JOURNAL

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES

FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874

PAR JOSEPH LIOUVILLE

M. A. BASOCO

On certain arithmetical functions due to M. Georges Humbert

Journal de mathématiques pures et appliquées 9e série, tome 26 (1947), p. 237-250.

<http://www.numdam.org/item?id=JMPA_1947_9_26__237_0>

On certain arithmetical functions due to M. Georges Humbert;

By M. A. BASOCO.

1. *Introduction.* — The contributions which M. Georges Humbert has made to that part of the theory of numbers which lies in close relationship to the theory of the elliptic and theta functions are well known (¹). They are rich in interesting results and in suggestions for research. The starting point for what follows in this paper, is to be found in a brief note which he published in the Comptes rendus de l'Academie des Sciences (²). In this note, M. Humbert has pointed out the existence of a certain class of entire functions which have some interesting arithmetical properties, and which are related to the theta functions of Jacobi by means of certain functional equations. These functional equations will be shown to have uniquė solutions and may, therefore, be used to define the functions of M. Humbert. When these functions are expressed in the form of Fourier Series, they resemble in structure, those for the Jacobian elliptic functions. They differ from these, however, in that their Fourier representations are valid throughout the entire finite complex plane. A further difference of interest lies in the fact that their arithmetical form involve incomplete numerical functions of the divisors of an integer (in the sense of Hermite).

(¹) See, for example, his important memoir in the *Journal de Mathématiques,* 6ᵉ serie, 1907.

(²) *Comptes rendus,* 158, 1914, p. 220 and 294.

In the present paper we wish to extend and supplement M. Humbert's note by giving a complete set of these functions as well as the functional equations which serve to characterize them. Advantage is taken of the fact that the functions of M. Humbert are special cases of certain others which the writer obtained a few years ago in a paper ([1]) dealing with a set of pseudo-periodic functions in two variables, to relate them to the theta functions by means of certain identities. The arithmetical equivalents of these identities are also obtained in the form of relations involving arbitrary functions in one variable. These relations, referred to as « paraphrases » or formulas of the Liouville type, are of interest because of their relative simplicity and also because the partitions involved refer to the representations of a number as the sum of five integral squares.

An immediate consequence of the analytical form of the functions under discussion is a series of relations between the greater integer function $E(x)$ and *incomplete* numerical functions ([2]). These are believed to be new; a partial list of these results is given in paragraph **6**.

2. *The functional Equations.* — In what follows, the notation is that ordinarily used in the theory of the Jacobi theta functions ([3]). The period $\pi\tau$ is such that $0 < \arg\tau < \pi$.

The set of functional equations considered has the form

$$(\mathrm{A})\qquad \begin{cases} h(z+\pi\ \ ) = (-1)^a\, h(z), \\ h(z+\pi\tau) = (-1)^b\, h(z) + \mathrm{F}_{ab}^{(\alpha)}(z), \end{cases}$$

where a, b may take the values zero or unity, and $\mathrm{F}_{ab}^{(\alpha)}(z)$, presently to be defined, is an expression which involves the theta function $\vartheta_\alpha(z)$. We shall denote the integral functions satisfying these equations by the symbol $\mathrm{H}_{ab}^{(\alpha)}(z)$. These are readily found on assuming series

---

([1]) M. A. Basoco, *The functions referred to appear as coefficients of the logarithmic derivative of* $\vartheta_\alpha(y)$ *in the expansion for* $\Theta_{\alpha\beta\gamma}(x, y)$ (*American Journal of Mathematics*, Vol. 54, 1932, p. 242-252).

([2]) M. A. Basoco, *In this paper some similar relations involving complete numerical functions are listed* (*Bull. Am. Math. Soc., Oct.* 1936, p. 720-726).

([3]) See, for example, Wittaker-Watson, *Modern Analysis* (Cambridge).

solutions of the form

$$H_{ab}^{(\alpha)}(z) = \sum_{k=-\infty}^{k=\infty} A_k e^{ikz} \qquad (i = \sqrt{-1}).$$

These solutions, for $(a, b) = (1, 0), (0, 1), (1, 1)$ are unique; for $(a, b) = (0, 0)$, the solution is completely determined to within an additive constant. For, suppose that for a given (a, b) two distinct solutions exist. Denote their difference by $D(z)$; this would, likewise, be an integral function and would satisfy periodicity relations of the form

$$D(z + \pi) = (-1)^a D(z),$$
$$D(z + \pi\tau) = (-1)^b D(z),$$

and would hence be an elliptic function. This would evidently reduce to a constant. From (A) it follows easily that this constant vanishes, except for the case $(a, b) = (0, 0)$, when it remains undetermined. In this case we have selected the solution which vanishes for $z = 0$.

3. *The function* $F_{ab}^{(\alpha)}(z)$ — Let $\lambda(z) = q^{-\frac{1}{4}} e^{-iz}$ and $\mu(z) = q^{-1} e^{-2iz}$, where $q = e^{i\pi\tau}$, $|q| < 1$. These expressions are the multipliers associated with the theta functions of arguments $z + \frac{\pi\tau}{2}$ and $z + \pi\tau$ respectively.

We define the functions $F_{ab}^{(\alpha)}(z)$ as follows

$$F_{00}^{(0)}(z) = i[1 - \mu(z)]\,\vartheta_0(z) - 2i, \qquad F_{11}^{(0)}(z) = 2i\lambda(z)\,\vartheta_0(z),$$
$$F_{00}^{(1)}(z) = 2i[i\lambda(z)\,\vartheta_1(z) - 1], \qquad F_{11}^{(1)}(z) = i[1 + \mu(z)]\,\vartheta_1(z),$$
$$F_{00}^{(2)}(z) = 2i[\lambda(z)\,\vartheta_2(z) - 1], \qquad F_{11}^{(2)}(z) = i[1 - \mu(z)]\,\vartheta_2(z),$$
$$F_{00}^{(3)}(z) = i[1 + \mu(z)]\,\vartheta_3(z) - 2i, \qquad F_{11}^{(3)}(z) = 2\lambda(z)\,\vartheta_3(z),$$
$$F_{01}^{(0)}(z) = i[1 + \mu(z)]\,\vartheta_0(z), \qquad F_{10}^{(0)}(z) = 2\lambda(z)\,\vartheta_0(z),$$
$$F_{01}^{(1)}(z) = 2i\lambda(z)\,\vartheta_1(z), \qquad F_{10}^{(1)}(z) = -i[1 - \mu(z)]\,\vartheta_1(z),$$
$$F_{01}^{(2)}(z) = 2\lambda(z)\,\vartheta_2(z), \qquad F_{10}^{(2)}(z) = i[1 + \mu(z)]\,\vartheta_2(z),$$
$$F_{01}^{(3)}(z) = i[1 - \mu(z)]\,\vartheta_3(z), \qquad F_{10}^{(3)}(z) = 2i\lambda(z)\,\vartheta_3(z).$$

These expressions satisfy the consistency condition

$$F_{ab}^{(\alpha)}(z + \pi) = (-1)^a F_{ab}^{(\alpha)}(z),$$

which is implied by equations (A).

4. *The solutions* $H_{ab}^{(\alpha)}(z)$. — If in equations (A) the preceding set of functions $F_{ab}^{(\alpha)}(z)$ be used, we find, following the method suggested in paragraph **2**, the trigonometric form of the solution of the corresponding functional equation. The solutions thus found are valid for all values of z. The integer n ranges over the values 1, 2, 3, 4, ..., while m ranges over the number 1, 3, 5, 7, ...

$$(1')\qquad H_{00}^{(3)}(z) = 2\sum_{n=1}^{\infty} \frac{q^{n^2+2n}+q^{n^2}}{1-q^{2n}} \sin 2nz,$$

$$(2')\qquad H_{00}^{(2)}(z) = 4\sum_{n=1}^{\infty} \frac{q^{n^2+n}}{1-q^{2n}} \sin 2nz,$$

$$(3')\qquad H_{01}^{(3)}(z) = 2\sum_{n=1}^{\infty} \frac{q^{n^2+2n}-q^{n^2}}{1+q^{2n}} \sin 2nz,$$

$$(4')\qquad H_{01}^{(2)}(z) = 1 + 4\sum_{n=1}^{\infty} \frac{q^{n^2+n}}{1+q^{2n}} \cos 2nz,$$

$$(5')\qquad H_{10}^{(3)}(z) = 4\sum_{m=1}^{\infty} \frac{q^{\frac{m^2+2m}{4}}}{1-q^{m}} \sin mz,$$

$$(6')\qquad H_{10}^{(2)}(z) = 2\sum_{m=1}^{\infty} \frac{q^{\frac{m^2+4m}{4}}+q^{\frac{m^2}{4}}}{1-q^{m}} \sin mz,$$

$$(7')\qquad H_{11}^{(3)}(z) = 4\sum_{m=1}^{\infty} \frac{q^{\frac{m^2+2m}{4}}}{1+q^{m}} \cos mz,$$

$$(8')\qquad H_{11}^{(2)}(z) = 2\sum_{m=1}^{\infty} \frac{q^{\frac{m^2+4m}{4}}-q^{\frac{m^2}{4}}}{1+q^{m}} \sin mz.$$

The remaining functions $H_{ab}^{(c)}(z)$, $c = 0, 1$ may be obtained from the preceding upon replacing z by $z+\frac{\pi}{2}$. In the interest of brevity we omit writing these in their analytical form; they are, however, listed below in their arithmetical form.

5. *Arithmetized form.* — In order to write the arithmetical form of the functions $H_{ab}^{(\alpha)}(z)$, we introduce the notation :

α and β are positive integers of the form $4k+1$ and $4k+3$ respectively; n and m are as in the preceding section. Σ' refers to the conjugate divisors (d, δ) of n and (t, τ) of α or β, such that $\delta < d$, $\tau < t$. Further restrictions on (d, δ) will be indicated as needed. $\varepsilon(n)$ is unity or zero according as n is or is not the square of an integer. $(-1\,|\,h) = (-1)^{\frac{h-1}{2}}$ h if is odd.

(1) $$H_{00}^{(3)}(z) = 2\sum_{n=1}^{\infty} q^n \varepsilon(n) \sin 2\sqrt{n}z + 4\sum_{(n)} q^n \left\{ \sum{}' \sin 2\delta z \right\}$$
$(\delta - d \equiv 0, \bmod 2)$;

(2) $$H_{00}^{(2)}(z) = 4\sum_{(n)} q^n \left\{ \sum{}' \sin 2\delta z \right\} \qquad (\delta - d \equiv 1, \bmod 2);$$

(3) $$H_{01}^{(3)}(z) = -2\sum_{n=1}^{\infty} q^n \varepsilon(n) \sin 2\sqrt{n}z - 4\sum_{(n)} q^n \left\{ \sum{}' (-1)^{\frac{d-\delta}{2}} \sin 2\delta z \right\}.$$
$(\delta - d \equiv 0, \bmod 2)$;

(4) $$H_{01}^{(2)}(z) = 1 + 4\sum_{(n)} q^n \left\{ \sum{}' (-1)^{\frac{\delta-d-1}{2}} \cos 2\delta z \right\} \qquad (\delta - d \equiv 1, \bmod 2);$$

(5) $$H_{10}^{(3)}(z) = 4\sum q^{\frac{\beta}{4}} \left\{ \sum{}' \sin \tau z \right\};$$

(6) $$H_{10}^{(2)}(z) = 2\sum q^{\frac{\alpha}{4}} \varepsilon(\alpha) \sin \sqrt{\alpha}z + 4\sum q^{\frac{\alpha}{4}} \left\{ \sum{}' \sin \tau z \right\};$$

(7) $$H_{11}^{(3)}(z) = 4\sum q^{\frac{\beta}{4}} \left\{ \sum{}' (-1)^{\frac{t-\tau-2}{2}} \cos \tau z \right\};$$

(8) $$H_{11}^{(2)}(z) = -2\sum q^{\frac{\alpha}{4}} \varepsilon(\alpha) \sin \sqrt{\alpha}z - 4\sum q^{\frac{\alpha}{4}} \left\{ \sum{}' (-1)^{\frac{t-\tau}{4}} \sin \tau z \right\};$$

(9) $$H_{00}^{(0)}(z) = 2\sum q^n \varepsilon(n) (-1)^{\sqrt{n}} \sin 2\sqrt{n}z + 4\sum q^n \left\{ \sum{}' (-1)^{\delta} \sin 2\delta z \right\}$$
$(\delta - d \equiv 0, \bmod 2)$;

(10) $$H_{00}^{(1)}(z) = 4\sum q^n \left\{ \sum{}' (-1)^{\delta} \sin 2\delta z \right\} \qquad (\delta - d \equiv 1, \bmod 2);$$

(11) $$H_{01}^{(0)}(z) = -2\sum q^n \varepsilon(n) (-1)^{\sqrt{n}} \sin 2\sqrt{n}z - 4\sum q^n \left\{ \sum{}' (-1)^{\frac{d+\delta}{2}} \sin 2\delta z \right\}$$
$(\delta - d \equiv 0, \bmod 2)$;

(12) $$H_{01}^{(1)}(z) = 1 + 4\sum q^n \left\{ \sum{}' (-1)^{\frac{d+\delta-1}{2}} \cos 2\delta z \right\} \qquad (d - \delta \equiv 1, \bmod 2);$$

$$(13)\quad H_{10}^{(0)}(z) = 4\sum q^{\frac{\beta}{4}}\left\{\sum{}'(-1\,|\,\tau)\cos\tau z\right\};$$

$$(14)\quad H_{10}^{(1)}(z) = 2\sum q^{\frac{\alpha}{4}}\varepsilon(\alpha)(-1\,|\sqrt{\alpha})\cos\sqrt{\alpha}\,z + 4\sum q^{\frac{\alpha}{4}}\left\{\sum{}'(-1\,|\,\tau)\cos\tau z\right\};$$

$$(15)\quad H_{11}^{(0)}(z) = -4\sum q^{\frac{\beta}{4}}\left\{\sum{}'(-1)^{\frac{t+\tau}{4}}\sin\tau z\right\};$$

$$(16)\quad H_{11}^{(1)}(z) = 2\sum q^{\frac{\alpha}{4}}\varepsilon(\alpha)(-1\,|\sqrt{\alpha})\cos\sqrt{\alpha}\,z + 4\sum q^{\frac{\alpha}{4}}\left\{\sum{}'(-1)^{\frac{t+\tau-2}{4}}\cos\tau z\right\}.$$

6. *Application to the function* $E(x)$. — The analytical form of the functions (1′) to (8′) and of the remaining eight deducible from them by increasing the argument by $\frac{\pi}{2}$, suggests the application of a devise due to Hermite ([1]), which yields identities involving the greatest integer function $E(x)$. Hermite's method depends on the following relations

$$\frac{u^b}{(1-u)(1-u^a)} = \sum_{(n)} E\left(\frac{n+a-b}{a}\right)u^n,$$

$$\frac{u^b}{(1-u)(1+u^a)} = \sum_{(n)} E_1\left(\frac{n+a-b}{2a}\right)u^n,$$

where a, b are positive integers and

$$E_1(x) = E(2x) - 2E(x) = E\left(x+\frac{1}{2}\right) - E(x).$$

We list below the identities deducible from the functions (1′) to (8′) and their equivalents (1) to (8). The remaining identities are analogous in form.

Let $F(z)$ be an arbitrary function; α and β are as before as well as (d, δ) and (t, τ); r is a positive fixed integer and $r = d\delta$. Define

([1]) HERMITE, *Acta Mathematica*, t. 5, p. 297-330; *J. für Math.* t. 100, p. 51-65; *Œuvres*, t. 4, p. 151-159.

$P_j(x, r)$, $Q_j(x, r)$, $R_j(x, r)$, $S_j(x, r)$ as follows

$$P_1(x,\ r) = \varepsilon(r)\,F(2\sqrt{r}x) + 2\sum{}' F(2\delta x) \qquad (r = d\delta,\ d - \delta \equiv 0,\ \mathrm{mod}\ 2);$$

$$P_2(x,\ r) = \sum{}' F(2\delta x) \qquad (r = d\delta,\ d - \delta \equiv 1,\ \mathrm{mod}\ 2);$$

$$Q_1(x,\ r) = \varepsilon(r)\,F(2\sqrt{r}x) + 2\sum{}' (-1)^{\frac{d-\delta}{2}} F(2\delta x) \qquad (r = d\delta,\ d - \delta \equiv 0,\ \mathrm{mod}\ 2);$$

$$Q_2(x,\ r) = \sum{}' (-1)^{\frac{\delta-d-1}{2}} F(2dx) \qquad (r = d\delta,\ d - \delta \equiv 1,\ \mathrm{mod}\ 2);$$

$$R_1(x,\ \beta) = \sum{}' F(\tau x) \qquad (\beta = t\tau);$$

$$R_2(x,\ \alpha) = \varepsilon(\alpha)\,F(\sqrt{\alpha}x) + 2\sum{}' F(\tau x) \qquad (\alpha = t\tau);$$

$$S_1(x,\ \beta) = \sum{}' (-1)^{\frac{t-\tau-2}{4}} F(\tau x) \qquad (\beta = t\tau);$$

$$S_2(x,\ \alpha) = \varepsilon(\alpha)\,F(\sqrt{\alpha}x) + 2\sum (-1)^{\frac{t-\tau}{4}} F(\tau x) \qquad (\alpha = t\tau).$$

The identities in question are as follows, n being an arbitrary fixed integers; s takes the values 1, 2, 3, 4, ..., while μ ranges over the positive odd integers 1, 3, 5, 7, ...

$$(a)\quad \sum_{r=1}^{n} P_1(x,\ r) = 2\sum_{s=1}^{[\sqrt{n+1}-1]} E\left(\frac{n-s^2}{2s}\right) F(2sx) + \sum_{s=1}^{[\sqrt{n}]} F(2sx);$$

$$(b)\quad \sum_{r=1}^{n} P_2(x,\ r) = \sum_{s=1} E\left(\frac{n+s-s^2}{2s}\right) F(2sx);$$

$$(c)\quad \sum_{r=1}^{n} Q_1(x,\ r) = \sum_{s=1} \left\{1 - 2E\left(\frac{n-s^2}{2s}\right) + 4E\left(\frac{n-s^2}{4s}\right)\right\} F(2sx);$$

$$(d)\quad \sum_{r=1}^{n} Q_2(x,\ r) = \sum_{s=1} \left\{E\left(\frac{n+s-s^2}{2s}\right) - 2E\left(\frac{n+s-s^2}{4s}\right)\right\} F(2sx);$$

$$(e)\quad \sum_{\mu=3}^{\beta \leq n} R_1(x,\ \beta) = \sum_{\mu=1} E\left(\frac{n+2\mu-\mu^2}{4\mu}\right) F(\mu x);$$

$$(f)\quad \sum_{\alpha=1}^{\alpha \leq n} R_2(x,\ \alpha) = \sum_{\mu=1}^{\mu \leq \sqrt{n}} \left\{1 + 2E\left(\frac{n-\mu^2}{4\mu}\right)\right\} F(\mu x);$$

$$(g) \quad \sum_{\beta=3}^{\beta \leqq n} S_1(x, \beta) = \sum_{\mu=1} \left\{ E\left(\frac{n+2\mu-\mu^2}{4\mu}\right) - 2E\left(\frac{n+2\mu-\mu^2}{8\mu}\right) \right\} F(\mu x);$$

$$(h) \quad \sum_{\alpha=1}^{\alpha \leqq n} S_2(x, \alpha) = \sum_{\mu=1}^{\mu \leqq \sqrt{n}} \left\{ 1 - 2E\left(\frac{n-\mu^2}{4\mu}\right) + 4E\left(\frac{n-\mu^2}{8\mu}\right) \right\} F(\mu x).$$

An interesting special case of the preceding formulae arises when we take $F(x) = x^k$, where k is a positive integer. Thus, for example, relations (a), (b) give rise to the following

$$(a') \qquad \sum_{r=1}^{n} X_1(r) = 2 \sum_{s=1}^{[\sqrt{n+1}-1]} E\left(\frac{n-s^2}{2s}\right) s^k + \sum_{s=1}^{[\sqrt{n}]} s^k,$$

$$(b') \qquad \sum_{r=1}^{n} X_2(r) = \sum_{s=1} E\left(\frac{n+s-s^2}{2s}\right) s^k,$$

where,

$$X_1(r) = \varepsilon(r) r^{\frac{k}{2}} + 2 \sum{}' \delta^k \qquad (r = d\delta, \delta < d, \delta - d \equiv 0, \bmod 2),$$

$$X_2(r) = \sum{}' \delta^k \qquad (r = d\delta, \delta < d, \delta - d \equiv 1, \bmod 2).$$

The incomplete numerical function $X_2(r)$ may, therefore, be expressed in terms of the greatest integer function

$$X_2(x) = \sum \left\{ E\left(\frac{n+s-s^2}{2s}\right) - E\left(\frac{n-s^2+s-1}{2s}\right) \right\} s^k.$$

Similarly for the others.

7. *Theta Identities.* — In a former paper (¹) the writer obtained the trigonometric developpement for sixteen theta quotients of the form

$$\vartheta_1'^2 \frac{\vartheta_\alpha(x+y)}{\vartheta_\beta^2(x)\vartheta_\gamma(y)}.$$

Regarded as functions of x these quotients are doubly periodic

(¹) *American Journal of Mathematics, loc. cit.*

of the Third Kind (Hermite's nomenclature) and may, therefore, be developed into a trigonometic series by the method of decomposition into simple elements due to M. Appell (¹). In a paper published elsewhere (²), the writer has given a simple analysis of the case when the functions have more poles than zero in a fundamental period cell; this is the case relevant to the theta quotients under discussion. We may obtain in this way, for example, the following expansion

$$\begin{aligned}\vartheta_1'^2 \frac{\vartheta_0(x+y)}{\vartheta_0^2(x)\,\vartheta_1(y)} = i\,\frac{\vartheta_1'(y)}{\vartheta_1(y)} \sum_{i=-\infty}^{\infty} (-1)^{r+1} q^{\frac{(2r+1)^2}{4}} e^{-i(2r+1)y} \operatorname{ctn}\left\{x-(2r+1)\frac{\pi\tau}{2}\right\} \\ + \sum_{i=-\infty}^{\infty} (-1)^{r} q^{\frac{(2r+1)^2}{4}} (2r+1)\, e^{-i(2r+1)y} \operatorname{ctn}\left\{x-(2r+1)\frac{\pi\tau}{2}\right\} \\ + i\sum_{i=-\infty}^{\infty} (-1)^{r+1} q^{\frac{(2r+1)^2}{4}} e^{-i(2r+1)y} \operatorname{cosec}^2\left\{x-(2r+1)\frac{\pi\tau}{2}\right\}.\end{aligned}$$

The result may readily be transformed into its arithmetical form; we find after a slight calculation that

$$\begin{aligned}\vartheta_1'^2 \frac{\vartheta_0(x+y)}{\vartheta_0^2(x)\,\vartheta_1(y)} = 2\sum_{m=1}(-1)^{\frac{m-1}{2}} m q^{\frac{m^2}{4}} \sin my + 2\sum_{\alpha=1} q^{\frac{\alpha}{4}} \left\{ \sum{}'(-1)^{\frac{\tau-1}{2}} (t+\tau)\sin\left(\frac{t-\tau}{2}x+\tau y\right)\right\} \\ + \frac{\vartheta_1'(y)}{\vartheta_1(y)}\left\{ 2\sum_{m=1}(-1)^{\frac{m-1}{2}} q^{\frac{m^2}{4}} \cos my + 4\sum_{\alpha=1} q^{\frac{\alpha}{4}}\left[\sum{}'(-1)^{\frac{\tau-1}{2}} \cos\left(\frac{t-\tau}{2}x+\tau y\right)\right]\right\},\end{aligned}$$

where m ranges over 1, 3, 5, 7, ...; α over 1, 5, 9, 13, ... and Σ' refers to the divisors t, τ, of α such that $\tau < t$. This form of the expansion is valid for all y and all x such that

$$-\frac{1}{2}\,\Im(\pi\tau) < \Im(x) < \frac{1}{2}\,\Im(\pi\tau).$$

If now, we set $x = 0$ and change y into z, we find the following identity which relates the function $H_{10}^{(1)}(z)$ to the theta function $\vartheta_1(z)$

(¹) P. Appell, *Annales Scientifiques de l'École Normale Supérieure*, Série 3, 1884-1885; *Mémorial des Sciences Mathématiques*, fascicule XXXVI, Paris, 1929.

(²) M. A. Basoco, *Acta Mathematica*, t. 57, 1930, p. 201.

and $\vartheta_1'(z)$

$$\vartheta_1'(z)\,H_{10}^{(1)}(z)=\vartheta_2^2\vartheta_3^2\vartheta_0(z)-2\vartheta_1(z)\sum_{\alpha=1}q^{\frac{\alpha}{4}}\left\{\varepsilon(\alpha)(-1)^{\frac{\sqrt{\alpha}-1}{2}}\sin\sqrt{\alpha}z+\sum{}'(-1)^{\frac{\tau-1}{2}}(t+\tau)\sin\tau z\right\}.$$

In a similar manner we may obtain the following identies; the notation is as in paragraph 5,

$$\vartheta_1'(z)\,H_{01}^{(1)}(z)=\vartheta_0^2\vartheta_3^2\vartheta_2(z)-4\vartheta_1(z)\sum q^n\left\{\sum{}'(-1)^{\frac{d-\delta-1}{2}}(d+\delta)\sin 2\delta z\right\}$$
$$(n=d\delta,\ \delta-d\equiv 1,\ \mathrm{mod}\,2);$$

$$\vartheta_2'(z)\,H_{01}^{(2)}(z)=-\vartheta_0^2\vartheta_3^2\vartheta_1(z)-4\vartheta_2(z)\sum q^n\left\{\sum{}'(-1)^{\frac{d+\delta-1}{2}}(d+\delta)\sin 2\delta z\right\}$$
$$(n=d\delta,\ \delta-d\equiv 1,\ \mathrm{mod}\,2);$$

$$\vartheta_3'(z)\,H_{10}^{(3)}(z)=-\vartheta_2^2\vartheta_3^2\vartheta_2(z)+2\vartheta_3(z)\sum q^{\frac{\beta}{4}}\left\{\sum{}'(t+\tau)\cos\tau z\right\};$$

$$\vartheta_3'(z)\,H_{11}^{(3)}(z)=\vartheta_0^2\vartheta_2^2\vartheta_1(z)-2\vartheta_3(z)\sum q^{\frac{\beta}{4}}\left\{\sum{}'(-1)^{\frac{t-\tau-2}{4}}(t+\tau)\sin\tau z\right\};$$

$$\vartheta_0'(z)\,H_{11}^{(0)}(z)=-\vartheta_0^2\vartheta_2^2\vartheta_2(z)-2\vartheta_3(z)\sum q^{\frac{\beta}{4}}\left\{\sum{}'(-1)^{\frac{t+\tau}{4}}(t+\tau)\cos\tau z\right\};$$

$$\vartheta_0'(z)\,H_{10}^{(0)}(z)=\vartheta_2^2\vartheta_3^2\vartheta_1(z)-2\vartheta_0(z)\sum q^{\frac{\beta}{4}}\left\{\sum{}'(-1\,|\,\tau)\cdot(t+\tau)\sin\tau z\right\};$$

$$\vartheta_1'(z)\,H_{10}^{(1)}(z)=\vartheta_2^2\vartheta_3^2\vartheta_0(z)-2\vartheta_1(z)\sum q^{\frac{\alpha}{4}}\left\{\varepsilon(\alpha)(-1\,|\,\sqrt{\alpha})\sqrt{\alpha}\sin\sqrt{\alpha}z+\sum{}'(-1\,|\,\tau)\,(t+\tau)\sin\tau z\right\};$$

$$\vartheta_1'(z)\,H_{11}^{(1)}(z)=\vartheta_0^2\vartheta_2^2\vartheta_3(z)-2\vartheta_1(z)\sum q^{\frac{\alpha}{4}}\left\{\varepsilon(\alpha)(-1\,|\,\sqrt{\alpha})\sqrt{\alpha}\sin\sqrt{\alpha}z+\sum{}'(-1)^{\frac{t+\tau-2}{4}}(t+\tau)\sin\tau z\right\};$$

$$\vartheta_2'(z)\,H_{10}^{(2)}(z)=-\vartheta_2^2\vartheta_3^2\vartheta_3(z)+2\vartheta_2(z)\sum q^{\frac{\alpha}{4}}\left\{\varepsilon(\alpha)\sqrt{\alpha}\cos\sqrt{\alpha}z+\sum{}'(t+\tau)\cos\tau z\right\};$$

$$\vartheta_2'(z)\,H_{11}^{(2)}(z)=-\vartheta_0^2\vartheta_2^2\vartheta_0(z)+2\vartheta_2(z)\sum q^{\frac{\alpha}{4}}\left\{\varepsilon(\alpha)\sqrt{\alpha}\cos\sqrt{\alpha}z+\sum{}'(-1)^{\frac{t-\tau}{4}}(t+\tau)\cos\tau z\right\};$$

$$\vartheta_0'(z)\,H_{01}^{(0)}(z)=\vartheta_0^2\vartheta_3^2\vartheta_3(z)-\vartheta_0(z)\left\{\psi_1(z)-4\sum q^n\left[\sum{}'(-1)^{\frac{d+\delta}{2}}(d+\delta)\cos 2\delta z\right]\right\};$$

$$\vartheta_3'(z)\,H_{01}^{(3)}(z)=-\vartheta_0^2\vartheta_3^2\vartheta_0(z)+\vartheta_3(z)\left\{\psi_2(z)-4\sum q^n\left[\sum{}'(-1)^{\frac{\delta-d}{2}}(d+\delta)\cos 2\delta z\right]\right\};$$

where, in the last two identies we have set

$$\psi_1(z) = 1 - 4\sum_{n=1}^{\infty}(-1)^n q^{n^2} n \cos 2nz, \qquad \psi_2(z) = 1 - 4\sum_{n=1}^{\infty} q^{n^2} n \cos 2nz.$$

The above set of relationships is incomplete, since no results involving $H_{00}^{(\alpha)}(z)$ are given. This case is exclused by the method which has been used to obtain this set of formulas.

8. *Paraphrases or formulas of the Liouville type.* — The theta identities given in the preceding section have rather simple arithmetical equivalents which may be obtained through the method of paraphrase. The essence of this method lies in the application of the following theorem concerning trigonometric identities : if $f(x)$ is an even function, and $g(x)$ is an odd function of x, and if $f(x)$, $g(x)$ are finite and single valued whenever x is a rational number (zero included), and if further $g(0) = 0$, then the folloving identities in z

$$a_0 + \sum_{i=1}^{M} b_i \cos c_i z = 0, \qquad \sum_{i=1}^{N} k_i \sin r_i z = 0,$$

where c_i, r_i, b_i, k_i are rational numbers, imply respectively

$$a_0 f(0) + \sum_{i=1}^{M} b_i f(c_i) = 0, \qquad \sum_{i=1}^{N} k_i g(r_i) = 0,$$

M, N being finite integers. Beyond the conditions stated $f(x)$ and $g(x)$ are entirely arbitrary. This is the simplest instance of a much more general theorem proved under very general bypotheses by Bell [1].

The application of this theorem to the trigonometric identities implicit in the relations given in paragraph **7** gives rise to certain formulas of the Liouville type of a rather simple structure and of some interest in as much as the partitions involved refer to the

---

[1] E. T. Bell, *Transactions American Mathematical Society*, Vol. **22**, 1921, p. 1-39 and 198-219; *see also his « Algebraic Arithmetic »*, *Colloqium Publications of the Am. Math. Soc.*, Vol. **7**.

representation of numbers in certain linear forms as the sum of five squares. The notations is a follows : α, β, m, m_1, n, n_1, t, τ, d, δ are positive integers; $\alpha \equiv 1$, mod 4; $\beta \equiv 3$, mod 4; m and m_1 are odd; n and n_1 are unrestricted integers. μ, $x_i \gtrless 0$ are odd integers; $w_i \gtrless 0$ are even integers; h, $z_i \gtreqless 0$ are unrestricted integers; (d, δ), (t, τ) are conjugate divisors such that $\delta < d$ and $\tau < t$, τ being always odd. Further restrictions on these divisions will be indicated as needed. $\varepsilon(n) = 1$ or 0, according as n is or is not the square of an integer. $a(n) = 1$ or 0, according as n is or is not the sum of two integer squares. $f(x) = f(-x)$ and $g(x) = -g(-x)$, but are otherwise arbitrary.

The partitions ower which the functions in the paraphrases are summed are

(I) $$\alpha = x_1^2 + w_1^2 + w_2^2 + w_3^2 + w_4^2 = \mu^2 + 4d\delta \quad (\delta - d \equiv 1, \bmod 2);$$

(II) $$\beta = x_1^2 + x_2^2 + x_3^2 + w_1^2 + w_2^2 = 4h^2 + t\tau,$$

(III) $$2m = x_1^2 + x_2^2 + w_1^2 + w_2^2 + w_3^2 = \mu^2 + t\tau = \mu^2 + m_1^2,$$

(IV) $$n = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = h^2 + d\delta = h^2 + n_1^2, \quad (\delta - d \equiv 0, \bmod 2).$$

The paraphrases follow, the Roman numeral on the left referring to the corresponding partition :

(I_1) $$\sum (-1)^{\frac{w_1+w_2}{2}} f(x_1) = 4\sum{}' (-1)^{\frac{d+\delta+\mu}{2}} (d+\delta-\mu) f(2\delta+\mu) + 2\varepsilon(\alpha)\sqrt{\alpha}(-1|\sqrt{\alpha}) f(\sqrt{\alpha});$$

(I_2) $$\sum (-1)^{\frac{w_1+w_2+x_1-1}{2}} g(x_1) = 4\sum{}' (-1)^{\frac{\delta-d+1}{2}} (d+\delta-\mu)\, g(2\delta+\mu) + 2\varepsilon(\alpha)\sqrt{\alpha}\, g(\sqrt{\alpha});$$

(II_1) $$\sum f(x_1) = 2\sum{}' (t+\tau-4h)\, f(\tau+2h);$$

(II_2) $$\sum (-1)^{\frac{w_1+w_2+x_1-1}{2}} g(x_1) = 2\sum{}' (-1)^{\frac{t-\tau-2}{4}} (t+\tau-4h)\, g(\tau+2h);$$

(II_3) $$\sum (-1)^{\frac{w_1+w_2}{2}} f(x_1) = -2\sum{}' (-1)^{\frac{t+\tau+4h}{4}} (t+\tau-4h)\, f(\tau+2h);$$

(II_4) $$\sum (-1)^{\frac{x_1-1}{2}} g(x_1) = 2\sum{}' (-1)^{\frac{\tau+2h-1}{2}} (t+\tau-4h)\, g(\tau+2h);$$

$$(\mathrm{III}_1)\quad \sum (-1)^{\frac{w_1}{2}} f(w_1) = 2\sum{}' (-1)^{\frac{\tau+\mu}{2}} (t+\tau-2\mu) f(\tau+\mu) + 2a(2m)\sum (-1)^{\frac{m_1+\mu}{2}} (m_1-\mu) f(m_1+\mu);$$

$$(\mathrm{III}_2)\quad \sum (-1)^{\frac{w_1+w_2}{2}} f(w_3) = 2\sum{}' (-1)^{\frac{t+\tau+2\mu}{4}} (t+\tau-2\mu) f(\tau+\mu) + 2a(2m)\sum (-1)^{\frac{m_1+\mu}{2}} (m_1-\mu) f(m_1+\mu);$$

$$(\mathrm{III}_3)\quad \sum f(w_1) = 2\sum{}' (t+\tau-2\mu) f(\tau+\mu) + 2a(2m)\sum (m_1-\mu) f(m_1+\mu);$$

$$(\mathrm{III}_4)\quad \sum (-1)^{\frac{w_1+w_2+w_3}{2}} f(w_1) = 2\sum{}' (-1)^{\frac{t-\tau}{4}} (t+\tau-2\mu) f(\tau+\mu) + 2a(2m)\sum (m_1-\mu) f(m_1+\mu);$$

$$(\mathrm{IV}_1)\quad \sum (-1)^{z_1+z_2} f(z_3) = f(0) + 2\varepsilon(n)(-1)^{\sqrt{n}} f(\sqrt{n}) + 4a(n)\sum (-1)^{h+n_1}(h-n_1) f(h+n_1) - 4\sum{}' (-1)^{\frac{d+\delta-2h}{2}} (d+\delta-2h) f(\delta+h);$$

$$(\mathrm{IV}_2)\quad \sum (-1)^{z_1+z_2+z_3} f(z_1) = f(0) + 2\varepsilon(n) f(\sqrt{n}) + 4a(n)\sum (h-n_1) f(h+n_1) - 4\sum{}' (-1)^{\frac{d-\delta}{2}} (d+\delta-2h) f(\delta+h).$$

9. *Conclusion*. — The preceding formulae with $f(x)\equiv 1$, yield enumerations relative to the number of representation of a number as the sum of five squares. The most interesting results are those deduced from (II_1) and (III_3). Thus we have the following theorems [1] :

Theorem (α). — *The number of representations of a number* $\beta\equiv 3$, mod 4, *in the form* $x^2+y^2+z^2+u^2+v^2$, *wherein* $x, y, z \gtrless 0$ *are odd and* u, $v \geqq 0$, *are even, is given by the expression*

$$2\,\Phi(\beta) + 4\,\Phi(\beta-4.1^2) + 4\,\Phi(\beta-4.2^2) + 4\,\Phi(\beta-4.3^2) + \ldots$$

---

[1] The results appear to be supplementary to the enumerations given by Hermite (*Œuvres*, 4, 1920, p. 237-238) and by Bell (*Am. J. Math.*, Vol. 42, 1920, p. 177).

the sum being continued so long as the argument remains positive; the function $\varphi(\beta)$ *is equal to the sum of all the divisors of* β.

THEOREM (β). — *The number of representations of a number* $2m$ *(*m *odd), in the form* $x^2+y^2+u^2+v^2+w^2$, *wherein* $x, y \gtrless 0$ *are odd and* $u, v, w \gtrless 0$ *are even is given by the expression*

$$4\Phi(2m-1^2)+4\Phi(2m-3^2)+4\Phi(2m-5^2)+\ldots$$

where $\varphi(n)$ *is as in the preceding theorem, provided* $2m$ *is* not *representable as the sum of two squares. If, however,* $2m$ *is so representable, the quantity* $G(2m)$ *must be added to the preceding expression, where*

$$G(2m)=4\sum x,$$

the sum being extended over all solutions of $2m=x^2+y^2$, $x, y>0$ *and odd.*