Johan Leray

Protoperads II: Koszul duality

PROTOPERADS II: KOSZUL DUALITY

by Johan Leray

Abstract. — In this paper, we construct a bar-cobar adjunction and a Koszul duality theory for protoperads, which are an operadic type notion encoding faithfully some categories of gebras with diagonal symmetries, like double Lie algebras ($\mathcal{DLie}$). We give a criterion to show that a binary quadratic protoperad is Koszul and we apply it successfully to the protoperad $\mathcal{DLie}$. As a corollary, we deduce that the properad $\mathcal{DPois}$ which encodes double Poisson algebras is Koszul. This allows us to describe the homotopy properties of double Poisson algebras which play a key role in non commutative geometry.

Résumé (Protopérades II : dualité de Koszul). — Dans cet article, on construit une adjonction bar-cobar et une dualité de Koszul pour les protopérades, qui encodent fidèlement des catégories de gèbres avec des symétries diagonales, comme les algèbres double Lie ($\mathcal{DLie}$). On donne un critère pour montrer qu'une protopérade quadratique binaire est de Koszul, critère que l'on applique avec succès à la protopérade $\mathcal{DLie}$. Comme corollaire, on en déduit que la propérade $\mathcal{DPois}$ qui encode les algèbres double Poisson est de Koszul. Cela nous permet de décrire les propriétés homotopiques des algèbres double Poisson, qui jouent un role clé en géométrie non commutative.

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INTRODUCTION

This paper develops the Koszul duality theory for *protoperads*, defined in [Ler19], which are an analog of properads (see [Val03, Val07]) with more symmetries. The main application of this theory is the proof of the Koszulness of the properad which encodes double Lie algebras, from which it follows that the properad encoding double Poisson algebras is Koszul.

The motivation for this work is to determine what is a double Poisson bracket up to homotopy. A double Poisson structure, as defined by Van den Bergh in [Van08a], gives a Poisson structure in noncommutative algebraic geometry (see [Gin05, Van08b]) under the *Kontsevich-Rosenberg principle*, i.e., if A is a double Poisson algebra, then the associated affine representation schemes $\mathrm{Rep}_n(A)$ have (classical) Poisson structures.

In order to determine the homotopical properties of a family of algebras, we use the classical strategy, which was already used to understand, for example, the homotopical properties of Gerstenhaber algebras (and also the homotopic properties of associative, commutative, Lie, Poisson, etc, algebras). The idea is to go to the upper level and understand the homological properties of the algebraic object that encodes the structure, such as the operad $\mathcal{G}erst$ for Gerstenhaber algebras. In the good case where the operad (or the properad) satisfies good properties, we can use Koszul duality in order to have a minimal cofibrant replacement of our operad. We can then go down to the level of algebras. Thanks to this cofibrant replacement (in the case of Gerstenhaber algebras, the operad $\mathcal{G}_\infty$), we obtain the associated notion of algebra up to homotopy: for example, Gerstenhaber algebras up to homotopy are encoded by $\mathcal{G}_\infty$ (see [Gin04] or [GCTV12, §2.1]). This structure has a good homotopical behaviour at the algebras' level, the homotopy transfer theorem (see [LV12, §10.3] for algebras over an operad), etc.

Double Poisson structures are *properadic* in nature as they are made up of operations with multiple inputs and multiples outputs. They are encoded by the properad $\mathcal{D}\mathcal{P}ois$, which is constructed with the properads $\mathcal{A}s$ and $\widetilde{\mathcal{D}\mathcal{L}ie}$ (see Lemma 5.9), where the properad $\widetilde{\mathcal{D}\mathcal{L}ie}$ encodes double Lie structure and the properad $\mathcal{A}s$ encodes associative algebra structure. The properad $\widetilde{\mathcal{D}\mathcal{L}ie}$ is a quadratic properad defined by generators and relations, with the generator $V_{\mathcal{D}\mathcal{L}ie}$ concentrated in arity $(2,2)$:

$$V_{\mathcal{D}\mathcal{L}ie}: \quad \overset{1 \quad 2}{\underset{1 \quad 2}{\rule{0pt}{0pt}\blacksquare}} \quad = - \quad \overset{2 \quad 1}{\underset{2 \quad 1}{\rule{0pt}{0pt}\blacksquare}}$$

and the relation in arity $(3,3)$

$$R_{DJ}: \quad \overset{1 \; 2 \; 3}{\underset{1 \; 2 \; 3}{\rule{0pt}{0pt}\blacksquare}} \quad + \quad \overset{2 \; 3 \; 1}{\underset{2 \; 3 \; 1}{\rule{0pt}{0pt}\blacksquare}} \quad + \quad \overset{3 \; 1 \; 2}{\underset{3 \; 1 \; 2}{\rule{0pt}{0pt}\blacksquare}} \; .$$

Thus double Lie bracket on a chain complex A is given by a morphism of properads $\widetilde{\mathcal{D}\mathcal{L}ie} \to \mathrm{End}_A$ where End_A is the properad of endomorphisms of A (see [Val07] for the definition).

The theory of properads is the good general algebraic framework to encode operations with several inputs and outputs. In certain cases, this framework can be simplified. For example, algebraic structures with several inputs and *one* output, like associative, commutative or Lie algebras, are encoded by *operads* (see [LV12]). In a certain sense, the operadic framework is the minimal one to study such structures. In this smaller framework, homotopical properties are much easier to study.

Similarly, protoperads form a special class of properads, which provide the appropriate framework for studying the double Lie properads. In the first article [Ler19], we have developed this minimal framework, such that there exists a protoperad $\mathcal{DLie}$ which encodes the double Lie structure. In [Ler19], we proved the existence of the free protoperad functor and gave an explicit combinatorial description of this, in terms of *bricks and walls*. An important property of protoperads is their compatibility with properads via the induction functor (see Definition/Proposition 1.16).

In this paper, we develop the homological algebra for protoperads. With the monoidal exact functor of induction, we prove the existence of a bar-cobar adjunction in the case of protoperads:

$$\Omega: \mathsf{coprotoperads}_k^{\mathrm{coaug}} \; \overrightarrow{\longleftarrow} \; \mathsf{protoperads}_k^{\mathrm{aug}}: \mathsf{B}.$$

We obtain also the following theorem, the protoperadic analogue of the criterion of Koszul of the properads [Val03, Th. 149],[Val07].

Theorem (see Theorem 2.25). — *Let $\mathcal{P}$ be a connected weight-graded protoperad. The following are equivalent:*

(1) *the inclusion $\mathcal{P}^{\mathsf{i}} \hookrightarrow \mathsf{B}\mathcal{P}$ is a quasi-isomorphism, i.e., the protoperad $\mathcal{P}$ is Koszul;*

(2) *the morphism of protoperads $\Omega\mathcal{P}^{\mathsf{i}} \to \mathcal{P}$ is a quasi-isomorphism, where $\mathcal{P}^{\mathsf{i}}$ is the Koszul dual of $\mathcal{P}$ (see Theorem 2.29)*

We give a useful criterion to show that a binary quadratic protoperad (i.e., a quadratic protoperad generated by a $\mathfrak{S}$-module concentrated in arity 2) is Koszul. Take a binary quadratic protoperad $\mathcal{P}$ given by generators and relations. We associate to $\mathcal{P}$ a family of associative algebras $\mathscr{A}(\mathcal{P}, n)$, for $n \geqslant 2$. The algebra $\mathscr{A}(\mathcal{P}, n)$ is constructed so that its bar construction splits and such that one of these factors is the n-th arity of the normalized simplicial bar construction of the protoperad $\mathcal{P}$.

Theorem (see Theorem 4.3). — *Let $\mathcal{P}$ be a binary quadratic protoperad. If, for all integers $n \geqslant 2$, the quadratic algebra $\mathscr{A}(\mathcal{P}, n)$ is Koszul, then the protoperad $\mathcal{P}$ is Koszul.*

This is a useful criterion because the study of the Koszulness of algebras is easier than for pro(to)perads. Many tools are available, such as PBW or Gröbner bases, or rewriting methods (see [LV12, Chap. 4]).

We use this criterion to show that the protoperad $\mathcal{DLie}$ is Koszul. As the functor of induction is exact and preserves the weight, so preserves the Koszulness, the properad $\widetilde{\mathcal{DLie}}$, which encodes double Lie algebras, is also Koszul.

Theorem (see Theorem 4.7 and Corollary 4.8). — *The protoperad $\mathcal{D}\mathcal{L}ie$ and the properad $\widetilde{\mathcal{D}\mathcal{L}ie}$ are Koszul.*

This theorem is very important: it is the first example of a Koszul properad with a generator not in arity $(1, 2)$ or $(2, 1)$. And so, with an argument of distributive law, we deduce the main theorem of this paper.

Theorem (see Theorem 5.11). — *The properad $\mathcal{D}\mathcal{P}ois$ is Koszul.*

In an future article, we will explain the homotopy transfer theorem for properadic algebras and we will use this in an other future work, where we will study the implications of Theorem 5.11 in derived noncommutative algebraic geometry à la Berest et al. (see [BCER12, BFP$^+$17, BFR14, CEEY17]). In particular, we will link it to pre-Calabi Yau structures as in [Yeu18, IK18]. We will also look at the cohomological theory of double Poison algebras. Indeed, the work of Merkulov and Vallette gives the notion of deformation theory of $\mathcal{P}$-algebras, for $\mathcal{P}$ a properad. We want to link the deformation complex defined in [MV09b] with the work of Pichereau et al. who defined the cohomology of *differential* double Poisson algebra (see [PVdW08]).

Organization of the paper. — After a review of definitions and some properties of protoperads (see [Ler19]) in Section 1, following the results on properads (see [Val03, Val07, MV09a]), we introduce the notion of shuffle protoperads in Section 1.3. In Section 2, we define the Koszul duality of protoperads. We transpose a part of the results on properads obtained by Vallette in [Val03, Val07] to the protoperatic framework thanks to the exactness of the induction functor Ind (see [Ler19, Prop. 4.4.]). In Section 3, we define the simplicial bar construction and the normalized one for protoperads and we described the levelisation morphism (see Definition/Proposition 3.6). In Section 4, we give a criterion to prove that a binary quadratic protoperad is Koszul and we use it to prove that the protoperad $\mathcal{D}\mathcal{L}ie$ is Koszul. Finally, in Section 5, we use results of Vallette on distributive laws to prove that the properad $\mathcal{D}\mathcal{P}ois$ is Koszul.

Notations. — We write $\mathbb{N}^*$ for the set $\mathbb{N} \setminus \{0\}$. In all this paper, k is a field with characteristic 0. We denote by Fin, the category with finite sets as objects and bijections as morphisms and Set, the category of all sets and all maps. For two integers a and b, we note by $[\![a, b]\!]$ the set $[a, b] \cap \mathbb{Z}$, and, for $n \in \mathbb{N}^*$, $\mathfrak{S}_n$ is the automorphism group of $[\![1, n]\!]$, i.e., $\mathfrak{S}_n = \mathrm{Aut}_{\mathsf{Fin}}([\![1, n]\!])$. We denote by Ch_k the category of $\mathbb{Z}$-graded chain complexes over the field k.

The goal of this paper is to study the Koszulness of the properad $\mathcal{D}\mathcal{P}ois$, using the protoperad $\mathcal{D}\mathcal{L}ie$. These two objects are not to be confused with $\mathcal{D}(\mathcal{P}ois)$ and $\mathcal{D}(\mathcal{L}ie)$ where $\mathcal{D}$ is the polydifferential functor on the category of props introduce by Merkulov and Willwacher.

1. RECOLLECTIONS ON PRO(TO)PERADS

We briefly recall the definition of protoperads and some results of [Ler19]. We denote by $\mathfrak{S}\text{-mod}_k^{\mathrm{red}}$, the category of contravariant functors from Fin to the category of chain complexes Ch_k such that $P(\varnothing) = 0$.

1.1. COMBINATORIAL FUNCTORS. — We recall two important functorial combinatorial constructions which are described in [Ler19, §1]: the functors $\mathcal{W}^{\mathrm{conn}}$ and $\mathfrak{X}^{\mathrm{conn}}$.

NOTATION. — For a poset $(P, \leqslant_P)$, we denote by $\mathrm{Succ}(P)$, the set of pairs $(r, s) \in P \times P$ such that $r <_P s$ and there does not exist $t \in P$ such that $r <_P t <_P s$.

DEFINITION 1.1 (The functor of walls). — The functor $\mathcal{W}_n : \mathsf{Fin}^{\mathrm{op}} \to \mathsf{Fin}^{\mathrm{op}}$ is defined, for every finite sets S, as follows. An element W of $\mathcal{W}_n(S)$ is a collection $\{W_\alpha\}_{\alpha \in A}$ of non-empty subsets of S, indexed by some finite set A of cardinality n, such that

(1) the union of these subsets is S, i.e., $\bigcup_{\alpha \in A} W_\alpha = S$;
(2) for every $s \in S$, the set $\Gamma(W, s) := \{W_\alpha \mid s \in W_\alpha\}$ is totally ordered by $\leqslant_s$;
(3) these orders are compatible on the intersection of the sets Γ: for every s and t in S, and $W_a, W_b \in \Gamma(W, s) \cap \Gamma(W, t)$, $W_a \leqslant_s W_b$ if and only if $W_a \leqslant_t W_b$;

the collection of partial orders $\{\leqslant_s\}_{s \in S}$ defines a canonical partial order on W (see [Ler19, Lem. 1.3.]). The action of $\sigma \in \mathrm{Aut}(S)$ on $\left(\{W_\alpha\}_{\alpha \in A}, \leqslant\right)$ in $\mathcal{W}_n$ is induced by the right action on S, i.e.,

$$\left(\{W_\alpha\}_{\alpha \in A}, \leqslant\right) \cdot \sigma = \left(W = \{W_\alpha \cdot \sigma\}_{\alpha \in A}, \leqslant^\sigma\right)$$

where $W_\alpha \cdot \sigma := \sigma^{-1}(W_\alpha)$ and the order $\leqslant^\sigma$ is induced by the total orders of $\Gamma_{s \cdot \sigma}^{W \cdot \sigma} := \{W_\alpha \cdot \sigma \mid s \cdot \sigma \in W_\alpha \cdot \sigma\}$. Using the collection of functors $\mathcal{W}_n$, one define *the functor of walls* by

$$\mathcal{W} : \mathsf{Fin}^{\mathrm{op}} \longrightarrow \mathsf{Set}^{\mathrm{op}}$$
$$S \longmapsto \coprod_{n \in \mathbb{N}^*} \mathcal{W}_n(S).$$

EXAMPLE 1.2. — One can graphically represent such wall. Consider the set $S = [\![1, 4]\!]$ and the wall $W = \{W_a, W_b, W_c\}$ in $\mathcal{W}([\![1, 4]\!])$ over S with the three bricks

$$W_a = \{1, 2\}, \quad W_b = \{3, 4\} \quad \text{and} \quad W_c = \{1, 2\}$$

with the partial order $W_a < W_c$. We represent this wall by
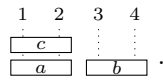
Given $(W = \{W_\alpha\}_{\alpha \in A}, \leqslant)$, a wall in $\mathcal{W}(S)$, we define *the equivalence relation of connectedness* $\overset{\mathrm{conn.}}{\sim}$ on W as follows. For two elements a and b of A, we say $W_a \overset{\mathrm{conn.}}{\sim} W_b$ if there exist an integer $n \geqslant 2$ and a sequence $W_0, W_1, \ldots, W_{n-1}, W_n$ of elements of W with $W_0 = W_a$ and $W_n = W_b$ such that, for all i in $[\![0, n-1]\!]$,

$$W_i \cap W_{i+1} \neq \varnothing \quad \text{and} \quad (W_i, W_{i+1}) \in \mathrm{Succ}(W) \quad \text{or} \quad (W_{i+1}, W_i) \in \mathrm{Succ}(W).$$

Definition 1.3 (Projection $\mathcal{K}$). — We define the natural projection $\mathcal{K}$ as follows: for a finite set S, we have

$$\mathcal{K}_S : \mathcal{W}(S) \longrightarrow \mathcal{Y}(S) \subset \mathcal{W}(-)$$
$$W \longmapsto \big\{\textstyle\bigcup_{B_\alpha \in \pi^{-1}([B])} B_\alpha \mid [B] \in \pi(W)\big\},$$

where π is the projection of W to its quotient by $\overset{\text{conn.}}{\sim}$.

Definition 1.4 (The functor of connected walls). — We also have the subfunctor $\mathcal{W}_n^{\text{conn}} \hookrightarrow \mathcal{W}_n$ of connected walls $\mathcal{W}_n^{\text{conn}} \colon \mathsf{Fin}^{\text{op}} \to \mathsf{Fin}^{\text{op}}$ which is given, for all finite sets S, by

$$\mathcal{W}_n^{\text{conn}}(S) := \big\{ \big(W = \{W_\alpha\}_{\alpha \in A}, \leqslant\big) \in \mathcal{W}_n(S) \mid \mathcal{K}_S(W) = \{S\} \big\}.$$

We also define the functor

$$\mathcal{W}^{\text{conn}} : \mathsf{Fin}^{\text{op}} \longrightarrow \mathsf{Set}^{\text{op}}$$
$$S \longmapsto \textstyle\coprod_{n \in \mathbb{N}^*} \mathcal{W}_n^{\text{conn}}(S).$$

An element W of $\mathcal{W}^{\text{conn}}(S)$ is called a *connected wall* over S, and an element of a wall W is called a *brick* of W.

Example 1.5. — Consider the wall $W \in \mathcal{W}(\llbracket 1, 5 \rrbracket)$ with 6 bricks:

$$W_1 = \{2,3\}, \ W_2 = \{4,5\}, \ W_3 = \{1,2\}, \ W_4 = \{3,4\}, \ W_5 = \{2,3\}, \ W_6 = \{4,5\},$$

with the partial order $W_1 \leqslant W_3 \leqslant W_5$, $W_1 \leqslant W_4 \leqslant W_5$ and $W_2 \leqslant W_4 \leqslant W_6$, graphically represented as follows:
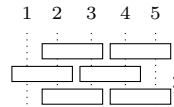
The wall W is connected.

Non-example 1.6. — The wall of Example 1.2 is not connected.

Hence, we have other important subfunctors of $\mathcal{W}^{\text{conn}}$.

– The functor $\mathcal{Y} : \mathsf{Fin}^{\text{op}} \to \mathsf{Set}^{\text{op}}$ is defined, for every finite set S, by

$$\mathcal{Y}(S) := \big\{ \big(\{K_\alpha\}_{\alpha \in A}, \leqslant\big) \in \mathcal{W}(S) \mid \forall s \in S, \ \big|\{K_\alpha \mid s \in K_\alpha\}\big| = 1 \big\}.$$

An element of $\mathcal{Y}(S)$ is a non-ordered partition of S, i.e., a wall composed with one row of bricks over S.

– The functor $\mathcal{X} = \mathcal{Y} \times \mathcal{Y}$ is defined, for every finite set S, by

$$\mathcal{X}(S) \cong \big\{ \big(\{K_\alpha\}_{\alpha \in A}, \leqslant\big) \in \mathcal{W}(S) \mid \forall s \in S, \ \big|\{K_\alpha \mid s \in K_\alpha\}\big| = 2 \big\}.$$

An element K of $\mathcal{X}(S)$ is an ordered pair of unordered partitions of the finite set S, so we also denote by (I, J) such a K. Graphically, an element of $\mathcal{X}$ has the form
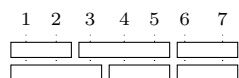
– The functor $\mathfrak{X}^{\mathrm{conn}}$ is the subfunctor of $\mathfrak{X}$ of connected walls, defined by

$$\mathfrak{X}^{\mathrm{conn}}(S) := \big\{\big(\{K_\alpha\}_{\alpha \in A}, \leqslant\big) \in \mathcal{W}^{\mathrm{conn}}(S) \mid \forall s \in S, \ \big|\{K_\alpha \mid s \in K_\alpha\}\big| = 2\big\}.$$

The functor $\mathfrak{X}^{\mathrm{conn}}$ encodes a new monoidal structure on the category of reduced $\mathfrak{S}$-modules, the connected composition product, as we will see in Definition/Proposition 1.12.

1.2. Monoidal structures and the induction functor. — We have three monoidal structures on $\mathfrak{S}\text{-mod}_k^{\mathrm{red}}$.

Definition/Proposition 1.7 (Composition product). — The *composition product* is the bifunctor

$$- \square - : \mathfrak{S}\text{-mod}_k^{\mathrm{red}} \times \mathfrak{S}\text{-mod}_k^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\mathrm{red}}$$

defined, for P, Q two reduced $\mathfrak{S}$-modules and S a finite set, by

$$\big(P \square Q\big)(S) := P(S) \otimes Q(S).$$

This bi-additive bifunctor gives $\mathfrak{S}\text{-mod}^{\mathrm{red}}$ a symmetric monoidal structure, with identity $I_\square$, defined, for all non empty sets S, by $I_\square(S) := k$ concentrated in degree 0.

Definition/Proposition 1.8 (Concatenation product). — The *concatenation product* is the bifunctor

$$- \otimes^{\mathrm{conc}} - : \mathfrak{S}\text{-mod}_k^{\mathrm{red}} \times \mathfrak{S}\text{-mod}_k^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\mathrm{red}}$$

defined, for all finite sets S and all reduced $\mathfrak{S}$-modules P and Q, by:

$$\big(P \otimes^{\mathrm{conc}} Q\big)(S) := \bigoplus_{\substack{S', S'' \in \mathrm{Ob}\,\mathsf{Fin} \\ S' \amalg S'' = S}} P(S') \otimes Q(S'').$$

This product is symmetric monoidal without unit (since we are working with reduced $\mathfrak{S}$-modules).

Definition/Proposition 1.9 (see [Ler19, §2.2.1]). — We denote by $\mathbb{S}$, the functor which sends a reduced $\mathfrak{S}$-module V to the free symmetric monoid *without unit* $\mathbb{S}(V)$ for the concatenation product. Moreover, it is isomorphic to

$$\mathbb{S}(V) \cong \bigoplus_{\{I_\alpha\}_{\alpha \in A} \in \mathcal{Y}(S)} \bigotimes_{\alpha \in A} V(I_\alpha),$$

where the notation $\bigotimes_{\alpha \in A} V(I_\alpha)$ is defined in [Ler19, Not. 2.7].

Remark 1.10. — We can extend the concatenation product:

(1) $$- \otimes^{\mathrm{conc}} - : \mathfrak{S}\text{-mod}_k \times \mathfrak{S}\text{-mod}_k^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\mathrm{red}}.$$

This extension is induced by the equivalence of categories $\mathfrak{S}\text{-mod}_k \cong \mathsf{Ch}_k \times \mathfrak{S}\text{-mod}_k^{\mathrm{red}}$, by the injection $(-)^{\mathfrak{S}} : \mathsf{Ch}_k \hookrightarrow \mathfrak{S}\text{-mod}_k$ defined, for all chain complexes C and all finite sets S, by $(C)^{\mathfrak{S}}(\varnothing) = C$ and $(C)^{\mathfrak{S}}(S) = 0$ when $|S| > 0$, and by the action of the category Ch_k on $\mathfrak{S}\text{-mod}_k^{\mathrm{red}}$ defined, for all chain complexes C and all finite sets S, by

$$\big(C \otimes^{\mathrm{conc}} V\big)(S) := C \otimes V(S).$$

This extension allows us to define the suspension of a $\mathfrak{S}$-module.

Definition 1.11 (Suspension of a $\mathfrak{S}$-module (see [Ler19, Def. 2.6])). — Let Σ (respectively Σ^{-1}) be the chain complex k concentrated in degree 1 (resp. in degree -1). For V a reduced $\mathfrak{S}$-module, the *suspension of V* (resp. *desuspension of V*) is the reduced $\mathfrak{S}$-module $\Sigma V \overset{\text{not.}}{:=} \Sigma \otimes^{\text{conc}} V$ (resp. $\Sigma^{-1} V \overset{\text{not.}}{:=} \Sigma^{-1} \otimes^{\text{conc}} V$).

Definition/Proposition 1.12 (Connected composition product of $\mathfrak{S}$-modules (see [Ler19, Def. 2.8]))

The *connected composition product* of reduced $\mathfrak{S}$-modules is the bifunctor

$$- \boxtimes_c - : \mathfrak{S}\text{-mod}_k^{\text{red}} \times \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\text{red}}$$

defined, for all reduced $\mathfrak{S}$-modules P, Q and for all non empty finite sets S, by:

$$P \boxtimes_c Q(S) := \bigoplus_{(I,J) \in \mathfrak{X}^{\text{conn}}(S)} \bigotimes_\alpha P(I_\alpha) \otimes \bigotimes_\beta Q(J_\beta),$$

where $\bigotimes_\alpha P(I_\alpha) \otimes \bigotimes_\beta Q(J_\beta)$ with (I, J) in $\mathfrak{X}^{\text{conn}}(S)$ is the notation for the quotient

$$\left(\bigotimes_{i=1}^r P(I_i) \otimes \bigotimes_{j=1}^s Q(J_j) \right) \Big/ {}_\sim$$

where the relation $\sim$ identifies $(p_1 \otimes \cdots \otimes p_r) \otimes (q_1 \otimes \cdots \otimes q_s)$ with

$$(-1)^{|\sigma(p)| + |\tau(q)|} (p_{\sigma^{-1}(1)} \otimes \cdots \otimes p_{\sigma^{-1}(r)}) \otimes (q_{\tau^{-1}(1)} \otimes \cdots \otimes q_{\tau^{-1}(s)})$$

for all σ in $\mathfrak{S}_r$, τ in $\mathfrak{S}_s$ with $(-1)^{|\sigma(p)|}, (-1)^{|\tau(q)|}$, the Koszul signs induced by permutations. We also denote by $I_\boxtimes$, the $\mathfrak{S}$-module given by

$$I_\boxtimes(S) := \begin{cases} k & \text{if } |S| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

which is the unit of the product $\boxtimes_c$. The category $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_\boxtimes)$ is a (symmetric) monoidal category. The monoids for this product are called *protoperads*.

We have a compatibility between these monoidal structures.

Proposition 1.13 (Compatibility between monoidal structures (see [Ler19, Prop. 3.19]))

Let P and Q be two reduced $\mathfrak{S}$-modules. There is a natural isomorphism of $\mathfrak{S}$-modules:

$$\mathbb{S}(P \boxtimes_c Q) \cong \mathbb{S}P \,\square\, \mathbb{S}Q.$$

In particular, for a protoperad $\mathcal{P}$, the $\mathfrak{S}$-module $\mathbb{S}\mathcal{P}$ is a monoid for the product $\square$.

We have a notion of free protoperad. The combinatorics of the free protoperad is described by the functor of connected walls $\mathcal{W}^{\text{conn}}$.

Proposition 1.14 (Free protoperad (see [Ler19, Th. 5.21])). — *Let V be a reduced $\mathfrak{S}$-module and ρ be a positive integer. There exists a free protoperad on V, denoted by $\mathscr{F}(V)$. For a finite set S, there is an isomorphism of weight-graded right $\text{Aut}(S)$-modules, given on each weight ρ, by*

$$\mathscr{F}^\rho(V)(S) \cong \bigoplus_{\substack{(\{K_\alpha\}_{\alpha \in A}, \leqslant) \\ \in \mathcal{W}_\rho^{\text{conn}}(S)}} \bigotimes_{\alpha \in A} V(K_\alpha),$$

where $\mathcal{W}^{\mathrm{conn}} : \mathsf{Fin}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ *is the weight-graded functor of connected walls. The functor* $\mathscr{F}$ *is the left adjoint to the forgetful functor*

$$\mathscr{F} : \mathfrak{S}\text{-}\mathsf{mod}_k^{\mathrm{red}} \rightleftarrows \mathsf{protoperads}_k : \mathrm{For}.$$

The notion of protoperad is compatible with the notion of properad, defined by Vallette in [Val03, Val07] and [MV09a], via the induction functor.

DEFINITION/PROPOSITION 1.15 (Properad – Free properad (see [Val03, Val07]))
The category of reduced $\mathfrak{S}$-bimodules, i.e., the category of functors

$$P : \mathsf{Fin} \times \mathsf{Fin}^{\mathrm{op}} \longrightarrow \mathsf{Ch}_k$$

such that, for all finite set S, $P(S, \varnothing) = 0 = P(\varnothing, S)$, is monoidal for the connected composition product denoted by $\boxtimes_c^{\mathrm{Val}}$. The monoids for this product are called *properads*. We have the *free properad functor*, which is denoted by $\mathscr{F}^{\mathrm{Val}}$ which is the left adjoint to the forgetful functor:

$$\mathscr{F}^{\mathrm{Val}} : \mathfrak{S}\text{-}\mathsf{bimod}_k^{\mathrm{red}} \rightleftarrows \mathsf{properads}_k : \mathrm{For}.$$

We define a monoidal adjunction between these categories.

DEFINITION/PROPOSITION 1.16 (Induction functor (see [Ler19, Def. 4.1]))
We define the *induction functor* $\mathrm{Ind} : \mathfrak{S}\text{-}\mathsf{mod}_k^{\mathrm{red}} \to \mathfrak{S}\text{-}\mathsf{bimod}_k^{\mathrm{red}}$ which is given, for all reduced $\mathfrak{S}$-modules V and, for all finite sets S and E, by:

$$\left(\mathrm{Ind}\, V\right)(S, E) \cong \begin{cases} 0 & \text{if } S \not\cong E \\ k[\mathrm{Aut}(E)] \otimes V(S) & \text{otherwise.} \end{cases}$$

This functor is exact, has a right adjoint which is the *functor of restriction* Res, and is monoidal. Hence, that induces the functor

$$\mathrm{Ind} : \mathsf{protoperads} \longrightarrow \mathsf{properads}.$$

Moreover, the induction functor commutes with the free monoid constructions, formally by adjunction, i.e., we have the natural isomorphism of reduced $\mathfrak{S}$-bimodules:

$$\mathrm{Ind}(\mathscr{F}(-)) \cong \mathscr{F}^{\mathrm{Val}}(\mathrm{Ind}(-)).$$

Then, for a protoperad $\mathcal{P}$ defined by generators and relations, i.e., $\mathcal{P} = \mathscr{F}(V)/\langle R \rangle$, the properad $\mathrm{Ind}(\mathcal{P})$ is given by

$$\mathrm{Ind}(\mathcal{P}) \cong \mathscr{F}^{\mathrm{Val}}(\mathrm{Ind}\, V)/\langle \mathrm{Ind}\, R \rangle.$$

The functor of induction also commutes with the cofree conilpotent comonoids constructions:

$$\mathrm{Ind}(\mathscr{F}^c(-)) \cong \mathscr{F}^{c,\mathrm{Val}}(\mathrm{Ind}(-)),$$

because the underlying functor of $\mathscr{F}^c(-)$ (resp. $\mathscr{F}^{c,\mathrm{Val}}$) is the same of the one of $\mathscr{F}(-)$ (resp. $\mathscr{F}^{\mathrm{Val}}$), and Ind respects the coproduct, because it is monoidal.

REMARK 1.17. — In the rest of this paper, we use the same name for the induction functor from $\mathfrak{S}\text{-}\mathsf{mod}_k^{\mathrm{red}}$ to $\mathfrak{S}\text{-}\mathsf{bimod}_k^{\mathrm{red}}$, and the one from $\mathsf{protoperads}$ to $\mathsf{properads}$.

1.3. Shuffle protoperad. — Here, we introduce *shuffle protoperads*. This notion is very similar to the notion of protoperad, without the actions of the symmetric groups; these notions are related by the functor defines in Proposition 1.24. In the operadic framework, shuffle operads are very useful to study Kozsulness. That permits to define the notion of PBW-basis or Gröbner basis (see [Hof10, DK10] or [LV12, Chap. 8] for definitions). Recall the PBW theorem: if a quadratic operad $\mathcal{O}$ has a PWB basis, then $\mathcal{O}$ is Koszul. As the author shows in [Ler17], the proof of a similar PBW theorem does not hold directly in the protoperadic case. In fact, the analogous of the spectral sequence appearing in [Hof10, §4] does not collapse in page 1, because the underlying combinatorial of protoperads is more complicated. Nevertheless, the notion of shuffle protoperad still holds some appeal, as we will see (see Remark 3.8).

We denote Ord, the category of totally ordered finite sets, with bijections. Analogously to [Ler19, §1.2], we define the combinatorial functors, $\mathcal{Y}^{\mathrm{sh}} : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$ and $\mathcal{X}^{\mathrm{sh}} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Fin}^{\mathrm{op}}$ which encode *shuffle protoperads*. The shuffle framework corresponds to choosing a representative for each wall. We define the functor $\mathcal{Y}^{\mathrm{sh}}$ as follows: for all finite, totally ordered sets S, we set

$$\mathcal{Y}^{\mathrm{sh}}_R(S) := \left\{ I = (I_j)_{j \in [\![1,R]\!]} \;\middle|\; \begin{array}{c} \forall r \in [\![1,R]\!],\ I_r \neq \varnothing;\ \bigcup_{r \in [\![1,R]\!]} I_r = S \\ \forall r \neq s \in [\![1,R]\!], I_r \cap I_s = \varnothing \\ \min(I_1) < \min(I_2) < \cdots < \min(I_R) \end{array} \right\}.$$

and $\mathcal{Y}^{\mathrm{sh}}(S) := \coprod_{r \in \mathbb{N}^*} \mathcal{Y}^{\mathrm{sh}}_r(S)$; we also have

$$\mathcal{X}^{\mathrm{sh}}_r(S) := \coprod_{i+j=r} \mathcal{Y}^{\mathrm{sh}}_i(S) \times \mathcal{Y}^{\mathrm{sh}}_j(S) \quad \text{and} \quad \mathcal{X}^{\mathrm{sh}}(S) := \coprod_{r \in \mathbb{N}^*} \mathcal{X}^{\mathrm{sh}}_r(S).$$

We have the natural isomorphism of functors $\mathcal{X}^{\mathrm{sh}} \cong \mathcal{Y}^{\mathrm{sh}} \times \mathcal{Y}^{\mathrm{sh}}$. As in the unshuffle case (see Definition/Proposition 1.9), using the functor $\mathcal{Y}^{\mathrm{sh}}$, one can define the following functor.

Definition 1.18 (The functor $\mathbb{S}^{\mathrm{sh}}$). — Let $V : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Ch}_k$ be a functor. We denote by $\mathbb{S}^{\mathrm{sh}}(V)$, the functor $\mathbb{S}^{\mathrm{sh}}(V) : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Ch}_k$, given, for a totally ordered set S, by

$$\mathbb{S}^{\mathrm{sh}}(V)(S) := \bigoplus_{(I_j)_{1 \leqslant j \leqslant R} \in \mathcal{Y}^{\mathrm{sh}}(S)} \bigotimes_{j=1}^{R} V(I_j).$$

The functors $\mathcal{Y}^{\mathrm{sh}}$ and $\mathcal{X}^{\mathrm{sh}}$ are compatible with their unshuffled analoguous through the forgetful functor $\underline{(-)} : \mathsf{Ord} \to \mathsf{Fin}$.

Lemma 1.19. — *We have the following commutative diagrams of functors up to natural isomorphisms*
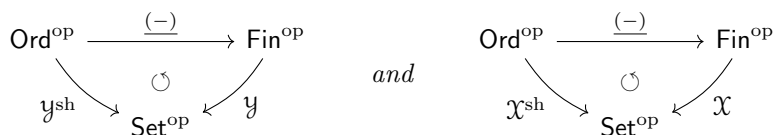
where $\mathcal{X}$ and $\mathcal{Y}$ are the functors defined in Section 1.1 (see also [Ler19, §1.2]).

We will define the shuffle-analoguous of the functor $\mathcal{X}^{\mathrm{conn}}$.

DEFINITION 1.20 (Projection $\mathcal{K}^{\mathrm{sh}}$). — We define the projection $\mathcal{K}^{\mathrm{sh}}$ as follows: for a totally ordered finite set $(S, <_S)$, we have

$$\mathcal{K}^{\mathrm{sh}}_S : \mathcal{X}^{\mathrm{sh}}(S) \longrightarrow \mathcal{Y}^{\mathrm{sh}}(S)$$
$$W \longmapsto \left\{ \textstyle\bigcup_{B_\alpha \in \pi^{-1}([B])} B_\alpha \mid [B] \in \pi(W) \right\},$$

where π is the projection of W to its quotient by $\overset{\mathrm{conn.}}{\sim}$ (cf. [Ler19, §1.4]), and the set $\mathcal{K}^{\mathrm{sh}}_S(W)$ is totally ordered by the order $<$ defined as follows: for B_α and B_β in $\mathcal{K}^{\mathrm{sh}}_S(W)$, $B_\alpha < B_\beta$ if $\min(B_\alpha) <_S \min(B_\beta)$.

As for $\mathcal{K}$ (cf. [Ler19, Lem. 1.10]), the product $\mathcal{K}^{\mathrm{sh}}$ on $\mathcal{Y}^{\mathrm{sh}}$ is associative. Let M and S be two totally ordered finite sets. Every monotone injection $j : M \hookrightarrow S$ induces a morphism

$$\iota_j : \mathcal{Y}^{\mathrm{sh}}(M) \longrightarrow \mathcal{Y}^{\mathrm{sh}}(S)$$

such that, for all $W = \{W_a\}_{a \in [\![1,r]\!]}$ in $\mathcal{Y}^{\mathrm{sh}}_r(M)$,

$$\iota_j(W) = \left\{ j(W_a) \right\}_{a \in [\![1,r]\!]} \amalg \coprod_{s \in S \smallsetminus j(M)} \{s\}.$$

Let M, N and S be three totally ordered finite sets and φ be the diagram of monotone injections $\varphi := \big(i : M \hookrightarrow S \hookleftarrow N : j \big)$ such that

$$\begin{cases} \operatorname{im}(i) \cup \operatorname{im}(j) = S \\ \operatorname{im}(i) \cap \operatorname{im}(j) \neq \varnothing, \end{cases}$$

then, we have the product $\mu_\varphi : \mathcal{Y}^{\mathrm{sh}}(M) \times \mathcal{Y}^{\mathrm{sh}}(N) \to \mathcal{Y}^{\mathrm{sh}}(S)$, given by the union of the images by i and j of the partitions of M and N, extended by singletons, i.e., defined by the following composition

$$\begin{array}{ccc} \mathcal{Y}^{\mathrm{sh}}(M) \times \mathcal{Y}^{\mathrm{sh}}(N) & \xrightarrow{\iota_i \times \iota_j} & \mathcal{Y}^{\mathrm{sh}}(S) \times \mathcal{Y}^{\mathrm{sh}}(S) \\ & & \Big\downarrow{\mathcal{K}^{\mathrm{sh}}_S} \\ \mu_\varphi := & & \mathcal{Y}^{\mathrm{sh}}(S) \end{array} \quad .$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X}^{\mathrm{sh}}(M) \times \mathcal{X}^{\mathrm{sh}}(N) & \xrightarrow{\mathcal{K}^{\mathrm{sh}}_M \times \mathcal{K}^{\mathrm{sh}}_N} & \mathcal{Y}^{\mathrm{sh}}(M) \times \mathcal{Y}^{\mathrm{sh}}(N) \\ \mu_\varphi^{\times 2} \Big\downarrow & & \Big\downarrow \mu_\varphi \\ \mathcal{X}^{\mathrm{sh}}(S) & \xrightarrow{\mathcal{K}^{\mathrm{sh}}_S} & \mathcal{Y}^{\mathrm{sh}}(S) \end{array} \quad .$$

Finally, we define the functor $\mathcal{X}^{\mathrm{conn,sh}} : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Fin}^{\mathrm{op}}$, for all totally ordered finite sets S, by

$$\mathcal{X}^{\mathrm{conn,sh}}(S) := \left\{ (I, J) \in \mathcal{X}^{\mathrm{sh}}(S) \mid \mathcal{K}^{\mathrm{sh}}_S(I, J) = \{S\} \right\}.$$

Definition/Proposition 1.21 (Connected shuffle product). — The *connected shuffle product* is the bifunctor

$$- \boxtimes_c^{\sqcup} - : \mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k) \times \mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k) \longrightarrow \mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k)$$

defined, for two objects P and Q of $\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k)$ and a finite totally ordered S, by

$$\left(P \boxtimes_c^{\sqcup} Q\right)(S) := \bigotimes_{(K,L) \in \mathfrak{X}^{\mathrm{conn,sh}}(S)} \bigotimes_{i=1}^m P(K_i) \otimes \bigotimes_{j=1}^n Q(L_j).$$

Proposition 1.22. — *The product $\boxtimes_c^{\sqcup}$ is associative. Also, for all objects A and B in the category $\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k)$, the endofuncteur*

$$\Phi_{A,B} : \mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k) \longrightarrow \mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k)$$
$$X \longmapsto A \boxtimes_c^{\sqcup} X \boxtimes_c^{\sqcup} B$$

is split analytic in the sense of [Val07, Val09]. *The category*

$$\left(\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k), \boxtimes_c^{\sqcup}, I_{\boxtimes}\right)$$

is an abelian (symmetric) monoidal category and the monoidal product preserves reflexive coequalizors and sequential colimits.

Proof. — Similar to the proof of [Ler19, Lem. 2.9] and [Ler19, Prop. 2.10].   □

Definition 1.23 (Shuffle protoperad). — The monoids of $\left(\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k), \boxtimes_c^{\sqcup}, I_{\boxtimes}\right)$ are called *shuffle protoperads*, and we denote $\mathsf{protoperads}^{\mathrm{sh}}$, the category of shuffle protoperads.

The forgetful functor $\underline{(-)} : \mathsf{Ord} \to \mathsf{Fin}$ induces the functor $(-)^{\mathrm{sh}}$ from $\mathfrak{S}\text{-}\mathsf{mod}_k^{\mathrm{red}}$ to $\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k)$.

Proposition 1.24. — *The functor*

$$(-)^{\mathrm{sh}} : \left(\mathfrak{S}\text{-}\mathsf{mod}_k^{\mathrm{red}}, \boxtimes_c, I_{\boxtimes}\right) \longrightarrow \left(\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k), \boxtimes_c^{\sqcup} I_{\boxtimes}\right)$$

is (strongly) monoidal. It induces the functor

$$(-)^{\mathrm{sh}} : \mathsf{protoperad}_k \longrightarrow \mathsf{protoperad}_k^{\mathrm{sh}}.$$

Proof. — Let S be a totally ordered finite set and P and Q be two reduced $\mathfrak{S}$-modules. We have the following isomorphisms:

$$\left(P \boxtimes_c Q\right)^{\mathrm{sh}}(S) = \left(P \boxtimes_c Q\right)(\underline{S}) \cong \bigoplus_{\substack{(K,L) \\ \in \mathfrak{X}^{\mathrm{conn}}(\underline{S})}} \bigotimes_{\alpha \in A} P(K_\alpha) \otimes \bigotimes_{\beta \in B} Q(L_\beta)$$

$$\cong \bigoplus_{\substack{(\widetilde{K}, \widetilde{L}) \\ \in \mathfrak{X}^{\mathrm{conn,sh}}(S)}} P(\widetilde{K}_1) \otimes \cdots \otimes P(\widetilde{K}_m) \otimes Q(\widetilde{L}_1) \otimes \cdots \otimes Q(\widetilde{L}_n)$$

$$\cong \left(P^{\mathrm{sh}} \boxtimes_c^{\sqcup} Q^{\mathrm{sh}}\right)(S). \qquad\qquad \square$$

As for the case of protoperads, we have a combinatorial description of the free shuffle protoperad.

Definition/Proposition 1.25. — Let V be a functor in $\mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k)$. The *free shuffle protoperad* $\mathscr{F}_{\sqcup\!\sqcup}(V)$ is given, for all totally ordered sets S, by:

$$\mathscr{F}_{\sqcup\!\sqcup}(V)(S) \coloneqq \bigoplus_{W \in \mathcal{W}^{\mathrm{conn}}(\underline{S})} \bigotimes_{\alpha \in A} V(W_\alpha).$$

The functor $\mathscr{F}_{\sqcup\!\sqcup}$ is the left adjoint to the forgetful functor

$$\mathscr{F}_{\sqcup\!\sqcup} \colon \mathrm{Func}(\mathsf{Ord}^{\mathrm{op}}, \mathsf{Ch}_k) \xrightarrow{\;\longleftarrow\;} \mathsf{protoperads}_k^{\mathrm{sh}} \colon \mathrm{For}.$$

By Proposition 1.24, we have the following.

Corollary 1.26. — *Let V be a reduced $\mathfrak{S}$-module. There is a natural isomorphism of shuffle protoperads*

$$\mathscr{F}_{\sqcup\!\sqcup}(V^{\mathrm{sh}}) \cong \left(\mathscr{F}(V)\right)^{\mathrm{sh}}.$$

Also, for $\langle R \rangle \subset \mathscr{F}(V)$ an ideal, $\langle R \rangle^{\mathrm{sh}}$ is an ideal of $\mathscr{F}_{\sqcup\!\sqcup}(V^{\mathrm{sh}})$, and there is a natural isomorphism of shuffle protoperads

$$\mathscr{F}_{\sqcup\!\sqcup}(V^{\mathrm{sh}})/\langle R \rangle^{\mathrm{sh}} \cong \left(\mathscr{F}(V)/\langle R \rangle\right)^{\mathrm{sh}}.$$

Moreover, if $\mathcal{P}$ is a weight graded protoperad, then so is the shuffle protoperad $\mathcal{P}^{\mathrm{sh}}$.

2. Koszul duality of protoperads

In this section, we adapt the constructions of [MV09a, §3] and [Val03, Val07] for properads to the protoperadic framework.

2.1. (Co)Augmentation, infinitesimal (co)bimodule and (co)derivation. —

Definition 2.1 (Augmented protoperad). — An *augmentation* of a protoperad $\mathcal{P}$ is a morphism of protoperads $\varepsilon \colon \mathcal{P} \to I_{\boxtimes}$, where $I_{\boxtimes}$ is the unit of the product $\boxtimes_c$. A protoperad with an augmentation is called *augmented*. We denote by $\mathsf{protoperads}_k^{\mathrm{aug}}$, the category of augmented protoperads. To an augmented protoperad $(\mathcal{P}, \varepsilon)$, we associate its *augmentation ideal* $\overline{\mathcal{P}}$, defined as the kernel of the augmentation ε, i.e., $\overline{\mathcal{P}} \coloneqq \mathrm{Ker}(\varepsilon)$.

For two reduces $\mathfrak{S}$-modules M and P, the $\mathfrak{S}$-module $P \boxtimes_c (P \oplus M)$ has a weight-grading, which we denote

$$P \boxtimes_c (P \oplus M) = \bigoplus_{r \in \mathbb{N}} (P \boxtimes_c (P \oplus M))^{(r)_M}.$$

Let $(\mathcal{P}, \varepsilon)$ be an augmented protoperad. Then, we have the isomorphism of reduced $\mathfrak{S}$-modules $\mathcal{P} \cong I_{\boxtimes} \oplus \overline{\mathcal{P}}$. Moreover, by the bigrading given by [Ler19, Lem. 5.16], we can decompose the connected composition product

$$\mu = \bigoplus_{(r,s) \in (\mathbb{N}^*)^2} \mu^{(r,s)} \text{ by } \mu^{(r,s)} \colon \left((I_{\boxtimes} \oplus \overline{\mathcal{P}}) \boxtimes_c (I_{\boxtimes} \oplus \overline{\mathcal{P}})\right)^{(r,s)_{\overline{\mathcal{P}}}} \longrightarrow \overline{\mathcal{P}}.$$

Definition 2.2 (Partial composition product). — Let $(\mathcal{P}, \varepsilon)$ be an augmented protoperad. The *partial composition product* is the restriction of the product $\mu : \mathcal{P} \boxtimes_c \mathcal{P} \to \mathcal{P}$ to

$$\mu^{(1,1)} : \left((I_\boxtimes \oplus \overline{\mathcal{P}}) \boxtimes_c (I_\boxtimes \oplus \overline{\mathcal{P}}) \right)^{(1,1)_{\overline{\mathcal{P}}}} \longrightarrow \overline{\mathcal{P}}.$$

Using the partial composition, we introduce the notion of an infinitesimal bimodule over a protoperad.

Definition 2.3 (Infinitesimal bimodule). — Let $(\mathcal{P}, \mu)$ be a protoperad. A $\mathfrak{S}$-module M is a $\mathcal{P}$-*infinitesimal bimodule* if M has two morphisms of $\mathfrak{S}$-modules, respectively called the *left and right actions*:

$$\lambda : \left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M) \right)^{(1)_M} \longrightarrow M \quad \text{and} \quad \rho : \left((\mathcal{P} \oplus M) \boxtimes_c \mathcal{P} \right)^{(1)_M} \longrightarrow M$$

such that the following compatibility diagrams commute:

(1) associativity of the left action λ:

$$
\begin{array}{ccc}
\left(\mathcal{P} \boxtimes_c \mathcal{P} \boxtimes_c (\mathcal{P} \oplus M) \right)^{(1)_M} & \xrightarrow{\mathcal{P} \boxtimes_c (\lambda + \mu)} & \left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M) \right)^{(1)_M} \\
{\scriptstyle \mu \boxtimes_c (\mathcal{P} \oplus M)} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \lambda} \\
\left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M) \right)^{(1)_M} & \xrightarrow{\quad\quad \lambda \quad\quad} & M
\end{array}
$$

(2) associativity of the right action ρ:

$$
\begin{array}{ccc}
\left((\mathcal{P} \oplus M) \boxtimes_c \mathcal{P} \boxtimes_c \mathcal{P} \right)^{(1)_M} & \xrightarrow{(\rho + \mu) \boxtimes_c \mathcal{P}} & \left((\mathcal{P} \oplus M) \boxtimes_c \mathcal{P} \right)^{(1)_M} \\
{\scriptstyle (\mathcal{P} \oplus M) \boxtimes_c \mu} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \rho} \\
\left((\mathcal{P} \oplus M) \boxtimes_c \mathcal{P} \right)^{(1)_M} & \xrightarrow{\quad\quad \rho \quad\quad} & M
\end{array}
$$

(3) the left and right actions commute:

$$
\begin{array}{ccc}
\left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M) \boxtimes_c \mathcal{P} \right)^{(1)_M} & \xrightarrow{(\lambda + \mu) \boxtimes_c \mathcal{P}} & \left((\mathcal{P} \oplus M) \boxtimes_c \mathcal{P} \right)^{(1)_M} \\
{\scriptstyle \mathcal{P} \boxtimes_c (\rho + \mu)} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \rho} \\
\left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M) \right)^{(1)_M} & \xrightarrow{\quad\quad \lambda \quad\quad} & M
\end{array}
$$

Remark 2.4. — We also have the dual definitions of co-augmented coprotoperad, partial coproduct and infinitesimal cobimodule.(see [MV09a] for properadic definition).

Remark 2.5. — For an augmented protoperad $(\mathcal{P}, \mu, \varepsilon : \mathcal{P} \to I_\boxtimes)$, the following definition is equivalent to the data of two actions $\lambda' : (\mathcal{P} \boxtimes_c M_+)^{(1)_{\overline{\mathcal{P}}}, (1)_M} \to M$ and $\rho' : (M_+ \boxtimes_c \mathcal{P})^{(1)_M, (1)_{\overline{\mathcal{P}}}} \to M$ where $M_+ = M \oplus I_\boxtimes$, compatible with the product

of the protoperad $\mathcal{P}$. In fact, if we consider the left action on M, then the injection $I_{\boxtimes} \hookrightarrow \mathcal{P}$ induces the following morphism of $\mathfrak{S}$-modules

$$
\begin{array}{ccc}
\left(\mathcal{P} \boxtimes_c M_+\right)^{(1)_{\overline{\mathcal{P}}}(1)_M} & & \\
\downarrow & \searrow^{\lambda'} & \\
\left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M)\right)^{(1)_M} & \xrightarrow{\ \lambda\ } & M
\end{array}
$$

compatible with the product μ of $\mathcal{P}$. Conversely, if we consider a $\mathfrak{S}$-module M with a morphism

$$
\lambda' : \left(\mathcal{P} \boxtimes_c (M_+)\right)^{(1)_{\overline{\mathcal{P}}}(1)_M} \longrightarrow M
$$

compatible with the product μ of $\mathcal{P}$, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
\left(\mathcal{P} \boxtimes_c \mathcal{P} \boxtimes_c M_+\right)^{(1)_{\overline{\mathcal{P}}},(1)_{\overline{\mathcal{P}}},(1)_M} & \xrightarrow{\ \mathcal{P} \boxtimes_c \left(\lambda' + \mu^{(1,1)}\right)\ } & \left(\mathcal{P} \boxtimes_c M_+\right)^{(1)_{\overline{\mathcal{P}}},(1)_M} \\
{\scriptstyle \mu^{(1,1)} \boxtimes_c M_+} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \lambda'} \\
\left(\mathcal{P} \boxtimes_c M_+\right)^{(1)_{\overline{\mathcal{P}}},(1)_M} & \xrightarrow{\hspace{3cm} \lambda' \hspace{3cm}} & M
\end{array}
$$

This compatibility and associativity of the product μ allows the extension of the morphism λ' to a morphism $\lambda : \left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M)\right)^{(1)_M} \to M$, which is the expected morphism. We have a similar equivalence for ρ and ρ'.

For the definition of infinitesimal bimodule in the properadic case, which is similar to the protoperadic case, the reader can refer to [MV09a].

Lemma 2.6. — *Let $\mathcal{P}$ be a protoperad and M be an infinitesimal $\mathcal{P}$-bimodule. The $\mathfrak{S}$-bimodule $\mathrm{Ind}(M)$ is an infinitesimal $\mathrm{Ind}(\mathcal{P})$-bimodule.*

Proof. — The functor Ind is monoidal for the products $\boxtimes_c$ and $\otimes^{\mathrm{conc}}$ (see [Ler19, Prop. 4.7, Th. 4.16]) and is additive, i.e., $\mathrm{Ind}(V \oplus W) \cong \mathrm{Ind}(V) \oplus \mathrm{Ind}(W)$, so preserves the weight grading:

$$
\mathrm{Ind}\left(\left(\mathcal{P} \boxtimes_c (\mathcal{P} \oplus M)\right)^{(1)_M}\right) \cong \left(\mathrm{Ind}\,\mathcal{P} \boxtimes_c (\mathrm{Ind}\,\mathcal{P} \oplus \mathrm{Ind}\,M)\right)^{(1)_{\mathrm{Ind}\,M}}. \qquad \square
$$

Definition 2.7 (Derivation, coderivation). — Let $(\mathcal{P}, \varepsilon)$ be an augmented protoperad and (M, λ, ρ) be an infinitesimal $\mathcal{P}$-bimodule. A morphism of $\mathfrak{S}$-modules $d : \mathcal{P} \to M$ of homological degree n is called a *homogeneous derivation* if the following diagram commutes:

$$
\begin{array}{ccc}
\left(\mathcal{P} \boxtimes_c \mathcal{P}\right)^{(1,1)} & \xrightarrow{\ \mu^{(1,1)}\ } & \mathcal{P} \\
{\scriptstyle d \boxtimes_c \mathcal{P} + \mathcal{P} \boxtimes_c d} \downarrow & & \downarrow {\scriptstyle d} \\
M \boxtimes_c \mathcal{P} \oplus \mathcal{P} \boxtimes_c M & \xrightarrow{\ \rho + \lambda\ } & M
\end{array}
$$

i.e., for all p and q in $\mathcal{P}$:

$$
d \circ \mu^{(1,1)}(p,q) = \rho(d(p),q) + (-1)^{n|p|}\lambda(p,d(q)).
$$

We denote $\underline{\mathrm{Der}}_n(\mathcal{P}, M)$, the k-module of derivations from $\mathcal{P}$ to M of homological degree n and the derivation complex is denoted by $\underline{\mathrm{Der}}_\bullet(\mathcal{P}, M)$, with the differential $[\partial, -]$ defined, for δ in $\underline{\mathrm{Der}}_n(\mathcal{P}, N)$, by $[\partial, \delta] := \partial_N \circ \delta - (-1)^{|n|} \delta \circ \partial_\mathcal{P}$.

Let $(\mathcal{C}, \nu)$ be a coaugmented coprotoperad and (N, λ, ρ), an infinitesimal $\mathcal{C}$-cobimodule. A morphism of $\mathfrak{S}$-modules $d : N \to \mathcal{C}$ of homological degree n is a *homogeneous coderivation* if the following diagram commutes:

$$
\begin{array}{ccc}
N & \xrightarrow{\quad\quad d \quad\quad} & \mathcal{C} \\
{\scriptstyle \lambda + \rho}\Big\downarrow & & \Big\downarrow{\scriptstyle \Delta^{(1,1)}} \\
(N \boxtimes_c \mathcal{C}) \oplus (\mathcal{C} \boxtimes_c N) & \xrightarrow{\ d \boxtimes_c \mathcal{C} + \mathcal{C} \boxtimes_c d\ } & \mathcal{C} \boxtimes_c \mathcal{C}
\end{array}
$$

We denote $\underline{\mathrm{Coder}}_n(\mathcal{C}, N)$, the k-module of homogeneous coderivations from $\mathcal{C}$ to N of degree n and $\underline{\mathrm{Coder}}_\bullet(\mathcal{C}, N)$, the coderivation complex.

PROPOSITION 2.8. — *Let $(\mathcal{P}, \varepsilon)$ be an augmented protoperad, $(\mathcal{C}, \nu)$ be a coaugmented coprotoperad, M be an infinitesimal $\mathcal{P}$-bimodule and N be an infinitesimal $\mathcal{C}$-cobimodule. We have the following natural isomorphisms:*

$$\mathrm{Ind}\big(\underline{\mathrm{Der}}^\bullet(\mathcal{P}, M)\big) \cong \underline{\mathrm{Der}}^\bullet\big(\mathrm{Ind}(\mathcal{P}), \mathrm{Ind}(M)\big);$$
$$\mathrm{Ind}\big(\underline{\mathrm{Coder}}^\bullet(\mathcal{C}, N)\big) \cong \underline{\mathrm{Coder}}^\bullet\big(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(N)\big).$$

Proof. — The functor Ind is additive monoidal and respects the grading on M (see Definition/Proposition 1.16, or [Ler19, Th. 4.16]). □

LEMMA 2.9. — *Let $\mathscr{F}(V)$ be the free protoperad on the $\mathfrak{S}$-module V. For a homogeneous morphism $\theta : V \to \mathscr{F}(V)$ of degree $|\theta|$, there exists a unique homogeneous derivation $d_\theta : \mathscr{F}(V) \to \mathscr{F}(V)$ of degree $|\theta|$, such that its restriction to V is θ: we have*

$$\underline{\mathrm{Der}}_{|\theta|}(\mathscr{F}(V), \mathscr{F}(V)) \cong \mathrm{Hom}_{|\theta|}(V, \mathscr{F}(V)).$$

Moreover, if $\theta(V) \subset \mathscr{F}^{(\rho)}(V)$ then we have $d_\theta(\mathscr{F}^{(s)}(V)) \subset \mathscr{F}^{(s+\rho-1)}(V)$.

Proof. — Let $\bigotimes_{j=1}^n (v_1^j \otimes \cdots \otimes v_{r_j}^j)$ be a representative of a class of $V_n = (V \oplus I_{\boxtimes})^{\boxtimes_c n}$ with each v_λ^j in $V \oplus I_{\boxtimes}$. We define the application d_θ by

$$
d_\theta\big(\textstyle\bigotimes_{j=1}^n (v_1^j \otimes \cdots \otimes v_{r_j}^j)\big)
$$
$$
:= \sum_{\substack{s \in [\![1,n]\!] \\ i \in [\![1,r_s]\!]}} \Big[(-1)^{\lambda_{s,i}} \bigotimes_{j=1}^{s-1} (v_1^j \otimes \cdots \otimes v_{r_j}^j) \otimes (v_1^s \otimes \cdots \otimes \theta(v_i^s) \otimes \cdots \otimes v_{r_s}^s)
$$
$$
\otimes \bigotimes_{j=s+1}^n (v_1^j \otimes \cdots \otimes v_{r_j}^j) \Big],
$$

where $\lambda_{s,i} = \big(\sum_{j=1}^{s-1} \sum_{l=1}^{r_j} |v_l^j| + |v_1^s| + \cdots + |v_{i-1}^s|\big)|\theta|$ and where we extend θ to $V \oplus I_{\boxtimes}$ by $\theta(I_{\boxtimes}) = 0$. The morphism d_θ is constant on the equivalence class of $\bigotimes_{j=1}^n (v_1^j \otimes \cdots \otimes v_{r_j}^j)$. We just need to verify that for $n = 1$ and for the transposition

which sends v_j to v_{j+1}:

$$d_\theta\big(v_1 \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes v_r\big) =$$

$$= \sum_{i \in [\![1,j-1]\!]} (-1)^{|\theta| \sum_{l=1}^{i-1} |v_l|} v_1 \otimes \cdots \otimes \theta(v_i) \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes v_r$$

$$+ (-1)^{|\theta| \sum_{l=1}^{j-1} |v_l|} v_1 \otimes \cdots \otimes \theta(v_{j+1}) \otimes v_j \otimes \cdots \otimes v_r$$

$$+ (-1)^{|\theta| \sum_{l=1}^{j-1} |v_l| + |\theta||v_{j+1}|} v_1 \otimes \cdots \otimes v_{j+1} \otimes \theta(v_j) \otimes \cdots \otimes v_r$$

$$+ \sum_{i \in [\![j+2,r]\!]} (-1)^{|\theta| \sum_{l=1}^{i-1} |v_l|} v_1 \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes \theta(v_i) \cdots \otimes v_r$$

$$\sim (-1)^{|v_j||v_{j+1}|} \Big(\sum_{i \in [\![1,j-1]\!]} (-1)^{|\theta| \sum_{l=1}^{i-1} |v_l|} v_1 \otimes \cdots \otimes \theta(v_i) \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_r$$

$$+ (-1)^{|\theta| \sum_{l=1}^{j-1} |v_l| + |\theta||v_j|} v_1 \otimes \cdots \otimes v_j \otimes \theta(v_{j+1}) \otimes \cdots \otimes v_r$$

$$+ (-1)^{|\theta| \sum_{l=1}^{j-1} |v_l|} v_1 \otimes \cdots \otimes \theta(v_j) \otimes v_{j+1} \otimes \cdots \otimes v_r$$

$$+ \sum_{i \in [\![j+2,r]\!]} (-1)^{|\theta| \sum_{l=1}^{i-1} |v_l|} v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes \theta(v_i) \cdots \otimes v_r \Big)$$

$$= (-1)^{|v_j||v_{j+1}|} d_\theta\big(v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_r\big).$$

Moreover, d_θ factorizes through $\widetilde{V}_n$ (see [Ler19, §5.2] for the definition); similarly, we show that, on the elements of the form

$$\overline{v} := (v_1 \otimes v_2 \otimes v_3) \otimes (v_1' \otimes 1 \otimes v_3') - (-1)^{|v_2|(|v_3| + |v_1'|)} (v_1 \otimes 1 \otimes v_3) \otimes (v_1' \otimes v_2 \otimes v_3'),$$

we have $d_\theta(\overline{v}) = 0$. Hence, we use the same arguments that the properadic case (cf. [Val03, Lem. 87]): the surjectivity of the product of the free protoperad $\mathscr{F}(V)$ gives us the uniqueness of the derivation d_θ and that all derivation are as above. $\square$

Dually we have the following lemma (which is the protoperadic analogue of [Val03, Lem. 88]).

Lemma 2.10. — *Let $\mathscr{F}^c(V)$ be the connected cofree coprotoperad on the $\mathfrak{S}$-module V. For all homogeneous morphisms of $\mathfrak{S}$-module $\theta : \mathscr{F}^c(V) \to V$ of homological degree $|\theta|$, there exists a unique homogeneous coderivation with the same degree $d_\theta : \mathscr{F}^c(V) \to \mathscr{F}^c(V)$ such that the composition*

$$\mathscr{F}^c(V) \xrightarrow{\ d_\theta\ } \mathscr{F}^c(V) \xrightarrow{\ \mathrm{proj}\ } V$$

is equal to θ. This correspondence is bijective; moreover, if θ is null on each weight component $\mathscr{F}^{c,(s)}(V)$ for $s \neq r$, then $d_\theta(\mathscr{F}^{c,(s+r-1)}(V)) \subset \mathscr{F}^{c,(s)}(V)$, for $s > 0$.

Definition 2.11 (Quasi-free protoperad/ quasi-cofree coprotoperad)
A protoperad $(\mathscr{F}(V), \partial = \partial_V + d_\theta)$ (resp. coprotoperad $(\mathscr{F}^c(V), \partial = \partial_V + d_\theta)$) is called *quasi-free* (resp. quasi-cofree).

Proposition 2.12. — *The projection $\mathscr{F}(V) \to V$ of a quasi-free protoperad on to its indecomposables is a morphism of $\mathfrak{S}$-modules if and only if $\theta(V) \subset \bigoplus_{r \geqslant 2} \mathscr{F}^{(r)}(V)$.*

Proof. — The proof is similar to the properadic one: see [Val03, Prop. 89].   □

2.2. Bar-cobar adjunction. — We introduce the bar construction of a protoperad. We denote by s the generator of the $\mathfrak{S}$-module Σ, the suspension (see Definition 1.11). Let $(\mathcal{P}, \mu, \varepsilon)$ be an augmented protoperad. The partial product $\mu^{(1,1)}$ of $\mathcal{P}$ induces a homogeneous morphism of $\mathfrak{S}$-modules of homological degree -1:

$$s\mu_2 : \mathscr{F}^{c,(2)}(\Sigma\overline{\mathcal{P}}) \longrightarrow \Sigma\overline{\mathcal{P}}$$

given by

$$s\mu_2\big(s.p_1 \otimes s.p_2\big) := (-1)^{|p_1|} s.\mu^{(1,1)}\big(p_1 \otimes p_2\big).$$

By Lemma 2.10, we can associate to $s\mu_2$, a homogeneous coderivation

$$d_{s\mu_2} : \mathscr{F}^c(\Sigma\overline{\mathcal{P}}) \longrightarrow \mathscr{F}^c(\Sigma\overline{\mathcal{P}}),$$

of homological degree -1. We consider the coderivation

$$\partial := \partial_{\mathcal{P}} + d_{s\mu_2} : \mathscr{F}^c(\Sigma\overline{\mathcal{P}}) \longrightarrow \mathscr{F}^c(\Sigma\overline{\mathcal{P}})$$

with $\partial_{\mathcal{P}}$ the coderivation induced by the internal differential of $\mathcal{P}$. We show that $\partial^2 = 0$, which is equivalent to showing that $\partial_{\mathcal{P}} d_{s\mu_2} + d_{s\mu_2}\partial_{\mathcal{P}} + d_{s\mu_2}^2 = 0$, because $\partial_{\mathcal{P}}$ is a differential. By Proposition 2.8, $\mathrm{Ind}(d_{s\mu_2})$ is a coderivation of homological degree -1 (in the properadic sense). As the functor Ind commutes with the cofree conilpotent comonoid functor $\mathscr{F}^c(-)$, with the suspension and Ind is exact (so commutes with the functor $\overline{(-)}$), we have the following isomorphism

$$\mathrm{Ind}\big(\big(\mathscr{F}^c(\Sigma\overline{\mathcal{P}}), \partial_{\mathcal{P}} + d_{s\mu_2}\big)\big) \cong \big(\mathscr{F}^c(\Sigma\overline{\mathrm{Ind}(\mathcal{P})}), \partial_{\mathrm{Ind}(\mathcal{P})} + \mathrm{Ind}(d_{s\mu_2})\big).$$

As the coderivation $d_{s\mu_2}$ is the suspension of the partial product $\mu^{(1,1)}$, and the functor Ind is compatible with the weight-bigrading in $\mathcal{P}$ of $\mathcal{P} \boxtimes_c \mathcal{P}$ and commutes with the suspension, then $\mathrm{Ind}(d_{s\mu_2^{\mathcal{P}}})$ is equal to $d_{s\mu_2^{\mathrm{Ind}(\mathcal{P})}}$, the coderivation induced by the partial product of the properad $\mathrm{Ind}(\mathcal{P})$.

This lends to the definition of the bar construction of a protoperad.

Definition/Proposition 2.13 (Bar construction). — Let $(\mathcal{P}, \mu, \partial_{\mathcal{P}}, \varepsilon)$ be an augmented protoperad. The *bar construction of* $\mathcal{P}$ is the following quasi-cofree coaugmented coprotoperad:

$$\big(\mathsf{B}\mathcal{P}, \partial\big) := \big(\mathscr{F}^c(\Sigma\overline{\mathcal{P}}), \partial_{\mathcal{P}} + d_{s\mu_2}\big),$$

which gives the functor $\mathsf{B}: \mathsf{protoperads}_k^{\mathrm{aug}} \to \mathsf{coprotoperads}_k^{\mathrm{coaug}}$. Moreover, the respective bar constructions commute with the induction functor:

$$\mathrm{Ind}\big(\mathsf{B}(-)\big) \cong \mathsf{B}^{\mathrm{Val}}\big(\mathrm{Ind}(-)\big),$$

where the functor $\mathsf{B}^{\mathrm{Val}}$ is the bar construction for properads defined in [Val03, Val07].

Proposition 2.14. — *Let V be a reduced $\mathfrak{S}$-module concentrated in homological degree 0. Then the homology of the chain complex given by the bar construction of the free protoperad over V is acyclic, i.e.,*

$$\mathrm{H}_{\bullet}\big(\mathsf{B}\mathscr{F}(V),\partial\big) \cong \Sigma V,$$

where Σ is the shifting of homological degree one.

Proof. — For this proof, we use the notion of *coloring* of a wall W and the *coloring complex* associated to W, defined in [Ler19, §6]. Let S be a totally ordered finite set. We have the following isomorphisms of chain complexes:

$$\big(\mathsf{B}\mathscr{F}(V)\big)(S) \cong \bigoplus_{K\in\mathcal{W}^{\mathrm{conn}}(S)} \bigoplus_{\varphi\in\mathcal{C}\mathrm{ol}(K)} \Sigma^{\#\varphi(K)}\Big(\bigotimes_{\alpha\in A} V(K_{\alpha}),\varphi\Big)$$

where the differential on the right side acts on the coloring as in the coloring complex. So we have:

$$\begin{aligned}
\big(\mathsf{B}\mathscr{F}(V)\big)(S) &\cong \bigoplus_{K\in\mathcal{W}^{\mathrm{conn}}(S)} \bigoplus_{\varphi\in\mathcal{C}\mathrm{ol}(K)} \Sigma^{\#\varphi(K)}\Big(\bigotimes_{\alpha\in A} V(K_{\alpha}),\varphi\Big) \\
&\cong \Sigma V \oplus \bigoplus_{\substack{K\in\mathcal{W}^{\mathrm{conn}}(S)\\ \#K\geqslant 2}} \bigoplus_{\varphi\in\mathcal{C}\mathrm{ol}(K)} \Sigma^{\#\varphi(K)}\Big(\bigotimes_{\alpha\in A} V(K_{\alpha}),\varphi\Big) \\
&\cong \Sigma V \oplus \bigoplus_{\substack{K\in\mathcal{W}^{\mathrm{conn}}(S)\\ \#K\geqslant 2}} \big(C_{\bullet}^{\mathcal{C}\mathrm{ol}}(K)\big)^{\oplus(\#K)\cdot\dim_k V}
\end{aligned}$$

then, by [Ler19, Th. 6.15], $\mathsf{B}\mathscr{F}(V) \simeq \Sigma V$. $\qquad\square$

We also have the cobar construction.

Definition/Proposition 2.15 (Cobar construction). — Let $(\mathcal{C},\Delta,\partial_{\mathcal{C}},\nu)$ be a coaugmented coprotoperad. The *cobar construction of $\mathcal{C}$* is the following quasi-free augmented protoperad:

$$\big(\Omega\mathcal{C},\partial\big) := \big(\mathscr{F}(\Sigma^{-1}\overline{\mathcal{C}}),\partial_{\mathcal{C}}+d_{s^{-1}\Delta_2}\big),$$

which gives the functor $\Omega\colon \mathsf{coprotoperads}_k^{\mathrm{coaug}} \to \mathsf{protoperads}_k^{\mathrm{aug}}$. Moreover, the respective cobar constructions commute with the induction functor:

$$\mathrm{Ind}\big(\Omega(-)\big) \cong \Omega^{\mathrm{Val}}\big(\mathrm{Ind}(-)\big),$$

where the functor Ω^{Val} is the cobar construction for properads defined in [Val03, Val07].

By the exactness of the functor Ind, we directly have the adjunction between bar and cobar construction.

Proposition 2.16. — *The functors B and Ω form a pair of adjoint functors:*

$$\Omega\colon \mathsf{coprotoperads}_k^{\mathrm{coaug}} \rightleftarrows \mathsf{protoperads}_k^{\mathrm{aug}} \colon \mathsf{B}.$$

Proof. — By the properties of the functor Ind (see [Ler19, Props. 4.4 & 5.20]) and [MV09a, Prop. 17]. $\qquad\square$

2.3. Koszul duality. — The result of this section are inspired by [Val03, Chap. 7]: as the results are very similar, we try to use the same notation as in [Val03].

2.3.1. *Definition of the Koszul dual.* — Let $(\mathcal{P}, \mu, \varepsilon)$ be an augmented protoperad, with a weight grading, $\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}^{[n]}$. This grading induced a new one on the bar construction of $\mathcal{P}$:

$$\mathsf{B}_{(r)}\mathcal{P} = \bigoplus_{\rho \in \mathbb{N}} \mathsf{B}_{(r)}(\mathcal{P})^{[\rho]},$$

where $\mathsf{B}_{(r)}\mathcal{P} = \mathscr{F}^{(r)}(\Sigma\overline{\mathcal{P}})$ is the grading described in [Ler19, Th. 5.21]. We interpret r as the number of elements of $\mathcal{P}$ and ρ as the total weight induced by the weight of each element of $\mathcal{P}$. As the product μ of $\mathcal{P}$ respects the weight grading, $d_{s\mu_2}$ respects the induced grading on $\mathsf{B}\mathcal{P}$; so we have

$$d_{s\mu_2}\big(\mathsf{B}_{(r)}(\mathcal{P})^{[\rho]}\big) \subset \mathsf{B}_{(r-1)}(\mathcal{P})^{[\rho]}.$$

Thus we have the following lemma.

Lemma 2.17. — *Let $\mathcal{P}$ (respectively $\mathcal{C}$) be a weight-graded, connected, protoperad (resp. coprotoperad), i.e., $\mathcal{P}^{[0]} = I_{\boxtimes}$ (resp. $\mathcal{C}^{[0]} = I_{\boxtimes}$). Then we have:*

$$\mathsf{B}_{(\rho)}(\mathcal{P})^{[\rho]} = \mathscr{F}^{c,(\rho)}(\Sigma\overline{\mathcal{P}}^{[1]}) \quad and \quad \mathsf{B}_{(r)}(\mathcal{P})^{[\rho]} = 0 \ for \ r > \rho$$

$$\big(resp. \ \Omega^{(\rho)}(\mathcal{C})^{[\rho]} = \mathscr{F}^{(\rho)}(\Sigma^{-1}\overline{\mathcal{C}}^{[1]}) \quad and \quad \Omega^{(r)}(\mathcal{C})^{[\rho]} = 0 \ for \ r > \rho\big).$$

Proof. — The proof is similar to the properadic one, see [Val03, §7.1]. □

Definition 2.18 (Koszul dual). — Let $\mathcal{P}$ (respectively $\mathcal{C}$) be a weight-graded, connected protoperad (resp. coprotoperad). We define the *Koszul dual* of $\mathcal{P}$ (resp. of $\mathcal{C}$), denoted by $\mathcal{P}^{\mathsf{i}}$ (resp. $\mathcal{C}^{\mathsf{i}}$) by the weight-graded $\mathfrak{S}$-module:

$$\mathcal{P}^{\mathsf{i}[\rho]} := \mathsf{H}_{(\rho)}\big(\mathsf{B}_*(\mathcal{P})^{[\rho]}, d_{s\mu_2}\big) \qquad \big(resp. \ \mathcal{C}^{\mathsf{i}[\rho]} := \mathsf{H}_{(\rho)}\big(\Omega_*(\mathcal{C})^{[\rho]}, d_{s^{-1}\Delta_2}\big)\big).$$

Remark 2.19. — One can remark that a weight-graded connected pro(to)perad $\mathcal{P}$ is augmented, the augmentation given by the projection on the weight 0 which is isomorphic to $I_{\boxtimes}$. Similarly, a weight-graded connected copro(to)perad $\mathcal{C}$ is coaugmented.

By Lemma 2.17, we have the equalities:

$$\mathcal{P}^{\mathsf{i}[\rho]} = \mathrm{Ker}\big(d_{s\mu_2} : \mathsf{B}_{(\rho)}(\mathcal{P})^{(\rho)} \longrightarrow \mathsf{B}_{(\rho-1)}(\mathcal{P})^{(\rho)}\big)$$

and

$$\mathcal{C}^{\mathsf{i}[\rho]} = \mathrm{Coker}\big(d_{s^{-1}\Delta_2} : \Omega_{(\rho-1)}(\mathcal{C})^{(\rho)} \longrightarrow \Omega_{(\rho)}(\mathcal{C})^{(\rho)}\big).$$

Moreover, if the protoperad $\mathcal{P}$ is concentrated in homological degree 0, then we have

$$\big(\mathsf{B}_{(r)}(\mathcal{P})^{[\rho]}\big)_m = \begin{cases} \mathsf{B}_{(r)}(\mathcal{P})^{[\rho]} & \text{if } m = r, \\ 0 & \text{otherwise.} \end{cases}$$

The dual coprotoperad $\mathcal{P}^{\mathsf{i}}$ is not concentrated in degree 0, but satisfies:

$$\big(\mathcal{P}^{\mathsf{i}[\rho]}\big)_m = \begin{cases} \mathcal{P}^{\mathsf{i}[\rho]} & \text{if } m = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.20. — *The functor* $(-)^{\mathsf{i}} : (\mathsf{co})\mathsf{protoperads}_k^{\mathrm{aug}} \to \mathfrak{S}\text{-}\mathsf{mod}_k^{gr}$ *commutes with the functor* Ind.

Proof. — By the exactness and the preservation of the weight grading of the functor Ind (see Definition/Proposition 1.16). □

We have a protoperadic equivalent of the proposition [Val03, Prop. 136].

PROPOSITION 2.21. — *Let* $\mathcal{P} = \bigoplus_n \mathcal{P}^{[n]}$ *(resp.* $\mathcal{C} = \bigoplus_n \mathcal{C}^{[n]}$*) be a weight-graded, connected protoperad (resp. coprotoperad). Then the Koszul dual of* $\mathcal{P}$ *is a sub weight-graded, connected, coaugmented coprotoperad of* $\mathscr{F}^c(\Sigma\mathcal{P}^{[1]})$ *(respectively, the Koszul dual of* $\mathcal{C}$ *is a connected, weight-graded, augmented protoperad quotient of* $\mathscr{F}(\Sigma^{-1}\mathcal{C}^{[1]})$*).*

2.3.2. *Koszul resolution*

DEFINITION 2.22 (Koszul protoperad, coprotoperad). — Let $\mathcal{P}$ and $\mathcal{C}$ be respectively a protoperad and a coprotoperad, each weight-graded and connected. The protoperad $\mathcal{P}$ is *Koszul* if the inclusion $\mathcal{P}^{\mathsf{i}} \hookrightarrow \mathsf{B}\mathcal{P}$ is a quasi-isomorphism. Dually, the coprotoperad $\mathcal{C}$ is *Koszul* if the projection $\Omega\mathcal{C} \twoheadrightarrow \mathcal{C}^{\mathsf{i}}$ is a quasi-isomorphism.

PROPOSITION 2.23. — *If* $\mathcal{P}$ *is a weight-graded, connected protoperad which is Koszul, then its dual* $\mathcal{P}^{\mathsf{i}}$ *is a Koszul coprotoperad, and* $\mathcal{P}^{\mathsf{i}\mathsf{i}} = \mathcal{P}$.

Proof. — By the properties of the functor Ind (see Definition/Proposition 1.16) and by [Val03, Prop. 141]. □

DEFINITION 2.24 (Koszul complex). — Let $\mathcal{P}$ be a weight-graded protoperad. The (right and left) *Koszul complexes* of $\mathcal{P}$ are the following complexes:

(1) the complex $\left(\mathcal{P}^{\mathsf{i}} \boxtimes_c \mathcal{P}, \partial = \partial_{\mathcal{P}} + d_{\Delta}^r\right)$, where the differential d_{Δ}^r is induced by the homogeneous morphism of homological degree -1:

$$\mathcal{P}^{\mathsf{i}} \xrightarrow{\Delta} \mathcal{P}^{\mathsf{i}} \boxtimes_c \mathcal{P}^{\mathsf{i}} \longrightarrow \left(\mathcal{P}^{\mathsf{i}} \boxtimes_c \left(I_{\boxtimes} \oplus \overline{\mathcal{P}}^{\mathsf{i}}\right)^{[1]}\right) \longrightarrow \mathcal{P}^{\mathsf{i}} \boxtimes_c \left(I_{\boxtimes} \oplus \mathcal{P}^{[1]}\right),$$

where the right morphism is induced by the isomorphism $(\overline{\mathcal{P}}^{\mathsf{i}})^{[1]} \cong \overline{\mathcal{P}}^{[1]}$;

(2) the complex $\left(\mathcal{P} \boxtimes_c \mathcal{P}^{\mathsf{i}}, \partial = \partial_{\mathcal{P}} + d_{\Delta}^l\right)$, where the differential d_{Δ}^l is induced by the homogeneous morphism of degree -1:

$$\mathcal{P}^{\mathsf{i}} \xrightarrow{\Delta} \mathcal{P}^{\mathsf{i}} \boxtimes_c \mathcal{P}^{\mathsf{i}} \longrightarrow \left(\left(I_{\boxtimes} \oplus \overline{\mathcal{P}}^{\mathsf{i}}\right)^{[1]} \boxtimes_c \mathcal{P}^{\mathsf{i}}\right) \longrightarrow \left(I_{\boxtimes} \oplus \mathcal{P}^{[1]}\right) \boxtimes_c \mathcal{P}^{\mathsf{i}}.$$

As in the properadic case, we have the following Koszul criterion:

THEOREM 2.25 (Koszul criterion). — *Let* $\mathcal{P}$ *be a connected weight-graded protoperad. The following are equivalent:*

(1) *the inclusion* $\mathcal{P}^{\mathsf{i}} \hookrightarrow \mathsf{B}\mathcal{P}$ *is a quasi-isomorphism, i.e., the protoperad* $\mathcal{P}$ *is Koszul;*

(2) *the Koszul complex* $\left(\mathcal{P}^{\mathsf{i}} \boxtimes_c \mathcal{P}, \partial = \partial_{\mathcal{P}} + d_{\Delta}^r\right)$ *is acyclic;*

(3) *the Koszul complex* $\left(\mathcal{P} \boxtimes_c \mathcal{P}^{\mathsf{i}}, \partial = \partial_{\mathcal{P}} + d_{\Delta}^l\right)$ *is acyclic;*

(4) *the morphism of protoperads* $\Omega\mathcal{P}^{\mathsf{i}} \to \mathcal{P}$ *is a quasi-isomorphism.*

Proof. — By the exactness of the functor Ind (see Definition/Proposition 1.16) and theorems [Val03, Th. 144, Th. 149]. □

Remark 2.26. — By Corollary 1.26, we have a bar-cobar adjunction and a Koszul duality for shuffle protoperads. Also, a properad $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^{\mathrm{sh}}$ is Koszul.

2.3.3. *The case of quadratic protoperads.* — This subsection is strongly inspired by [Val08, §2] which described the notion of a quadratic properad. We adapt the notion to the protoperadic framework. Let V be a $\mathfrak{S}$-module and $R \subset \mathscr{F}^{(2)}(V)$: such a pair (V, R) is called a *quadratic datum*.

As the underlying $\mathfrak{S}$-modules of the free protoperad $\mathscr{F}(V)$ and the cofree coprotoperad $\mathscr{F}^c(V)$ are isomorphic, we consider the following morphisms of $\mathfrak{S}$-modules:

$$R \hookrightarrow \mathscr{F}^{(2)}(V) \hookrightarrow \mathscr{F}(V) \overset{(*)}{\cong} \mathscr{F}^c(V) \longrightarrow \mathscr{F}^{c,(2)}(V) \longrightarrow \mathscr{F}^{c,(2)}(V)/R =: \overline{R},$$

where the isomorphism $(*)$ is an isomorphism of $\mathfrak{S}$-modules. Using this, we naturally define a quotient protoperad of $\mathscr{F}(V)$ or a sub-coprotoperad of $\mathscr{F}^c(V)$.

Definition 2.27 (Quadratic (co)protoperad). — The *(homogeneous) quadratic protoperad generated by V and R* is the quotient protoperad of $\mathscr{F}(V)$ by the ideal generated by $R \subset \mathscr{F}^{(2)}(V)$. We denote this protoperad by $\mathscr{P}(V, R) := \mathscr{F}(V)/\langle R \rangle$. Dually, the *(homogeneous) quadratic coprotoperad cogenerated by V and $\overline{R}$* is the sub-coprotoperad of $\mathscr{F}^c(V)$ cogenerated by $\mathscr{F}^{c,(2)}(V) \twoheadrightarrow \overline{R}$. We denote this coprotoperad by $\mathscr{C}(V, \overline{R})$.

Remark 2.28. — All quadratic protoperads $\mathscr{P}(V, R)$ and all quadratic coprotoperads $\mathscr{C}(V, \overline{R})$ have a weight-grading by V, as for properads (see [Val03, Prop. 55]).

Theorem 2.29 (Kosul dual $(-)^{\mathsf{i}}$). — *Let (V, R) be a quadratic datum. We denote by $\Sigma^2 R$, the image of R in $\mathscr{F}^{(2)}(\Sigma V)$ and $\Sigma^{-2}\overline{R}$, the quotient of $\mathscr{F}^{c,(2)}(\Sigma^{-1}V)$ by $\Sigma^{-2}R$. The* Koszul dual *of the protoperad $\mathscr{P}(V, R)$, denoted by $\mathscr{P}(V, R)^{\mathsf{i}}$, is the coprotoperad given by*

$$\mathscr{P}(V, R)^{\mathsf{i}} = \mathscr{C}(\Sigma V, \Sigma^2 \overline{R}).$$

Dually, the Koszul dual *of the coprotoperad $\mathscr{C}(V, \overline{R})$, denoted by $\mathscr{C}(V, \overline{R})^{\mathsf{i}}$, is the protoperad given by*

$$\mathscr{C}(V, \overline{R})^{\mathsf{i}} = \mathscr{P}(\Sigma^{-1}V, \Sigma^{-2}R).$$

Also, we have $\mathscr{P}(V, R)^{\mathsf{ii}} = \mathscr{P}(V, R)$ and $\mathscr{C}(V, \overline{R})^{\mathsf{ii}} = \mathscr{C}(V, \overline{R})$.

Proof. — It is a similar proof as [Val08, Th. 8]. □

Proposition 2.30. — *Let (V, R) be a locally finite quadratic datum, i.e., for all finite sets S, $V(S)$ has a finite dimension. The linear dual of the coprotoperad $\mathscr{C}(V, \overline{R})$ is the quadratic protoperad*

$$\left(\mathscr{C}(V, \overline{R}) \right)^* = \mathscr{F}(V^*)/\langle R^\perp \rangle$$

with $R^\perp \subset \mathscr{F}^{(2)}(V)^* \cong \mathscr{F}^{(2)}(V^*)$. *In particular, we have*

$$\left(\mathscr{P}(V,R)^{\text{¡}}\right)^* = \mathscr{P}(\Sigma^{-1}V^*, \Sigma^{-2}R^\perp).$$

Proof. — It is the same proof as [Val03, Cor. 154] or [Val08, Prop. 9]. □

3. Simplicial bar construction for protoperads

We construct the simplicial bar complex for protoperads, as in the properadic case (see [Val07, §6]). Recall that Σ^n denotes the homological suspension of degree n (see Definition 1.11).

Definition 3.1 ((Reduced) Simplicial bar construction). — Let $(\mathcal{P}, \mu, \eta, \varepsilon)$ be an augmented protoperad. We denote

$$\mathsf{C}_n(\mathcal{P}) := \Sigma^n I_{\boxtimes} \boxtimes_c \mathcal{P}^{\boxtimes_c n} \boxtimes_c I_{\boxtimes}.$$

The face maps $d_i : \mathsf{C}_n(\mathcal{P}) \to \mathsf{C}_{n-1}(\mathcal{P})$ are induced by:

- the augmentation $\mathcal{P} \boxtimes_c I_{\boxtimes} \xrightarrow{\varepsilon \boxtimes_c I_{\boxtimes}} I_{\boxtimes} \boxtimes_c I_{\boxtimes} \cong I_{\boxtimes}$ for $i = 0$;
- the composition μ of the i-th and the $(i+1)$-th row,
- the augmentation $I_{\boxtimes} \boxtimes_c \mathcal{P} \xrightarrow{I_{\boxtimes} \boxtimes_c \varepsilon} I_{\boxtimes} \boxtimes_c I_{\boxtimes} \cong I_{\boxtimes}$ for $i = n$.

The degeneracy maps $s_i : \mathsf{C}_n(\mathcal{P}) \to \mathsf{C}_{n+1}(\mathcal{P})$ are given by the insertion of the unit $\eta : I_{\boxtimes} \to \mathcal{P}$ of the protoperad $s_i := \Sigma^n I_{\boxtimes} \boxtimes_c \mathcal{P}^{\boxtimes_c i} \boxtimes_c \eta \boxtimes_c \mathcal{P}^{\boxtimes_c n-i} \boxtimes_c I_{\boxtimes}$. The differential $\partial_{\mathsf{C}(\mathcal{P})}$ is defined by

$$\partial_{\mathsf{C}(\mathcal{P})} := \partial_{\mathcal{P}} + \sum_{i=0}^{n} (-1)^{i+1} d_i.$$

One can check that $\partial_{\mathsf{C}(\mathcal{P})}^2 = 0$. This chain complex is called the (reduced) *simplicial bar construction* of $\mathcal{P}$.

Definition 3.2 (Normalized bar construction). — The *normalized bar construction* is given by the quotient of the simplicial bar construction by the image of the degeneracy maps. We denote by $\mathsf{N}(\mathcal{P})$ the following graded $\mathfrak{S}$-module, given in grading n, by:

$$\mathsf{N}_n(\mathcal{P}) := \Sigma^n \operatorname{Coker}\left(\bigoplus_{i=0}^{n} I_{\boxtimes} \boxtimes_c \mathcal{P}^{\boxtimes_c i} \boxtimes_c \eta \boxtimes_c \mathcal{P}^{\boxtimes_c n-i} \boxtimes_c I_{\boxtimes}\right).$$

We define the functor $\mathcal{W}_{n\uparrow}^{\mathrm{conn,lev}} : \mathsf{Fin}^{\mathrm{op}} \to \mathsf{Fin}^{\mathrm{op}}$ of n-level connected wall given, for all finite set S, by

$$\mathcal{W}_{n\uparrow}^{\mathrm{conn,lev}}(S) = \left\{ W = (W^1, \ldots, W^n) \,\middle|\, \begin{array}{l} \forall i \in [\![1,n]\!], \exists I \in \mathcal{Y}(S) \\ \varnothing \neq W^i = \{W_\alpha^i\}_{\alpha \in A_i} \subseteq I \\ \bigcup_{i=1}^{n} \bigcup_{\alpha \in A_i} W_\alpha^i = S; \mathcal{K}_S(W) = \{S\} \end{array} \right\},$$

where $\mathcal{K}_S$ is the natural projection defined in Definition 1.3 (see also [Ler19, Def. 1.9]). We denote the label of the number of levels by $n \uparrow$, because $\mathcal{W}_{n\uparrow}^{\mathrm{conn,lev}}(S)$ is also weight-graded by the number of bricks: an element $(W^1, \ldots, W^n)$ lives in $\mathcal{W}_{n\uparrow,b}^{\mathrm{conn,lev}}(S)$ with $b = |W^1| + \cdots + |W^n|$. The graded functor of level connected wall is denoted by

$\mathcal{W}^{\mathrm{conn,lev}} = \mathrm{II}_n \mathcal{W}^{\mathrm{conn,lev}}_{n\uparrow} : \mathsf{Fin}^{\mathrm{op}} \to \mathsf{Set}^{\mathrm{op}}$. We have also the natural projection of *unlevelisation*

$$\mathrm{unl} \colon \mathcal{W}^{\mathrm{conn,lev}} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{W}^{\mathrm{conn}}$$

which sends an element $(\{W^1_\alpha\}_{\alpha \in A_1}, \dots, \{W^n_\alpha\}_{\alpha \in A_n})$ in $\mathcal{W}^{\mathrm{conn,lev}}(S)$ to the connected wall W over S which contains W^i_α, for all i in $[\![1, n]\!]$ and all α in A_i and such that, for all s in S, the total order of $\Gamma_s = \{W^i_\alpha | s \in W^i_\alpha\}$ is defined by levels: for W^i_α and W^j_β in Γ_s, $W^i_\alpha <_s W^j_\beta$ if $i < j$.

Remark that the unlevelisation morphism projects the functor of n-leveled connected wall with n bricks to $\mathcal{W}^{\mathrm{conn}}_n$. We denote by $\pi_{n\uparrow}$ the restriction of the unlevelisation morphism to $\mathcal{W}^{\mathrm{conn,lev}}_{n\uparrow,n}$

$$\pi_{n\uparrow} : \mathcal{W}^{\mathrm{conn,lev}}_{n\uparrow,n} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{W}^{\mathrm{conn}}_n$$

Proposition 3.3. — *Let $\mathcal{P}$ be an augmented protoperad, and its augmentation ideal $\overline{\mathcal{P}}$, and S be a finite set. We have the following isomorphism*

$$\mathsf{N}_n(\mathcal{P})(S) \cong \Sigma^n \bigoplus_{\substack{(W^1,\dots,W^n) \\ \in \mathcal{W}^{\mathrm{conn,lev}}_{n\uparrow}(S)}} \bigotimes_{i=1}^n \bigotimes_{\alpha \in A} \overline{\mathcal{P}}(W^i_\alpha).$$

Proof of Proposition 3.3. — As in the properadic case (see [Val07, §6.1.3, 1rst rem.]). An element $W = (W^1, \dots, W^n)$ in $\mathcal{W}^{\mathrm{conn,lev}}_{n\uparrow}(S)$ describes the position of non-trivial elements in each level. In the definition of the normalized bar construction, the cokernel ensures that there is a non-trivial element in each level: this is the condition $W^i \neq \varnothing$. The conditions $\bigcup_{i=1}^n \bigcup_{\alpha \in A_i} W^i_\alpha = S$ and $\mathcal{K}_S(\mathrm{unl}(W)) = \{S\}$ ensure that we have the connectedness of the product. □

Proposition 3.4. — *The simplicial bar construction and the normalized simplicial bar construction commute with the induction functor* Ind*:*

$$\mathrm{Ind}(\mathsf{C}(-)) = \mathsf{C}^{\mathrm{Val}}(\mathrm{Ind}(-)) \text{ and } \mathrm{Ind}(\mathsf{N}(-)) = \mathsf{N}^{\mathrm{Val}}(\mathrm{Ind}(-)),$$

where the functors $\mathsf{C}^{\mathrm{Val}}$ and $\mathsf{N}^{\mathrm{Val}}$ are respectively, the reduced simplicial bar construction and the normalized simplicial bar construction for properads (see [Val07]).

Proof. — The functor Ind is monoidal and exact (see Definition/Proposition 1.16). □

Proposition 3.5

(1) *The simplicial bar construction and the normalized bar construction preserve quasi-isomorphisms.*

(2) *Let $\mathcal{P}$ be a quasi-free protoperad on a weight-graded $\mathfrak{S}$-module V, i.e., $\mathcal{P}$ has underlying $\mathfrak{S}$-module $\mathscr{F}(V)$, such that $V^{(0)} = 0$ and concentrated in homological degree 0. The natural projection $\mathsf{N}(\mathcal{P}) \to \Sigma V$ is a quasi-isomorphism.*

Proof

(1) As for the bar construction of protoperads. The functors Res, Ind and $\mathsf{C}^{\mathrm{Val}}$, (cf. [Val07, Prop. 6.1]) preserve quasi-isomorphisms. Let φ be a quasi isomorphism of $\mathfrak{S}$-modules, then $\mathrm{Res}(\mathsf{C}^{\mathrm{Val}}(\mathrm{Ind}(\varphi))) = \mathrm{Res} \circ \mathrm{Ind}\, \mathsf{C}(\varphi) = \mathsf{C}(\varphi)$ is a quasi-isomorphism.

(2) Similar to [Val07, Prop. 6.5]   □

We define the levelisation morphism as in the operadic and the properadic case (see [Val07, §6.2]).

DEFINITION/PROPOSITION 3.6. — Let $\mathcal{P}$ be an augmented protoperad. The *levelisation morphism* is the injective morphism of $\mathfrak{S}$-modules

$$e : \mathsf{B}(\mathcal{P}) \longrightarrow \mathsf{N}(\mathcal{P})$$

which, for a finite set S, and a wall $W = \{W_\alpha\}_{\alpha \in A}$ in $\mathcal{W}^{\mathrm{conn}}(S)$, sends

$$\bigotimes_{\alpha \in A} \Sigma \overline{\mathcal{P}}(W_\alpha) \overset{e}{\longmapsto} \bigoplus_{\widetilde{W} \in \pi_{n\uparrow}^{-1}(W)} \bigotimes_{i=1}^{|A|} \Sigma \overline{\mathcal{P}}(\widetilde{W}_i) \subset \mathsf{N}(\mathcal{P});$$

the map e sends each element of $\bigotimes_{\alpha \in A} \Sigma \overline{\mathcal{P}}(W_\alpha)$ to the sum of representatives (with signs induced by the Koszul sign of the symmetry).

THEOREM 3.7. — *Let $\mathcal{P}$ be a weight-graded augmented protoperad. The levelisation morphism $e : \mathsf{B}(\mathcal{P}) \to \mathsf{N}(\mathcal{P})$ is a quasi-isomorphism.*

Proof. — Let $\mathcal{P}$ be a weight-graded, augmented protoperad, and consider the levelisation morphism $e : \mathsf{B}(\mathcal{P}) \to \mathsf{N}(\mathcal{P})$. The induction functor sends e to e^{Val}, the levelisation morphism for properads, defined by Vallette in [Val07, §6], which is a quasi isomorphism (see [Val07, Th. 6.7]):

$$\mathrm{Ind}(\mathsf{B}\mathcal{P}) \cong \mathsf{B}^{\mathrm{Val}}\mathrm{Ind}(\mathcal{P}) \xrightarrow[\sim]{\mathrm{Ind}(e) = e^{\mathrm{Val}}} \mathsf{N}^{\mathrm{Val}}\mathrm{Ind}(\mathcal{P}) \cong \mathrm{Ind}(\mathsf{N}\mathcal{P})$$

We apply the functor Res to this map, which is an exact functor, and which satisfies $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}$, then the map e is a quasi-isomorphism. We just use the same arguments that for the properadic case (see [Val07, §6]).   □

REMARK 3.8 (About shuffle protoperads). — All these constructions, the bar construction, the simplicial bar construction and the normalized simplicial bar construction, have their shuffle analoguous.

– As we have a free shuffle protoperad $\mathscr{F}_{\sqcup\sqcup}(-)$ and an associated cofree conilpotent shuffle coprotoperad $\mathscr{F}_{\sqcup\sqcup}^c(-)$, one has the shuffle bar construction $\mathsf{B}_{\sqcup\sqcup}$. Moreover, let $\mathcal{P}$ be a quadratic protoperad. For every ordered finite set S, we have the isomorphism of chain complexes

$$(\mathsf{B}\mathcal{P})(\underline{S}) \cong (\mathsf{B}_{\sqcup\sqcup}\mathcal{P}^{\mathrm{sh}})(S).$$

These isomorphisms justify the construction of shuffle protoperads (see the proof of Theorem 4.3).

– As we have a shuffle product $\boxtimes_c^{\sqcup\!\sqcup}$, one can construct the shuffle simplicial bar construction $\mathsf{C}_{\sqcup\!\sqcup}$ and the shuffle normalized simplicial bar construction $\mathsf{N}_{\sqcup\!\sqcup}$.

Moreover, these constructions commute with the functor $(-)^{\mathrm{sh}}$, by Corollary 1.26 and Proposition 1.24.

4. Studying Koszulness of binary quadratic protoperad

In this section, we describe a criterion to study the Koszulness of binary quadratic protoperad, which are protoperads given by a quadratic datum (V, R) such that V is concentrated in arity 2, $V(S) = 0$ for all finite sets S with $|S| \neq 2$.

4.1. A useful criterion. — We give an algebraic criterion for a binary quadratic protoperads concentrated in homological degree 0 to be Koszul.

Definition 4.1 (Binary protoperad). — A protoperad $\mathcal{P}$, given by generators and relations, i.e., $\mathcal{P} = \mathscr{F}(V)/\langle R \rangle$ is *binary* if the $\mathfrak{S}$-module V is concentrated in arity 2, i.e., $V(S) = 0$ for all finite sets S with $|S| \neq 2$.

Let $\mathcal{P}$ be a binary quadratic protoperad concentrated in homological degree 0, given by the quadratic datum (V, R), then $V = V_0(2)$. We associate to $\mathcal{P}$, a family of quadratic algebras $\{\mathscr{A}(\mathcal{P}, n)\}_{n \geqslant 2}$, defined by

$$\mathscr{A}(\mathcal{P}, n) := \mathbb{S}^{\mathrm{sh}}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, n \rrbracket),$$

where $\mathbb{S}^{\mathrm{sh}}$ is defined in Definition 1.18. We will see that the algebras $\mathscr{A}(\mathcal{P}, n)$ are quadratic. Fix $n \geqslant 2$, we consider the decomposition of V in irreducible representations:

$$V = \bigoplus_{\nu=1}^{m} V_\nu$$

where $V_\nu = k \cdot v_\nu$ is the trivial representation or the signature representation of $\mathfrak{S}_2$ (recall that the characteristic of k is different to 2). To V, we associate the set $V(\mathcal{P}, n)$ of generators of $\{\mathscr{A}(\mathcal{P}, n)\}_{n \geqslant 2}$:

$$V(\mathcal{P}, n) := \{(v_\nu)_{ij} \mid 1 \leqslant i < j \leqslant n,\, 1 \leqslant \nu \leqslant m\}$$

Thus $V(\mathcal{P}, n)$ corresponds to the generators of $\mathbb{S}(\mathcal{P})(\llbracket 1, n \rrbracket)$ as algebra for the product $\mu_\square$ (see Proposition 1.13), i.e.,

(2) $$(v_\nu)_{ij} \rightsquigarrow \begin{array}{c} \scriptstyle 1 \quad\; \scriptstyle i \quad\;\; \scriptstyle j \quad\; \scriptstyle n \\ \boxed{}\cdots\boxed{\nu}\cdots\boxed{\nu}\cdots\boxed{} \end{array} .$$

As $\mathcal{P}$ is binary and quadratic, the set of relations R is concentrated in arity 2 and 3. Each relation in $R(2)$ is given by a linear combination of terms as

$$\begin{array}{c} \scriptstyle 1 \;\; \scriptstyle 2 \\ \boxed{} \\[-4pt] \boxed{} \end{array}$$

where each brick is labeled by a generator v_ν. To a such relation r in $R(2)$, we associate a family of quadratic relations $\{r_{ij}\}_{1 \leqslant i < j \leqslant n}$ in terms of $V(\mathcal{P}, n)$, where r_{ij} is given by replacing a monomial indexed by v_α for the bottom brick and v_β for the upper brick, with v_α and v_β two generators, by the monomial $(v_\alpha)_{ij}(v_\beta)_{ij}$ in $V(\mathcal{P}, n)^{\otimes 2}$, as

in Figure 1. We denote by $R(2)_{ij}$, the set of relations in $V(\mathcal{P}, n)^{\otimes 2}$ which are obtained

$$\begin{array}{c} {}^{1\ \ 2} \\ \boxed{} \\ \boxed{} \end{array} \overset{ij}{\rightsquigarrow} (v_\alpha)_{ij}(v_\beta)_{ij}.$$

FIGURE 1. Labelled procedure for $R(2)$

by the labeled procedure $\overset{ij}{\rightsquigarrow}$ (see Figure 1). Similarly, by connectivity, each relation in $R(3)$ is given by a linear combination of terms as follow:

where each brick is labeled by a generator v_ν. If $n \geqslant 3$, for all relation r in $R(3)$, we associate a family of quadratic relations $\{r_{ijk}\}_{1\leqslant i<j<k\leqslant n}$ with $r_{ijk} \in V(\mathcal{P}, n)^{\otimes 2}$, where r_{ijk} is given by replacing all monomial indexed by v_α for the bottom brick and v_β for the upper brick, with v_α and v_β two generators, as in Figure 2.

FIGURE 2. Labelled procedure for $R(3)$

We denote by $R(3)_{ijk}$, the set of relations in $V(\mathcal{P}, n)^{\otimes 2}$ which are obtained by the labeled procedure $\overset{ijk}{\rightsquigarrow}$. We consider the quadratic algebra

$$\frac{\mathrm{T}\left(V(\mathcal{P}, n)\right)}{\left\langle R(2)_{ij}, R(3)_{ijk}, \left[(v_\alpha)_{ij}, (v_\beta)_{ab}\right] \mid \begin{smallmatrix} \forall 1\leqslant i<j<k\leqslant n, \\ \forall 1\leqslant a<b\leqslant n, \{i,j\}\cap\{a,b\}=\varnothing \end{smallmatrix}\right\rangle}.$$

The new relations given by the commutator $[(v_\alpha)_{ij}, (v_\beta)_{ab}]$ correspond to the "parallelism commutativity" which is present in the protoperadic structure:

(see [Val07] for the properadic case).

LEMMA 4.2. — *Let $\mathcal{P}$ be a binary quadratic protoperad. For all integer $n \geqslant 2$, we have the isomorphism of algebras*

$$(3) \qquad \mathscr{A}(\mathcal{P}, n) \cong \frac{\mathrm{T}\left(V(\mathcal{P}, n)\right)}{\left\langle R(2)_{ij}, R(3)_{ijk}, \left[(v_\alpha)_{ij}, (v_\beta)_{ab}\right] \mid \begin{smallmatrix} \forall 1\leqslant i<j<k\leqslant n, \\ \forall 1\leqslant a<b\leqslant n, \{i,j\}\cap\{a,b\}=\varnothing \end{smallmatrix}\right\rangle}.$$

Proof. — We recall that, for a protoperad $(\mathcal{P}, \mu)$, the product on $\mathbb{S}(\mathcal{P})(\llbracket 1, n \rrbracket)$ is given by

$$\mathbb{S}(\mathcal{P})(\llbracket 1, n \rrbracket) \,\square\, \mathbb{S}(\mathcal{P})(\llbracket 1, n \rrbracket) \cong \mathbb{S}(\mathcal{P} \boxtimes_c \mathcal{P})(\llbracket 1, n \rrbracket) \xrightarrow{\ \mathbb{S}(\mu)\ } \mathbb{S}(\mathcal{P})(\llbracket 1, n \rrbracket).$$

As $V(\mathcal{P}, n)$ is, by construction, a set of generators of the algebra $\mathbb{S}(\mathscr{F}^{\mathrm{sh}}(V))(\llbracket 1, n \rrbracket)$, we have the following morphism of algebras $\mathrm{T}(V(\mathcal{P}, n)) \to \mathbb{S}(\mathscr{F}^{\mathrm{sh}}(V))(\llbracket 1, n \rrbracket)$, which factorizes as follows:

$$
\begin{array}{ccc}
\mathrm{T}(V(\mathcal{P}, n)) & \longrightarrow & \mathbb{S}(\mathscr{F}^{\mathrm{sh}}(V))(\llbracket 1, n \rrbracket) \ \ ; \\
\downarrow & \ \ {}^{\varphi}\nearrow & \\
& \ \ {}_{\cong} & \\
\dfrac{\mathrm{T}(V(\mathcal{P}, n))}{\left\langle \left[(v_\alpha)_{ij}, (v_\beta)_{ab}\right] \,\middle|\, \begin{subarray}{c} \forall v_\alpha, v_\beta \\ \forall 1 \leqslant a < b \leqslant n, \forall 1 \leqslant i < j \leqslant n \\ \{i,j\} \cap \{a,b\} = \varnothing \end{subarray} \right\rangle} & &
\end{array}
$$

the isomorphism φ induces the isomorphism (3). $\qquad\square$

Theorem 4.3 (Criterion of Koszulness). — *Let $\mathcal{P}$ be a binary quadratic protoperad. If, for all integer $n \geqslant 2$, the quadratic algebra $\mathscr{A}(\mathcal{P}, n)$ is Koszul, then the protoperad $\mathcal{P}$ is Koszul.*

Proof. — Fix n an integer such that $n \geqslant 2$. By Lemma 4.2, the bar constructions of the algebras $\mathscr{A}(\mathcal{P}, n)$ are isomorphic, so we have the isomorphism of chain complexes

$$(4) \qquad \mathsf{B}^{\mathsf{Alg}}(\mathscr{A}(\mathcal{P}, n)) \cong \mathsf{B}^{\mathsf{Alg}}\big(\mathbb{S}^{\mathrm{sh}}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, n \rrbracket)\big),$$

where $\mathsf{B}^{\mathsf{Alg}}$ is the bar construction for algebras (see [LV12, §2.2]). To a monomial m of $\mathsf{B}^{\mathsf{Alg}}(\mathscr{A}(\mathcal{P}, n))$, we associate the partition which is induced by the set of pairs (i, j) of generator indices which appear in m, as explain below. We have the surjection $\mathsf{B}^{\mathsf{Alg}}(\mathrm{T}(V(\mathcal{P}, n))) \to \mathsf{B}^{\mathsf{Alg}}(\mathscr{A}(\mathcal{P}, n))$, so choose a representative $\overline{m}$ of m in $\mathsf{B}^{\mathsf{Alg}}(\mathrm{T}(V(\mathcal{P}, n)))$ and consider the set of pairs (i, j) of generator indices which appear in $\overline{m}$, completed by singletons $\{k\}$ if k in $\llbracket 1, n \rrbracket$ does not appear in any of the pairs. Such sets can be viewed as elements of $\mathcal{W}^{\mathrm{conn}}(\llbracket 1, n \rrbracket)$, with the partial order induced by the lexicographic order. Then, by the natural transformation $\mathcal{K} : \mathcal{W} \to \mathcal{Y}$ (see Definition 1.3), we associate $\overline{m}$, a partition $\mathfrak{p}$ of $\llbracket 1, n \rrbracket$.

All relations in $\mathscr{A}(\mathcal{P}, n)$ are given by r_{ab} and r'_{ijk} for $1 \leqslant i < j < k \leqslant n$, $1 \leqslant a < b \leqslant n$, r in $R(2)$ and r' in $R(3)$. So, as we see in Figure 2, any choice of representative $\overline{m}$ for m gives us the same partition, then the partition $\mathfrak{p}$ does not depend of the choice of the representative $\overline{m}$. By the same argument, as the differential of $\mathsf{B}^{\mathsf{Alg}}(\mathscr{A}(\mathcal{P}, n))$ is induced by the product of $\mathscr{A}(\mathcal{P}, n)$, the bar complex splits:

$$(5) \qquad \mathsf{B}^{\mathsf{Alg}}(\mathscr{A}(\mathcal{P}, n)) \cong \bigoplus_{\mathfrak{p} \in \mathcal{Y}(\llbracket 1, n \rrbracket)} \mathsf{B}^{\mathsf{Alg}}_{\mathfrak{p}}(\mathscr{A}(\mathcal{P}, n)).$$

For convenience, we denote by $\mathfrak{p}_0$, the trivial partition with one element of $\llbracket 1, n \rrbracket$. Through the isomorphism in Equation (4), we identify the complex $\mathsf{B}^{\mathsf{Alg}}_{\mathfrak{p}_0}(\mathscr{A}(\mathcal{P}, n))$ with the normalized simplicial bar construction $\mathsf{N}_{\sqcup\sqcup}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, n \rrbracket)$ (see Remark 3.8).

Let $p \geqslant 1$, a monomial w in $\mathsf{N}^{(p)}_{\sqcup}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, n \rrbracket)$ is given by a leveled connected wall where bricks are labeled by monomials of $\mathcal{P}^{\mathrm{sh}}$. To such a monomial w, we associate directly an element of $\mathsf{B}^{\mathrm{Alg},(p)}_{\mathfrak{p}_0}(\mathscr{A}(\mathcal{P}, n))$ where each level w_i of w is sent to a monomial m_i in $\mathscr{A}(\mathcal{P}, n)$, as in Figure 3. It is clear that this application is an isomorphism

$$\mathsf{N}^{(2)}_{\sqcup}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, 5 \rrbracket) \ni \; \raisebox{-1em}{[wall diagram]} \; \longleftrightarrow (v_\alpha)_{13}(v_\beta)_{23} \otimes (v_\gamma)_{45}(v_\alpha)_{34} \in \mathsf{B}^{\mathrm{Alg},(2)}_{\mathfrak{p}_0}(A(\mathcal{P}, 5))$$

FIGURE 3. Example

of chain complexes:

(6) $$\mathsf{N}^{(p)}_{\sqcup}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, n \rrbracket) \cong \mathsf{B}^{\mathrm{Alg},(p)}_{\mathfrak{p}_0}(\mathscr{A}(\mathcal{P}, n)).$$

As the algebras $\mathscr{A}(\mathcal{P}, n)$ are Koszul by hypothesis, for all $n \geqslant 2$, then the homology of $\mathsf{B}^{\mathrm{Alg},(p)}(\mathscr{A}(\mathcal{P}, n))$ is concentrated in degree p. As this complex splits (see Equation (5)), then the homology of $\mathsf{B}^{\mathrm{Alg},(p)}_{\mathfrak{p}_0}(\mathscr{A}(\mathcal{P}, n))$ is also concentrated in degree p. Then, by the isomorphism in Equation (6), the homology of $\mathsf{N}^{(p)}_{\sqcup}(\mathcal{P}^{\mathrm{sh}})(\llbracket 1, n \rrbracket)$ is also concentrated in degree p. So, by Theorem 3.7, the shuffle protoperad $\mathcal{P}^{\mathrm{sh}}$ is Koszul, then $\mathcal{P}$ too, because we have, for every ordered finite set S, the following isomorphisms of chain complexes:

$$\mathsf{B}_{\sqcup}\mathcal{P}^{\mathrm{sh}}(S) \cong (\mathsf{B}\mathcal{P})^{\mathrm{sh}}(S) \cong \mathsf{B}\mathcal{P}(\underline{S}),$$

the first one given by Corollary 1.26 and the second one by definition of the functor $(-)^{\mathrm{sh}}$. $\qquad \square$

4.2. THE MAIN EXAMPLE: THE PROTOPERAD $\mathcal{DLie}$. — In this section, we define the protoperad $\mathcal{DLie}$ and we show that it is Koszul by Theorem 4.3.

DEFINITION 4.4 (The protoperad $\mathcal{DLie}$). — The protoperad $\mathcal{DLie}$ is the quadratic protoperad

$$\mathcal{DLie} := \mathscr{F}\Big(\; \raisebox{-0.5em}{[brick diagram]} = - \; \raisebox{-0.5em}{[brick diagram]} \; \Big) \Big/ \Big\langle \; \raisebox{-0.5em}{[brick diagram]} + \; \raisebox{-0.5em}{[brick diagram]} + \; \raisebox{-0.5em}{[brick diagram]} \; \Big\rangle.$$

REMARK 4.5. — We associate to the protoperad $\mathcal{DLie}$, the shuffle protoperad

$$\mathcal{DLie}^{\mathrm{sh}} = \frac{\mathscr{F}^{\mathrm{sh}}\Big(\; \raisebox{-0.5em}{[brick diagram]} \; \Big)}{\Big\langle \; \raisebox{-0.5em}{[diagram]} - \; \raisebox{-0.5em}{[diagram]} - \; \raisebox{-0.5em}{[diagram]}; \; \raisebox{-0.5em}{[diagram]} - \; \raisebox{-0.5em}{[diagram]} - \; \raisebox{-0.5em}{[diagram]} \; \Big\rangle},$$

by Corollary 1.26.

To the protoperad $\mathcal{DLie}$, we associate the family of quadratic algebras, denoted by $\mathscr{A}(\mathcal{DLie}, n)$ for $n \geqslant 2$, given by the quadratic datum $(V(\mathcal{DLie}, n), R(\mathcal{DLie}, n))$, with generators

$$V(\mathcal{DLie}, n) = \{x_{ij} \mid 1 \leqslant i < j \leqslant n\}$$

and relations

$$R(\mathcal{DL}ie, 2) = 0 \ ; \ R(\mathcal{DL}ie, 3) = \begin{Bmatrix} x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12} \\ x_{23}x_{12} - x_{13}x_{23} - x_{12}x_{13} \end{Bmatrix};$$

and, for $n \geqslant 4$,

$$R(\mathcal{DL}ie, n) = \left\{ \begin{matrix} x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} \\ x_{jk}x_{ij} - x_{ik}x_{jk} - x_{ij}x_{ik} \\ x_{ab}x_{uv} - x_{uv}x_{ab} \end{matrix} \middle| \begin{matrix} 1 \leqslant i < j < k \leqslant n \\ 1 \leqslant u < v \leqslant n \\ 1 \leqslant a < b \leqslant n \\ \{a, b\} \cap \{u, v\} = \varnothing \end{matrix} \right\}.$$

Proposition 4.6. — *For all $n \geqslant 2$, the quadratic algebra $\mathscr{A}(\mathcal{DL}ie, n)$ is Koszul.*

Proof. — See Proposition A.1 for the proof. □

Theorem 4.7. — *The protoperad $\mathcal{DL}ie$ is Koszul.*

Proof. — By Proposition 4.6 and Theorem 4.3 □

Corollary 4.8. — *The properad $\mathrm{Ind}(\mathcal{DL}ie)$ is Koszul.*

Proof. — The monoidal functor Ind is exact by Definition/Proposition 1.16, and preserves the graduation by the weight. □

This corollary is very important: it is the first example of a Koszul properad with a generator not in arity $(1, 2)$ or $(2, 1)$.

5. $\mathcal{DP}ois$ is Koszul

In this section, we study the Koszul dual of the protoperad $\mathcal{DL}ie$, which is called $\mathcal{DC}om$, by analogy of the case of operads $\mathcal{L}ie$ and $\mathcal{C}om$.

5.1. The Koszul dual of $\mathcal{DL}ie$. — To the protoperad $\mathcal{DL}ie$, we associate its Koszul dual, which we will called $\mathcal{DC}om$:

$$\mathcal{DC}om := \mathscr{F}(V^*_{\mathcal{DL}ie}) / \langle R^\perp_{DJ} \rangle,$$

where V^* is the linear dual of V and, for all $R \subset \mathscr{F}^{(2)}(V)$, $R^\perp$ is the orthogonal of R in $\mathscr{F}^{(2)}(V^*)$. The $\mathfrak{S}$-module $V^*_{\mathcal{DL}ie}$ is identified to

$$V^*_{\mathcal{DL}ie}(\llbracket 1, m \rrbracket) = \begin{cases} \mathrm{sgn}(\mathfrak{S}_2) & \text{if } m = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then, as in the case of the protoperad $\mathcal{DL}ie$, we can diagrammatically interpret $V^*_{\mathcal{DL}ie}(\llbracket 1, 2 \rrbracket)$ as follow

$$V^*_{\mathcal{DL}ie}(\llbracket 1, 2 \rrbracket) \cong \left\langle \overset{1 \quad 2}{\longleftrightarrow} = -\overset{2 \quad 1}{\longleftrightarrow} \right\rangle.$$

We also have the following relations:

$$R^\perp_{DJ} : \quad \overset{1 \ 2 \ 3}{\vdash\!\!\!\longleftrightarrow} \ - \ \overset{2 \ 3 \ 1}{\vdash\!\!\!\longleftrightarrow} \ ; \ \equiv.$$

By the second relation in $\mathcal{R}^{\perp}_{\mathcal{D}\mathcal{L}ie}$, we directly have that

$$\mathcal{D}\mathcal{C}om([\![1,2]\!]) = \mathcal{D}\mathcal{C}om_{(1)}([\![1,2]\!]) = V^{*}_{\mathcal{D}\mathcal{L}ie}.$$

For the $\mathfrak{S}$-module $\mathcal{D}\mathcal{C}om([\![1,3]\!])$, we have:

$$\text{(diagram with labels 1 2 3)} = \text{(diagram with labels 3 2 1)}.$$

If we consider the elements of weight 3 in $\mathcal{D}\mathcal{C}om([\![1,3]\!])$, we have, by the first relation in $R^{\perp}_{DJ}$, the two following equality:

$$\text{(diagram 1 2 3)} = \text{(diagram 3 1 2)} = 0 \; ; \; \text{(diagram 1 3 2)} = -\text{(diagram 3 2 1)} = 0.$$

That implies that the $\mathfrak{S}$-module $\mathcal{D}\mathcal{C}om([\![1,3]\!])$ is reduced to its component of weight 2, i.e., $\mathcal{D}\mathcal{C}om([\![1,3]\!]) = \mathcal{D}\mathcal{C}om_{(2)}([\![1,3]\!])$. This equality is a more general thing, as we will see, i.e., we will prove that, for all $n \geqslant 2$, we have $\mathcal{D}\mathcal{C}om([\![1,n]\!]) = \mathcal{D}\mathcal{C}om_{(n-1)}([\![1,n]\!])$.

LEMMA 5.1. — *Every stairway of arity n is invariant up to the sign by the diagonal action of $\mathbb{Z}/n\mathbb{Z}$, that is, for all $n \geqslant 2$*

$$\text{(stairway diagram, labels 1,2,...,n-2,n-1,n)} = (-1)^{n-1} \text{(stairway diagram, labels 2,3,...,n-1,n,1)}.$$

Proof. — We prove this result by induction on the arity n. By the definition of the protoperad $\mathcal{D}\mathcal{C}om$, we have:

$$\text{(diagram 1 2)} = -\text{(diagram 2 1)} \quad \text{and} \quad \text{(diagram 1 2 3)} = \text{(diagram 2 3 1)}.$$

Suppose that, for a fixed integer n, we have the following equality:

$$\text{(stairway diagram, labels 1,2,...,n-2,n-1,n)} = (-1)^{n-1} \text{(stairway diagram, labels 2,3,...,n-1,n,1)}.$$

Then, we have

$$\text{(diagram, labels 1,2,...,n-1,n,n+1)} = (-1)^{n-1} \text{(diagram n n+1, 1, 2 3 ...)} = (-1)^{n} \text{(diagram, labels 1,n-1,...,2 3,n n+1)}.$$

Then, we have

$$\text{(stairway diagram, labels 1,2,...,n-1,n,n+1)} = (-1)^{n} \text{(stairway diagram, labels 2 3 ...,n-1,n,n+1 1)}.$$

$\square$

Lemma 5.2. — *For all integer n greater than 2, we have the following equality*

$$\text{(diagram)} \;=\; (-1)^{n+1} \; \text{(diagram)} .$$

Proof. — We prove this result by induction on the arity. We also have

$$\text{(diagram)} \;=\; \text{(diagram)} .$$

Suppose that, for a fixed integer $n \geqslant 2$, we have

$$\text{(diagram)} \;=\; (-1)^n \; \text{(diagram)} .$$

Then, we have

$$\text{(diagram)} \;=\; - \; \text{(diagram)} \;=\; -(-1)^n \; \text{(diagram)} .$$

$\square$

Lemma 5.3. — *Every monomial of $\mathcal{D}\mathcal{C}om$ such that the underlying non-oriented graph does not have cycles can be rewritten as a stairway.*

Proof. — We prove this result by induction on the weight of monomials, i.e., the number of vertices of the underlying graph. By Lemma 5.2, we have that this lemma holds for a monomial of weight 2. Let n be an integer strictly greater than 2. Suppose the lemma holds for every monomial of weight $w < n$. We consider Φ, a monomial of weight n and we denote by $\overline{\Phi}$, its underlying non-oriented graph. As $\mathcal{D}\mathcal{C}om$ is a properad, the graph $\overline{\Phi}$ is connected: we label its n vertices by $v_1, \ldots, v_n$. There exists α in $[\![1, n]\!]$ such that the subgraph $\overline{\Phi}^* := \overline{\Phi} \smallsetminus v_\alpha$ is connected. By the induction hypothesis, we can rewrite Φ^* as a stairway, then we can rewrite Φ as a one of these two following monomials:

$$\text{(diagram)} \qquad \text{or} \qquad \text{(diagram)}$$

and, by invariance of stairways under the cyclic group action, we have our result. $\square$

Lemma 5.4. — *Every monomial of $\mathcal{D}\mathcal{C}om$ such that the underlying non-oriented graph has a cycle is null.*

Proof. — We prove the result by induction on the weight of monomials. We limit ourselves to considering only monomials whose underlying non-oriented graph is a cycle, i.e., monomials whose each elementary block is linked by two edges to another block. We have the relation

$$\overline{\underline{\underline{\overline{}}}} \; = \; 0$$

which initialize our induction. Suppose that every cycle of weight $\leqslant n-1$ is null. We consider a cycle Φ of weight n and, we isolate one of the blocs v_α in the cycle (i.e., one of the vertex of the underlying graph) such that its two outputs are linked with an other bloc. In a cycle, such a bloc already exists. We denote by Φ^*, the monomial obtained by the forgetfulness of the bloc v_α in the initial cycle. The monomial Φ^* does not contain a cycle, then, by Lemma 5.3, Φ^* can be rewriting in a stairway. Finally, the monomial Φ can be rewrite as one of the two following monomials in Figure 4 and Figure 5. By the invariance of stairways under the diagonal action of the cyclic
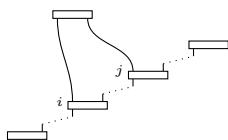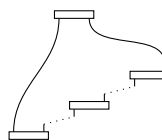

FIGURE 4. Monomial of form 1

FIGURE 5. Monomial of form 2

group (cf. Lemma 5.1), a monomial with the form 2 (see Figure 5) can be rewrite as a monomial with the form 1 (see Figure 4). Then, Φ can be rewrite as a monomial which contains a smaller cycle, then Φ is null. $\qquad\square$

PROPOSITION 5.5. — *For all $n \geqslant 2$, we have*

$$\mathcal{DC}om(\llbracket 1, n \rrbracket) = \mathcal{DC}om_{(n-1)}(\llbracket 1, n \rrbracket).$$

with $\mathcal{DC}om_{(n-1)}(\llbracket 1, n \rrbracket)$ generated by φ_n, the stairway with n inputs, which is stable under the diagonal action of the cyclic group. In terms of group representations, $\mathcal{DC}om(n, n)$ is given by

$$\mathcal{DC}om(\llbracket 1, n \rrbracket) = \begin{cases} \mathrm{sgn}(\mathbb{Z}/n\mathbb{Z}) \uparrow^{\mathfrak{S}_n}_{\mathbb{Z}/n\mathbb{Z}} & \text{if } n \text{ even,} \\ \mathrm{triv}(\mathbb{Z}/n\mathbb{Z}) \uparrow^{\mathfrak{S}_n}_{\mathbb{Z}/n\mathbb{Z}} & \text{if } n \text{ odd.} \end{cases}$$

Proof. — We already have that $\mathcal{DC}om_{(k)}(\llbracket 1, n \rrbracket) = 0$ for all k in $\llbracket 0, n-2 \rrbracket$. We also have that the $\mathfrak{S}$-module $\mathcal{DC}om_{(n-1)}(\llbracket 1, n \rrbracket)$ is generated by the stairway with n inputs. Finally, monomials in $\mathcal{DL}ie^!_{(k)}(\llbracket 1, n \rrbracket)$ for $k \geqslant n$ have a cycle, thus they are null, by Lemma 5.4. $\qquad\square$

NOTATION 5.6. — For all integers $n \geqslant 2$, we denote:

$$\widetilde{\mathrm{sgn}}(\mathbb{Z}/n\mathbb{Z}) \overset{\text{not.}}{:=} \mathrm{sgn}(\mathbb{Z}/n\mathbb{Z})^{\otimes n} = \begin{cases} \mathrm{sgn}(\mathbb{Z}/n\mathbb{Z}) & \text{if } n \text{ even,} \\ \mathrm{triv}(\mathbb{Z}/n\mathbb{Z}) & \text{if } n \text{ odd.} \end{cases}$$

By Theorem 2.29, the dual coprotoperad of $\mathcal{D}\mathcal{L}ie$ is given by

$$\mathcal{D}\mathcal{L}ie^{\mathfrak{i}} = \big(\mathscr{P}(V_{\mathcal{D}\mathcal{L}ie}, R_{DJ})\big)^{\mathfrak{i}} = \mathscr{C}(\Sigma V_{\mathcal{D}\mathcal{L}ie}, \Sigma^2 \overline{R_{DJ}}).$$

We have seen, in Proposition 5.5, that the protoperad $\mathcal{D}\mathcal{C}om = \mathscr{P}(V^*_{\mathcal{D}\mathcal{L}ie}, R^{\perp}_{DJ})$ satisfies

$$\mathcal{D}\mathcal{C}om([\![1, n]\!]) = \mathcal{D}\mathcal{C}om_{(n-1)}([\![1, n]\!]) = \widetilde{\mathrm{sgn}}(\mathbb{Z}/n\mathbb{Z}) \uparrow^{\mathfrak{S}_n^{\mathrm{op}} \times \mathfrak{S}_n}_{\mathbb{Z}/n\mathbb{Z}}$$

By Proposition 2.30, we have the isomorphism

$$\mathscr{C}(\Sigma V_{\mathcal{D}\mathcal{L}ie}, \Sigma^2 \overline{R_{DJ}})^* \cong \mathscr{P}(\Sigma^{-1} V^*_{\mathcal{D}\mathcal{L}ie}, \Sigma^{-2} R^{\perp}_{DJ}),$$

so, for all integer $n > 0$, we have

$$\mathcal{D}\mathcal{L}ie^{\mathfrak{i}}([\![1, n]\!]) = \Sigma^{n-1} \widetilde{\mathrm{sgn}}(\mathbb{Z}/n\mathbb{Z}) \uparrow^{\mathfrak{S}_n}_{\mathbb{Z}/n\mathbb{Z}}.$$

So we have the following proposition.

Proposition 5.7. — *The properad* $\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)_{\infty} := \Omega(\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)^{\mathfrak{i}})$*, which is a cofibrant resolution of the properad* $\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)$*, is the quasi-free properad* $(\mathscr{F}^{\mathrm{Val}}(\Sigma^{-1}W), \partial_{\Delta})$ *with* $\Sigma^{-1}W$*, the* $\mathfrak{S}$*-bimodule defined by* $\Sigma^{-1}W([\![1, m]\!], [\![1, n]\!]) = 0$ *for* $m \neq n$ *in* $\mathbb{N}$ *and*

$$\Sigma^{-1}W([\![1, n]\!], [\![1, n]\!]) := (\Sigma^{-1}W)_{n-2}([\![1, n]\!], [\![1, n]\!]) = \Sigma^{n-2} \mathrm{Ind}\big(\widetilde{\mathrm{sgn}}(\mathbb{Z}/n\mathbb{Z}) \uparrow^{\mathfrak{S}_n}_{\mathbb{Z}/n\mathbb{Z}}\big)$$

$$= \left\langle \Sigma^{-1}\varphi_n := \Sigma^{-1} \underset{\substack{1 \ 2 \ \cdots \ n}}{\overset{1 \ 2 \ \cdots \ n}{\rule[0.5ex]{4em}{1.2pt}}} = (-1)^{n+1} \Sigma^{-1} \underset{\substack{2 \ 3 \ \cdots \ 1}}{\overset{2 \ 3 \ \cdots \ 1}{\rule[0.5ex]{4em}{1.2pt}}} \right\rangle$$

with $\Sigma^{-1}W([\![1, n]\!], [\![1, n]\!])$ *concentrated in homological degree* $n - 2$ *and with the differential* ∂_{Δ} *induced by the coproduct* Δ *of the coproperad* $\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)^{\mathfrak{i}}$*, which sends*

$$\underset{\substack{1 \ 2 \ \cdots \ n}}{\overset{1 \ 2 \ \cdots \ n}{\rule[0.5ex]{4em}{1.2pt}}} \xrightarrow{\Delta} \sum_{\substack{2 \leqslant i \leqslant n-1 \\ \sigma \in \mathbb{Z}/n\mathbb{Z}}} \pm \ \underset{\substack{\sigma(1) \ \sigma(2) \ \sigma(i) \ \sigma(n)}}{\overset{\sigma(1) \ \sigma(2) \ \sigma(i) \ \sigma(n)}{\rule[0.5ex]{4em}{1.2pt}}} \ .$$

We exhibit the action of the differential on generators of degree 2, 3 and 4.

– The element φ_2 in $\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)^{\mathfrak{i}}$ is primitive, i.e., $\Delta(\varphi_2) = (1 \otimes 1)\varphi_2 + \varphi_2(1 \otimes 1)$ then

$$\partial_{\Delta}(s^{-1}\varphi_2) = -\big((1 \otimes 1)s^{-1}\varphi_2 + (-1)^{|\varphi_2|}s^{-1}\varphi_2(1 \otimes 1)\big) = 0.$$

– We have seen that the $\mathfrak{S}$-bimodule $\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)^{\mathfrak{i}}([\![1, 3]\!], [\![1, 3]\!])$ is generated by φ_3, the stairway of arity 3 which is stable under the diagonal action of the cyclic group, so

$$\Delta(\varphi_3) = \sum_{i=0}^{2} \sigma^{-i}_{(123)}(\varphi_2, 1)(1, \varphi_2)\sigma^{i}_{(123)} \ ;$$

then

$$\partial_\Delta(s^{-1}\varphi_3) = -\left(\sum_{i=0}^{2}(-1)^{|\varphi_2|}\sigma_{(123)}^{-i}(s^{-1}\varphi_2,1)(1,s^{-1}\varphi_2)\sigma_{(123)}^{i}\right)$$

$$= \sum_{i=0}^{2}\sigma_{(123)}^{-i}(s^{-1}\varphi_2,1)(1,s^{-1}\varphi_2)\sigma_{(123)}^{i},$$

which is exactly the double Jacobi relation.

– $\mathrm{Ind}(\mathcal{DLie})^{\mathbf{i}}(\llbracket 1,4\rrbracket,\llbracket 1,4\rrbracket)$ is generated by φ_4, with

$$\Delta(\varphi_4) = \sum_{i=0}^{3}\sigma_{(1234)}^{-i}\big((\varphi_2,1)(1,\varphi_3)+(\varphi_3,1)(1,\varphi_2)\big)\sigma_{(1234)}^{i} \; ;$$

then

$$\partial_\Delta(s^{-1}\varphi_4) = -\sum_{i=0}^{3}\sigma_{(1234)}^{-i}\Big((-1)^{|\varphi_2|}(s^{-1}\varphi_2,1)(1,s^{-1}\varphi_3)$$

$$+ (-1)^{|\varphi_3|}(s^{-1}\varphi_3,1)(1,s^{-1}\varphi_2)\Big)\sigma_{(1234)}^{i}$$

$$= \sum_{i=0}^{3}\sigma_{(1234)}^{-i}\Big((s^{-1}\varphi_2,1)(1,s^{-1}\varphi_3)-(s^{-1}\varphi_3,1)(1,s^{-1}\varphi_2)\Big)\sigma_{(1234)}^{i}.$$

5.2. THE PROPERAD $\mathcal{DPois}$. — We define the properad $\mathcal{DPois}$ which encodes the structure of double Poisson algebra. $\mathcal{DPois}$ is the quadratic properad gives as follows:

$$\mathcal{DPois} := \mathscr{F}^{\mathrm{Val}}\big(V\oplus W\big)\big/\big\langle R_{\mathcal{As}}\oplus D_\lambda\oplus R_{DJ}\big\rangle$$

with generators concentrated in homological degree 0:

$$V := V_{\mathcal{As}} = \mu.k\otimes k[\mathfrak{S}_2] = \bigvee_{\mu}^{1\quad 2} \otimes k[\mathfrak{S}_2]$$

and

$$W := V_{\mathcal{DLie}} = \overset{1\quad 2}{\rule{1cm}{2pt}}_{1\quad 2} \otimes \; \mathrm{sgn}(\mathfrak{S}_2)\uparrow_{\mathfrak{S}_2}^{\mathfrak{S}_2\times\mathfrak{S}_2^{\mathrm{op}}}$$

and the relations

– of associativity for the product μ :

$$R_{\mathcal{As}} := \bigvee \quad - \quad \bigvee \quad ;$$

– double Jacobi for the double bracket:

$$R_{DJ} := \rule{1cm}{2pt} + \rule{1cm}{2pt} + \rule{1cm}{2pt} \; ;$$

– of derivation:

$$D_\lambda :=\;\; \text{[diagram]} \;-\; \text{[diagram]} \;-\; \text{[diagram]}.$$

We recall the following result of Vallette

Proposition 5.8 (see [Val03, Lem. 155, Props. 156 & 158]). — *Let $\mathcal{P}$ be a properad of the form $\mathcal{P} := \mathscr{P}(V \oplus W, R \oplus D_\lambda \oplus S)$, with λ, a compatible distributive law. Then we have the following isomorphism of $\mathfrak{S}$-bimodules*

$$\mathcal{P} \cong \mathcal{A} \boxtimes_c^{\mathrm{Val}} \mathcal{B}$$

with $\mathcal{A} := \mathscr{P}(V, R)$ and $\mathcal{B} := \mathscr{P}(W, S)$. Also, if the sum

$$\sum_{m,n} \dim_k\big((V \oplus W)(\llbracket 1, m \rrbracket, \llbracket 1, n \rrbracket)\big)$$

is finite and W is concentrated in homological degree 0, then we have the isomorphism of $\mathfrak{S}$-bimodules $\mathcal{P}^{\mathsf{i}} \cong \mathcal{B}^{\mathsf{i}} \boxtimes_c^{\mathrm{Val}} \mathcal{A}^{\mathsf{i}}$ with $\mathcal{A} := \mathscr{P}(V, R)$ and $\mathcal{B} := \mathscr{P}(W, S)$. Moreover, if the properads $\mathcal{A}$ and $\mathcal{B}$ are Koszul, then the properad $\mathcal{P}$ is also a Koszul properad.

For $\mathcal{DP}ois$, the relation of derivation D_λ is given by a compatible replacement law (see [Val03, Val07]), with λ the following morphism of $\mathfrak{S}$-bimodules:

$$\lambda : (I_{\boxtimes} \oplus W) \boxtimes_c^{\mathrm{Val}} (I_{\boxtimes} \oplus V)^{(1)_W,(1)_V} \longrightarrow (I_{\boxtimes} \oplus V) \boxtimes_c^{\mathrm{Val}} (I_{\boxtimes} \oplus W)^{(1)_V,(1)_W}$$

given by

$$\lambda :\;\; \text{[diagram]} \;\; \longmapsto \;\; \text{[diagram]} \;+\; \text{[diagram]}.$$

Lemma 5.9. — *The morphisms of $\mathfrak{S}$-bimodules*

$$\mathcal{A}s \boxtimes_c^{\mathrm{Val}} \mathrm{Ind}(\mathcal{DL}ie)^{(1),(2)} \longrightarrow \mathcal{DP}ois \quad and \quad \mathcal{A}s \boxtimes_c^{\mathrm{Val}} \mathrm{Ind}(\mathcal{DL}ie)^{(2),(1)} \longrightarrow \mathcal{DP}ois$$

are injectives.

Proof. — We start by considering the morphism $\mathcal{A}s \boxtimes_c^{\mathrm{Val}} \mathrm{Ind}(\mathcal{DL}ie)^{(2),(1)} \to \mathcal{DP}ois$: in $\mathcal{DP}ois$, we consider the terms

$$\text{[diagram]} \quad \text{and} \quad \text{[diagram]}.$$

In the properad $\mathcal{DP}ois$, by the relation D_λ, we have the following equalities

$$\text{[diagram]} \;=\; \text{[diagram]} \;+\; \text{[diagram]} \;+\; \text{[diagram]} \;=\; \text{[diagram]},$$

then $\mathcal{A}s \boxtimes_c^{\mathrm{Val}} \mathrm{Ind}(\mathcal{DL}ie)^{(2),(1)} \to \mathcal{DP}ois$ is injective. As the double jacobiator is a multiderivation (see [Van08a]), then the morphism $\mathcal{A}s \boxtimes_c^{\mathrm{Val}} \mathrm{Ind}(\mathcal{DL}ie)^{(1),(2)} \to \mathcal{DP}ois$ is injective   □

Corollary 5.10. — *We have the following isomorphism of properads:*

$$\mathcal{D}\mathcal{P}ois \cong \mathcal{A}s \boxtimes_c^{\mathrm{Val}} \mathrm{Ind}(\mathcal{D}\mathcal{L}ie).$$

As the properads $\mathcal{A}s$ and $\mathrm{Ind}(\mathcal{D}\mathcal{L}ie)$ are Koszul (see [LV12, Chap. 9] for the case of $\mathcal{A}s$), we obtain the main theorem of this paper.

Theorem 5.11. — *The properad $\mathcal{D}\mathcal{P}ois$ is Koszul.*

Proof. — Directly by Proposition 5.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Appendix. The algebras $\mathscr{A}(\mathcal{D}\mathcal{L}ie, n)$ are Koszul

In this section, $\mathcal{D}\mathcal{L}ie$ is the protoperad of double Lie algebras.
We consider the family of quadratic algebras $\mathscr{A}(\mathcal{D}\mathcal{L}ie, n)$, for $n \geqslant 2$, given by the quadratic datum $(V(\mathcal{D}\mathcal{L}ie, n), R(\mathcal{D}\mathcal{L}ie, n))$, with

$$V(\mathcal{D}\mathcal{L}ie, n) = \{x_{ij} \mid 1 \leqslant i < j \leqslant n\}$$

and, for n in $\mathbb{N}$,

$$R(\mathcal{D}\mathcal{L}ie, n) = \left\{ \begin{array}{c} x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} \\ x_{jk}x_{ij} - x_{ik}x_{jk} - x_{ij}x_{ik} \\ x_{ab}x_{uv} - x_{uv}x_{ab} \end{array} \middle| \begin{array}{c} 1 \leqslant i < j < k \leqslant n \\ 1 \leqslant u < v \leqslant n \\ 1 \leqslant a < b \leqslant n \\ \{a, b\} \cap \{u, v\} = \varnothing \end{array} \right\}.$$

Proposition A.1. — *For all $n \geqslant 2$, the algebra $\mathscr{A}(\mathcal{D}\mathcal{L}ie, n)$ is Koszul.*

Proof. — The algebra $\mathscr{A}(\mathcal{D}\mathcal{L}ie, 2)$ is isomorphic to $k[x]$, which is Koszul. We denote by W^n, the Koszul dual of $\mathscr{A}(\mathcal{D}\mathcal{L}ie, n)$; this quadratic algebra is given by the quadratic datum $(V(\mathcal{D}\mathcal{L}ie, n)^\vee, R(\mathcal{D}\mathcal{L}ie, n)^\perp)$:

$$V(\mathcal{D}\mathcal{L}ie, n)^\vee = \{x_{ij} \mid 1 \leqslant i < j \leqslant n\}$$

$$R(\mathcal{D}\mathcal{L}ie, n)^\perp = \left\{ \begin{array}{c} x_{ij}^2 \\ x_{ij}x_{jk} + x_{jk}x_{ik} \\ x_{ij}x_{jk} + x_{ik}x_{ij} \\ x_{ij}x_{ik} + x_{jk}x_{ij} \\ x_{ij}x_{ik} - x_{ik}x_{jk} \\ x_{ab}x_{uv} + x_{uv}x_{ab} \end{array} \middle| \begin{array}{c} 1 \leqslant i < j < k \leqslant n \\ 1 \leqslant u < v \leqslant n \\ 1 \leqslant a < b \leqslant n \\ \{a, b\} \cap \{u, v\} = \varnothing \end{array} \right\}.$$

We prove that the algebra W_n is Koszul by the rewriting method; we will follow [LV12, Chap. 4, Sect 4.1].

Step 1. — We totally order the set of generators of W^n by the right lexicographic order on indices:

$$x_{ij} < x_{k\ell} \text{ if } j < \ell \text{ or } j = \ell \text{ and } i < k.$$

Step 2. — We extend this order to the set of monomials by the left lexicographic order.

Step 3. — We obtain the following rewriting rules:

$$x_{ij}^2 \xrightarrow{\ 1\ } 0, \quad x_{jk}x_{ik} \xrightarrow{\ 2\ } -x_{ij}x_{jk}, \quad x_{ik}x_{ij} \xrightarrow{\ 3\ } -x_{ij}x_{jk}$$

$$x_{jk}x_{ij} \xrightarrow{\ 4\ } -x_{ij}x_{ik}, \quad x_{ik}x_{jk} \xrightarrow{\ 5\ } x_{ij}x_{ik}, \quad x_{uv}x_{ij} \xrightarrow{\ 6\ } -x_{ij}x_{uv}.$$

Step 4. — We test the confluence of rewriting rules for all critical monomials. Recall that a *critical monomial* is a monomial $x_{ij}x_{k\ell}x_{uv}$ such that monomials $x_{ij}x_{k\ell}$ and $x_{k\ell}x_{uv}$ can be rewrite by rewriting rules. Any critical monomial gives an oriented graph under the rewriting rules which is *confluent* if it has only one terminal vertex.

We denote by α-β the confluence diagram associated to the monomial $x_{ij}x_{k\ell}x_{uv}$ where $x_{ij}x_{k\ell}$ is the leading term (the term of the left side) of the rewriting rule α and $x_{k\ell}x_{uv}$, the leading term of the rewriting rule β. We adopt the following notation: for a monomial $x_{ij}x_{k\ell}x_{uv}$, when we use the rewriting rule α on $x_{ij}x_{k\ell}$, we denote that by

$$x_{ij}x_{k\ell}x_{uv} \xrightarrow[\alpha]{} \widetilde{x_{ab}x_{cd}}x_{uv}$$

and when we use the rewriting rule α on $x_{k\ell}x_{uv}$, we denote that by

$$x_{ij}x_{k\ell}x_{uv} \xrightarrow{\ \alpha\ } x_{ij}\widetilde{x_{ab}x_{cd}}.$$

We start with the case of $1 \leqslant i < j < k \leqslant n$ and $x_{uv} < x_{ij}$ to study diagrams of the form 1-β:
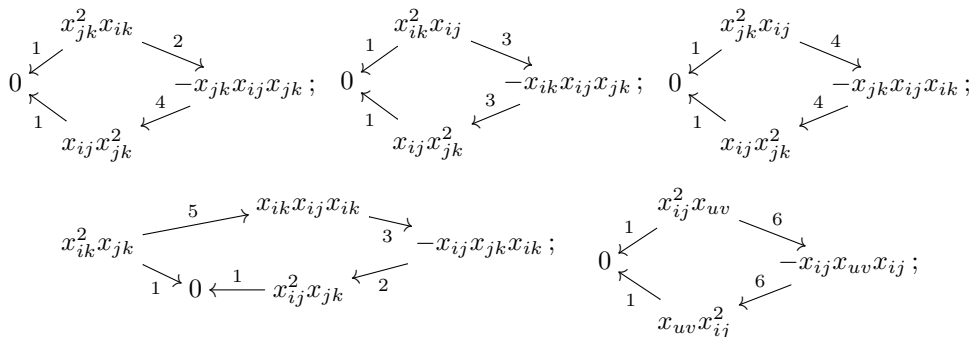
all diagrams for a critical monomial with the leading term of 1 on the left are confluent. Similarly, all diagrams α-1 are confluent.

Now, we study the diagrams for a critical monomial with the leading term of 2 on the left. We start with 2-2: let $u < i < j < k$:
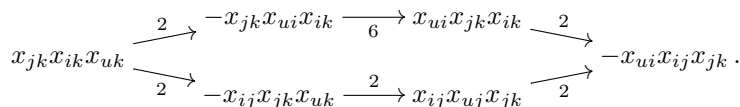
For 2-3, there are three cases: we begin with $i < j < u < k$:

for the case $i < j = u < k$, we have:

$$x_{jk}x_{ik}x_{ij} \xrightarrow{3} -x_{jk}x_{ij}x_{jk} \xrightarrow{4} x_{ij}x_{ik}x_{jk} \xrightarrow{5} x_{ij}x_{ij}x_{ik} \xrightarrow{1} 0 ;$$
$$x_{jk}x_{ik}x_{ij} \xrightarrow{2} -x_{ij}x_{jk}x_{ij} \xrightarrow{4} x_{ij}x_{ij}x_{ik} \xrightarrow{1} 0 ;$$

and, for $i < u < j < k$, we have:

$$x_{jk}x_{ik}x_{iu} \xrightarrow{3} -x_{jk}x_{iu}x_{uk} \xrightarrow{6} x_{iu}x_{jk}x_{uk} \xrightarrow{2} -x_{iu}x_{uj}x_{jk} .$$
$$x_{jk}x_{ik}x_{iu} \xrightarrow{2} -x_{ij}x_{jk}x_{iu} \xrightarrow{6} x_{ij}x_{iu}x_{jk} \xrightarrow{3} -x_{iu}x_{uj}x_{jk} .$$

For 2-4, there are only one case: let $u < i < j < k$

$$x_{jk}x_{ik}x_{ui} \xrightarrow{4} -x_{jk}x_{ui}x_{uk} \xrightarrow{6} x_{ui}x_{jk}x_{uk} \xrightarrow{4} -x_{ui}x_{uj}x_{uk} ;$$
$$x_{jk}x_{ik}x_{ui} \xrightarrow{2} -x_{ij}x_{jk}x_{ui} \xrightarrow{4} x_{ij}x_{ui}x_{uk} \xrightarrow{4} -x_{ui}x_{uj}x_{uk} ;$$

For 2-5, there is three cases: we begin with $i < u < j < k$:

$$x_{jk}x_{ik}x_{uk} \xrightarrow{5} x_{jk}x_{iu}x_{ik} \xrightarrow{6} -x_{iu}x_{jk}x_{ik} \xrightarrow{2} -x_{iu}x_{ij}x_{jk} ;$$
$$x_{jk}x_{ik}x_{uk} \xrightarrow{2} -x_{ij}x_{jk}x_{uk} \xrightarrow{2} x_{ij}x_{uj}x_{jk} \xrightarrow{5} -x_{iu}x_{ij}x_{jk} ;$$

for the case $i < j = u < k$, we have:

$$x_{jk}x_{ik}x_{jk} \xrightarrow{5} x_{jk}x_{ij}x_{ik} \xrightarrow{4} -x_{ij}x_{ik}x_{ik} \downarrow 1$$
$$x_{jk}x_{ik}x_{jk} \xrightarrow{2} -x_{ij}x_{jk}x_{jk} \xrightarrow{1} 0 \quad ;$$

and, for $i < u < j < k$, we have:

$$x_{jk}x_{ik}x_{uk} \xrightarrow{5} x_{jk}x_{iu}x_{ik} \xrightarrow{6} x_{iu}x_{jk}x_{ik} \xrightarrow{2} -x_{iu}x_{ij}x_{jk} .$$
$$x_{jk}x_{ik}x_{uk} \xrightarrow{2} -x_{ij}x_{jk}x_{uk} \xrightarrow{5} x_{ij}x_{iu}x_{jk} \xrightarrow{3} -x_{iu}x_{ij}x_{jk} .$$

For 2-6, there are three cases: we begin with $u < i < j < k$, $v < k$ and $v \neq j$:

$$x_{jk}x_{ik}x_{uv} \xrightarrow{6} -x_{jk}x_{uv}x_{ik} \xrightarrow{6} x_{uv}x_{jk}x_{ik} \xrightarrow{2} -x_{uv}x_{ij}x_{jk} ;$$
$$x_{jk}x_{ik}x_{uv} \xrightarrow{2} -x_{ij}x_{jk}x_{uv} \xrightarrow{6} x_{ij}x_{uv}x_{jk} \xrightarrow{6} -x_{uv}x_{ij}x_{jk} ;$$

and for $u < i < j < k$ and $v = j$:

$$x_{jk}x_{ik}x_{uj} \xrightarrow{6} -x_{jk}x_{uj}x_{ik} \xrightarrow{4} x_{uj}x_{uk}x_{ik} \xrightarrow{5} x_{uj}x_{ui}x_{uk} \xrightarrow{3} -x_{ui}x_{ij}x_{uk} .$$
$$x_{jk}x_{ik}x_{uj} \xrightarrow{2} -x_{ij}x_{jk}x_{uj} \xrightarrow{4} x_{ij}x_{uj}x_{uk} \xrightarrow{2} -x_{ui}x_{uk}x_{ij} \xrightarrow{6} -x_{ui}x_{ij}x_{uk} .$$

For 3-2, let $u < i < j < k$:

$$x_{ik}x_{ij}x_{uj} \xrightarrow{2} -x_{ik}x_{ui}x_{ij} \xrightarrow{2} x_{ui}x_{uk}x_{ij} \xleftarrow{6} $$
$$x_{ik}x_{ij}x_{uj} \xrightarrow{3} -x_{ij}x_{jk}x_{uj} \xrightarrow{3} x_{ij}x_{uj}x_{uk} \xrightarrow{2} -x_{ui}x_{ij}x_{uk} \, .$$

Consider the case 3-3, let $i < v < j < k$:

$$x_{ik}x_{ij}x_{iv} \xrightarrow{3} -x_{ik}x_{iv}x_{vj} \xrightarrow{3} x_{iv}x_{vk}x_{vj} \xrightarrow{3} $$
$$x_{ik}x_{ij}x_{iv} \xrightarrow{3} -x_{ij}x_{jk}x_{iv} \xrightarrow{6} x_{ij}x_{iv}x_{jk} \xrightarrow{3} -x_{iv}x_{vj}x_{jk} \, .$$

For the case 3-4, let $u < i < j < k$:

$$x_{ik}x_{ij}x_{ui} \xrightarrow{4} -x_{ik}x_{ui}x_{uj} \xrightarrow{4} x_{ui}x_{uk}x_{uj} \xrightarrow{3} $$
$$x_{ik}x_{ij}x_{ui} \xrightarrow{3} -x_{ij}x_{jk}x_{ui} \xrightarrow{6} x_{ij}x_{ui}x_{jk} \xrightarrow{4} -x_{ui}x_{uj}x_{jk} \, .$$

Consider the case 3-5, let $i < u < j < k$:

$$x_{ik}x_{ij}x_{uj} \xrightarrow{5} x_{ik}x_{iu}x_{ij} \xrightarrow{3} -x_{iu}x_{uk}x_{ij} \xrightarrow{6} $$
$$x_{ik}x_{ij}x_{uj} \xrightarrow{3} -x_{ij}x_{jk}x_{uj} \xrightarrow{4} x_{ij}x_{uj}x_{uk} \xrightarrow{5} x_{iu}x_{ij}x_{uk} \, .$$

For 3-6, there is three cases: we begin with $u < i < j < k$, $v \neq j$ and $v \neq k$:

$$x_{ik}x_{ij}x_{uv} \xrightarrow{6} -x_{ik}x_{uv}x_{ij} \xrightarrow{6} x_{uv}x_{ik}x_{ij} \xrightarrow{3} $$
$$x_{ik}x_{ij}x_{uv} \xrightarrow{3} -x_{ij}x_{jk}x_{uv} \xrightarrow{6} x_{ij}x_{uv}x_{jk} \xrightarrow{6} -x_{uv}x_{ij}x_{jk} \, ;$$

and for $u < i < j < k$, $v \neq j$ and $v = k$:

$$x_{ik}x_{ij}x_{uk} \xrightarrow{6} -x_{ik}x_{uk}x_{ij} \xrightarrow{2} x_{ui}x_{ik}x_{ij} \xrightarrow{3} $$
$$x_{ik}x_{ij}x_{uk} \xrightarrow{3} -x_{ij}x_{jk}x_{uv} \xrightarrow{2} x_{ij}x_{uj}x_{jk} \xrightarrow{2} -x_{ui}x_{ij}x_{jk} \, .$$

For 4-2, let $u < i < j < k$:

$$x_{jk}x_{ij}x_{uj} \xrightarrow{2} -x_{jk}x_{ui}x_{ij} \xrightarrow{6} x_{ui}x_{jk}x_{ij} \xrightarrow{4} $$
$$x_{jk}x_{ij}x_{uj} \xrightarrow{4} -x_{ij}x_{ik}x_{uj} \xrightarrow{6} x_{ij}x_{uj}x_{ik} \xrightarrow{2} -x_{ui}x_{ij}x_{ik} \, .$$

Consider the case 4-3, let $i < u < j < k$:

$$x_{jk}x_{ij}x_{iu} \xrightarrow{3} -x_{jk}x_{iu}x_{uj} \xrightarrow{6} x_{iu}x_{jk}x_{uj} \xrightarrow{4} $$
$$x_{jk}x_{ij}x_{iu} \xrightarrow{4} -x_{ij}x_{jk}x_{iu} \xrightarrow{3} x_{ij}x_{iu}x_{uk} \xrightarrow{3} -x_{iu}x_{uj}x_{uk} \, .$$

Consider the case 4-4, let $u < i < j < k$:

$$x_{jk}x_{ij}x_{ui} \xrightarrow{4} -x_{jk}x_{ui}x_{uj} \xrightarrow{6} x_{ui}x_{jk}x_{uj} \xrightarrow{4} -x_{ui}x_{uj}x_{uk} .$$
$$x_{jk}x_{ij}x_{ui} \xrightarrow{4} -x_{ij}x_{ik}x_{ui} \xrightarrow{4} x_{ij}x_{ui}x_{uk} \xrightarrow{4} -x_{ui}x_{uj}x_{uk} .$$

Consider the case 4-5, let $i < u < j < k$:

$$x_{jk}x_{ij}x_{uj} \xrightarrow{5} -x_{jk}x_{iu}x_{ij} \xrightarrow{6} x_{iu}x_{jk}x_{ij} \xrightarrow{5} -x_{iu}x_{ij}x_{ik} .$$
$$x_{jk}x_{ij}x_{uj} \xrightarrow{4} -x_{ij}x_{ik}x_{uj} \xrightarrow{6} x_{ij}x_{uj}x_{ik} \xrightarrow{5} -x_{iu}x_{ij}x_{ik} .$$

For 4-6, there are three cases: we begin with $i < u < j < k$, $v \neq j$ and $v \neq k$:

$$x_{jk}x_{ij}x_{uv} \xrightarrow{6} -x_{jk}x_{uv}x_{ij} \xrightarrow{6} x_{uv}x_{jk}x_{ij} \xrightarrow{3} -x_{uv}x_{ij}x_{ik} ;$$
$$x_{jk}x_{ij}x_{uv} \xrightarrow{4} -x_{ij}x_{ik}x_{uv} \xrightarrow{6} x_{ij}x_{uv}x_{ik} \xrightarrow{6} -x_{uv}x_{ij}x_{ik} ;$$

and for $i < u < j < k$, $v \neq j$ and $v = k$:

$$x_{jk}x_{ij}x_{uk} \xrightarrow{6} -x_{jk}x_{uk}x_{ij} \xrightarrow{2} x_{uj}x_{jk}x_{ij} \xrightarrow{4} -x_{uj}x_{ij}x_{ik}$$
$$x_{jk}x_{ij}x_{uk} \xrightarrow{4} -x_{ij}x_{ik}x_{uv} \xrightarrow{2} x_{ij}x_{ui}x_{ik} \xrightarrow{4} -x_{ui}x_{uj}x_{ik}$$
with a vertical arrow $5\downarrow$ between $-x_{uj}x_{ij}x_{ik}$ and $-x_{ui}x_{uj}x_{ik}$.

For 5-2, we have three cases. We begin with the case where $u < i < j < k$:

$$x_{ik}x_{jk}x_{uk} \xrightarrow{2} -x_{ik}x_{uj}x_{jk} \xrightarrow{6} x_{uj}x_{ik}x_{jk} \xrightarrow{5} x_{uj}x_{ij}x_{ik}$$
$$x_{ik}x_{jk}x_{uk} \xrightarrow{5} x_{ij}x_{ik}x_{uk} \xrightarrow{2} -x_{ij}x_{ui}x_{ik} \xrightarrow{4} x_{ui}x_{ij}x_{ik}$$
with a vertical arrow $5\downarrow$ between $x_{uj}x_{ij}x_{ik}$ and $x_{ui}x_{ij}x_{ik}$;

we continue with $u = i < j < k$:

$$x_{ik}x_{jk}x_{ik} \xrightarrow{2} -x_{ik}x_{ij}x_{jk} \xrightarrow{3} x_{ij}x_{jk}x_{jk}$$
$$x_{ik}x_{jk}x_{ik} \xrightarrow{5} x_{ij}x_{ik}x_{ik} \xrightarrow{1} 0$$
with a vertical arrow $\downarrow 1$ between $x_{ij}x_{jk}x_{jk}$ and 0 ;

and we finish by $i < u < j < k$:

$$x_{ik}x_{jk}x_{uk} \xrightarrow{2} -x_{ik}x_{uj}x_{jk} \xrightarrow{6} x_{uj}x_{ik}x_{jk} \xrightarrow{5} x_{uj}x_{ij}x_{ik}$$
$$x_{ik}x_{jk}x_{uk} \xrightarrow{5} x_{ij}x_{ik}x_{uk} \xrightarrow{5} x_{ij}x_{iu}x_{ik} \xrightarrow{3} -x_{iu}x_{uj}x_{ik}$$
with a vertical arrow $2\downarrow$ between $x_{uj}x_{ij}x_{ik}$ and $-x_{iu}x_{uj}x_{ik}$.

For 5-3, let $i < j < u < k$:

$$x_{ik}x_{jk}x_{ju} \xrightarrow{3} -x_{ik}x_{ju}x_{uk} \xrightarrow{6} x_{ju}x_{ik}x_{uk} \xrightarrow{5} x_{ju}x_{iu}x_{ik} .$$
$$x_{ik}x_{jk}x_{ju} \xrightarrow{5} x_{ij}x_{ik}x_{ju} \xrightarrow{6} -x_{ij}x_{ju}x_{ik} \xrightarrow{2} x_{ju}x_{iu}x_{ik} .$$

For 5-4, we have three cases. We begin with the case where $u < i < j < k$:

$$x_{ik}x_{jk}x_{uj} \xrightarrow{4} -x_{ik}x_{uj}x_{uk} \xrightarrow{6} x_{uj}x_{ik}x_{uk} \xrightarrow{2} -x_{uj}x_{uj}x_{ik}$$
$$x_{ik}x_{jk}x_{uj} \xrightarrow{5} x_{ij}x_{ik}x_{uj} \xrightarrow{6} -x_{ij}x_{uj}x_{ik} \xrightarrow{2} x_{ui}x_{ij}x_{ik}$$

with the vertical arrow labeled $3\downarrow$;

we continue with $u = i < j < k$:

$$x_{ik}x_{jk}x_{ij} \xrightarrow{4} -x_{ik}x_{ij}x_{ik} \xrightarrow{3} x_{ij}x_{jk}x_{ik} \xrightarrow{2} -x_{ij}x_{ij}x_{jk}$$
$$x_{ik}x_{jk}x_{ij} \xrightarrow{5} x_{ij}x_{ik}x_{ij} \xrightarrow{3} x_{ij}x_{ij}x_{jk} \xrightarrow{1} 0$$

with the vertical arrow labeled $1\downarrow$;

and we finish by $i < u < j < k$:

$$x_{ik}x_{jk}x_{uj} \xrightarrow{4} -x_{ik}x_{uj}x_{uk} \xrightarrow{6} x_{uj}x_{ik}x_{uk} \xrightarrow{5} x_{uj}x_{iu}x_{ik}$$
$$x_{ik}x_{jk}x_{uj} \xrightarrow{5} x_{ij}x_{ik}x_{uj} \xrightarrow{6} -x_{ij}x_{uj}x_{ik} \xrightarrow{5} -x_{iu}x_{ij}x_{ik}$$

with the vertical arrow labeled $4\downarrow$.

For 5-5, we have three cases. We begin with the case where $i < j < u < k$:

$$x_{ik}x_{jk}x_{uk} \xrightarrow{5} -x_{ik}x_{ju}x_{uk} \xrightarrow{6} x_{ju}x_{ik}x_{uk} \xrightarrow{5}$$
$$x_{ik}x_{jk}x_{uk} \xrightarrow{5} x_{ij}x_{ik}x_{uk} \xrightarrow{5} x_{ij}x_{iu}x_{ik} \xrightarrow{4} x_{ju}x_{ij}x_{ik} .$$

For 5-6, there are many cases: we begin with $i < j < k$, $u \neq i$ and $j < v < k$:

$$x_{ik}x_{jk}x_{uv} \xrightarrow{6} -x_{ik}x_{uv}x_{jk} \xrightarrow{6} x_{uv}x_{ik}x_{jk} \xrightarrow{5}$$
$$x_{ik}x_{jk}x_{uv} \xrightarrow{5} x_{ij}x_{ik}x_{uv} \xrightarrow{6} -x_{ij}x_{uv}x_{ik} \xrightarrow{6} -x_{uv}x_{ij}x_{ik} ;$$

for $i < j < k$, $u, v \neq i$ and $v < j$:

$$x_{ik}x_{jk}x_{uv} \xrightarrow{6} -x_{ik}x_{uv}x_{jk} \xrightarrow{6} x_{uv}x_{ik}x_{jk} \xrightarrow{5}$$
$$x_{ik}x_{jk}x_{uv} \xrightarrow{5} x_{ij}x_{ik}x_{uv} \xrightarrow{6} -x_{ij}x_{uv}x_{ik} \xrightarrow{6} -x_{uv}x_{ij}x_{ik} ;$$

for $u = i < j < v < k$:

$$x_{ik}x_{jk}x_{iv} \xrightarrow{6} -x_{ik}x_{iv}x_{jk} \xrightarrow{3} x_{iv}x_{vk}x_{jk} \xrightarrow{2}$$
$$x_{ik}x_{jk}x_{iv} \xrightarrow{5} x_{ij}x_{ik}x_{iv} \xrightarrow{3} x_{ij}x_{iv}x_{vk} \xrightarrow{5} -x_{iv}x_{jv}x_{vk} ;$$

for $u = i < v < j < k$:

$$x_{ik}x_{jk}x_{iv} \xrightarrow{6} -x_{ik}x_{iv}x_{jk} \xrightarrow{3} x_{iv}x_{vk}x_{jk} \xrightarrow{5}$$
$$x_{ik}x_{jk}x_{iv} \xrightarrow{5} x_{ij}x_{ik}x_{iv} \xrightarrow{3} -x_{ij}x_{iv}x_{vk} \xrightarrow{3} x_{iv}x_{vj}x_{vk} ;$$

for $u < i = v < j < k$:

$$x_{ik}x_{jk}x_{ui} \xrightarrow{6} -x_{ik}x_{ui}x_{jk} \xrightarrow{4} x_{ui}x_{uk}x_{jk} \xrightarrow{5} x_{ui}x_{uj}x_{uk}\,.$$
$$x_{ik}x_{jk}x_{ui} \xrightarrow{5} x_{ij}x_{ik}x_{ui} \xrightarrow{4} -x_{ij}x_{ui}x_{uk} \xrightarrow{4} x_{ui}x_{uj}x_{uk}\,.$$

Finally, consider the diagrams 6-β. We start with the case 6-2; there are two cases: $i < j < k < v$ and $u \neq i, j, k$:

$$x_{uv}x_{jk}x_{ik} \xrightarrow{2} -x_{uv}x_{ij}x_{jk} \xrightarrow{6} x_{ij}x_{uv}x_{jk} \xrightarrow{6} -x_{ij}x_{jk}x_{uv}\,;$$
$$x_{uv}x_{jk}x_{ik} \xrightarrow{6} -x_{jk}x_{uv}x_{ij} \xrightarrow{6} x_{jk}x_{ij}x_{uv} \xrightarrow{2} -x_{ij}x_{jk}x_{uv}\,;$$

and the case $i < j < k < v$ and $u = i$:

$$x_{iv}x_{jk}x_{ik} \xrightarrow{2} -x_{iv}x_{ij}x_{jk} \xrightarrow{3} x_{ij}x_{jv}x_{jk} \xrightarrow{3} -x_{ij}x_{jk}x_{kv}\,.$$
$$x_{iv}x_{jk}x_{ik} \xrightarrow{6} -x_{jk}x_{iv}x_{ik} \xrightarrow{3} x_{jk}x_{ik}x_{kv} \xrightarrow{2} -x_{ij}x_{jk}x_{kv}\,.$$

For the case 6-3, we begin with the case where $i < j < k < v$ and $u \neq i, j, k$:

$$x_{uv}x_{ik}x_{ij} \xrightarrow{3} -x_{uv}x_{ij}x_{jk} \xrightarrow{6} x_{ij}x_{uv}x_{jk} \xrightarrow{6} -x_{ij}x_{jk}x_{uv}\,;$$
$$x_{uv}x_{ik}x_{ij} \xrightarrow{6} -x_{ik}x_{uv}x_{ij} \xrightarrow{6} x_{ik}x_{ij}x_{uv} \xrightarrow{3} -x_{ij}x_{jk}x_{uv}\,;$$

and the case $i < j < k < v$ and $u = j$:

$$x_{iv}x_{jk}x_{ik} \xrightarrow{3} -x_{iv}x_{ij}x_{jk} \xrightarrow{3} x_{ij}x_{jv}x_{jk} \xrightarrow{3} -x_{ij}x_{jk}x_{kv}\,.$$
$$x_{iv}x_{jk}x_{ik} \xrightarrow{6} -x_{jk}x_{iv}x_{ik} \xrightarrow{3} x_{jk}x_{ik}x_{kv} \xrightarrow{2} -x_{ij}x_{jk}x_{kv}\,.$$

For the case 6-4, we begin with the case where $i < j < k < v$ and $u \neq i, j, k$:

$$x_{uv}x_{jk}x_{ij} \xrightarrow{4} -x_{uv}x_{ij}x_{ik} \xrightarrow{6} x_{ij}x_{uv}x_{ik} \xrightarrow{6} -x_{ij}x_{ik}x_{uv}\,;$$
$$x_{uv}x_{jk}x_{ij} \xrightarrow{6} -x_{jk}x_{uv}x_{ij} \xrightarrow{6} x_{jk}x_{ij}x_{uv} \xrightarrow{4} -x_{ij}x_{ik}x_{uv}\,;$$

and the case $i < j < k < v$ and $u = i$:

$$x_{iv}x_{jk}x_{ij} \xrightarrow{5} -x_{iv}x_{ij}x_{ik} \xrightarrow{3} x_{ij}x_{jv}x_{ik} \xrightarrow{6} -x_{ij}x_{ik}x_{jv}\,.$$
$$x_{iv}x_{jk}x_{ij} \xrightarrow{6} -x_{jk}x_{iv}x_{ij} \xrightarrow{3} x_{jk}x_{ij}x_{jv} \xrightarrow{4} -x_{ij}x_{ik}x_{jv}\,.$$

For the case 6-5, we begin with the sub-case where $i < j < k < v$ and $u \neq i, j, k$:

$$x_{uv}x_{ik}x_{jk} \xrightarrow{5} x_{uv}x_{ij}x_{ik} \xrightarrow{6} -x_{ij}x_{uv}x_{ik} \xrightarrow{6} x_{ij}x_{ik}x_{uv}\,;$$
$$x_{uv}x_{ik}x_{jk} \xrightarrow{6} -x_{ik}x_{uv}x_{jk} \xrightarrow{6} x_{ik}x_{jk}x_{uv} \xrightarrow{5} x_{ij}x_{ik}x_{uv}\,;$$

and the case $i < j < k < v$ and $u = j$:

$$x_{jv}x_{ik}x_{jk} \xrightarrow{5} x_{jv}x_{ij}x_{ik} \xrightarrow{4} -x_{ij}x_{iv}x_{ik} \xrightarrow{3} x_{ij}x_{ik}x_{kv} .$$
$$x_{jv}x_{ik}x_{jk} \xrightarrow{6} -x_{ik}x_{jv}x_{jk} \xrightarrow{3} x_{ik}x_{jk}x_{kv} \xrightarrow{5} x_{ij}x_{ik}x_{kv} .$$

For the case 6-6, there are two sub-cases: the first one is for $b < j < v$ and $u \neq a$:

$$x_{uv}x_{ij}x_{ab} \xrightarrow{6} x_{uv}x_{ab}x_{ij} \xrightarrow{6} -x_{ab}x_{uv}x_{ij} \xrightarrow{6} -x_{ab}x_{ij}x_{uv} ;$$
$$x_{uv}x_{ij}x_{ab} \xrightarrow{6} -x_{ij}x_{uv}x_{ab} \xrightarrow{6} x_{ij}x_{ab}x_{uv} \xrightarrow{6} -x_{ab}x_{ij}x_{uv} ;$$

and the second is for $b < j < v$ and $u = a$:

$$x_{av}x_{ij}x_{ab} \xrightarrow{6} x_{av}x_{ab}x_{ij} \xrightarrow{3} -x_{ab}x_{bv}x_{ij} \xrightarrow{6} x_{ab}x_{ij}x_{bv} .$$
$$x_{av}x_{ij}x_{ab} \xrightarrow{6} -x_{ij}x_{av}x_{ab} \xrightarrow{3} x_{ij}x_{ab}x_{bv} \xrightarrow{6} x_{ab}x_{ij}x_{bv} .$$

Since all diagrams are confluent, the algebra W_n is Koszul. Hence, for all integers $n \geqslant 2$, the algebra $\mathscr{A}(\mathcal{DLie}, n)$ is Koszul. $\qquad\square$

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