JOURNAL
DE
MATHÉMATIQUES
PURES ET APPLIQUÉES
FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874
PAR JOSEPH LIOUVILLE

DURGA PROSAD BANERJIE

On the harmonics associated with an ellipsoid and its application to the electrification of two parallel coaxial elliptic discs

Journal de mathématiques pures et appliquées 9e série, tome 26 (1947), p. 269-282.
<http://www.numdam.org/item?id=JMPA_1947_9_26__269_0>

On the harmonics associated with an ellipsoid and its application to the electrification of two parallel coaxial elliptic discs;

By Durga Prosad BANERJIE,

Professor A. M. College Mymensingh, Bengal (India).

Dr. S. K. Banerjie ([1]) has written a paper on Harmonics associated with an ellipsoid. In part I : I shall consider certain new and interesting properties of the function. On part II : I shall define certain functions suitable to the Boundary conditions of the elliptic discs and show that these functions reduce to spheroidal Harmonics if the eccentricity of the ellipse be zero *i.e* if the disc be circular. Then applying these functions I shall solve the problem of the electrification of two parallel coaxial elliptic discs and shall show that when the eccentricity of the elliptic dises be zero my result reduces to that arrived at by Dr Nicholson ([2]) in his paper on Electrification of two coaxial parallel circular discs. My sincere thanks are due to Principal B. M. Sen M. A. M. Se (cal. cantab) for kind suggestion and interest in the paper.

([1]) Dr. S. K. Banerjie, *On the Harmonics associated with an Ellipsoid* (*Bull. cal. Math. Soc.*, vol. X, nos 2, 3).

([2]) Dr. Nicholson, *On the electrification of two parallel coaxial circular discs* (*Phil. Trans. Royal soc.*, series A, vol. 224, p. 309).

PART I.

The following results will be required (1)

$$x = a\rho \sin\theta \cos\varphi,$$
$$y = b\rho \sin\theta \sin\varphi,$$
$$z = c\rho \cos\theta.$$

$\rho =$ constant denotes a set of similar and similarly situated ellipsoid.

$$\rho^n C_n^m(\theta, \varphi) = \rho^n C_n^m(c\cos\theta)$$
$$= \rho^n C_n^m = \frac{\lfloor n+m}{2\pi \lfloor n i^m} \int_0^{2\pi} (z + ix\cos u + iy\sin u)^n \cos mu \, du,$$
$$\rho^n S_n^m(\theta, \varphi) = \rho^n S_n^m(c\cos\theta)$$
$$= \rho^n S_n^m = \frac{\lfloor n+m}{2\pi \lfloor n i^m} \int_0^{2\pi} (z + ix\cos u + iy\sin u)^n \sin mu \, du,$$
$$\frac{F_n^m(\theta, \varphi)}{\rho^{n+1}} = \frac{F_n^m(c\cos\theta)}{\rho^{n+1}} = \frac{(-1)^m \lfloor n}{2\pi \lfloor n-m} \int_0^{2\pi} \frac{\cos mu \, du}{(z + ix\cos u + iy\sin u)^{n+1}},$$
$$\frac{G_n^m(\theta, \varphi)}{\rho^{n+1}} = \frac{G_n^m(c\cos\theta)}{\rho^{n+1}} = \frac{(-1)^m \lfloor n}{2\pi \lfloor n-m} \int_0^{2\pi} \frac{\sin mu \, du}{(z + ix\cos u + iy\sin u)^{n+1}},$$

satisfy the differential equation $\Delta^2 v = 0$.

$$\frac{\partial}{\partial z} \rho^n C_n^m(\theta, \varphi) = (n+m)\rho^{n-1} C_{n-1}^m(\theta, \varphi),$$
$$\frac{\partial \rho}{\partial z} = \frac{\cos\theta}{c}, \qquad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{c}.$$

Hence

$$(1-\mu^2)\frac{dC_n^m}{d\mu} + n\mu C_n^m = c(n+m)C_{n-1}^m,$$

where $\mu = \cos\theta$.

Similarly

$$(1-\mu^2)\frac{dS_n^m}{d\mu} + n\mu S_n^m = c(n+m)S_{n-1}^m.$$
$$\frac{\partial}{\partial z}\frac{F_n^m}{\rho^{n+1}} = -(n+m+1)\frac{F_{n+1}^m}{\rho^{n+2}},$$
$$(1-\mu^2)\frac{dF_n^m}{d\mu} - c\mu(n+1)F_n^m + (n-m+1)F_{n+1}^m = 0.$$

(1) Dr. S. K. Banerjie, *loc. cit.*

Similarly

$$(1-\mu^2)\frac{dG_n^m}{d\mu} - c\mu(n+1)G_n^m + (n-m+1)G_{n+1}^m = 0.$$

Let

$$\varphi_n^m = \rho^n C_n^m,$$

$$x\frac{\partial \varphi_n^m}{\partial x} + y\frac{\partial \varphi_n^m}{\partial y} + z\frac{\partial \varphi_n^m}{\partial z} = n\varphi_n^m,$$

$$2\frac{\partial}{\partial x}\rho^n C_n^m = \frac{\underline{|n+m}}{\underline{|n-m}\, i^{m-1}}\int_0^{2\pi}(z + ix\cos u + iy\sin u)^{n+1}$$

$$\times[\cos(m+1)n + \cos(m-1)n]\,du$$

$$= -\rho^{n-1}C_{n-1}^{m+1} + (n+m)(n+m-1)\rho^{n-1}C_{n-1}^{m-1},$$

$$\frac{\partial \rho}{\partial x} = \frac{\sin\theta\cos\varphi}{a}, \qquad \frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\varphi}{a\rho}, \qquad \frac{\partial \varphi}{\partial x} = \frac{\sin\varphi}{a\rho\sin\theta}.$$

Hence

$$2(1-\mu^2)\mu\mu'\frac{\partial C_n^m}{\partial \mu} + 2(1-\mu'^2)\frac{\partial C_n^m}{\partial \mu'}$$

$$= \sqrt{1-\mu^2}[2nC_n^m + C_{n-1}^{m+1} - (n+m)(n+m-1)C_{n-1}^{m-1}],$$

where

$$\mu' = \cos\varphi'.$$

Similarly

$$2(1-\mu^2)\mu\mu'\frac{\partial S_n^m}{\partial \mu} - 2(1-\mu'^2)\frac{\partial S_n^m}{\partial \mu'}$$

$$= \sqrt{1-\mu^2}[2nS_n^m + S_{n-1}^{m+1} - (n+m)(n+m-1)S_{n-1}^{m-1}],$$

$$2\frac{\partial}{\partial x}\frac{F_n^m}{\rho^{n+1}} = i\frac{F_{n+1}^{m+1}}{\rho^{n+2}} + (n-m+2)(n-m+1)i\frac{F_{n+1}^{m-1}}{\rho^{n+2}}.$$

Hence

$$2(1-\mu^2)\mu\mu'\frac{\partial C_n^m}{\partial \mu} - 2(1-\mu'^2)\frac{\partial C_n^m}{\partial \mu'}$$

$$= \sqrt{1-\mu^2}[2nC_n^m - C_{n-1}^{m+1} + (n+m)(n+m-1)C_{n-1}^{m-1}],$$

where

$$\mu' = \cos\varphi'.$$

Similarly

$$2(1-\mu^2)\mu\mu'\frac{dS_n^m}{\partial\mu} - 2(1-\mu'^2)\frac{\partial S_n^m}{\partial\mu'}$$
$$=\sqrt{1-\mu^2}[2nS_n^m - S_{n-1}^{m+1} + (n+m)(n+m-1)S_{n-1}^{m-1}],$$

$$2\frac{\partial}{\partial x}\frac{F_n^m}{\rho^{n+1}} = \frac{(-1)^{m+1}i\lfloor n+1}{2\pi\lfloor n-m}\int_0^{2\pi}\frac{2\cos mu\cos u\,du}{(z+ix\cos u+iy\sin u)^{n+2}}$$
$$= i\frac{F_{n+1}^{m+1}}{\rho^{n+2}} + (n-m+2)(n-m+1)i\frac{F_{n+1}^{m-1}}{\rho^{n+2}}:$$

$$2\left[(n+1)F_n^m + (1-\mu^2)\mu\mu'\frac{\partial F_n^m}{\partial\mu} - (1-\mu'^2)\frac{\partial F_n^m}{\partial\mu'}\right]$$
$$+ i[F_{n+1}^{m+1} + (n-m+2)(n-m+1)F_{n+1}^{m-1}] = 0.$$

Similarly

$$2\left[(n+1)G_n^m + (1+\mu^2)\mu\mu'\frac{\partial G_n^m}{\partial\mu} - (1-\mu'^2)\frac{\partial G_n^m}{\partial\mu'}\right]$$
$$+ i[G_{n+1}^{m+1} + (n-m+2)(n-m+1)G_{n+1}^{m-1}] = 0.$$

Following the method of Dr Banerjie we can prove that

$$F_n(c\cos\theta\cos u + a\sin\theta\cos\varphi\sin u\cos\nu + b\sin\theta\sin\varphi\sin u\sin\nu)$$
$$= F_n(c\cos\theta)\,P_n(\cos u) + 2\sum_{m=1}^{n}\frac{\lfloor n-m}{\lfloor n+m}$$
$$\times[F_n^m(c\cos\theta)P_n^m(\cos u)\cos m\nu + G_n^m(c\cos\theta)P_n^m(\cos u)\sin m\nu].$$

Since $C_0^0(c\cos\theta) = 1$,

$$\int_0^{2\pi}\int_0^{\pi}\frac{C_n(c\cos\theta)\sin\theta\,d\theta\,d\varphi}{p_0^2} = 0,$$

where

$$\frac{1}{p_0^2} = \frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta\cos^2\varphi}{a^2} + \frac{\sin^2\theta\sin^2\varphi}{b^2}.$$

Since

$$\rho^n C_n^m(\theta,\varphi) = r^n P_n^m(\cos\theta_1)\cos m\varphi_1,$$

where

$$x = r\sin\theta_1\cos\varphi_1,\qquad y = r\sin\theta_1\sin\varphi_1,\qquad z = r\cos\theta_1.$$

Following the corresponding series in Spherical Harmonics we have the following results

$$e^{c\rho\cos\theta}\,J_0\left(\rho\sin\theta\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right)=\sum_0^\infty\frac{\rho^n\,C^n(c\cos\theta)}{\lfloor n},$$

$$\sum_0^\infty(2n+1)\,C_n(c\cos\theta)\,\rho^n=\frac{1-\rho^2(c^2\cos^2\theta+a^2\sin^2\theta\cos^2\varphi+b^2\sin^2\theta\sin^2\varphi)}{X^{\frac{3}{2}}},$$

where

$$X=1-2\rho\cos\theta+\rho^2[c^2\cos^2\theta+a^2\sin^2\theta\cos^2\varphi+b^2\sin^2\theta\sin^2\varphi],$$

we know

$$\frac{P_n(\mu)}{\mu^{n+1}}=\frac{1}{\lfloor n}\int_0^\infty e^{-\lambda z}\,J_0(\lambda u)\lambda^n\,d\lambda,$$

where

$$u=\sqrt{x^2+y^2},\qquad z=r\cos\theta;$$

Hence

$$\frac{F_n(c\cos\theta)}{\rho^{n+1}}=\frac{1}{\lfloor n}\int_0^\infty e^{-\lambda c\rho\cos\theta}\,J_0(\lambda\rho\sin\theta p)\,\lambda^n\,d\lambda,$$

where

$$p=\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi};$$

we know

$$\frac{1}{\sqrt{\mu^2-2\mu\mu_1\cos\gamma+\mu_1^2}}=\sum_0^\infty\frac{\mu_1^n\,P_n(\cos\gamma)}{\mu^n}$$

if $\mu>\mu_1$, where

$$\cos\gamma=\cos\theta\cos\theta_1+\sin\theta\sin\theta_1\cos(\varphi-\varphi_1).$$

Hence

$$\frac{1}{\sqrt{\mu^2+\mu_1^2-2[zz_1+xx_1+yy_1]}}$$

$$=\sum_0^\infty\left[\mu_n^1\frac{P_n(\cos\theta)\,P_n(\cos\theta_1)}{\mu^{n+1}}\right.$$

$$\left.+2\sum_{m=1}^{n}\frac{\lfloor n-m}{\lfloor n+m}P_n^m(\cos\theta)\,P_n^m(\cos\theta_1)\cos m(\varphi-\varphi_1)\right]$$

$$=\sum_0^\infty\left\{\rho_1^n\,C_n(\theta',\varphi')\frac{F_n(\theta'',\varphi'')}{\rho^{n+1}}\right.$$

$$\left.+2\sum_{m=1}^{n}\frac{\lfloor n-m}{\lfloor n+m}\left[\rho_1^n\,C_n^m(\varphi',\theta')\frac{F_n^m(\theta'',\varphi'')}{\rho^{n+1}}+\rho_1^n\,S_n^m(\theta',\varphi')\frac{G_n^m(\theta'',\varphi'')}{\rho^{n+1}}\right]\right\},$$

where

$$z = c\rho_1 \cos\theta', \qquad z' = c\rho\cos\theta'',$$
$$x = a\rho_1 \sin\theta' \cos\varphi', \qquad x' = a\rho\sin\theta''\cos\varphi'',$$
$$y = b\rho_1 \sin\theta' \sin\varphi', \qquad y' = b\rho\sin\theta''\sin\varphi'',$$

$$\sum_0^\infty \rho^n C_n(c\cos\theta) = \frac{1}{\sqrt{1 - 2c\rho\cos\theta + \rho^2(c^2\cos^2\theta + a^2\sin^2\theta\cos^2\varphi + b^2\sin^2\theta\sin^2\varphi)}}.$$

PART II.

$$L_n^m(\rho, \theta, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} q_n\left(\frac{z - i(x\cos u + y\sin u)}{a}\right)\cos mu\, du,$$

$$M_n^m(\rho, \theta, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} q_n\left(\frac{z - i(x\cos u + y\sin u)}{a}\right)\sin mu\, du,$$

$$R_n^m(\rho, \theta, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} p_n\left(\frac{z - i(x\cos u + y\sin u)}{a}\right)\cos mu\, du,$$

$$T_n^m(\rho, \theta, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} p_n\left(\frac{z - i(x\cos u + y\sin u)}{a}\right)\sin mu\, du,$$

where

$$z = a\rho\cos\theta$$
$$x = a\rho\sin\theta\cos\varphi,$$
$$y = b\rho\sin\theta\sin\varphi, \qquad \text{and} \qquad q_n(t),\quad p_n(t),$$

are the solutions of the differential equation

$$\frac{d}{dt}(1 + t^2)\frac{dy}{dt} + n(n+1)y = 0.$$

Then L_n^m, M_n^m, R_n^m, T_n^m are the solutions of the differential equation $\Delta v = 0$.

$$L_n^m,\quad M_n^m = 0 \quad \text{at} \quad \rho = \infty.$$

It is well known that if $\rho > \mu$

$$\frac{|\underline{n-m}}{|\underline{n+m}} Q_n^m(\rho)\, P_n^m(\mu)\cos m\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} Q_n\left\{\mu\rho + \sqrt{(1-\mu^2)(1-\rho^2)}\cos(u-\varphi)\right\}\cos mu\, du.$$

The corresponding formulæ ([1]) in q functions will be

$$\frac{\lfloor n-m}{i^m \lfloor n+m} q_n^m(\rho)\,P_n^m(\mu)\cos m\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} q_n\left\{\mu\rho - i\sqrt{(1-\mu^2)(1+\rho^2)}\cos(u-\varphi)\right\}\cos mu\,du,$$

for unrestricted value of S.

Similarly

$$\frac{\lfloor n-m}{i^m \lfloor n+m} q_n^m(\rho)\,P_n^m(\mu)\sin m\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} q_n\left\{\mu\rho - i\sqrt{(1-\mu^2)(1+\rho^2)}\cos(u-\varphi)\right\}\sin mu\,du.$$

Hence when the eccentricity of the ellipse be zero $i.e$, $a=b$ and consequently the discs circular we have

$$L_n^m(\rho, 0, \varphi) = \frac{\lfloor n-m}{i^m \lfloor n+m} q_n^m(\rho)\,P_n^m(\mu)\cos m\varphi,$$

$$M_n^m(\rho, \theta, \varphi) = \frac{\lfloor n-m}{i^m \lfloor n+m} q_n^m(\rho)\,P_n^m(\mu)\sin m\varphi,$$

$$L_n^m\left(1, \frac{\pi}{2}, \varphi\right) = \frac{1}{2\pi}\int_0^{2\pi} q_n - i\left(\cos\varphi\cos u + \frac{b}{a}\sin\varphi\sin u\right)\cos mu\,du$$
$$= \frac{1}{2\pi}\int_0^{2\pi} q_n - i\left\{\sqrt{1-e^2\sin^2\varphi}\cos(u-\psi)\right\}\cos mu\,du$$
$$= \frac{\lfloor n-m}{i^m \lfloor n+m} q_n^m(0)\,P_n^m(e\sin\varphi)\cos m\psi,$$

where

$$\tan\psi = \frac{b}{a}\tan\varphi, \qquad b^2 = a^2(1-e^2).$$

Similarly

$$M_n^m\left(1, \frac{\pi}{2}, \varphi\right) = \frac{\lfloor n-m}{i^m \lfloor n+m} q_n^m(0)\,P_n^m(e\sin\varphi)\sin m\psi.$$

([1]) Dr. J. W. Nicholson, *loc. cit.*, p. 309.

ELECTRIFICATION OF ONE ELLIPTIC DISC. — Let the disc be situated with its centre at the origin. Let V be the potential. $V=1$, when $\rho=1$, $\theta=\frac{\pi}{2}$.

$$V=\sum_{0}^{\infty}\sum_{m=1}^{n} a_n^m L_n^m(\rho,\theta,\varphi)+\sum_{1}^{\infty}\sum_{m=1}^{n} b_n^m M_n^m(\rho,\theta,\varphi).$$

Since $V=1$ and consequently does not contain

$$P_\mu^m(r\sin\varphi)\frac{\cos}{\sin}m\psi \quad \text{for} \quad \mu=1,2,\ldots,\infty;\ m=1,2,\ldots,n;$$

When $\rho=1$, $\theta=\frac{\pi}{2}$,

$$b_n^m=0=a_n^m \quad \text{for} \quad n=1,2,\ldots,\infty;\ m=1,2,\ldots,n.$$

Hence

$$V=\frac{2}{\pi}L_0(\rho,\theta,\varphi).$$

The surface density $\underset{z=0}{Lt}\frac{dv}{dt}=$

$$\frac{1}{\pi^2}\int_0^{2\pi} q'_0-i\left(\frac{x\cos u+y\sin u}{a}\right)du=\frac{1}{\pi^2}\int_0^{2\pi} q'_0-\left(i\rho\sqrt{1-e^2\sin^2\varphi}\cos\psi\right)d\psi'$$

When the eccentricity of the disc be zero [1],

$$V=\frac{2}{\pi}q_0(\rho)=\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\rho}\right).$$

The surface density is

$$\frac{2}{\pi^2}\int_0^{\pi} q'_0-(i\rho\cos\psi')\,d\psi'=\frac{2}{\pi}\frac{1}{\sqrt{1-\frac{x^2+y^2}{a^2}}}.$$

ELECTRIFICATION OF TWO ELLIPTIC DISCS. — Let O_1 and O_2 be two origins on the axis of z. O_1 being at $z=-d$ on the left and O_2 at $z=0$. Let (x,y,z), $(x'y'z')$ be the coordinates of a point with

---

[1] This result agrees with that given by DR. J. H. JEANS's, *Electricity and Magnetism*, p. 249.

respect to O_2 and O_1 respectively.

$$x' = x, \qquad y' = y, \qquad z' = z + d,$$

$$\begin{aligned} L_n^m(\rho', \theta', \varphi') &= \frac{1}{2\pi}\int_0^{2\pi} q_n\left(\frac{z' - ix'\cos u - iy'\sin u}{a}\right)\cos mu\, du \\ &= \frac{1}{2\pi}\int_0^{2\pi} q_n\left(\frac{z + d - ix\cos u - iy\sin u}{a}\right)\cos mu\, du \\ &= \frac{1}{2\pi}\int_0^{2\pi} q_n\left(\frac{d}{a} + X\right)\cos mu\, du, \end{aligned}$$

where

$$X = z - ix\cos u - iy\sin u,$$

$$\begin{aligned} L_n^m(\rho', \theta', \varphi') &= \frac{2^n(\underline{|n})^2}{2\pi\underline{|2n+1}}\int_0^{2\pi} \\ &\times \left\{\frac{1}{\left(\frac{d}{a} + X\right)^{n+1}} - \frac{(n+1)(n+2)}{2(2n+3)}\frac{1}{\left(\frac{d}{a} + X\right)^{n+3}} + \dots\right\}\cos mu\, du \\ &= (-1)^n\sqrt{\frac{\pi}{2D}}J_{n+\frac{1}{2}}(D)\int_0^{2\pi}\frac{du\cos mu}{\frac{d}{a} + X}. \end{aligned}$$

where

$$D = \frac{\partial}{\partial\frac{d}{a}},$$

we know (¹)

$$\frac{1}{t - z} = \sum_0^\infty (2n+1)P_n(z)Q_n(t),$$

if

$$|t| > |z|.$$

Putting

$$t = it_1, \qquad z_1 = iz,$$

we have

$$\frac{1}{t_1 + z_1} = \sum_0^\infty (-1)^n(2n+1)p_n(z_1)q_n(t_1).$$

(¹) *Whittaker Modern Analysis*, p. 322.

Hence

$$L_n^{m\prime}(\rho', \theta', \varphi') = \frac{(-1)^n}{2\pi}\left(\frac{\pi}{2D}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(D)$$

$$\times \sum_0^\infty (-1)^\mu (2\mu+1)\, q_\mu\left(\frac{d}{a}\right) \int_0^{2\pi} p_\mu(X)\cos mu\, du$$

$$= \sum_0^\infty (-1)^\mu (2\mu+1)\, R_\gamma^{m\prime}(\rho, \theta, \varphi)\,(-1)^n \sqrt{\frac{\pi}{2D}}\, J_{n+\frac{1}{2}}(D)\, q_\gamma\left(\frac{d}{a}\right).$$

Let

$$\frac{\pi}{2} K_\mu^n\left(\frac{d}{a}\right) = (-1)^n \sqrt{\frac{\pi}{2}}\, \frac{J_{n+\frac{1}{2}}(D)}{\sqrt{D}}\, q_\gamma\left(\frac{d}{a}\right)$$

then

$$L_n^{m\prime}(\rho', \theta', \varphi') = \frac{\pi}{2}\sum_0^\infty (-1)^\mu (2\mu+1)\, R_\gamma^{m\prime}(\rho, \theta, \varphi)\, K_\mu^n\left(\frac{d}{a}\right).$$

Similarly

$$M_n^{m\prime}(\rho', \theta', \varphi') = \frac{\pi}{2}\sum_0^\infty (-1)^\mu (2\mu+1)\, T_\mu^{m\prime}(\rho, \theta, \varphi)\, K_\mu^n\left(\frac{d}{a}\right).$$

We know (1) that

$$p_n\left\{xx' - i\sqrt{(1+x^2)(1-x'^2)}\cos\omega\right\} = p_n(x)\, P_n(x')$$

$$+ 2\sum_{m=1}^{n} \frac{\lfloor n-m}{i^m \lfloor n+m} P_n^m(x')\, p_n^m(x)\cos m\omega.$$

Hence

$$\lim_{(\rho=1,\, \theta=\frac{\pi}{2})} L_n^{m\prime}(\rho', \theta', \varphi')$$

$$= \frac{\pi}{2}\sum^\infty (-1)^\mu (2\mu+1)\, \frac{\lfloor n-m}{i^m \lfloor n+m}\, p_n^m(0)\, P_n^m(e\sin\varphi)\cos m\psi\, K_\mu^n\left(\frac{d}{a}\right).$$

---

(1) WHITTAKER, *loc. cit.*, p. 328.

Similarly

$$\lim_{(\rho=1,\,\theta=\frac{\pi}{2})} \mathrm{M}_n^m(\rho', \theta', \varphi')$$

$$=\frac{\pi}{2}\sum_0^\infty(-1)^\mu(2\mu+1)\frac{|\underline{n-m}}{i^m\,|\underline{n+m}}p_n^m(0)\,\mathrm{P}_n^m(e\sin\varphi)\sin m\psi\,\mathrm{K}_\mu^n\left(\frac{d}{a}\right)$$

$$\lim_{(\rho'=1,\,\theta'=\frac{\pi}{2})} \mathrm{L}_n^m(\rho, \theta, \varphi)$$

$$=\frac{\pi}{2}\sum_0^\infty(-1)^\mu(2\mu+1)\frac{|\underline{n-m}}{i^m\,|\underline{n+m}}p_n^m(0)\,\mathrm{P}_n^m(e\sin\varphi)\cos m\psi\,\mathrm{K}_\mu^n\left(\frac{d}{a}\right)$$

$$\lim_{(\rho'=1,\,\theta'=\frac{\pi}{2})} \mathrm{M}_n^m(\rho, \theta, \varphi)$$

$$=\frac{\pi}{2}\sum_0^\infty(-1)^\mu(2\mu+1)\frac{|\underline{n-m}}{i^m\,|\underline{n+m}}p_n^m(0)\,\mathrm{P}_n^m(e\sin\varphi)\sin m\psi\,\mathrm{K}_\mu^n\left(\frac{d}{a}\right)$$

Let the discs be equallycharged. Omitting the actual charge for general convenience the general expression for potential is

$$\begin{aligned}\mathrm{V} = {} & \mathrm{L}_0(\rho, \theta, \varphi) + \mathrm{L}_0(\rho', \theta', \varphi') \\ & + \sum_{n=1}^{\infty}\mathrm{A}_n^m[\mathrm{L}_n^m(\rho, \theta, \varphi) + (-1)^n\,\mathrm{L}_n^m(\rho', \theta', \varphi')] \\ & + \sum_{n=1}^{\infty}\mathrm{B}_n^m[\mathrm{M}_n^m(\rho, \theta, \varphi) + (-1)^n\,\mathrm{M}_n^m(\rho', \theta', \varphi')]\end{aligned}$$

by reason of symmetry.

Since $\mathrm{V} =$ constant over the discs and does not contain any term

$$\mathrm{P}_n^m(e\sin\varphi)\,{}^{\cos}_{\sin}\,m\psi \qquad \text{for } n=1, 2, \ldots, \infty; \quad m=1, 2, \ldots, n;$$

we have

$$\mathrm{A}_n^m = \mathrm{B}_n^m = 0.$$

Let

$$\mathrm{V}=\mathrm{V}_1 \quad \text{when } \rho=1,\ \theta=\frac{\pi}{2};$$

$$\mathrm{V}=\mathrm{V}_2 \quad \text{when } \rho'=1,\ \theta'=\frac{\pi}{2};$$

$$V_1 = \sum_0^\infty A_n q_n(0) P_n(e\sin\varphi)$$

$$+ \frac{\pi}{2} \sum_0^\infty (-1)^\mu (2\mu + 1) P_\mu(e\sin\varphi) p_\mu(0) K_\mu^0\left(\frac{d}{a}\right)$$

$$+ \frac{\pi}{2} \sum_{n=1}^\infty \sum_{m=0}^\infty (-1)^{\mu+n} A_n (2\mu + 1) P_\mu(e\sin\varphi) p_\mu(0) K_\mu^n\left(\frac{d}{a}\right),$$

$$A_0 = 1,$$

$$A_\mu = (-1)^\mu (2\mu + 1) \frac{\pi}{2} \frac{p_\mu(0)}{q_\mu(0)} \sum_0^\infty A_n (-1)^n K_\mu^n\left(\frac{d}{a}\right),$$

we know ([1])

$$\frac{p_{2\mu}(0)}{q_{2\mu}(0)} = \frac{2}{\pi}, \qquad \frac{p_{2\mu+1}(0)}{q_{2\mu+1}(0)} = 0.$$

Hence

$$A_{2n+1} = 0 \qquad (n = 0, 1, \ldots, \infty),$$

$$A_{2\mu} = -(4\mu + 1) \sum_{n=0}^\infty A_{2n} K_{2\mu}^{2n}\left(\frac{d}{a}\right).$$

We know ([2])

$$K_\mu^n\left(\frac{d}{a}\right) = \int_0^\infty e^{-\lambda\frac{d}{a}} J_{\mu+\frac{1}{2}}(\lambda) J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\lambda},$$

we have

$$A_{2\mu} = -(4\mu + 1) \int_0^\infty e^{-\lambda\frac{d}{a}} \frac{d\lambda}{\lambda} J_{2\mu+\frac{1}{2}}(\lambda) \sum_0^\infty A_{2n} J_{2n+\frac{1}{2}}(\lambda)$$

$$= -(4\mu + 1) \int_0^\infty e^{-\lambda\frac{d}{a}} \frac{d\lambda}{\lambda} J_{2\mu+\frac{1}{2}}(\lambda) f(\lambda)$$

where

$$f(\lambda) = \sum_0^\infty A_{2n} J_{2n+\frac{1}{2}}(\lambda),$$

$$\sum_1^\infty A_{2\mu} J_{2\mu+\frac{1}{2}}(y) = -\int_0^\infty e^{-\frac{d}{a}\lambda} f(\lambda) \frac{d\lambda}{\lambda} S(\lambda),$$

---

(1) Dr. Nicholson, *loc. cit.*, p. 320.

(2) Dr. Nicholson, *loc. cit.*, p. 312.

and (1)

$$S(\lambda) = \frac{\sqrt{\lambda y}}{\pi}\left\{\frac{\sin(\lambda+y)}{\lambda+y} + \frac{\sin(\lambda-y)}{\lambda-y}\right\}.$$

Hence

$$f(y) - J_{\frac{1}{2}}(y) = -\frac{\sqrt{y}}{\pi}\int_0^\infty e^{-\frac{d}{a}} f(\lambda)\frac{d\lambda}{\sqrt{\lambda}}\left\{\frac{\sin(\lambda+y)}{\lambda+y} + \frac{\sin(\lambda-y)}{\lambda-y}\right\}.$$

The solution of this integral equation will determine $f(\lambda)$ and consequently A_{2n} will be found out. This integral equation has been solved by Dr. Nicholson. We know (2) from his result

$$A_{2n} = \frac{(-1)^n \int_{-1}^{1} Q_{2n}(\rho)\, Q_0\left(\dfrac{\tanh \frac{\pi\rho}{4q}}{\tanh \frac{\pi}{4q}}\right) d\rho}{\int_{-1}^{1} [Q_{2n}(\rho)]^2\, d\rho}$$

where $q = \dfrac{d}{2a}$.

$$\varphi(\lambda) = \frac{f(\lambda)}{\sqrt{\lambda}} = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\int_0^\infty \sin\lambda t\, dt\, Q_0\left(\frac{\tanh\frac{\pi t}{4q}}{\tanh\frac{\pi}{4q}}\right)$$

$$+\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\int_1^\infty \sin\lambda t\, dt\, \tanh^{-1}\left(\frac{\tanh\frac{\pi}{4q}}{\tanh\frac{\pi t}{4q}}\right);$$

for approximate solution

$$\frac{f(\lambda)}{\lambda} = \sum_0^\infty b_{2m}\lambda^{2m},$$

$$b_{2m} - \frac{(-1)^m}{\lfloor 2m+1}\sqrt{\frac{2}{\pi}} = \frac{(-1)^m}{\lfloor 2m}\,\frac{2}{\pi}\sum_0^\infty b_{2n}\left\{\frac{\lfloor 2n}{2m+1}\,\frac{1}{k^{2n+1}}\cdots\right\}$$

where

$$k = \frac{d}{a}.$$

When the eccentricity of the discs be zero *i.e* when the discs become

(1) Dr. Nicholson, *loc. cit.*, p. 322.
(2) Dr. Nicholson, *loc. cit.*, p. 324, 346 et 347.

circular we have

$$V = q_0(\rho) + q_0(\rho') + \sum_0^\infty A_{2n}[P_{2n}(\mu)\, q_{2n}(\rho) + P_{2n}(\mu')\, q_{2n}(\rho')].$$

This agrees exactly with the result given by Dr. Nicholson in his paper on Electrification of two parallel coaxial circular discs.

We know ([1])

$$q_\gamma(x) = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\lambda x} J_{\mu+\frac{1}{2}}(\lambda) \frac{d\lambda}{\sqrt{\lambda}}.$$

Hence

$$I_n = \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} \int_0^\infty e^{-\frac{\lambda}{a}(z - i\, r\cos u - i y \sin u)} J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\sqrt{\lambda}}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{\lambda z}{a}} J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\sqrt{\lambda}} \int_0^{2\pi} e^{i\rho\lambda\sqrt{1-e^2\sin^2\varphi}\cos\psi'}\, du$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\frac{\lambda z}{a}} J_0(\lambda\rho\sqrt{1 - e^2\sin^2\varphi})\, J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\sqrt{\lambda}}$$

where $\psi' = u - \psi$.

$$V = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\frac{\lambda z}{a}} \left[1 + e^{-\frac{d\lambda}{a}}\right] \varphi(\lambda) J_0(\rho\lambda\sqrt{1 - e^2\sin^2\varphi})\, d\lambda.$$

Following Dr. Nicholson ([2]) we have the surface density on the outer side of either disc

$$\sigma_1 = \frac{a}{4\pi}\sqrt{\frac{\pi}{2}} \int_0^\infty \lambda(1 + e^{-\lambda d}) J_0(a\lambda\rho\sqrt{1 - e^2\sin^2\varphi})\, \varphi(a\lambda)\, d\lambda.$$

The surface density on the inner side

$$\sigma^2 = \frac{a}{4\pi}\sqrt{\frac{\pi}{2}} \int_0^\infty \lambda(1 - e^{-\lambda d})\, \varphi(a\lambda)\, d\lambda\, J_0(\lambda a\rho\sqrt{1 - e^2\sin^2\varphi})$$

which tends to zero as $d \to 0$.

---

(1) Dr. Nicholson, *loc. cit.*, p. 312.
(2) Dr. Nicholson, *loc. cit.*, p. 348 et 349.