars, say $G_{x_1}, G_{x_2}, \ldots, G_{x_q}$, whose union covers F. Hence F is contained in a finite number of open simplexes and the proof is complete. Q. E. D.

THEOREM **9.5**. — *The simplicial boundednes of a polytope* X *admits a countable basis, if and only if the polytope* X *is countable.*

PROOF. — *Necessity.* — Suppose that the simplicial boundedness admits a countable basis $\{B_n\} = \{B_1, B_2, \ldots\}$. By definition, B_n is contained in the union X_n of a finite number of open simplexes. Since $\{B_n\}$ is a basis of the simplicial boundedness, each open simplex σ is contained in some B_n and hence in X_n. Therefore, there are only a countable number of open simplexes $\sigma \in X$.

Sufficiency. — Suppose that X be countable and let $\sigma_1, \sigma_2, \ldots$ be its open simplexes. Let $X_n = \bigcup_{i=1}^{n} \sigma_i$, then the family $\{X_n\}$ forms a countable basis of the simplicial boundedness. Q. E. D.

THEOREM **9.6**. — *For a given polytope* X, *the following conditions are equivalent :*

(**9.61**) *The polytope* X *is finite.*
(**9.62**) *The simplicial universe* X *is bounded.*
(**9.63**) *The simplicial boundedness of* X *has a maximal bounded set.*

PROOF. — The implications (**9**.61) → (**9**.62), (**9**.62) → (**9**.63) are trivial. It remains to prove (**9**.63) → **9**.61). Let B_* be the maximal bounded set. By the definition of the simplicial boundedness, there a finite number of open simplexes $\sigma_1, \sigma_2, \ldots \sigma_n$ such that $B_* \subset \bigcup_{i=1}^{n} \sigma_i$. Since $\bigcup_{i=1}^{n} \sigma_i$ is bounded and B_* is the maximal bounded set, we have $B_* = \bigcup_{i=1}^{n} \sigma_i$. If there is an open simplex $\sigma \in X$ different from each σ_i $(i = 1, 2, \ldots, n)$, then $B_* \cup \sigma$ would be a bounded set. Hence $B_* = X$. Q. E. D.

10. METRIZABLE UNIVERSES. — In a metric space X with the metric ρ, there is a natural boundedness (called *the boundedness defined by the metric* ρ) described as follows. We call a set $B \subset X$ bounded if its diameter $\delta(B)$ is finite.

DEFINITION **10**.1. — *A metric space with the boundedness defined by its metric is called a metric universe. A universe* X *is said to be metrizable if there can be introduced a metric* ρ *which defines both the topology and the boundedness of* X.

THEOREM **10**.2. — *Every metrizable universe is proper, locally bounded, and with a countable basis of its boundedness.*

PROOF. — Let X be a metrizable universe, and ρ be one of its defining metrics. Let B by an arbitrary bounded set. Since $\delta(\bar{B}) = \delta(B)$, $\bar{B}$ is bounded. Further, the ε-neighbourhood of B is an open set with diameter $\leq \delta(B) + 2\varepsilon$. Hence X is proper.

Let $x \in X$ be an arbitrary point, then the ε-neighbourhood U_x is a bounded open set containing x. Hence X is locally bounded.

Choose a fixed point $x_0 \in X$. Let G_n denote the n-neighbourhood of x_0 $(n = 1, 2, \ldots)$. It remains to prove that the family $\mathcal{G} = \{G_1, G_2, \ldots\}$ form a basis of the boundedness of X defined by the metric ρ. Let B be an arbitrary bounded set. Choose a point $x_1 \in B$, and let $d = \rho(x_0, x_1)$, $\delta = \delta(B)$. Then $\rho(x_0, x) \leq d + \delta$ for each $x \in B$. Hence B is contained in G_n if $n > d + \delta$. Q. E. D.

Let X be a metric space with metric ρ, and M be a subset of X. We use the notations :

$$d_\rho(x_0, \mathrm{M}) = \inf_{x \in \mathrm{M}} \rho(x_0, x), \qquad \delta_\rho(x_0, \mathrm{M}) = \sup_{x \in \mathrm{M}} \rho(x_0, x).$$

DEFINITION **10**.3. — *Two metrics ρ_1, ρ_2 in a set X are said to be completely equivalent, if they define the same topology and same boundedness in X.*

THEOREM **10**.4. — *Two metrics ρ_1, ρ_2 in X are completely equivalent, if and only if the following two conditions are both satisfied for each point $x \in \mathrm{X}$ and each subset $\mathrm{M} \subset \mathrm{X}$:*

(**10**.41) *$d_{\rho_1}(x, \mathrm{M})$ and $d_{\rho_2}(x, \mathrm{M})$ are both zero or not;*
(**10**.42) *$\delta_{\rho_1}(x, \mathrm{M})$ and $\delta_{\rho_2}(x, \mathrm{M})$ are both finite or not.*

PROOF. — Since (**10**.41) is the necessary and sufficient condition for ρ_1, ρ_2 to define the same topology, it remains to prove that (**10**.42) is the necessary and sufficient condition for ρ_1, ρ_2 to define the same boundedness.

Necessity. — Suppose $\delta_{\rho_1}(x, \mathrm{M})$ be finite for a given pair x and M. Since $\delta_{\rho_1}(\mathrm{M}) \leqq 2\,\delta_{\rho_1}(x, \mathrm{M})$, M is bounded. Let $x_0 \in \mathrm{M}$, then $\delta_{\rho_2}(x, \mathrm{M}) \leqq \rho_2(x, x_0) + \delta_{\rho_2}(\mathrm{M})$. Hence $\delta_{\rho_2}(x, \mathrm{M})$ is also finite.

Sufficiency. — Suppose M be bounded in the boundedness defined by ρ_1. Then, as above, $\delta_{\rho_1}(x, \mathrm{M})$ is finite for an arbitrary point $x \in \mathrm{X}$. By (**10**.42), $\delta_{\rho_2}(x, \mathrm{M})$ is also finite. Since $\delta_{\rho_2}(\mathrm{M}) \leqq 2\,\delta_{\rho_2}(x, \mathrm{M})$, M is also bounded in the boundedness defined by ρ_2. Q. E. D.

DEFINITION **10**.5. — *The euclidean n-space R^n (the Hilbert space R^ω) with the boundedness defined by its metric is called the euclidean n-universe R^n (the Hilbert universe R^ω).*

The following theorem is trivial.

THEOREM **10**.6. — *For the euclidean n — space R^n, the compact boundedness coincides with the boundedness defined by its metric; hence the euclidean n — universe R^n is boundedly compact.*

11. Boundedness of transformations. — Definition **11**.1. — *A transformation f of a univers X into a universe Y is said to be bounded, if the image of every bounded set of X is a bounded set of Y; it is said to be bounding, if the inverse image of every bounded set of Y is a bounded set of X.*

Definition **11**.2. — *A homeomorphism h of a universe X onto a universe Y is said to be complete, if h is both bounded and bounding. X and Y are said to be completely homeomorphic, if there exists a complete homeomorphism h of X onto Y.*

The notion of complete homeomorphism defined above is due to J. W. Alexander [1]. The following statement is trivial.

Theorem **11**.3. — *Both topology and boundedness are invariant properties under complete homeomorphisms.*

Now let f be a transformation of a universe X into topological space Y, and denote by $\mathcal{B} = \{B\}$ the boundedness of X.

Definition **11**.4. — *The boundedness in Y generated by the family $\{f(B)\}$ is called the image of $\mathcal{B}$, denoted by $f(\mathcal{B})$.*

Theorem **11**.5. — $\{f(B)$ *is a basis of* $f(\mathcal{B})$.

Proof. — Our theorem follows from the relation

$$f(B_1 \cup B_2) = f(B_1) \cup f(B_2).$$

The following theorem is trivial.

Theorem **11**.6. — *$f(\mathcal{B})$ is the weakest boundedness that can be introduced in Y so that f becomes a bounded transformation.*

Next let f be a transformation of a topological space X into a universe Y, and denote by $\mathcal{C} = \{C\}$ the boudedness of Y.

Definition **11**.7. — *The boundedness in X generated by the family $\{f^{-1}(C)\}$ is called the inverse image of $\mathcal{C}$ under f, denoted by $f^{-1}(\mathcal{C})$.*

THEOREM 11.8. — $\{f^{-1}(\mathcal{C})\}$ *is a basis of* $f^{-1}(\mathcal{C})$.

PROOF. — Our theorem follows from the relation

$$f^{-1}(C_1 \cup C_2) = f^{-1}(C_1) \cup f^{-1}(C_2).$$

The following theorem is trivial.

THEOREM 11.9. — $f^{-1}(\mathcal{C})$ *is the weakest boundedness that can be introduced in* X *so that* f *becomes a bounding transformation. Further, if* f *is a mapping, then* $f^{-1}(\mathcal{C})$ *is closed, open, or proper, according as* $\mathcal{C}$ *is closed, open, or proper.*

12. RELATIVIZATION OF BOUNDEDNESS. — Let X be a universe with boundedness $\mathcal{B} = \{B\}$, and X_* be a subspace of X provided with the topology obtained by relativization.

THEOREM 12.1. — *The family of subsets* $\{B \cap X_*\}$ *is a boundedness in* X_* *which will be called the boundedness* $\mathcal{B}_*$ *relative to* $\mathcal{B}$.

PROOF. — Let $B_1, B_2 \in \mathcal{B}$, then we have

$$(B_1 \cap X_*) \cup (B_2 \cap X_*) = (B_1 \cap B_2) \cap X_*.$$

Hence $\{B \cap X_*\}$ satisfies (1.12). Let C be a subset of $B \cap X_*$, then $C \in \mathcal{B}$. It follows that $C = C \cap X_*$ is a bounded set of $\{B \cap X_*\}$. Hence $\{B \cap X_*\}$ satisfies (1.11). Q. E. D.

DEFINITION 12.2. — *The subspace* $X_* \subset X$ *with the boundedness* $\mathcal{B}_*$ *relative to* $\mathcal{B}$ *is called the subuniverse* X_*.

THEOREM 12.3. — *A subuniverse* X_* *of a closed, open, or proper universe* X *is closed, open, or proper.*

PROOF. — Suppose X be a closed (open) universe. Let B be an arbitrary bounded set of X, then there exists a bounded closed set F (a bounded open set G) which contains B. By relativization, $F \cap X_*$ is a bounded closed set ($G \cap X_*$ is a bounded open set) of the subuniverse X_* which contains $B \cap X_*$. Hence X_* is a closed (open) universe. Q. E. D.

THEOREM **12.4**. — *If a point $x \in X_* \subset X$ is a finite point of the universe X, then x is also a finite point of the subuniverse X_*; hence a subuniverse X_* of a locally bounded universe X is locally bounded.*

PROOF. — Since x is a finite point of X, there is a bounded open set G of X which contains x. Then $G \cap X_*$ is a bounded open set of X_* containing x, hence x is a finite of X_*. Q. E. D.

THEOREM **12.5**. — *If $\mathfrak{A} = \{A\}$ is a basis of the boundedness $\mathfrak{B}$ of a universe X, then the family $\mathfrak{A}_* = \{A \cap X_*\}$ is a basis of the boundedness $\mathfrak{B}$ in X_* relative to $\mathfrak{B}$.*

PROOF. — Let $B \cap X_*$ be an arbitrary bounded set X_*. Since $\mathfrak{A}$ is a basis of $\mathfrak{B}$, there is a set $A \in \mathfrak{A}$ which contains B. Hence $B \cap X_* \subset A \cap X_*$, and $\{A \cap X_*\}$ is a basis of $\mathfrak{B}_*$. Q. E. D.

COROLLARY **12.6**. — *If the boundedness $\mathfrak{B}$ of a universe X admits a countable (finite) basis, so does the boundedness $\mathfrak{B}_*$ of every subuniverse X_* of X.*

THEOREM **12.7**. — *The identity mapping of a subuniverse X_* of a universe X into X is both bounded and bounding.*

PROOF. — Let B_* be an arbitrary bounded set of X_*; then by the definition of $\mathfrak{B}_*$, there is a bounded set B of X such that $B_* = B \cap X_*$, Hence $B_* \subset B$ is also a bounded set of X, i. e. the identity mapping is bounded. Conversely, let $M \subset X_*$ be a bounded set of X; then $M \cap X_*$ is also a bounded set of X_*. Hence the identity mapping is bounding. Q. E. D.

Hereafter, the subspaces L, W, Λ, Ω of a universe X will be provided with the boundedness obtained by relativization; and, therefore, they are subuniverses of X.

THEOREM **12.8**. — *The kernel Λ of a universe X is the greatest locally bounded subuniverse of X which is an open set of X.*

PROOF. — Let $x \in \Lambda$, then there is a bounded open set G of X which contains x. Then $G \cap \Lambda$ is a bounded open set of Λ containing x; hence Λ is locally bounded. Λ is an open set of X by (**6**.4).

On the other hand, let an open set X_* of X be a locally bounded subuniverse of X. Let x be an arbitrary point of X_*. Since X_* is locally bounded, there exists a bounded open set G of X_* which contains x. Then G is also a bounded set of X. Since X_* is an open set of X, G is also an open set of X. Hence x is a finite point of X, and $X_* \subset \Lambda$. Q. E. D.

THEOREM **12**.9. — *The kernel* Λ *of an open universe* X *is the greatest locally bounded subuniverse of* X.

PROOF. — By (**12**.8), Λ is a locally bounded subuniverse of X. On the other hand, let X_* be a locally bounded subuniverse of X and let $x \in X_*$. Then x is a bounded point of X_* and therefore a bounded point of X. Since X is an open universe, x is a finite point. Hence $X_* \subset \Lambda$. Q. E. D.

13. METRIZATION. — DEFINITION **13**.1. — *A characteristic function of a universe* X *is a real-valued continuous function* $\psi \geq 0$ *defined over* X *in such a way that a subset* M *of* X *is bounded if and only if the least upper bound of* ψ *over* M *is finite.*

THEOREM **13**.2. — *A normal universe* X *admits a characteristic function* ψ, *if and only if* X *is proper, locally bounded, and with a countable basis of its boundedness.*

PROOF. — *Necessity*. — Suppose X be a universe which admits a characteristic function ψ. Let $x \in X$ and $\psi(x) = y$. Choose $b > y$, and denote by B_b the subset of X which consists of the totality of the points $\xi \in X$ for which $\psi(\xi) < b$. From the continuity of ψ, it follows that B_b is an open set. By the condition of a characteristic function, B_b is bounded. Since $x \in B_b$, X is locally bounded.

Next let B be an arbitrary bounded set, choose b so that

$$b > \sup_{x \in B} \psi(x).$$

Then B_b is a bounded open set and $\overline{B}_b$ is a bounded closed set both containing B. Hence X is proper.

Now let B_n denote the subset of X which consists of the totality of the points $\xi \in X$ for which $\psi(\xi) < n$ ($n = 1, 2, \ldots$). Then $\{B_n\}$ is an increasing sequence of bounded open sets. Let B be an arbitrary bounded set, and choose a positive integer $n > \sup_{x \in B} \psi(x)$. Then $B \subset B_n$. Hence $\{B_n\}$ forms a countable basis of the boundedness of X.

Sufficiency. — Suppose X be a normal universe satisfying the condition of the theorem. If X is bounded, then $\psi(x) = 0$ is a characteristic function of X. Hereafter, we assume X to be non-bounded. Since X is locally bounded, X cannot have a maximal bounded set. It follows from (5.6) that the boundedness of X has a basis $\mathcal{G}$ which consists of a strictly increasing sequence of bounded open sets $G_1, G_2, \ldots, G_n, \ldots$ such that $\overline{G}_n$ is contained in G_{n+1} for each $n = 1, 2, \ldots$.

Since $\overline{G}_n$ and $X - G_{n+1}$ are disjoint closed sets of a normal space, it follows from Urysohn's lemma that there exists a continuous function ψ_n defined over X such that : $\psi_n(x) = n - 1$, if $x \in \overline{G}_n$; $\psi_n(x) = n$, if $x \in X - G_{n+1}$; and $n - 1 \leq \psi_n(x) \leq n$ for each $n \in X$. Now define ψ by taking $\psi(x) = \psi_n(x)$ if $x \in G_{n+1} - G_n$. It remains to prove that ψ is a characteristic function of X.

The continuity of ψ follows from the fact that $\psi_{n-1}(x) = n - 1 = \psi_n(x)$ for each $x \in \overline{G}_n - G_n$. Now let M be an arbitrary set of X. If M is a bounded set, then there is a set G_{n+1} containing M; hence $\psi(x) \leq n$ for each $x \in M$. Conversely, suppose that $\sup_{x \in M} \psi(x)$ is finite. Choose n so large that $\psi(x) < n$ for each $x \in M$; then we have $M \subset G_{n+1}$ and M is bounded. Q. E. D.

Let R^{ω} be the Hilbert space with coordinates system $y_0, y_1, y_2, \ldots, y_n, \ldots$. Let H denote the half-line defined by $y_0 \geq 0$ and $y_n = 0$ ($n = 1, 2, \ldots$); and I^{ω}, the Hilbert cube defined by $y_0 = 0$ and $0 \leq y_n \leq \frac{1}{n}$ ($n = 1, 2, \ldots$). We shall denote by H^{ω} the single-winged Hilbert cube defined by $y_0 \geq 0$ and $0 \leq y_n \leq \frac{1}{n}$ ($n = 1, 2 \ldots$).

THEOREM **13**. 3. — *A necessary and sufficient condition for a universe* X *to be completely homeomorphic with a subset of the Hilbert universe* R^{ω} *is that be normal, proper, locally bounded, with a countable base of its topology and a countable basis of its boundedness.*

PROOF. — The necessity follows immediately from Urysohn's imbedding theorem and (**10**.2). It remains to prove the sufficiency.

By Urysohn's imbedding theorem, there exists a homeomorphism h of the space X onto a subset of the Hilbert cube I^{ω}. Let $\psi_n(x)$ $(n=1, 2, \ldots)$ denote the n^{th} coordinate of $h(x)\in I^{\omega}$. By (**13**.2) there is a characteristic function ψ_0 defined over X. Now let ψ be the mapping of X into the single-winged Hilbert cube H^{ω}, defined by

$$\psi(x)=\psi_0(x),\quad \psi_1(x),\quad \psi_2(x),\ \ldots,\quad \psi_n(x),\ \ldots \qquad (x\in X).$$

Clearly ψ is a homeomorphism of X onto a subset of H^{ω}. Next let B be an arbitrary bounded set of X; by (**13**.1) there exists a positive number b such that $\psi_0(x)\in b$ for each $x \leq B$. Therefore the image $\psi(B)$ is contained in the bounded set $<0, b> \times I^{\omega}$ of the Hilbert universe R^{ω}. Hence ψ is bounded. On the other hand, let $A\subset H^{\omega}$ be an arbitrary bounded set, then there is a positive number c such that the 0^{th} coordinate y_0 of every point $y\in A$ is less than c. Now let B_c denote the subset of X which consists of the totality of the points $x\in X$ such that $\psi_0(x)<c$. By (**13**.1) B_c is a bounded set of X. Since the inverse image $\psi^{-1}(A)$ is contained in B_c, it is bounded. Hence ψ is boundeding. Q. E. D.

THEOREM **13**.4. — *A separable metrizable universe* X *of dimension n is completely homeomorphic with a subset of the euclidien universe* R^{2n+2}.

PROOF. — The proof is almost exactly the same as that of (**13**.3), by using Menger-Noebeling theorem [4, p. 60] instead of Urysohn's imbedding theorem.

THEOREM **13**.5. — *If* X *be a proper universe with a countable basis of its boundedness and if the kernel* Λ *be a separable metrizable space, then a metric* ρ *can be introduced in* Λ *in such a way that :*

(**13**.51) *a subset* $M \subset A$ *is bounded, if and only if the diameter* $\delta(M)$ *is finite;*

(**13**.52) *for each pair of points* $x_0 \in A, x_* \in \bar{A} - A$ *and each positive number* b, *there exists an open set* $G \ni x_*$ *such that* $\rho(x_0, x)$ *for each* $x \in G \cap A$.

Proof. — Since X is proper, the subuniverse A is also proper by (**12**.3). By (**12**.9) A is locally bounded. By (**12**.6) A admits a countable basis of its boundedness. Hence it follows from (**13**.3) that there exists a complete homeomorphism ψ of A onto a subset of the Hilbert universe R^{ω}. Let a metric ρ be defined over A by means of $\rho(x_1, x_2) = \rho[\psi(x_1), \psi(x_2)]$ for each pair x_1, x_2 of A. Since ψ is a complete homeomorphism, (**13**.51) is satisfied.

To prove (**13**.52), let $x_0 \in A$, $x_* \in \bar{A} - A$, and $b > 0$ be arbitrarily given. Let B denote the set of points x of A such that $\rho(x_0, x) \leqq b$, then A is a bounded closed set of A. Since A is proper, there is a bounded open set U of A which contains B. Since A is itself an open set of X, U is a bounded open set of X. Since X is proper $\bar{U}$ is a bounded closed set of X and $\bar{U} \subset A$ by (**6**.5). Hence the open set $G = X - \bar{U}$ contains x, and $G \cap A$ is contained in $A - B$. Therefore, it follows thas $\rho(x_0, x) > b$ for each $x \in \cap A$ and (**13**.52) is proved.

Q. E. D.

14. Connectivity theory of universes. — The Čech theory of homology and cohomology groups of a universe can be naturally defined, a sketch of which is the object of the present paragraph.

Let X be an arbitrary universe, and $\mathcal{B} = \{B\}$ be the boundedness of X. Throughout the present paragraph, a finite open covering of X will be simply called a *covering*.

Let $\alpha = \{a_1, a_2, \ldots, a_{i_\alpha}\}$ be an arbitrary covering of X, and let N_α denote the *nerve* of α. A vertex $a_i \in N_\alpha$ is said to be *special*, if the open set $a_i \in \alpha$ is not bounded; a simplex $\sigma \in N_\alpha$ is said to be *special*, if all of its vertices are special. The special simplexes of N_α constitute a closed subcomplex M_α, called *the special subcomplex* of N_α.

A covering $\beta = \{b_1, b_2, \ldots, b_{i_\beta}\}$ is said to be a *refinement* of the covering α (denoted by $\alpha < \beta$), if each $b_i \in \beta$ is contained in some $a_j \in \alpha$.

Let Σ be the set of all covering of X, partially ordered by the relation $\alpha < \beta$. Σ is a direct set since any two coverings α and β have a common refinement, obtained by mutual intersections of the elements of α with those of β.

Suppose $\alpha < \beta$. Let us select for each member of β a member of α containing it. This gives a simplicial mapping $\Phi_{\beta\alpha}$ of N_β into N_α which is called a *projection* of N_β into N_α. It is trivial that $\Phi_{\beta\alpha}(M_\beta) \subset M_\alpha$. For any two projections $\Phi_{\beta\alpha}$ and $\psi_{\beta\alpha}$ of N_β into N_α, it is easily seen that for each simplex $\sigma \in N_\beta$ the simplexes $\Phi_{\beta\alpha}(\sigma)$ and $\psi_{\beta\alpha}(\sigma)$ are faces of some simplex $\tau \in N_\alpha$ and that τ can be selected from M_α if $\sigma \in M_\beta$. Further, if $\alpha < \beta < \gamma$ and $\Phi_{\beta\alpha}$, $\psi_{\gamma\beta}$ are projections, $\Phi_{\beta\alpha}\psi_{\gamma\beta}$ is a projection of N_γ into N_α.

For a given commutative coefficient group G, let us denote by

$$H_n(\alpha) = H_n(N_\alpha \bmod M_\alpha,\ G),$$
$$H^n(\alpha) = H^n(N_\alpha \bmod M_\alpha,\ G),$$

the n^{th} homology and the n^{th} cohomology groups of N_α modulo M_α [4, p. 116]. The projections $\Phi_{\beta\alpha}$ of N_β into N_α induce unique homomorphisms

$$\varpi_{\beta\alpha}:\quad H_n(\beta) \to H_n(\alpha),$$
$$\pi_{\alpha\beta}:\quad H^n(\alpha) \to H^n(\beta),$$

further, if $\alpha < \beta < \gamma$, the following relations hold :

$$\varpi_{\beta\alpha}\varpi_{\gamma\beta} = \varpi_{\gamma\alpha}, \qquad \pi_{\beta\gamma}\pi_{\alpha\beta} = \pi_{\alpha\gamma}.$$

This shows that for any integer n and coefficient group G, $\{H_n(\alpha),\ \alpha \in \Sigma\}$ is an inverse system of groups with homomorphisms $\varpi_{\beta\alpha}$ and $\{H^n(\alpha),\ \alpha \in \Sigma\}$ is a direct system of groups with homomorphisms $\pi_{\alpha\beta}$.

DEFINITION **14.1**. — *Let* X *be a universe,* G *a commutative group, n a non-negative integer, and* $\Sigma = \{\alpha\}$ *the collection of all the coverings of* X. *The limit group of the inverse system* $\{H_n(\alpha),\ \alpha \in \Sigma\}$ *is defined to be the* n^{th} *homology group* $H_n(X, G)$ *of the universe* X *with coefficient group* G. *The limit group of the direct system* $\{H^n(\alpha),\ \alpha \in \Sigma\}$ *is defined to be the* n^{th} *cohomology group* $H^n(X, G)$ *of the universe* X *with coefficient group* G.

If X is a compactum and with the trivial boundedness, then our definition reduces to the usual Čech theory [4, p. 135]. If X is locally compact, with the compact boundedness, and homeomorphic to an open set of some normal space, then the above definition reduces to the definition given by P. Alexandroff [2, p. 82].

THEOREM **14.2**. — *Let* G *be a discrete group and* $G^\star$ *its compact character group; then for each universe* X *and each integer* $n \geqq 0$, $H_n(X, G^\star)$ *is the character group of* $H^n(X, G)$.

PROOF. — For each $\alpha \in \Sigma$, the compact homology group $H_n(\alpha, G^\star)$ is the character group of the discrete cohomology group $H^n(\alpha, G)$ [2, p. 53]. Further, it can be easily seen that the homomorphisms $\varpi_{\beta\alpha}$ and $\pi_{\alpha\beta}$ are dual to each other; hence our theorem follows from a statement of [4, p. 134]. Q. E. D.

THEOREM **14.3**. — *Let* X *be a proper universe and* Λ *be its kernel, then for each coefficient group* G *and each integer* $n \geqq 0$ *the following homomorphisms hold :*

(**14.31**) $$H_n(X, G) \approx H_n(\Lambda, G),$$

(**14.32**) $$H^n(X, G) \approx H^n(\Lambda, G).$$

PROFF. — The argument given below is an analoque of that used by P. Alexandroff [2, p. 87] in his proof of the Kolmogoroff duality theorem.

Let $\alpha = \{a_1, \ldots, a_p, a_{p+1}, \ldots, a_q, a_{q+1}, a_r\}$ be a covering of X, where $\alpha_0 = \{a_1, \ldots, a_p\}$ denotes all the bounded elements of α and $\alpha_\star = \{a, \ldots, a_q\}$ denotes all the elements of α which are contained in Λ. Let Q_α denote the bounded closed set $\bigcup_{i=1}^{p} \bar{a}_i$. The covering α is said to de *regular*, if (1) every element of α which meets Q_α belongs to $\alpha_\star$, and (2) $\alpha_\star$ is a covering of Λ.

The regular coverings form a cofinal subset R *of the set of all coverings of* X. Indeed, let α be an arbitrary covering of X. Denote all the non-void sets of the form $\Lambda \cap a_i$, $a_i \in \alpha$ by $\beta_\star = \{b_1, \ldots, b_h, b_{h+1}, \ldots, b_k\}$ where $\beta_0 = \{b_1, \ldots, b_h\}$ are the bounded elements. $\beta_\star$ is evidently a covering of Λ. Let P denote the set $\bigcup_{i=1}^{h} \bar{b}_i$, and let $b_{k+i} = a_{q+i} - \mathrm{P}$ $(i = 1, 2, \ldots, r-q)$; then the covering

$$\beta = \{ b_1, \ldots, b_h, b_{h+1}, \ldots, b_k, b_{k+1}, b_{k+r-q} \}$$

is a refinement of α. Since $\mathrm{Q}_\beta = \mathrm{P}$, β is regular.

For every covering $\delta = \{ d_1, \ldots, d_k \}$ *of* Λ *there is a regular covering* α *of* X *with* $\alpha_* = \delta$. In fact, let $\delta_0 = \{ d_1, \ldots, d_h \}$ denote the bounded elements of δ and let $\mathrm{P} = \bigcup_{i=1}^{h} \bar{d}_i$. The covering α consisting of all the elements of δ and the $\mathrm{X} - \mathrm{P}$ is a required one.

Any two regular coverings α *and* β *of* X *have a common refinement* γ *such that* γ_* *is a common refinement of* α_* *and* β_*. In fact, let δ be a regular covering which is a common refinement of α and β. Denote all the non-void sets of the form $a_i \cap b_j \cap d_k$ where $a_i \in \alpha_*$, $b_j \in \mathrm{B}_*$, $d_h \in \delta_*$, by $\gamma_* = \{ c_1, \ldots, c_m, c_{m+1}, \ldots, c_n \}$, where $\gamma_0 = \{ c_1, \ldots, c_m \}$ denotes the bounded elements. Let P denote the set $\bigcup_{i=1}^{m} \bar{c}_i$, and let $c_{n+1}, \ldots, c_s$ denote all the sets of the form $d_i - \mathrm{P}$, where $d_i \in \delta - \delta_*$. The covering

$$\gamma = \{ c_1, \ldots, c_m, c_{m+1}, \ldots, c_n, c_{n+1}, \ldots, c_s \}$$

is a required one.

In the set R of all regular coverings of X, we define a partial order by the statement : $\alpha < \beta$ if (1) β is a refinement of α and (2) β_* is a refinement of α_*. It follows from the foregoing preliminary considerations that $\mathrm{H}_n(\mathrm{X}, \mathrm{G})$ *is isomorphic with the limit group of the inverse system* $\{ \mathrm{H}_n(\mathrm{N}_\alpha \bmod \mathrm{M}_\alpha, \mathrm{G}), \alpha \in \mathrm{R} \}$ *with* $\varpi_{\beta\alpha}$ *as the homomorphisms and that* $\mathrm{H}^n(\mathrm{X}, \mathrm{G})$ *is isomorphic with the limit group of the direct system* $\{ \mathrm{H}^n(\mathrm{N}_\alpha \bmod \mathrm{M}_\alpha, \mathrm{G})\ \alpha \in \mathrm{R} \}$ *with* $\pi_{\alpha\beta}$ *as the homomorphisms.*

Now let $\alpha \in \mathrm{R}$, then α_* is a covering of Λ. Let N_α, N_{α_*} denote the nerves of α, α_*; and let M_α, M_{α_*} denote the special subcomplexes of N_α, N_{α_*}. Since each vertex of N_{α_*} is a vertex of N_α, it follows that $\mathrm{N}_{\alpha_*} \subset \mathrm{N}_\alpha$. Since for a proper universe X the bounded sets of X coincide with those of Λ, it follows that $\mathrm{M}_{\alpha_*} \subset \mathrm{M}_\alpha$.

$\mathrm{N}_\alpha - \mathrm{M}_\alpha = \mathrm{N}_{\alpha_*} - \mathrm{M}_{\alpha_*}$ *for each* $\alpha \in \mathrm{R}$. In fact, let $\sigma \in \mathrm{N}_\alpha - \mathrm{M}_\alpha$;

then σ has a bounded set a_i of X as one of its vertices. It follows from the regularity of α that all the vertices of σ belong to α_α, i. e. $\sigma \in N_{\alpha_*}$. Since a_i is also a bounded set of A, $\sigma \in N_{\alpha_*} - M_{\alpha_*}$. Conservely, suppose $\sigma \in N_{\alpha_*} - M_{\alpha_*}$; then σ has at least one vertex a_i which is a bounded set of A and hence a bounded set of X. Therefore, $\sigma \in N_\alpha - M_\alpha$.

Since $N_\alpha - M_\alpha = N_{\alpha_*} - M_{\alpha_*}$, then the identity mapping $N_{\alpha_*} \to N_\alpha$ induces isomorphisms φ_α of $H_n(N_{\alpha_*} \bmod M_{\alpha_*}, G)$ onto $H_n(N_\alpha \bmod M_\alpha, G)$ and ψ_α of $H^n(N_\alpha \bmod M_\alpha G)$ onto $H^n(N_{\alpha_*} \bmod M_{\alpha_*}, G)$. Further, the following relations can be verified.

$$\varphi_\alpha \varpi_{\beta_*\alpha_*} = \varpi_{\beta\alpha} \varphi_\beta, \qquad \psi_\beta \pi_{\alpha\beta} = \pi_{\beta_*\alpha_*} \psi_\alpha$$

for each pair $\alpha, \beta \in R$ with $\alpha < \beta$.

Let S denote the set of all covering of A. We have proved that the limit groups of the two inverse systems

$$\{ H_n(N_\alpha \bmod M_\alpha, G), \alpha \in R \}, \qquad \{ H_n(N_{\alpha_*} \bmod M_{\alpha_*}, G), \alpha_* \in S \}$$

are isomorphic and the limit groups of the two direct systems

$$\{ H^n(N_\alpha \bmod M_\alpha, G), \alpha \in R \}, \qquad \{ H^n(N_{\alpha_*} \bmod M_{\alpha_*}, G), \alpha_* \in S \}$$

are isomorphic. This completes the proof. Q. E. D.

By the aid of (**14**.3), the connectivity theory of a proper universe reduces to that of a locally bounded proper universe. It happens to me that the whole theory of homology and cohomology of a topological space can be re-constructed in a more desirable form for a proper universe. For instance, we shall formulate a generalization of the Kolmogoroff duality theorem as follows.

DEFINITION **14**.4. — *The weak relative boundedness in a subset* X_* *of a universe* X *consists of the totality of the bounded sets* B *of* X *with* $\overline{B} \subset X_*$.

In particular, if X_* is a closed subset of X, then the weak relative boundedness coincides with the relative boundedness defined in paragraph **12**. The following generalized Kolmogoroff duality theorem can be proved by the method of P. Alexandroff [2] with some trivial modifications.

THEOREM **14**.5. — *Suppose* X *be a locally bounded proper normal universe,* X_0 *a closed subset of* X *and* X_* *the open complement* $X - X_0$, *both with the weak relative boundedness. Then for each coefficient group* G *and each integer* $n \geq 0$, $H^n(X, G) = 0 = H^{n+1}(X, G)$ *implies* $H^n(X_0, G) \approx H^{n+1}(X_*, G)$.

In particular, if X is a locally compact normal topological space, then (**14**.5) reduces to Alexandroff's formulation of the Kolmogoroff duality theorem [2, p. 86] by giving X the compact boundedness; for, in this case, the weak relative boundedness in X_0 and in X_* coincide with compact boundedness.

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