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On the elementary solution of the general linear differential equation of the second order with analytic coefficients

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On the elementary solution of the general linear differential equation of the second order with analytic coefficients;

By T. Y. THOMAS and E. W. TITT.

Introduction.

Consider the second order differential equation [1]

$$\text{(A)} \qquad F(u) \equiv \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} + \sum_{\alpha=1}^{n} f^\alpha \frac{\partial u}{\partial x^\alpha} + bu = 0,$$

where the g, f and b are analytic functions of the real variables $x^1, \ldots, x^n$ in a region R of the (real) n-dimensional number space $(n \geq 2)$. Or, more generally, owing to its invariance under coordinate transformations, this equation may be considered to be defined with analytic coefficients g, f and b over an analytic manifold $\mathfrak{M}$ [2]. We assume that the determinant $|g^{\alpha\beta}|$ does not vanish at any point of the space under consideration.

The following paper is largely, although not entirely, expository. It is concerned largely with Hadamard's elementary solution of the equation (A) as treated in Chapter III of his Lectures on Cauchy's

(1) Hereafter the summation convention will be used.

(2) By definition an analytic manifold $\mathfrak{M}$ is a Hausdorff space with coordinate neighborhoods, homeomorphic to the interior of a spherical surface of $n-1$ dimensions in a Euclidean space of n dimensions, and such that the coordinate relationships between the coordinates of two intersecting neighborhoods are analytic.

Problem in Linear Partial Differential Equations [1]. We have made direct use of the invariant form ot the equation (A), the coefficients of which are the components of tensors, the well known properties of normal coordinates, and have so treated the case where n is even that the convergence proof can be given without resort to Cauchy's existence theorem [2]. We believe that our treatment of this problem will greatly facilitate the reading of Chapter III of Hadamard's Lecture by present day students of Tensor Analysis.

In paragraph 4 we have proved a new result of general interest concerning normal coordinates, namely that a Riemann space is flat if the determinant of the components of its fundamental metric tensor is constant in any system of normal coordinates. We have shown the significance of this result for the elementary solution of the equation (A).

Before proceeding to the discussion of our problem let us first state precisely what is meant by a real analytic function of n real variables $x^1, \ldots, x^n$. A real function $f(x^1, \ldots, x^n)$ defined in a region R(open point set) of the real n-dimensional number space will be said to be analytic at a point $x^\alpha = a^\alpha$ of R if there exists a neighborhood $\mathrm{N} \subset \mathrm{R}$ of this point such that in N the function can be represented by a series of the form

$$\sum_{\alpha_1 \ldots \alpha_n = 0}^{\infty} \mathrm{A}_{\alpha_1 \ldots \alpha_n} (x^1 - a^1)^{\alpha_1} (x^2 - a^2)^{\alpha_2} \ldots (x^n - a^n)^{\alpha_n},$$

which is absolutely convergent in N, the coefficients A of the series being real. The requirement of absolute convergente is no essential restriction for our purposes. We recall that if there exists a point $x^\alpha = b^\alpha$ where $b^\alpha \neq a^\alpha$ for $\alpha = 1, \ldots, n$ such that for $x^\alpha = b^\alpha$ the absolute values of the terms of the above series are bounded (necessary condition for the series to converge as a simple series for some ordering

(1) Yale University Press, 1923. A French translation of this book has also appeared, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Hermann, Paris, 1932. Hadamard's lectures may be consulted for references to the original researches on the elementary solution of the above equation.

(2) This method is in fact suggested by Hadamard on p. 101 of his Lectures.

of its terms) then the series will be absolutely convergent in the domain $|x^\alpha - a^\alpha| < |b^\alpha - a^\alpha|$ and the convergence will be uniform in any closed set contained in this domain. The function $f(x^1, \ldots, x^n)$ is analytic in R if it is analytic at every point in R. In the following when we say that a series is convergent we shall always mean that it is absolutely convergent.

1. Invariant formulation of the differential equation. — We assume that u is a scalar function in the expression $F(u)$ and likewise that this expression is itself of scalar character under analytic coordinate transformations in R or $\mathfrak{M}$. It follows immediately from these assumptions that under such a transformation $x \to \bar{x}$ the coefficients g, f and b in $F(u)$ transform in the following manner

$$(1.1)\qquad \begin{cases} \bar{g}^{\sigma\tau} = g^{\alpha\beta} \dfrac{\partial \bar{x}^\sigma}{\partial x^\alpha} \dfrac{\partial \bar{x}^\tau}{\partial x^\beta}, \\ \bar{f}^\sigma = f^\alpha \dfrac{\partial \bar{x}^\sigma}{\partial x^\alpha} + g^{\alpha\beta} \dfrac{\partial^2 \bar{x}^\sigma}{\partial x^\alpha \partial x^\beta}, \\ \bar{b} = b. \end{cases}$$

The first set of the above equations expresses the fact that the quantities $g^{\alpha\beta}$ are the components of a contrevariant tensor. Hence it is possible to define a covariant tensor with components $g_{\alpha\beta}(x)$ which are the normalized cofactors of the $g^{\alpha\beta}$ and thus in the usual manner introduce into the space a metric defined by

$$dr^2 = g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta.$$

We make no assumptions on this quadratic differential form other than those above staded, namely that the coefficients $g_{\alpha\beta}(x)$ are analytic and that the determinant $|g_{\alpha\beta}|$ does not vanish in the region R or the manifold $\mathfrak{M}$. The differential form defining the metric may be positive definite in which case we speak of R or $\mathfrak{M}$ as a Riemann space, or the form may be indefinite and we shall then call R or $\mathfrak{M}$ a pseudo-Riemann space [if the differential form is negative definite it may be replaced by one which is positive definite by multiplying the differential equation (A) by -1]. While this distinction in terminology is not customary, it is convenient, as we shall find that certain results can be proved only when the quadratic differential form defining the metric is positive definite, i.e. when we deal with a Riemann space in the strict sense in which we employ that word.

On the basis of the above metric we can now define the Christoffel symbols, the curvature tensor, etc. and in fact the various quantities which one is accustomed to consider in a Riemann or pseudo-Rieman space.

Let us now denote by $u_{,\alpha\beta}$ the components of the second covariant derivative of the scalar u and let us put

$$\Delta u \equiv g^{\alpha\beta} u_{,\alpha\beta} \equiv g^{\alpha\beta}\left[\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} - \frac{\partial u}{\partial x^\sigma}\left\{ {\sigma \atop \alpha\beta} \right\}\right]. \tag{1.2}$$

We now see that the expression $F(u)$ can be written in the form

$$F(u) \equiv \Delta u + a^\alpha u_{,\alpha} + bu, \tag{1.3}$$

where $u_{,\alpha}$ denotes the partial derivative of u with respect to x^α and where

$$a^\alpha = f^\alpha + g^{\mu\nu}\left\{ {\alpha \atop \mu\nu} \right\}.$$

Since the right member of (1.3) is a scalar and since the first and last terms of this expression are also scalars it follows that the middle term $a^\alpha u_{,\alpha}$ is likewise a scalar. In fact since $a^\alpha u_{,\alpha}$ is a scalar for arbitrary values of the $u_{,\alpha}$ at a point P of the space it follows that the coefficients a^α are the components of a contravariants vector (quotient law of tensors). A direct proof of the vector character of the a^α can be obtained from the second set of equations (1.1) by eliminating the second derivatives by means of the equations of transformation of the Christoffel symbols.

In particular if the coefficients a^α and b are identically zero in the expression $F(u)$ the equation (A) reduces to be generalized Laplace equation

$$\Delta u = 0. \tag{B}$$

In most of the following discussion it is immaterial whether or not the above quadratic differential form definess a Riemann or pseudo-Riemann space. For definiteness however in our discussion (§ 2-

§ **10**) *we shall assume that this differential form is positive definite, i.e. that it defines the metric of a Riemann space. The modifications necessary in case this differential form is not positive definite will be briefly discussed in* paragraph **11**.

2. Application of normal coordinates. — We shall now derive the form which the equation (A) assumes in a system of normal coordinates with origin at an arbitrary point P. We recall that these coordinates are defined in a neighborhood of the point P and are related to the underlying coordinates x^α by analytic transformations. As a result of such a tranformation to normal coordinates y^α let us suppose that

$$u(x) \to u(y), \quad g^{\alpha\beta}(x) \to h^{\alpha\beta}(y), \quad a^\alpha(x) \to c^\alpha(y), \quad b(x) \to d(y).$$

The first of these indicated transformations is of course not intended to imply that the $u(x)$ and $u(y)$ are the same functions of the variables x^α and y^α in any sense, the same letter u being used in each case since it is convenient to have a single letter to represent the solution of the equation (A). When referred to the y coordinate system the equation (A) therefore becomes

$$F(u) \equiv \Delta u + c^\alpha u_{,\alpha} + du = 0, \tag{2.1}$$

where Δu and $u_{,\alpha}$ are defined as above with respect to the y system.

As is well known the equations of the geodesics through the origin of the normal coordinate system have the form

$$y^\alpha = \xi^\alpha s, \tag{2.2}$$

where the ξ^α are constants. If the $\xi's$ are the components of a unit vector then the parameter s occurring in the above equations represents the arc length measured from the origin. We recall that the transformation $x \to y$ is such that the derivatives $\partial x^\alpha/\partial y^\beta$ have the values δ^α_β at the origin of the normal coordinates. Thus $h_{\alpha\beta}(0) = (g_{\alpha\beta})_P$ and hence the function

$$\Gamma = (g_{\alpha\beta})_P y^\alpha y^\beta$$

gives the square of the geodesic distance of a point with coordinates y^α from the origin of the normal coordinate system. Let us recall also that in the normal coordinate system the following identities are satisfied

$$(2.3)\qquad \begin{cases} h_{\alpha\sigma}(y)\,y^{\sigma} = h_{\alpha\sigma}(0)\,y^{\sigma}, \\ \left\{ {k \atop ij} \right\} y^{i} y^{j} = 0. \end{cases}$$

We shall make frequent use of these identities ([1]).

It will greatly facilitate matters if we now calculate the result of substituting a function $u(y, \Gamma)$ into the equation (2.1). If we look upon Γ as a function of the y^{α} we have

$$\frac{du}{dy^{\alpha}} = \frac{\partial u}{\partial y^{\alpha}} + \frac{\partial u}{\partial \Gamma}\frac{\partial \Gamma}{\partial y^{\alpha}}.$$

Making this substitution and the corresponding substitution for the second derivatives of u into the equation (2.1) this equation becomes

$$(2.4)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial \Gamma^2}\Delta_1\Gamma + 2\Delta_1^{*}\left(\dfrac{\partial u}{\partial \Gamma}, \Gamma\right) + \dfrac{\partial u}{\partial \Gamma}\Delta\Gamma \\ \qquad + \Delta^{*}u + c^{\alpha}\left(\dfrac{\partial u}{\partial y^{\alpha}} + \dfrac{\partial u}{\partial \Gamma}\dfrac{\partial \Gamma}{\partial y^{\alpha}}\right) + du = 0 \end{cases}$$

in terms of the following differential quantities :

$$\Delta_1\Gamma = h^{\alpha\beta}\frac{\partial \Gamma}{\partial y^{\alpha}}\frac{\partial \Gamma}{\partial y^{\beta}}, \qquad \Delta_1^{*}\left(\frac{\partial u}{\partial \Gamma}, \Gamma\right) = h^{\alpha\beta}\frac{\partial^2 u}{\partial \Gamma\,\partial y^{\alpha}}\frac{\partial \Gamma}{\partial y^{\beta}},$$

$$\Delta\Gamma = h^{\alpha\beta}\left(\frac{\partial^2 \Gamma}{\partial y^{\alpha}\,\partial y^{\beta}} - \frac{\partial \Gamma}{\partial y^{\sigma}}\left\{ {\sigma \atop \alpha\beta} \right\}\right), \qquad \Delta^{*}u = h^{\alpha\beta}\left(\frac{\partial^2 u}{\partial y^{\alpha}\,\partial y^{\beta}} - \frac{\partial u}{\partial y^{\sigma}}\left\{ {\sigma \atop \alpha\beta} \right\}\right),$$

where the Christoffel symbols appearing in these expressions refer of course to the normal coordinate system. By use of the first identity (2.3) we have

$$(2.5)\qquad \Delta_1\Gamma = 4h^{\alpha\beta}h_{\alpha\sigma}(0)\,y^{\sigma}h_{\beta\tau}(0)\,y^{\tau} = 4\Gamma,$$

$$(2.6)\qquad \Delta_1^{*}\left(\frac{\partial u}{\partial \Gamma}, \Gamma\right) = 2h^{\alpha\beta}\frac{\partial^2 u}{\partial \Gamma\,\partial y^{\alpha}}h_{\beta\sigma}(0)\,y^{\sigma} = 2y^{\mu}\frac{\partial^2 u}{\partial \Gamma\,\partial y^{\mu}}.$$

When we replace the Christoffel symbols in the above expression for

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([1]) For a discussion of normal coordinates and their properties the reader may be referred to T. Y. Thomas, *The Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934, p. 84-87.

$\Delta\Gamma$ by their values in terms of the $h_{\alpha\beta}$ and their derivatives we have

$$\Delta\Gamma = 2h^{\alpha\beta}h_{\alpha\beta}(0) - 2h^{\alpha\beta}y^{\sigma}\frac{\partial h_{\alpha\sigma}}{\partial y^{\beta}} + h^{\alpha\beta}y^{\sigma}\frac{\partial h_{\alpha\beta}}{\partial y^{\sigma}}.$$

If we differentiate the first relation (2.3) with respect to y^{β}, multiply the result by $h^{\alpha\beta}$ and sum on the indices α and β, we find that

$$h^{\alpha\beta}y^{\sigma}\frac{\partial h_{\alpha\sigma}}{\partial y^{\beta}} = -n + h^{\alpha\beta}h_{\alpha\beta}(0).$$

Hence the expression for $\Delta\Gamma$ becomes

$$\Delta\Gamma = 2n + y^{\sigma}\frac{\partial \log h}{\partial y^{\sigma}}, \tag{2.7}$$

where h is the determinant of the components $h_{\alpha\beta}$. When we replace the quantities appearing in (2.4) by their values given by (2.5), (2.6) and (2.7) we find that $u(y, \Gamma)$ must satisfy the equation

$$4\Gamma\frac{\partial^2 u}{\partial\Gamma^2} + 4y^{\sigma}\frac{\partial^2 u}{\partial\Gamma\,\partial y^{\sigma}} + M\frac{\partial u}{\partial\Gamma} + F^{*}(u) = 0 \tag{2.8}$$

if it is to be a solution of (2.1); here

$$M = 2n + y^{\sigma}\frac{\partial \log h}{\partial y^{\sigma}} + c^{\alpha}\frac{\partial\Gamma}{\partial y^{\alpha}},$$

$$F^{*}(u) = \Delta^{*}u + c^{\alpha}\frac{\partial u}{\partial y^{\alpha}} + du.$$

3. CONDITION FOR THE DIFFERENTIAL EQUATION TO ADMIT A EUCLIDEAN SOLUTION. — Our purpose will be to obtain a generalization of the ordinary potential for the general (analytic) linear differential equation of the second order (A). With this end in view let us first consider the possibility of obtaining a solution of (A) or (2.1) of the form

$$\left\{\begin{array}{ll} u = \log\Gamma & (n = 2), \\ u = \Gamma^{p} & (n \geqq 3), \end{array}\right. \tag{3.1}$$

where p is a constant which we assume to be different from zero and Γ, as defined in paragraph **2**, is the square of the geodesic distance from an arbitrary but fixed point P. The functions u defined by (3.1) for $p = \frac{(2-n)}{2}$ are *direct* generalizations of the well known solutions of the ordinary Laplace equation, i.e. the equation (B) for which the associated Riemann space in the Euclidean. We shall therefore refer to a solution of the form (3.1) as a *Euclidean solution.* For the case under consideration the equation (2.8) becomes

$$4\Gamma\frac{d^2u}{d\Gamma^2} + M\frac{du}{d\Gamma} + du = 0. \tag{3.2}$$

Making the substitutions (3.1) we now have ([1])

$$(3.3)\quad \begin{cases} -4 + 2n + y^\sigma \dfrac{\partial \log h}{\partial y^\sigma} + 2c^\alpha h_{\alpha\beta}(0)y^\beta + d\Gamma\log = 0 & (n=2), \\ p\left[4p - 4 + 2n + y^\sigma\dfrac{\partial \log h}{\partial y^\sigma} + 2c^\alpha h_{\alpha\beta}(0)y^\beta\right] + d\Gamma = 0 & (n \geqq 3). \end{cases}$$

Evaluating the second of equations (3.3) at the origin we obtain $p(4p - 4 + 2n) = 0$ so that $p = \frac{2-n}{2}$ as in the case of the ordinary solution. Hence (3.3) becomes

$$(3.4)\quad \begin{cases} y^\sigma\dfrac{\partial \log h}{\partial y^\sigma} + 2c^\alpha h_{\alpha\beta}(0)y^\beta + d\Gamma\log\Gamma = 0 & (n=2), \\ p\left[y^\sigma\dfrac{\partial \log h}{\partial y^\sigma} + 2c^\alpha h_{\alpha\beta}(0)y^\beta\right] + d\Gamma = 0 & \left[\begin{matrix} n \geqq 3 \\ p = \dfrac{(2-n)}{2}\end{matrix}\right]. \end{cases}$$

In other words *the differential equation* (A) *will admit a non-trivial Euclidean solution* (3.1) *if, and only if,* $p = \frac{(2-n)}{2}$ *for* $n \geqq 3$ *and the corresponding condition* (3.4) *is satisfied in a system of normal coordinates.* While this condition is stated in normal coordinates it is in reality a condition on the coefficients of the original differential equation (A). In the particular case that the equation (A) reduces to the generalized Laplace equation (B) the condition (3.4) becomes

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([1]) In writing équations (3.3) we have neglected the factor Γ^{-1} in case $n = 2$ and the factor Γ^{p-1} in case $n \geqq 3$ which is legitimate since $\Gamma = 0$ only at the origin at which point the left members of these equations are continuous.

$y^\sigma \frac{\partial \log h}{\partial y^\sigma} = 0$ for all values of $n \geqq 2$. In consequence of (2.2) this latter condition becomes $s \frac{d \log h}{ds} = 0$ along an arbitrary geodesic through the origin. Hence *the differential equation* (B) *will admit a non-trivial Euclidean solution* (3.1) *if, and only if,* $p = \frac{(2-n)}{2}$ *for* $n \geqq 3$ *and the determinant* $|h_{\alpha\beta}|$ *is constant in a system of normal coordinates.*

It is evident from the foregoing discussion that (3.1) will define solutions when the right members of these equations are constructed with reference to a particular initial point P if, and only if, the above conditions are satisfied in the normal coordinate system having its origin at the point P. If (3.1) is to define solutions for arbitrary initial points P these conditions must hold in every normal coordinate system.

4. A NEW CHARACTERIZATION OF FLAT RIEMANN SPACES. — Let us now consider the significance of the condition that the determinant $|h_{\alpha\beta}|$ be constant in every system of normal coordinates. We shall in fact prove the following

LEMMA. — *A Riemann space of dimensionality* $n \geqq 2$ *is flat if, and only if, the determinant* $|h_{\alpha\beta}|$ *of the components of the fundamental metric tensor is constant in every system of normal coordinates* (1).

We know that for any Riemann space the determinant $|h_{\alpha\beta}|$ is a relative invariant of weight 2 and that its first derivatives vanish at

(1) The proof of this lemma does not require that the components $g_{\alpha\beta}$ of the metric tensor be analytic functions. In fact it will be seen to be sufficient for the components $h_{\alpha\beta}$ of this tensor in any system of normal coordinates to have continuous derivatives to the third order inclusive. We can then deduce the equation (4.9) from (4.8) by differentiating this latter equation twice along an arbitrary geodesic issuing from the origin instead of differentiating four times with respect to the coordinates y^α as is done above. For the existence of these continuous derivatives of the $h_{\alpha\beta}$ it is sufficient that the components $g_{\alpha\beta}$ of the metric tensor in the underlying x coordinate system have continuous derivatives to the order five inclusive. *See* T. Y. THOMAS, *On normal coordinates* (*Proc. Nat. Acad. Sci.*, Vol. **22**, 1936, p. 309).

the origin of normal coordinates. The second derivatives of $|h_{\alpha\beta}|$ with respect to $y^k y^l$ when evaluated at the origin will define the components G_{kl} of a relative tensor of weight 2 (T. Y. THOMAS, *loc. cit.*, p. 97). In fact it readily follows that

$$G_{kl} = |g_{\mu\nu}| g^{ij} g_{ij,kl}, \tag{4.1}$$

where $g_{ij,kl}$ are the components of the second extension of the fundamental metric tensor. Now between the components of this extension and the components B_{ijkl} of the curvature tensor we have the relations

$$B_{ijkl} = g_{ik,jl} - g_{jk,il}, \qquad 3g_{ij,kl} = B_{ikjl} + B_{jkil}, \tag{4.2}$$

(*loc. cit.*, p. 131). From (4.1) and the second set of equations (4.2) we now have

$$G_{kl} = -\frac{2}{3}|g_{\mu\nu}|B_{kl} \qquad (B_{kl} = g^{ij}B_{iklj}), \tag{4.3}$$

where B_{kl} are the components of the contracted curvature tensor. Since the determinant $|h_{\alpha\beta}|$ is constant by our hypothesis it follows that $G_{kl} = 0$ and hence by (4.3) we have that $B_{kl} = 0$. Now it is well known that the vanishing of the contracted curvature tensor for $n = 2$ or 3 is the condition for a Riemann (or pseudo-Riemann) space to be flat. This proves part of the above lemma.

To investigate the case $n > 3$ we shall need to make explicit use of the expressions defining the components of the contracted curvature tensor, namely the expressions

$$B_{ij} = \frac{\partial}{\partial y^k}\left\{ {k \atop ij} \right\} - \frac{\partial}{\partial y^j}\left\{ {k \atop ki} \right\} + \left\{ {k \atop km} \right\}\left\{ {m \atop ij} \right\} - \left\{ {k \atop mj} \right\}\left\{ {m \atop ik} \right\}. \tag{4.4}$$

Now the left member of this equation vanishes as we have proved in consequence of the fact that $|h_{\alpha\beta}|$ is constant in any normal coordinate system and it likewise follows from this same fact that the second and third terms of the right member of the above equation vanish since

$$\left\{ {k \atop ki} \right\} = \frac{1}{2}\frac{\partial \log h}{\partial y^i}. \tag{4.4}$$

By differentiation we obtain from the second set of identities (2.3) the relation

$$\frac{\partial}{\partial y^k}\left\{ {k \atop ij} \right\} y^i y^j + 2\left\{ {k \atop ik} \right\} y^i = 0. \tag{4.6}$$

Hence if we multiply the right member of (4.4) by $y^i y^j$ and sum on repeated indices the first term vanishes by (4.6) and the fact that the left member of (4.5) is equal to zero. Hence only the term obtained from the last set of terms in the right member of (4.4) remains, i. e. we have

$$\left\{ {k \atop mi} \right\}\left\{ {m \atop jk} \right\} y^i y^j = 0. \tag{4.7}$$

Now by making use of the equations obtained by differentiating the first set of equations (2.3) it follows readily that

$$\left\{ {i \atop mk} \right\} y^k = \frac{1}{2} y^k h^{ij} \frac{\partial h_{jm}}{\partial y^k}.$$

Hence from (4.7) we have

$$h^{kp} h^{mq} \frac{\partial h_{pm}}{\partial y^i} \frac{\partial h_{qk}}{\partial y^j} y^i y^j = 0. \tag{4.8}$$

Now differentiate the last set of equations with respect to $y^a y^b y^c y^d$ and evaluate at the origin of normal coordinates. There results the set of equations

$$g^{lm} g^{hk}[g_{km,ab} g_{hl,cd} + g_{km,ac} g_{hl,bd} + g_{km,ad} g_{hl,bc}] = 0. \tag{4.9}$$

Now in consequence of the vanishing of the contracted curvature tensor we have that

$$g^{kl} g_{ij,kl} = 0.$$

Hence if we multiply (4.9) through by $g^{ac} g^{bd}$ and sum on repeated indices we find that

$$g^{lm} g^{hk} g^{ac} g^{bd} g_{lh,ab} g_{mk,cd} = 0. \tag{4.10}$$

In a Riemann space this condition implies the vanishing of the second extension of the fundamental metric tensor. In fact we can make a coordinate transformation such that at an arbitrary point P of the space the components g_{ij} will have the values δ^i_j. Then (4.10) becomes

$$g_{lh,ab}\,g_{lh,ab} = 0,$$

where the summation extends over all possible values of the indices l, h, a, b. Since all terms in this summation are either positive or zero it follows that all components $g_{lh,ab}$ vanish. Hence all components B_{ijkl} of the curvature tensor vanisch and this is the well known condition for the space to be flat.

Conversely if the space is flat it follows immediately that the determinant $|h_{\alpha\beta}|$ is constant in any system of normal coordinates (¹). This completes the proof of the above lemma.

The following result has now been proved : *A necessary and sufficient condition that the differential equation* (B) *defined in a Riemann space of dimensionality* $n \geqq 2$ *have the Euclidean solution*

$$u = \log \Gamma \qquad (n = 2),$$
$$u = \Gamma^{\frac{(2-n)}{2}} \qquad (n \geqq 3),$$

where the initial point P *used in the construction of the function* Γ *is arbitrary, is that the space be flat.*

5. Fundamental formulæ. — Let us now return to the consideration of the general differential equation (A). We seek solutions of (2.8) which are valid in a neighborhood of the origin of the normal coordinate system. Since the coefficient of $\frac{\partial^2 u}{\partial \Gamma^2}$ in equation (2.8) vanishes at the origin and since the coefficient of $\frac{\partial u}{\partial \Gamma}$ does not vanish at the origin we are led by analogy with ordinary differential equations to look for a solution of the form

$$(5.1) \qquad u = \Gamma^p [u_0 + u_1 \Gamma + u_2 \Gamma^2 + \ldots],$$

(¹) In fact if the space is flat we can introduce coordinates $\bar{x}^\alpha$ in the neighborhood of any point such that with respect to the $\bar{x}$ system the $g_{\alpha\beta}$ have constant values. If P is any point having coordinates q^α with respect to the $\bar{x}$ system then the normal coordinates y^α with origin at P are related to the $\bar{x}^\alpha$ coordinates by $\bar{x}^\alpha = q^\alpha + y^\alpha$. Hence the components of the fundamental metric tensor are constant in the y system and so in particular the determinant of these components is constant.

where the u_i are analytic functions of the y^α in a neighborhood of $y^\alpha = 0$ and p is a constant. Substituting (5.1) into (2.8) we have

$$\sum_{k=0}^{\infty}(p+k)\Gamma^{p+k-1}\left\{4y^\sigma\frac{\partial u_k}{\partial y^\sigma} + u_k(4k+4p-4+M)\right\} + \sum_{k=0}^{\infty}\Gamma^{p+k}F(u_k) = 0.$$

Equating to zero the coefficients of like powers of Γ this gives

$$(5.2)\quad \left\{\begin{array}{l} p\left[4y^\sigma\dfrac{\partial u_0}{\partial y^\sigma} + u_0(4p-4+M)\right] = 0, \\ (p+k)\left[4y^\sigma\dfrac{\partial u_k}{\partial y^\sigma} + u_k(4k+4p-4+M)\right] + F(u_{k-1}) = 0 \\ \qquad\qquad (k = 1, 2, 3, \ldots). \end{array}\right.$$

When we make use of the equations (2.2) of the geodesics through the origin, the equations (5.2) go over into the following system of ordinary differential equations

$$(5.3)\quad \left\{\begin{array}{l} p\left[4s\dfrac{du_0}{ds} + u_0(4p_1 + M^*)\right] = 0, \\ (p+k)\left[4s\dfrac{du_k}{ds} + u_k(4k+4p_1+M^*)\right] + F(u_{k-1}) = 0 \\ \qquad\qquad (k = 1, 2, 3, \ldots), \end{array}\right.$$

where $4p_1 = 4p - 4 + 2n$ and M^* is used to denote the function $M - 2n$.

If we equate the bracket expression in the first equation (5.3) to zero we have a differential equation for which the variables are separable. Integrating we obtain

$$(5.4)\qquad u_0 = \frac{Q}{G}s^{-p_1},$$

where Q is an arbitrary constant depending in general on the geodesic of integration and G is defined by

$$G = e^{\frac{1}{4}\int_0^s \frac{M^*}{s}ds}.$$

We shall now show that G is an analytic function of the coordinates y^α in a neighborhood of the values $y^\alpha = 0$. Since M^* is such a function and since $M^* = 0$ at the origin we have

$$M^* = a_\alpha y^\alpha + a_{\alpha\beta}y^\alpha y^\beta + a_{\alpha\beta\gamma}y^\alpha y^\beta y^\gamma + \ldots.$$

Making use of the substitution (2.2) we have

$$\int_0^s \frac{M^*}{s}\,ds = \int_0^s [a_\alpha \xi^\alpha + a_{\alpha\beta}\xi^\alpha\xi^\beta s + a_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma s^2 + \ldots]\,ds$$
$$= a_\alpha y^\alpha + \frac{1}{2} a_{\alpha\beta} y^\alpha y^\beta + \frac{1}{3} a_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma + \ldots,$$

and obviously this series will converge whenever the above series for M^* converges. Hence the function G is analytic as above stated and has the value 1 at the origin.

Since $G = 1$ for $y^\alpha = 0$ we can expand the function $\frac{1}{G}$ in a power series about the values $y^\alpha = 0$. Similar power series expansions exist for the coefficients in the expression $F(u)$ defined by (2.1). Hence we can find a neighborhood U(P) of the origin of normal coordinates in which the expansions about $y^\alpha = 0$ of the functions G, $\frac{1}{G}$ and the coefficients of the expression $F(u)$ are all convergent.

An integrating factor for the second set of equations (5.3) is

$$\frac{s^{k+p_1-1}G}{4(k+p)}$$

and the general solution is

$$(5.5)\qquad u_k s^{k+p_1} G = -\frac{1}{4(k+p)}\int_0^s s^{k+p_1-1} G\,F(u_{k-1})\,ds + R_k \qquad (k = 1, 2, 3, \ldots),$$

where R_k is an arbitrary constant. We shall leave the discussion of the analyticity of u_0 and u_k as defined by (5.4) and (5.5) until we find what values may be given to the constant p.

6. A SOLUTION WITHOUT SINGULARITIES. — Let us first consider the case $p = 0$ in equations (5.3). We can then take u_0 to be an arbitrary analytic function of the coordinates y^α with a power series expansion about $y^\alpha = 0$ convergent in a spherical neighborhood U_1 (P) defined by $\Sigma y^\alpha y^\alpha < \rho$ where ρ is a positive constant such that $U_1(P)$ is contained in the neighborhood U(P) defined in paragraph 5. If $u_0, u_1, \ldots, u_{k-1}$ are expandable in power series about $y^\alpha = 0$ which are convergent in the neighborhood U_1 (P) we shall prove that u_k as given by (5.5) admits a similar power series expansion provided that $R_k = 0$. By the definition of the neighborhood $U(P)$, the fact that $U_1(P) \subset U(P)$ and the theorems on differentiation and multiplication of power series it follows that the function $GF(u_{k-1})$ can be expanded in a series of the form

$$GF(u_{k-1}) = c + c_\alpha y^\alpha + c_{\alpha\beta} y^\alpha y^\beta + c_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma + \dots$$

convergent in $U_1(P)$.

In the following we shall use q to denote the value $\frac{(n-2)}{2}$. When $p = 0$ we then have $p_1 = q$. Noting that q is positive or zero, making use of the equations of the geodesics (2.2) and the above expansion for $GF(u_{k-1})$ we have

$$\begin{aligned} &-\frac{1}{4ks^{k+q}} \int_0^s s^{k+q-1} GF(u_{k-1})\, ds \\ &= -\frac{1}{4k}\left[\frac{c}{k+q} + \frac{c_\alpha y^\alpha}{k+q+1} + \frac{c_{\alpha\beta} y^\alpha y^\beta}{k+q+2} + \dots\right] \qquad (k \geqq 1), \end{aligned}$$

and this series converges in $U_1(P)$. Observe here that in evaluating the above integral we have made use of the fact that the geodesic given by (2.2) which issues from the origin and ends at an arbitrary point of $U_1(P)$ is contained entirely in this neighborhood in consequence of its spherical character. Since $k + p_1$ is positive in (5.5) it follows that $u_k G$ is analytic in a neighborhood of the values $y^\alpha = 0$ if, and only if, the constant $R_k = 0$. Hence

$$(6.1) \qquad u_k = -\frac{1}{4ks^{k+q}G} \int_0^s s^{k+q-1} GF(u_{k-1})\, ds \qquad (k = 1, 2, 3, \dots),$$

is expandable in a power series about $y^\alpha = 0$ which is convergent in the neighborhood $U_1(P)$.

Since u_0 and u_k as given by (6.1) are defined and are in fact analytic in the neighborhood $U_1(P)$ it follows that these functions satisfy the equations (5.2). Hence (5.1) with $p = 0$ and the above functions u_0 and u_k as coefficients is a formal solution of (2.8) and so of (2.1) in the neighborhood $U_1(P)$. It will be proved in paragraph **10** that this formal solution is convergent in a neighborhood $U_2(P) \subset U_1(P)$ and in fact that the function u admits a power series expansion in the y^α about $y^\alpha = 0$ which is convergent in $U_2(P)$.

THEOREM I. — *If y^α are the coordinates of a system of normal coordinates with origin at the point P and if u_0 is an arbitrary analytic function of the y^α defined by a power series convergent in a spherical neighborhood $U_1(P) \subset U(P)$ where $U(P)$ is the neighborhood specified in paragraph 5, then analytic functions u_k of the coordinates y^α, for $k = 1, 2, 3, \ldots$, are given by (6.1) such that the u_k admit power series expansions about $y^\alpha = 0$ convergent in the neighborhood $U_1(P)$ and the differential equation (A) or more specifically (2.1) has the solution*

$$u = u_0 + u_1\Gamma + u_2\Gamma^2 + \ldots \tag{6.2}$$

which admits a power series expansion in the y^α about $y^\alpha = 0$ convergent in a neighborhood $U_2(P) \subset U_1(P)$.

7. THE ELEMENTARY SOLUTION FOR ODD VALUES OF n. — Suppose that $p \neq 0$ in (5.3). Then u_0 has the form given by (5.4). Since G is an analytic function of the coordinates in the neighborhood of $y^\alpha = 0$, it is evident that we must take p_1 to be an integer $\leqq 0$ if u_0 is to be an analytic function of the y^α in the neighborhood of $y^\alpha = 0$. In the following we consider only the case $p_1 = 0$; then $-p$ has the value $\frac{(n-2)}{2}$ and so is equal to the number q introduced in paragraph **6**.

Now suppose that Q in (5.4) is a fixed constant independent of the geodesic of integration and for definiteness let us take $Q = 1$. Since $p_1 = 0$ by the above assumption, the equation (5.4) becomes $u_0 = G^{-1}$. Hence u_0, G and the coefficients in the expression $F(u)$ defined by (2.1) admit power series expansions, convergent in a spherical neighborhood $U_1(P)$ contained in the neighborhood $U(P)$ defined in paragraph 5.

If n is an odd integer the expression $k+p$ appearing in the denominator of the first term of the right member of (5.5) which has the value

$$k - q = \frac{2k + 2 - n}{2}$$

will always be different from zero. Since here $p_1 = 0$ it will follow by the argument used in paragraph **6** that u_k as given by (5.5) will be an analytic function in a neighborhood of $y^\alpha = 0$ if, and only if, the constant $R_k = 0$. Also by the argument of paragraph **6** the functions u_k given by

$$(7.1)\qquad u_k = -\frac{1}{4(k-q)s^k G}\int_0^s s^{k-1} G F(u_{k-1})\, ds \qquad (k = 1, 2, 3, \ldots),$$

admit power series expansions about $y^\alpha = 0$, convergent in the neighborhood $U_1(P)$. Corresponding to Theorem I we can now state the following theorem.

THEOREM II. — *For odd values of the integer n the differential equation* (A) *referred to a system of normal coordinates y^α with origin at the point* P, *i. e. the equation* (2.1), *will admit the solution*

$$(7.2)\qquad u = \frac{G^{-1} + u_1\Gamma + u_2\Gamma^2 + u_3\Gamma^3 + \ldots}{\Gamma^{\frac{(2-n)}{2}}}$$

where the u_k are defined by (7.1) *and are analytic functions of the coordinates y^α having power series expansions about $y^\alpha = 0$ convergent in a neighborhood* $U_1(P)$ *contained in the neighborhood* $U(P)$ *defined in paragraph* **5**. *The numerator of this solution admits a power series expansion in the y^α about $y^\alpha = 0$ convergent in a neighborhood* $U_2(P) \supset U_1(P)$.

The proof of the convergence of the series mentioned in the above theorem will be given in paragraph **10**. The solution u given by Theorem II will be analytic at all points of the neighborhood $U_2(P)$ for which Γ is different from zero (i. e., with the exception of the origin of normal coordinates for the case of the Riemann space under consideration).

8. SIMPLIFICATION OF THE ELEMENTARY SOLUTION BY THE INTRODUCTION OF SPECIAL COORDINATES. — We shall now observe an interesting fact regarding the solution (7.2) of the differential equation (2.1) for the case when n is an odd integer. Let us put

$$(8.1)\qquad \frac{1}{\theta^{n-2}} = G^{-1} + u_1\Gamma + u_2\Gamma^2 + u_3\Gamma^3 + \ldots$$

which defines θ as an analytic function of the normal coordinates y^α

in a neighborhood N of the origin. In fact since the exponent $n-2$ is an odd integer θ will be defined uniquely in N and will have the value 1 at the origin since $G(O)=1$. Hence $w^\alpha = \theta(y)y^\alpha$ will define an analytic coordinate transformation in a neighborhood V(P) of the origin P of the normal coordinate system. We observe that the derivatives $\frac{\partial w^\alpha}{\partial y^\beta}$ have the values δ^α_β at the origin P. With respect to the coordinates w^α in V(P) the solution u given by (7.2) will become

$$u = \frac{1}{\left[(g_{\alpha\beta})_P w^\alpha w^\beta\right]^{\frac{(n-2)}{2}}}, \tag{8.2}$$

where the coefficients $(g_{\alpha\beta})_P$ are the components of the metric tensor at the point P.

The above coordinates w^α have a unique determination when the underlying x coordinate system and the origin P are specified [in fact the only arbitrariness entering into this determination is involved in the arbitrary constant Q in (5.4) and we have fixed the value of this constant to be 1 in paragraph **7**]. To deduce the behaviour of the coordinates w^α having a fixed point P as origin when the underlying coordinates x^α undergo arbitrary analytic transformations we observe first of all that the quantity Γ is a scalar under such transformations, i. e. Γ is a scalar function of the coordinates with respect to arbitrary analytic coordinate transformations in a neighborhood V(P) when the initial point P entering into the definition of the function Γ is kept fixed. Similarly $\Delta\Gamma$ is a scalar under such transformations by the definition of the operator Δ and the scalar character of the function Γ. A similar remark applies to the quantity $c^\alpha \frac{\partial \Gamma}{\partial y^\alpha}$ since the c^α are the components of a contravariant vector. Hence

$$M^* = \Delta\Gamma + c^\alpha \frac{\partial \Gamma}{\partial y^\alpha} - 2n$$

is a scalar function. It follows that the function G defined in paragraph **5** and hence its inverse G^{-1} are scalars under the above coordinate transformations. In a similar manner we observe the scalar character of the successive coefficients u_k defined by (7.1) which occur in the solution (7.2). Hence u defined by (7.2) is a scalar function of the coordinates with respect to arbitrary analytic coordinate transformations in a neighborhood V(P) and hence the above function $\theta(y)$ is a scalar under such transformations.

Now under an arbitrary analytic transformation $x \to \bar{x}$ of the underlying coordinates x^α in a neighborhood V(P), the normal coordinates y^α and $\bar{y}^\alpha$ which are determined by the x^α and $\bar{x}^\alpha$ coordinates and which have the fixed poind P as origin undergo the transformation

$$y^\alpha = a^\alpha_\beta \bar{y}^\beta, \qquad \left[a^\alpha_\beta = \left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right)_{\mathrm{P}}\right].$$

Hence if we multiply each member of this equation by the above scalar θ we have

$$\theta(y) y^\alpha = a^\alpha_\beta \bar{\theta}(\bar{y}) \bar{y}^\beta;$$

or

$$(8.3) \qquad w^\alpha = a^\alpha_\beta \bar{w}^\beta.$$

The results which we have now obtained are stated in the following theorem.

THEOREM III. — *If n is an odd integer it is possible to define in a neighborhood* V(P) *of an arbitrary point* P *a system of coordinates w^α with origin at the point* P *and related analytically to the underlying coordinates x^α in* V(P) *such that the equation* (A) *with respect to the w^α coordinate system has the solution*

$$u = \frac{1}{[(g_{\alpha\beta})_{\mathrm{P}} w^\alpha w^\beta]^{\frac{(n-2)}{2}}},$$

where the coefficients $(g_{\alpha\beta})_{\mathrm{P}}$ are the components of the fundamental metric tensor with respect to the w^α coordinate system and evaluated at the origin (or these coefficients are the components of the fundamental metric tensor with respect to the underlying x^α coordinate system evaluated at the point P*). The coordinates w^α which can be so introduced are determined uniquely by the underlying x^α coordinate system and the point* P *as origin and when the coordinates x^α are subjected to an arbitrary analytic transformation $x \to \bar{x}$ in the neighborhood* V(P) *the coordinates w^α undergo the linear homogeneous transformation* (8.3) *with coefficients a^α_β equal to the derivatives $\frac{\partial x^\alpha}{\partial \bar{x}^\beta}$ evaluated at the point* P.

9. THE CASE OF EVEN VALUES OF n. — If $n \geqq 4$ and even we note that (7.1) fails to define the coefficient u_q. For these of n we are again led by analogy with ordinary equations to look for a solution of the form

$$(9.1) \qquad u = \left(\sum_{k=0}^{\infty} v_k \Gamma^k\right) \log \Gamma + \left(\sum_{k=0}^{\infty} u_k \Gamma^k\right) \Gamma^{-q}.$$

where the u_i and v_i are analytic functions of y^α in a neighborhood of $y^\alpha = 0$. When we substitute a function of the form (9.1) into (2.8) the terms containing $\log\Gamma$ as a factor are

$$(9.2) \qquad \left\{ \sum_{k=0}^{\infty} k \Gamma^{k-1} \left[4 y^\sigma \frac{\partial v_k}{\partial y^\sigma} + v_k(4k - 4 + \mathrm{M})\right] + \sum_{k=0}^{\infty} \Gamma^k \mathrm{F}(v_k) \right\} \log \Gamma.$$

The rest of the terms are given by

$$(9.3) \qquad \sum_{k=0}^{\infty} (k-q)\Gamma^{k-q-1} \left\{ 4 y^\sigma \frac{\partial u_k}{\partial y^\sigma} + u_k(4k - 4q - 4 + \mathrm{M}) \right\}$$
$$+ \sum_{k=0}^{\infty} \Gamma^{k-q} \mathrm{F}(u_k) + \sum_{k=0}^{\infty} \Gamma^{k-1} \left\{ 4 y^\sigma \frac{\partial v_k}{\partial y^\sigma} + v_k(8k - 4 + \mathrm{M}) \right\}.$$

When we equate to zero the coefficients of like powers of Γ in the bracket expression in (9.2) and also the coefficients of like powers of Γ in (9.3) we obtain

$$(9.4) \qquad 4 y^\sigma \frac{\partial u_0}{\partial y^\sigma} + u_0 \mathrm{M}^* = 0,$$

$$(9.5) \qquad (k-q)\left[4 y^\sigma \frac{\partial u_k}{\partial y^\sigma} + u_k(4k + \mathrm{M}^*)\right] + \mathrm{F}(u_{k-1}) = 0$$
$$(k = 1, 2, 3, \ldots, q-1),$$

$$(9.6) \qquad 4 y^\sigma \frac{\partial v_0}{\partial y^\sigma} + v_0(4q + \mathrm{M}^*) + \mathrm{F}(u_{q-1}) = 0,$$

$$(9.7) \qquad k\left[4 y^\sigma \frac{\partial v_k}{\partial y^\sigma} + v_k(4k + 4q + \mathrm{M}^*)\right] + \mathrm{F}(v_{k-1}) = 0 \qquad (k = 1, 2, 3, \ldots),$$

$$(9.8) \qquad (k-q)\left[4 y^\sigma \frac{\partial u_k}{\partial y^\sigma} + u_k(4k + \mathrm{M}^*)\right] + \mathrm{F}(u_{k-1})$$
$$+ 4 y^\sigma \frac{\partial v_{k-q}}{\partial y^\sigma} + v_{k-q}[8(k-q) + 4q + \mathrm{M}^*] = 0$$
$$(k = q+1, q+2, \ldots).$$

where the quantity $M^* = M - 2n$ is introduced. When use is made of equations (9.7) these last equations become

$$(9.9)\quad (k-q)\left[4y^\sigma \frac{\partial u_k}{\partial y^\sigma} + u_k(4k + M^*)\right] + F\left(u_{k-1} - \frac{v_{k-q-1}}{k-q}\right) + 4(k-q)v_{k-q} = 0 \qquad (k = q+1, q+2, \dots).$$

Now equation (9.4) is identical with the first equation (5.2) with $p \neq 0$ and the equations (9.5) are identical with the first $q-1$ equations of the second set of equations (5.2). Hence the solution of (9.4) is given by $u = G^{-1}$ where G is the function defined in paragraph **5**. We now define the spherical neighborhood $U_1(P)$ as in paragraph **7**. Also as in paragraph **7** we use the equations (9.5) to define functions $u_1, \dots, u_{q-1}$ of the y^α which admit power series expansions about $y^\alpha = 0$ convergent in $U_1(P)$. In particular if $n = 4$ we have $q = 1$ so that there are no equations in the set (9.5) and this step is to be omitted. In integrating the equations (9.6), (9.7) and (9.9) it will follow by augument used in paragraph **6** that we must take zero for the values of our constants of integration. We then have

$$(9.10)\qquad v_0 = -\frac{1}{4s^q G}\int_0^s s^{q-1}\, G\, F(u_{q-1})\, ds,$$

$$(9.11)\qquad v_k = -\frac{1}{4ks^{k+q}G}\int_0^s s^{k+q-1}\, G\, F(v_{k-1})\, ds \qquad (k = 1, 2, 3, \dots),$$

$$(9.12)\qquad u_k = -\frac{1}{4(k-q)s^k G}\int_0^s s^{k-1}\, G\, F\left(u_{k-1} - \frac{v_{k-q-1}}{k-q}\right) ds - \frac{1}{s^k G}\int_0^s s^{k-1}\, G\, v_{k-q}\, ds \qquad (k = q+1, q+2, \dots).$$

From (9.10) and (9.11) we determine v_0 and the v_k as analytic functions of the y^α admitting power series expansions about $y^\alpha = 0$ convergent in $U_1(P)$. Defining u_q by an arbitrary power series about $y^\alpha = 0$ convergent in $U_1(P)$ we arrive at a similar conclusion regarding the functions u_k determined by (9.12). In view of the convergence proof in paragraph **10** we are now able to state the following theorem.

THEOREM IV. — *Let the y^α be normal coordinates with origin at the point* P *and let u_q where $q = \frac{(n-2)}{2}$ be an arbitrary analytic function of the y^α admitting a power series expansion about $y^\alpha = 0$ convergent in a spherical neighborhood $U_1(P) \subset U(P)$ where $U(P)$ is defined in paragraph* **5**. *Then for even values of $n \geqq 4$ the differential equation* (2.1) *will admit the solution*

$$U = J \log \Gamma + \frac{K}{\Gamma^{\frac{(n-2)}{2}}}, \tag{9.13}$$

where

$$J = \sum_{k=0}^{\infty} v_k \Gamma^k \qquad \text{and} \qquad K = G^{-1} + \sum_{k=1}^{\infty} u_k \Gamma^k,$$

the coefficients v_k and u_k being analytic function of the coordinates y^α with power series expansions about $y^\alpha = 0$ convergent in $U_1(P)$. The v_k are determined by (9.10) *and* (9.11), *the u_k $(k = 1, \ldots, q-1)$ are determined by* (7.1) *and the u_k $(k = q+1, q+2, \ldots)$ are determined by* (9.12). *The above functions* J *and* K *admit power series expansions in the y^α about $y^\alpha = 0$ convergent in a neighborhood $U_2(P) \subset U_1(P)$.*

If $n = 2$ the value of p_1 considered in paragraph **7**, namely $p_1 = 0$, leads to $p = 0$. Hence (5.1) assumes the form (6.2) and we are back to the theory of paragraph **6**. However for $n = 2$ we can have a solution of the form (9.1) where $q = 0$. Taking $q = 0$ and equating to zero the coefficients of like powers of Γ in (9.3) and in the bracket expression in (9.2) we now have

$$4y^\sigma \frac{\partial v_0}{\partial y^\sigma} + v_0 M^* = 0, \tag{9.14}$$

$$k\left[4y^\sigma \frac{\partial v_k}{\partial y^\sigma} + v_k(4k + M^*)\right] + F(v_{k-1}) = 0 \qquad (k = 1, 2, 3, \ldots), \tag{9.15}$$

$$k\left[4y^\sigma \frac{\partial u_k}{\partial y^\sigma} + u_k(4k + M^*)\right] + F(u_{k-1}) + 4y^\sigma \frac{\partial v_k}{\partial y^\sigma} + v_k(8k + M^*) = 0. \tag{9.16}$$
$$(k = 1, 2, 3, \ldots).$$

From (9.14) we have $v_0 = G^{-1}$. The function u_0 can now be taken arbitrarily; we take u_0 to be defined by an arbitrary power series convergent in the spherical neighborhood $U_1(P) \subset U(P)$ where P is the origin of the system of normal coordinates y^α and U(P) is determined as in paragraph **5**. Since (9.15) and (9.16) are the same as (9.7) and (9.8) with $q = 0$ it follows that the $v_1, v_2, \ldots,$ and the $u_1, u_2, \ldots,$ will be determined by (9.11) and (9.12) with $q = 0$ and will be given by power series about $y^\alpha = 0$ convergent in $U_1(P)$.

THEOREM V. — *If $n = 2$ the differential equation* (2.1) *will admit the solution*

$$u = J \log \Gamma + K, \tag{9.17}$$

where

$$J = G^{-1} + \sum_{k=1}^{\infty} v_k \Gamma^k \qquad \text{and} \qquad K = \sum_{k=0}^{\infty} u_k \Gamma^k,$$

the coefficients v_k and u_k being analytic functions of the normal coordinates y^α with origin at the point P *and having power series expansions about $y^\alpha = 0$ convergent in a suitably chosen neighborhood* $U_1(P)$. *The above functions* J *and* K *admit power series expansions in the y^α about $y^\alpha = 0$ convergent in a neighborhood* $U_2(P) \subset U_1(P)$.

The solution u given by Theorem IV and Theorem V will be analytic at all points of the neighborhood $U_2(P)$ with the exception of the origin of normal coordinates at which Γ is equal to zero. Since $G(O) = 1$ the neighborhood $U_2(P)$ may be chosen so that in this neighborhood $K \neq 0$ for even values of $n \geqq 4$. Also for even values of $n \geqq 4$ the value of J is uniquely determined by the choice of the arbitrary constant Q (taken to be equal to 1 in paragraph **7**). For the ordinary Laplace equation (with $n \geqq 4$ and even) the value of J is easily seen to be zero from the equations determining the coefficients v_k in the expression for J; moreover taking the arbitrary function u_q to be identically zero the funktion K in the solution (9.13) has the value one and hence the solution (9.13) reduces to the solution (3.1) treated in paragraph **3**.

Similary for $n = 2$ the neighborhood $U_2(P)$ can be taken so that $J \neq 0$ in $U_2(P)$ and J is uniquely determined by the choice of the constant Q. When the equation (A) is the ordinary Laplace equation ($n = 2$) and the arbitrary function u_0 is taken to be identically zero, the function K will be identically zero and J will have the value one; hence (9.17) reduces to the first equation (3.1).

10. CONVERGENCE PROOFS. — Before proving the convergence of the series appearing in the solutions (6.2), (7.2), (9.13), (9.17), it will be convenient to make a change of unknown u in the differential equation (2.1), which will give us a new differential equation with its corresponding function G equal to unity. Let us consider the differential equation

$$(10.1) \qquad F_1(\bar{u}) = G\, F\left(\frac{\bar{u}}{G}\right) = 0,$$

where F is defined by (2.1) and G as in paragraph **5**. The equation (10.1) can be written

$$(10.2) \qquad F_1(\bar{u}) \equiv \Delta \bar{u} + \left(c^\alpha + 2h^{\alpha\beta}\frac{\partial \log G^{-1}}{\partial y^\beta}\right)\frac{\partial \bar{u}}{\partial y^\alpha} + d_1 \bar{u} = 0,$$

where d_1 depends on the coefficients of $F(u)$, the function G, and their derivatives. Note that the metric associated with (10.1) or (10.2) is the same as that associated with (2.1). Let the functions corresponding to M^* and G when constructed for the equation (10.1) be denoted by M_1^* and G_1. Referring to the definition of $M(= M^* + 2n)$ as given in paragraph **2** we have

$$M_1^* = y^\sigma \frac{\partial \log h}{\partial y^\sigma} + \left(c^\alpha + 2h^{\alpha\beta}\frac{\partial \log G^{-1}}{\partial y^\beta}\right) 2h_{\alpha\sigma}(0)\, y^\sigma$$

or

$$(10.3) \qquad M_1^* = M^* + 4y^\sigma \frac{\partial \log G^{-1}}{\partial y^\sigma}.$$

Along any geodesic issuing from the origin of the normal coordinate system the second term in the right member of (10.3) is equal to $4s \dfrac{d \log G^{-1}}{ds}$ and this in turn is equal to $-M^*$. Hence this term is equal to $-M^*$ in a neighborhood of the origin and hence in this neighborhood $M_1^* = 0$. From this it follows that $G_1 = 1$.

Let us now examine the relationship between the solutions (6.2) constructed for the equation (2.1) and the corresponding solution for the equation (10.1). In (6.2) the function u_0 is arbitrary. We define the corresponding function $\bar{u}_0$ for the solution of the equation (10.1) to be given by $\bar{u}_0 = G u_0$. The formulæ (6.1) can be writen in the form

$$(10.4) \qquad u_k = \frac{1}{G}\left\{ -\frac{1}{4ks^{k+q}} \int_0^s s^{k+q-1}\, G\, F\left(\frac{G u_{k-1}}{G}\right) ds \right\}.$$

We shall use $\bar{u}_k$ to denote the functions of the form (6.1) when constructed for (10.1) with the assigned arbitrary function $\bar{u}_0$. Assuming that $\bar{u}_{k-1} = G u_{k-1}$ it can be seen from (10.4) that $\bar{u}_k = G u_k$. Making use of the definition of $F_1(\bar{u})$ and the fact that $G_1 = 1$, the equation (10.4) can be written

$$G u_k = -\frac{1}{4ks^{k+q}} \int_0^s s^{k+q-1}\, F_1(\bar{u}_{k-1})\, ds = \bar{u}_k.$$

Hence mere multiplication of all of the coefficients in the series defining the solution (6.2) by the quantity G yields the corresponding solution for the equation (10.1). A similar statement can be made for each of the solutions (7.2), (9.13), (9.17).

In the following we treat the series appearing in the solutions (6.2), (7.2), (9.13), (9.17) when constructed for the equation (10.1). We show that these series define functions of y^α which can be represented by power series convergent in a neighborhood $U_2(P)$ of the origin of normal coordinates; the same can be said for the corresponding series constructed for the equation (2.1) since G^{-1} has a power series expansion convergent in $U_2(P)$ and by the above the series for (2.1) can be obtained from the series for (10.1) by multiplication by the function G^{-1}.

For convenience in what follows we drop the subscript on $F_1(\ )$ and the bar on $\bar{u}_k$, i.e. we treat (2.1) under the assumption that $G = 1$.

Let us recall that a power series $\varphi(y^\alpha)$ is said to be dominated by a power series $\Psi(y^\alpha)$ having positive coefficients, if each coefficient of φ is in absolute value less than or equal to the corresponding coefficient of ψ. If we put

$$\sigma = y^1 + y^2 + y^3 + \ldots + y^n,$$

the positive constants m and r can be so chosen that all the coefficients of $F(u)$ will by dominated by

$$\frac{m}{\left(1-\frac{\sigma}{r}\right)}.$$

Let $w(y^\alpha)$ be any function analytic in the neighborhood of $y^\alpha = 0$ such that

$$(10.5) \qquad w \ll \frac{M}{\left(1-\frac{\sigma}{r}\right)^h},$$

where h is zero or a positive integer and where the sign $\ll$ means « dominated by ». Then it follows that

$$\frac{\partial w}{\partial y^\alpha} \ll \frac{Mh}{r\left(1-\frac{\sigma}{r}\right)^{h+1}}, \qquad \frac{\partial^2 w}{\partial y^\alpha \partial y^\beta} \ll \frac{M\,h(h+1)}{r^2\left(1-\frac{\sigma}{r}\right)^{h+2}}.$$

These relations and the relation (10.5) will still hold if we increase the exponent h in the right member for none of the coefficients in the power series expansions of these right members will be decreased by this operation. When we take account of this fact and also the number of terms in the expression $F(w)$ we see that we can write

$$(10.6) \qquad F(w) \ll \frac{m' M\,h(h+1)}{\left(1-\frac{\sigma}{r}\right)^{h+2}},$$

where

$$m' = m\left(1+\frac{n}{r}+\frac{n^2}{r^2}\right).$$

Let us now consider the question of the convergence of the numerator in the formal solution (7.2) of the equation (2.1) for odd values of the integer n. In this numerator the functions $u_1(y)$, $u_2(y)$, $u_3(y)$, ..., admit power series expansions about the values $y^\alpha = 0$ convergent in the neighborhood $U_1(P)$ as shown in paragraph **7**. Let us now assume that

$$(10.7) \qquad u_{k-1} \ll \frac{M_{k-1}}{\left(1-\frac{\sigma}{r}\right)^{2(k-1)}}$$

for some value of the integer $k \geqq 1$. Since $u_0 = 1$ the relation (10.7) can obviously be satisfied for $k = 1$ by taking $M_0 = 1$. We are going to show that we can write

$$u_k \leqq \frac{M_k}{\left(1-\frac{\sigma}{r}\right)^{2k}}, \tag{10.8}$$

where the constant M_k will be defined later. If in (10.5) we take $w = u_{k-1}$, $M = M_{k-1}$ and $h = 2(k-1)$ we have from (10.6) that

$$F(u_{k-1}) \leqq \frac{m' M_{k-1}(2k-2)(2k-1)}{\left(1-\frac{\sigma}{r}\right)^{2k+1}}. \tag{10.9}$$

Hence from (7.1), recalling that $G = 1$, we have

$$u_k \leqq \frac{m' M_{k-1}(2k-2)(2k-1)}{4(k-q)s^k} \int_0^s \frac{s^{k-1}}{\left(1-\frac{\sigma}{r}\right)^{2k+1}} . \tag{10.10}$$

Let us now consider only geodesics (2.2) along which $\eta = \Sigma_\alpha \xi^\alpha$ is different from zero. Along such a geodesic we have $\sigma = \eta s$. To avoid confusion we introduce the letter t as the variable of integration in the right member of (10.10). Then along such a geodesic we have

$$\frac{1}{s^k}\int_0^s \frac{s^{k-1}}{\left(1-\frac{\sigma}{r}\right)^{2k+1}}\,ds = \frac{1}{\sigma^k}\int_0^s \frac{(\eta t)^{k-1}}{\left(1-\frac{\eta t}{r}\right)^{2k+1}}\,\eta\,dt. \tag{10.11}$$

Upon making the substitution $\tau = \eta t$ the right member of (10.11) becomes

$$\frac{1}{\sigma^k}\int_0^\sigma \frac{\tau^{k-1}}{\left(1-\frac{\tau}{r}\right)^{2k+1}}\,d\tau. \tag{10.12}$$

Now consider the following relation involving the integrand function of (10.12)

$$\frac{\tau^{k-1}}{\left(1-\frac{\tau}{r}\right)^{2k+1}} < \frac{\tau^{k-1}}{\left(1-\frac{\tau}{r}\right)^{2k+1}}\left(1+\frac{\tau}{r}\right). \tag{10.13}$$

Using differentiation it is easily seen that the integral of the right member of (10.13) is given by

$$\int_0^\sigma \frac{\tau^{k-1}}{\left(1-\frac{\tau}{r}\right)^{2k+1}}\left[\left(1-\frac{\tau}{r}\right)+2\frac{\tau}{r}\right]d\tau = \frac{\sigma^k}{k\left(1-\frac{\sigma}{r}\right)^{2k}}.$$

Combining the above results we can now replace (10.10) by the following relation

$$(10.14) \qquad u_k \leqq \frac{m' M_k(2k-2)(2k-1)}{4(k-q)k\left(1-\frac{\sigma}{r}\right)^{2k}}.$$

In deriving (10.14) let us notice that we have shown that along geodesics for which $\eta \neq 0$, the right member of (10.10) is actually equal to the function (10.12) multiplied by a certain constant, and this in turn is dominated by the right member of (10.14). Since the dominating property depends only on the coefficients of power series, the relation (10.14) is generally valid irrespective of whether or not the condition $\eta \neq 0$ is satisfied. Defining M_k by

$$(10.15) \qquad M_k = \frac{m' M_{k-1}(2k-2)(2k-1)}{4(k-q)k},$$

the relation (10.14) yields the desired relation (10.8).

This completes the induction. Hence (10.8) is valid for $k = 0, 1, 2, 3, \ldots$ with $M_0 = 1$ and the successive values of the constants M_k given by (10.15).

Let us now select a set of n positive constants a^α satisfying the conditions $\Sigma_\alpha a^\alpha < r$ and

$$(10.16) \qquad \frac{m' \,| h_{\alpha\beta}(0) a^\alpha a^\beta |}{\left(1-\frac{\Sigma_\alpha a^\alpha}{r}\right)^2} \leqq \frac{m' \,| h_{\alpha\beta}(0) |\, a^\alpha a^\beta}{\left(1-\frac{\Sigma_\alpha a^\alpha}{r}\right)^2} = \rho < 1;$$

such a choice of the a^α is evidently possible. In view of (10.8), each term of the power series expansion for u_k will be in absolute value less than the corresponding term of the expansion of

$$\frac{M_k}{\left(1-\frac{\Sigma_\alpha a^\alpha}{r}\right)^{2k}}$$

for values of y^α satisfying $|y^\alpha| < a^\alpha$. Hence for the same values of y^α, each term of the power series expansion for $u_k\Gamma^k$(k not summed) will be in absolute value less than the corresponding term of the series for

$$(10.17) \qquad \frac{M_k |h_{\alpha\beta}(0)a^\alpha a^\beta|^k}{\left(1 - \frac{\Sigma_\sigma a^\sigma}{r}\right)^{2k}}.$$

The series of positive constants, the terms of which are the constants (10.17) for $k = 0, 1, 2, \dots$, is convergent in view of (10.16) and the fact that $\lim\limits_{k \to} \frac{M_k}{M_{k-1}} = m'$ (ratio test).

Now let us consider the double array, that has for its k *th* row the terms of the power series for $u_k\Gamma^k$(k not summed) arranged in some order, it being understood that the terms in any one column involve the same y's raised to the same powers. Due to the convergence of the series of constants (10.17), this double array is absolutely convergent when summed by rows for values of $|y^\alpha| < a^\alpha$. Making use of the theorem on double arrays (1), we will also have absolute convergence when summing by columns. However this is nothing more than a statement of the fact that the series appearing in the numerator of the solution (7.2) defines a function of y^α which can be represented by a power series convergent in the neighborhood $y^\alpha| < a^\alpha$. By taking the positive constants a^α sufficiently small the inequalities $|y^\alpha| < a^\alpha$ will define a neighborhood $U_2(P) \subset U_1(P)$ as defined in paragraph 7.

Let us now consider the case of even values of $n \geq 4$ i.e. the convergence of the series mentioned in Theorem IV. As before for $k = 1, \dots, q-1$ the relation (10.8) is valid; in particular

$$u_{q-1} \ll \frac{M_{q-1}}{\left(1 - \frac{\sigma}{r}\right)^{2(q-1)}}.$$

Making use of (10.6) and (9.10) we have

$$v_0 \ll \frac{m' M_{q-1}(2q-2)(2q-1)}{4s^q} \int_0^s \frac{s^{q-1}}{\left(1 - \frac{\sigma}{r}\right)^{2q+1}} ds.$$

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(1) E. Goursat, *Cours d'Analyse mathématique*, 1927, p. 409-414.

Using the process to obtain a dominant function for the above integral which was previously used to obtain a dominant function for the integral in (10.10) we have

$$(10.13)\qquad v_0 \ll \frac{N_0}{\left(1-\frac{\sigma}{r}\right)^{2q}},$$

where

$$N_0 = \frac{m' M_{q-1}(2q-2)(2q-1)}{4q}.$$

Making use of (10.18), (10.6), and (9.11) we have

$$(10.19)\qquad v_k \ll \frac{N_k}{\left(1-\frac{\sigma}{r}\right)^{2(k+q)}} \qquad (k=0,\ 1,\ 2,\ \ldots),$$

provided we choose

$$(10.20)\qquad N_k = \frac{m' N_{k-1}\, 2(k+q-1)[2(k+q)-1]}{4k(k+q)} \qquad (k=1,\ 2,\ \ldots).$$

Let us now select the positive constant M_q such that the arbitrary function u_q satisfies the condition

$$(10.21)\qquad u_q \ll \frac{M_q}{\left(1-\frac{\sigma}{r}\right)^{2q}}.$$

From (10.19) and (10.6) we have

$$(10.22)\qquad \left\{\begin{aligned} &F(v_{k-q-1}) \ll \frac{m' N_{k-q-1}(2k-2)(2k-1)}{\left(1-\frac{\sigma}{r}\right)^{2k+1}},\\ &v_{k-q} \ll \frac{N_{k-q}}{\left(1-\frac{\sigma}{r}\right)^{2k+1}} \qquad (k=q+1,\ q+2,\ \ldots).\end{aligned}\right.$$

Making use of (10.21), (10.22), (10.9) and (9.12) we have for $k=q+1$ that

$$u_k \ll \left(M_{k-1}+\frac{N_{k-q-1}}{k-q}\right)\frac{m'(2k-2)(2k-1)}{4(k-q)s^k}\int_0^s \frac{s^{k-1}}{\left(1-\frac{\sigma}{r}\right)^{2k+1}}\,ds$$
$$+\frac{N_{k-q}}{s^k}\int_0^s \frac{s^{k+1}}{\left(1-\frac{\sigma}{r}\right)^{2k+1}}\,ds.$$

Using the above process to obtain dominant functions for the integrals in the above expression we are led to (10.8) for $k=q+1$. In fact it is readily seen that we can continue our induction to obtain (10.8) for all values of k provided we take

$$(10.23) \qquad M_k=\left(M_{k-1}+\frac{N_{k-q-1}}{k-q}\right)\frac{m'(2k-2)(2k-1)}{4k(k-q)}+\frac{N_{k-q}}{k}$$
$$(k=q+1,\ q+2,\ \ldots).$$

From (10.23) we deduce that $\frac{M_k}{M_{k-1}} \to m'$ as $k\to\infty$ making use of the fact that $\frac{N_k}{N_{k-1}}\to m'$ as $k\to\infty$ from (10.20). Using (10.8) and (10.19) and the theorem on double arrays as above, it therefore follows that the functions J and K in (9.13) admit powers series expansions in y^α about $y^\alpha=0$ convergent in a neighborhood $U_2(P)$ of the origin of normal coordinates. This completes the convergence proof required for Theorem IV.

With slight modifications the above proof for even values of $n\geq 4$ will include the case $n=2$ covered by Theorem V, the formulae (9.11) and (9.12) now being the only ones involved in the determination of the functions v_k and u_k. The proof of the convergence of the series (6.2) is very similar to the one given above for odd values of n. We might point out that in each of the solutions just mentioned u_0 is arbitrary and this leads in place of (10.8) to a dominant relation of the form

$$u_k \ll \frac{M_k}{\left(1-\frac{\sigma}{r}\right)^{2k+1}} \qquad (k=0,\ 1,\ 2,\ \ldots).$$

14. Remarks on the indefinite case. — If R or $\mathfrak{M}$ is a pseudo-Riemann space in which case the fundamental quadratic form is indefinite we cannot choose the vector ξ in equations (2.2) in such a manner that the parameter s will represent the arc length measured from the origin along all geodesics passing through the origin. These exceptional curves are the so-called geodesics of zero length. Any neighborhood N of the origin of normal coordinates will contain a real hypersurface $\Gamma=0$ which will separate N into a portion N_p over which Γ is positive and a portion N_n over which Γ is negative. By multiplying the differential equation (A) through by -1 the two portions N_p and N_n will be interchanged.

If we take the neighborhood N to be spherical (Euclidean definition of distance in the normal coordinate system) any point in N_p can be joined to the origin of normal coordinates by a geodesic lying entirely in N_p and this geodesic can be determined by a vector of unit length as in paragraph **2** and hence admits a parameter s which can be interpreted as arc length. Confining our attention to the portion N_p of the neighborhood N all expressions in the preceeding sections will retain their real meaning, viz. log Γ and Γ^p in (3.1), etc. will be real functions. An inspection of the formulæ defining G, the u_k and the v_k reveals that for the indefinite case under consideration they are likewise expressible as analytic functions of the y^α admitting power series expansions about $y^\alpha = 0$ convergent in some neighborhood of the origin of normal coordinates. Also the convergence results of paragraph **10** remain valid without modification. Hence Theorem I is valid for the indefinite case as are also Theorem II, IV and V these latter theorems giving real solutions of the differential equation (A) in N_p. It is evident also that the special coordinates w^α in paragraph **8** can be defined for the indefinite case and lead to the special form of the elementary solution (real in N_p) which is given by Theorem III.