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Some infinite integrals involving parabolic cylinder functions

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Some infinite integrals involving parabolic cylinder functions;

By R. S. VARMA.

1. The object of this paper is to evaluate some infinite integrals involving parabolic cylinder functions. Thus in paragraphs **2-4** infinite integrals involving a Bessel function and a parabolic cylinder function are deduced. Again in a recent paper ([1]) I have proved that the functions

$$(1.1) \qquad x^{\nu+\frac{1}{2}} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x) \qquad [R(\nu) > -1]$$

and

$$(1.2) \qquad x^{\nu-\frac{1}{2}} e^{\frac{1}{4}x^2} D_{-2\nu}(x) \qquad \left[R(\nu) > -\frac{1}{2}\right],$$

are self-reciprocal ([2]) in the Hankel-transform of order ν. In para graph **4** a kernel is also discovered which transform (**1**.2) into (**1**.1). Further we know that a generalization of the parabolic cylinder function is given by Sonine's polynomial ([3])

$$(1.3) \qquad T_m^n(x) = \sum_{r=0}^{n} \frac{(-)^r x^{n-r}}{r!\,(n-r)!\,\Gamma(n+m-r+1)},$$

([1]) R. S. VARMA, *Some functions which are self-reciprocal in the Hankel-transform* (*Proc. Lond. Math. Soc.* (2), **42**, 1937, p. 9-17).

([2]) Following Hardy and Littlewood we shall henceforth say that a function is R_ν as an abbreviation of the statement that it is self-reciprocal in the Hankel-transform of order ν.

([3]) SONINE, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries* (*Math. Annalen*, **16**, 1880, p. 1-80).

since, for $m = -\frac{1}{2}$ and $m = \frac{1}{2}$, (1.3) reduces to

$$T^n_{-\frac{1}{2}}(x^2) = \frac{2^n}{2n!\sqrt{\pi}} e^{\frac{1}{2}x^2} D_{2n}(x\sqrt{2})$$

and

$$xT^n_{\frac{1}{2}}(x^2) = \frac{2^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}} e^{\frac{1}{2}x^2} D_{2n+1}(x\sqrt{2})$$

respectively. In paragraph **5-6** infinite integrals involving Sonine's polynomial are evaluated. It is interesting to note that the integrals of paragraphs **4-6** give functions which are R_ν.

2. Now Watson (¹) has shown that, if $|\arg\alpha| < \frac{1}{2}\pi$.

$$\int_\infty^{(0+)} e^{\left(\frac{1}{4}-\alpha\right)z^2} z^m D_n(z)\, dz$$
$$= \frac{\pi^{\frac{3}{2}} 2^{\frac{1}{2}n-m} e^{\pi i\left(m-\frac{1}{2}\right)}}{\Gamma(-m)\Gamma\left(\frac{1}{2}m - \frac{1}{2}n + 1\right)\alpha^{\frac{1}{2}(m+1)}}$$
$$\times F\left(-\frac{1}{2}n, \frac{1}{2}m + \frac{1}{2}; \frac{1}{2}m - \frac{1}{2}n + 1; 1 - \frac{1}{2}\alpha^{-1}\right).$$

It is easy to see that this reduces, when $\alpha = \frac{1}{2}$ to

$$(2.1) \qquad \int_0^\infty x^m e^{-\frac{1}{4}x^2} D_n(x)\, dx = \sqrt{\pi}\,.\,2^{\frac{1}{2}n-\frac{1}{2}m-\frac{1}{2}} \frac{\Gamma(m+1)}{\Gamma\left(\frac{1}{2}m - \frac{1}{2}n + 1\right)}$$
$$(m > -1).$$

Since

$$(2.2) \qquad J_n(x) = \sum_{r=0}^\infty \frac{(-)^r x^{n+2r}}{2^{n+2r} r!\,\Gamma(n+r+1)},$$

(¹) G. N. Watson, *The Harmonic functions associated with the parabolic cylinder* (*Proc. Lond. Math. Soc.* (2), 8, 1910, p. 393-421).

et

$$\int_0^\infty x^l e^{-\frac{1}{4}x^2} D_m(x) J_n(ax)\,dx$$

$$=\int_0^\infty x^l e^{-\frac{1}{4}x^2} D_m(x) \sum_{r=0}^{\infty} \frac{(-)^r (ax)^{n+2r}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}\,dx$$

$$=\sum_{r=0}^{\infty} \frac{(-)^r a^{n+2r}}{2^{n+2r}\, r!\, \Gamma(n+r+1)} \int_0^\infty x^{l+n+2r} e^{-\frac{1}{4}x^2} D_m(x)\,dx$$

$$=\sum_{r=0}^{\infty} \sqrt{\pi} \frac{(-)^r a^{n+2r}\Gamma(l+n+2r+1)}{2^{\frac{3}{2}n+3r-\frac{1}{2}m+\frac{1}{2}l+\frac{1}{2}}\, r!\, \Gamma(n+r+1)\Gamma\left(\frac{1}{2}l+\frac{1}{2}n+r-\frac{1}{2}m+1\right)}$$

$$(n>0;\ l+n+1>0)$$

by the help of (**2.** 1)

$$=\frac{\sqrt{\pi}\, a^n \Gamma(l+n+1)}{2^{\frac{3}{2}n-\frac{1}{2}m+\frac{1}{2}l+\frac{1}{2}}\Gamma(n+1)\Gamma\left(\frac{1}{2}l+\frac{1}{2}n-\frac{1}{2}m+1\right)}$$

$$\times\sum_{r=0}^{\infty}\left(-\frac{1}{2}a^2\right)^r \frac{\left(\frac{1}{2}l+\frac{1}{2}n+\frac{1}{2},\, r\right)\left(\frac{1}{2}l+\frac{1}{2}n+1,\, r\right)}{r!\,(n+1,\, r)\left(\frac{1}{2}l+\frac{1}{2}n-\frac{1}{2}m+1,\, r\right)},$$

where

$$(n,\, r)=n(n+1)\ldots(n+r-1).$$

The step (**2.**2) is justified since for positive values of n, $J_n(y)$ represents a uniformly convergent series in $y \geqq 0$ and since the resulting series is absolutely convergent. Hence, when $n>0$ and $l+n+1>0$,

$$(2.3)\quad \int_0^\infty x^l e^{-\frac{1}{4}x^2} D_m(x) J_n(ax)\,dx$$

$$=\frac{\sqrt{\pi}\, a^n \Gamma(l+n+1)}{2^{\frac{3}{2}n-\frac{1}{2}m+\frac{1}{2}l+\frac{1}{2}}\Gamma(n+1)\Gamma\left(\frac{1}{2}l+\frac{1}{2}n-\frac{1}{2}m+1\right)}$$

$$\times\, {}_2F_2\left\{\begin{matrix}\frac{1}{2}l+\frac{1}{2}n+\frac{1}{2}, & \frac{1}{2}l+\frac{1}{2}n+1, \\ n+1, & \frac{1}{2}l+\frac{1}{2}n-\frac{1}{2}m+1, \end{matrix}\quad -\frac{1}{2}a^2\right\}.$$

In particular, when $l = n+1$, this gives

$$\int_0^\infty x^{n+1} e^{-\frac{1}{4}x^2} D_m(x) J_n(ax)\, dx$$

$$= \frac{2^{\frac{1}{2}m} a^n \Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n-\frac{1}{2}m+\frac{3}{2}\right)} {}_1F_1\left(n+\frac{3}{2},\ n-\frac{1}{2}m+\frac{3}{2},\ -\frac{1}{2}a^2\right)$$

$$(n > 0).$$

3. Again we know that

$$J_m(x) J_n(x) = \sum_{r=0}^{\infty} \frac{(-)^r \Gamma(m+n+2r+1)}{r!\,\Gamma(m+r+1)\Gamma(n+r+1)\Gamma(m+n+r+1)} \left(\frac{x}{2}\right)^{m+n+2r}.$$

Hence

$$\int_0^\infty x^l e^{-\frac{1}{4}x^2} D_p(x) J_m(x) J_n(x)\, dx$$

$$= \int_0^\infty x^l e^{-\frac{1}{4}x^2} D_p(x) \sum_{r=0}^{\infty} \frac{(-)^r \Gamma(m+n+2r+1)}{r!\,\Gamma(m+r+1)\Gamma(n+r+1)\Gamma(m+n+r+1)} \left(\frac{x}{2}\right)^{m+n+2r} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-)^r \Gamma(m+n+2r+1)}{r!\,\Gamma(m+r+1)\Gamma(n+r+1)\Gamma(m+n+r+1)} \frac{1}{2^{m+n+2r}} \int_0^\infty x^{l+m+n+2r} e^{-\frac{1}{4}x^2} D_p(x)\, dx$$

$$= \sqrt{\pi} \sum_{r=0}^{\infty} (-)^r \frac{2^{\frac{1}{2}p-\frac{1}{2}l-\frac{3}{2}m-\frac{3}{2}n-3r-\frac{1}{2}} \Gamma(m+n+2r+1)\Gamma(l+m+n+2r+1)}{r!\,\Gamma(m+r+1)\Gamma(n+r+1)\Gamma(m+n+r+1)\Gamma\left(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}p+r+1\right)}$$

$$(m+n > 0;\ l+m+n+1 > 0),$$

by the help of (**2**.1), term by term integration being obviously justifiable.

It follows therefore that, when $m+n > 0$ and $l+m+n+1 > 0$,

$$\int_0^\infty x^l e^{-\frac{1}{4}x^2} D_p(x) J_m(x) J_n(x)\, dx$$

$$= \frac{\sqrt{\pi}\,\Gamma(l+m+n+1)}{2^{\frac{1}{2}l+\frac{3}{2}m+\frac{3}{2}n-\frac{1}{2}p+\frac{1}{2}} \Gamma(m+1)\Gamma(n+1)\Gamma\left(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}p+1\right)}$$

$$\times {}_4F_4\left\{\begin{matrix} \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}; & \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n+1, & \frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}, & \frac{1}{2}m+\frac{1}{2}n+1 \\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}p+1, & m+1, & n+1, & m+n+1 \end{matrix} ;\ -2\right\}$$

4. Bailey has shown [1] that if $f(x)$ is R_μ, then

$$(4.1)\qquad \int_0^\infty (xt)^{\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}} J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(xt) f(t)\,dt$$

is R_ν. In this, taking

$$f(x) = x^{\mu-\frac{1}{2}} e^{\frac{1}{4}x^2} D_{-2\mu}(x),$$

which is R_μ, we obtain that

$$(4.2)\qquad x^{\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}} \int_0^\infty t^{\frac{1}{2}\mu+\frac{1}{2}\nu} e^{\frac{1}{4}t^2} D_{-2\mu}(t) J_{\frac{1}{2}\nu+\frac{1}{2}\mu}(xt)\,dt$$

is R_ν.

Evaluating (4.2) by the help of the known integral [2]

$$\int_0^\infty x^{n-\frac{1}{2}} e^{\frac{1}{4}x^2} J_{n-\frac{1}{2}}(ax) D_{-m}(x)\,dx$$

$$= \frac{(2a)^{n-\frac{1}{2}}\Gamma(n)}{2\sqrt{\pi}\,\Gamma(m)} \left[2^{\frac{1}{2}m-n}\Gamma\left(\frac{1}{2}m-n\right) {}_1F_1\left(n;\ 1+n-\frac{1}{2}m;\ \frac{1}{2}a^2\right) + a^{m-2n}\frac{\Gamma\left(n-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}m\right)}{\Gamma(n)}\,{}_1F_1\left(\frac{1}{2}m;\ 1-n+\frac{1}{2}m;\ \frac{1}{2}a^2\right)\right]$$

$$[R(n)>0,\ R(m)>R(n-1)],$$

we arrive at the result that

$$(4.3)\qquad 2^{\frac{1}{2}\mu+\frac{1}{2}\nu-1} x^{\nu+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(2\mu)}$$

$$\times\left[2^{\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}}\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\right) {}_1F_1\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2};\ \frac{1}{2}\nu-\frac{1}{2}\mu+\frac{3}{2};\ \frac{1}{2}x^2\right) + x^{\mu-\nu-1}\frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\mu\right)\Gamma(\mu)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}\,{}_1F_1\left(\mu;\ \frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2};\ \frac{1}{2}x^2\right)\right]$$

is R_ν.

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[1] W. N. Bailey, *On the solutions of some integral equations* (*Journ. Lond. Math. Soc.*, 6, 1931, p. 242-247).

[2] R. S. Varma, *An infinite integral involving Bessel Functions and parabolic cylinder functions* (*Proc. Camb. Phil. Soc.*, 33, 1937, p. 210-211).

Since ([1]) for all values of n

$$D_n(x) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}n\right)} 2^{\frac{1}{2}n} e^{-\frac{1}{4}x^2} {}_1F_1\left(-\frac{1}{2}n; \frac{1}{2}; \frac{1}{2}x^2\right)$$

$$+\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}n\right)} 2^{\frac{1}{2}n-\frac{1}{2}} x e^{-\frac{1}{4}x^2} {}_1F_1\left(\frac{1}{2}-\frac{1}{2}n; \frac{3}{2}; \frac{1}{2}x^2\right),$$

(4.3) gives as a particular case, when $\mu = \nu + 2$, that

$$2^{\nu} x^{\nu+\frac{1}{2}} \frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(2\nu+4)} \left[2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) {}_1F_1\left(\nu+\frac{3}{2}; \frac{1}{2}; \frac{1}{2}x^2\right) \right.$$

$$\left. + x \frac{\Gamma\left(-\frac{1}{2}\right)\Gamma(\nu+2)}{\Gamma\left(\nu+\frac{3}{2}\right)} {}_1F_1\left(\nu+2; \frac{3}{2}; \frac{1}{2}x^2\right) \right]$$

$$= \frac{1}{2\nu+3} x^{\nu+\frac{1}{2}} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x)$$

is R_ν.

It follows from (4.1) that if

$$(4.4) \qquad f(x) = x^{\nu+\frac{3}{2}} e^{\frac{1}{4}x^2} D_{-2\nu-4}(x)$$

is $R_{\nu+2}$, then

$$(4.5) \quad \int_0^\infty (xt)^{-\frac{1}{2}} J_{\nu+1}(xt)\, t^{\nu+\frac{3}{2}} e^{\frac{1}{4}t^2} D_{-2\nu-4}(t)\, dt = \frac{1}{2\nu+3} x^{\nu+\frac{1}{2}} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x)$$

is R_ν; in other words, the kernel $(xt)^{-\frac{1}{2}} J_{\nu+1}(xt)$ transforms (4.4) into (4.5).

5. In (4.1), take $f(x) = x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} T_\mu^n(x^2)$ which we know ([2]),

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([1]) WHITTAKER and WATSON, *Modern Analysis* (fourth Edition), p. 347.

([2]) WILSON, *On an extension of Milne's integral equation* (*Messenger of Math.*, **53**, 1923-1924, p. 157-160).

is R_μ or skew R_μ according as n is even or odd. We then obtain that

$$(5.1) \qquad x^{\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}(\nu+\mu+2)} e^{-\frac{1}{2}t^2} T^n_\mu(t^2)\, J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(xt)\, dt$$

is R_ν or skew R_ν according as n is even or odd.

We shall now show that (**5**.1) can be expressed in terms of Kummer's function. By the help of (**1**.3) and the integral (¹)

$$\int_0^\infty t^l e^{-\frac{1}{2}t^2} J_n(tx)\, dt = \frac{x^n \Gamma\left(\frac{1}{2}l+\frac{1}{2}n+\frac{1}{2}\right)}{2^{\frac{1}{2}n-\frac{1}{2}l+\frac{1}{2}}\Gamma(n+1}\, {}_1F_1\left(\frac{1}{2}l+\frac{1}{2}n+\frac{1}{2};\, n+1;\, -\frac{1}{2}x^2\right)$$

$$[R(l+n+1)>0]$$

we obtain that

$$\int_0^\infty t^{\frac{1}{2}(\nu+\mu+2)} e^{-\frac{1}{2}t^2} T^n_\mu(t^2)\, J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(xt)\, dt$$

$$=\sum_{r=0}^{n}\frac{(-1)^r}{r!\,(n-r)!\,\Gamma(n+\mu-r+1)}\int_0^\infty t^{\frac{1}{2}\nu+\frac{1}{2}\mu+1+2n-2r} e^{-\frac{1}{2}t^2} J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(xt)\, dt$$

$$=\frac{x^{\frac{1}{2}\mu+\frac{1}{2}\nu}}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)}\sum_{r=0}^{n}\frac{(-)^r 2^{n-r}\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+n-r+1\right)}{r!\,(n-r)!\,\Gamma(n+\mu-r+1)}$$

$$\times {}_1F_1\left(\frac{1}{2}\nu+\frac{1}{2}\mu+n-r+1;\, \frac{1}{2}\mu+\frac{1}{2}\nu+1;\, -\frac{1}{2}x^2\right)$$

$$[R(\mu+\nu+2)>0].$$

We deduce therefore that, when $R(\mu+\nu+2)>0$,

$$\frac{2^n x^{\nu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)}\sum_{r=0}^{n}\left(-\frac{1}{2}\right)^r\frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+n-r+1\right)}{r!\,(n-r)!\,\Gamma(n+\mu-r+1)}$$

$$\times {}_1F_1\left(\frac{1}{2}\nu+\frac{1}{2}\mu+n-r+1;\, \frac{1}{2}\mu+\frac{1}{2}\nu+1;\, -\frac{1}{2}x^2\right).$$

is R_ν or skew R_ν according as n is even or odd.

(¹) It is interesting to note that this known integral can be obtained from our integral (**2**.3) as a particular case by putting $m=0$.

In particular, when $\nu = \mu$, we get that

$$\frac{2^n x^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \sum_{r=0}^{n} \frac{\left(-\frac{1}{2}\right)^r}{r!\,(n-r)!}\, {}_1F_1\left(\nu+n-r+1 : \nu+1 : -\frac{1}{2}x^2\right)$$

$$[R(\nu+1) > 0]$$

is R_ν or skew R_ν according as n is even or odd.

6. Another theorem [1] connecting different classes of self-reciprocal functions given by Bailey is that, if $f(x)$ is R_μ, then

$$(6.1) \qquad \int_0^\infty \frac{t^{\mu+\frac{1}{2}} f(xt)}{(1+t^2)^{\frac{1}{2}\mu+\frac{1}{2}\nu+1}}\, dt$$

is R_ν.

Taking $f(x) = x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} T_\mu^n(x^2)$ as in paragraph **5**, we obtain from (**6**.1) that

$$x^{\mu+\frac{1}{2}} \int_0^\infty \frac{t^{2\mu+1} e^{-\frac{1}{2}x^2 t^2} T_\mu^n(x^2 t^2)}{(1+t^2)^{\frac{1}{2}\mu+\frac{1}{2}\nu+1}}\, dt$$

is R_ν or skew R_ν according as n is even or odd.

By the help of (**1**.3) and the integral [2]

$$2\int_0^\infty \frac{x^{m-1} e^{-\frac{1}{2}x^2}}{(x^2+a^2)^n}\, dx$$

$$= 2^{\frac{1}{2}m-n}\,\Gamma\left(\frac{1}{2}m-n\right) {}_1F_1\left(n ; 1+n-\frac{1}{2}m ; \frac{1}{2}a^2\right)$$

$$+ a^{m-2n} \frac{\Gamma\left(n-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}m\right)}{\Gamma(n)}\, {}_1F_1\left(\frac{1}{2}m ; 1-n+\frac{1}{2}m ; \frac{1}{2}a^2\right)$$

$$[R(m) > 0],$$

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(1) W. N. Bailey, *loc. cit.*

(2) R. S. Varma, *An infinite integral involving Bessel functions and parabolic cylinder functions* (*loc. cit.*).

we obtain that

$$\int_0^\infty \frac{t^{2\mu+1} e^{-\frac{1}{2}x^2t^2} T_\mu^n(x^2t^2)}{(1+t^2)^{\frac{1}{2}\mu+\frac{1}{2}\nu+1}} dt$$

$$= \sum_{r=0}^{n} \frac{(-)^r x^{2n-2r}}{r!(n-r)!\Gamma(n+\mu-r+1)} \int_0^\infty \frac{t^{2\mu+2n-2r+1} e^{-\frac{1}{2}x^2t^2}}{(1+t^2)^{\frac{1}{2}\mu+\frac{1}{2}\nu+1}} dt$$

$$= \frac{x^{\nu-\mu}}{2} \sum_{r=0}^{n} \frac{(-)^r}{r!(n-r)!\Gamma(n+\mu-r+1)}$$

$$\times \Bigg[2^{\frac{1}{2}\mu-\frac{1}{2}\nu+n-r} \Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+n-r\right) {}_1F_1\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1; \frac{1}{2}\nu-\frac{1}{2}\mu-n+r+1; \frac{1}{2}x^2\right)$$

$$+ x^{\mu-\nu+2n-2r} \frac{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu-n+r\right)\Gamma(\mu+n-r+1)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)}$$

$$\times {}_1F_1\left(\mu+n-r+1; \frac{1}{2}\mu-\frac{1}{2}\nu+n-r+1; \frac{1}{2}x^2\right)\Bigg]$$

$$[R(2\mu+1)>0].$$

We infer therefore that

$$\frac{x^{\nu+\frac{1}{2}}}{2} \sum_{r=0}^{n} \frac{(-)^r \psi_r}{r!(n-r)!\Gamma(n+\mu-r+1)} \qquad [R(2\mu+1)>0].$$

is R_ν or skew R_ν according as *n* is even or odd, where

$$\psi_r = 2^{\frac{1}{2}\mu-\frac{1}{2}\nu+n-r} \Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+n-r\right)$$

$$\times {}_1F_1\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1; -\frac{1}{2}\mu+\frac{1}{2}\nu-n+r+1; \frac{1}{2}x^2\right)$$

$$+ x^{\mu-\nu+2n-2r} \frac{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu-n+r\right)\Gamma(\mu+n-r+1)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)}$$

$$\times {}_1F_1\left(\mu+n-r+1; \frac{1}{2}\mu-\frac{1}{2}\nu+n-r+1; \frac{1}{2}x^2\right).$$

In particular, putting $\mu = \nu$, we obtain that when $R(2\nu + 1) > 0$,

$$\frac{x^{\nu+\frac{1}{2}}}{2} \sum_{r=0}^{n} \frac{(-)^r \psi'_r}{r!\,(n-r)!\,\Gamma(n+\nu-r+1)},$$

is R_ν or skew R_ν according as n is even or odd, where

$$\psi'_r = 2^{n-r}\Gamma(n-r)\,{}_1F_1\left(\nu+1\,;\,-n+r+1\,;\,\frac{1}{2}x^2\right)$$
$$+\,2^{2n-2r}\frac{\Gamma(r-n)\,\Gamma(\nu+n-r+1)}{\Gamma(\nu+1)}\,{}_1F_1\left(\nu+n-r+1\,;\,n-r+1\,;\,\frac{1}{2}x^2\right).$$