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Abstract

We deal with the incompressible Navier-Stokes equations, in two and three dimensions, when some vortex patches are prescribed as initial data i.e. when there is an internal boundary across which the vorticity is discontinuous. We show -thanks to an asymptotic expansion- that there is a sharp but smooth variation of the fluid vorticity into a internal layer moving with the flow of the Euler equations; as long as this later exists and as $t << 1/\nu$, where ν is the viscosity coefficient.

1. Introduction

The equations of incompressible fluid mechanics read

$$\partial_t v^\nu + v^\nu \cdot \nabla v^\nu + \nabla p^\nu = \nu \Delta v^\nu \qquad (1)$$

$$\operatorname{div} v^\nu = 0, \qquad (2)$$

where v^ν and p^ν respectively denote the velocity and the pressure of the fluid and $\nu \geqslant 0$ is the viscosity coefficient. When $\nu = 0$ the equations (1)-(2) are referred as the **Euler** equations whereas $\nu > 0$ corresponds to the **Navier-Stokes** equations. Here we will consider the academic case where the spatial derivative x runs into the whole space $\mathbb{R}^d$ for $d = 2$ or 3.

We deal with the class of initial data of the **vortex patches**. More precisely let be given an open subset $\mathcal{O}_{0,+}$ of $\mathbb{R}^d$ of Holder class $C^{s+1,r}$ where s is in $\mathbb{N}$ and $0 < r < 1$. This means that there exists a function $\varphi_0 \in C^{s+1,r}(\mathbb{R}^d, \mathbb{R})$ such that an equation of the boundary $\partial \mathcal{O}_{0,+}$ is given by $\partial \mathcal{O}_{0,+} = \{\varphi_0 = 0\}$, such that $\mathcal{O}_{0,\pm} = \{\pm\varphi_0 > 0\}$, where $\mathcal{O}_{0,-}$ is the interior of the complementary of $\mathcal{O}_{0,+}$, and $\nabla\varphi_0 \neq 0$ in a neighborhood of $\partial \mathcal{O}_{0,+}$. We assume that the boundary $\partial \mathcal{O}_{0,+}$ is bounded. We consider a divergence free initial velocity v_0 in $L^2_{loc}(\mathbb{R}^d)$ whose vorticity $\omega_0 := \operatorname{curl} v_0$ is in the Holder space $C^{s,r}_c(\mathcal{O}_{0,\pm})$, that is a compactly supported vorticity which is $C^{s,r}$ on each side of $\partial \mathcal{O}_{0,+}$.

Numerous results about the existence of solutions for such data (and even for more general ones) are available either for the case of the Euler equations and for

the Navier-Stokes equations. The main goal of this paper is to obtain an expansion for the solutions of the Navier-Stokes equations which describes as well as possible their behaviour with respect not only to the variables t, x but also to the viscosity coefficient ν, in the limit $\nu t \to 0$. However it is useful to gather first from the literature the following compendium of results regarding the inviscid case $\nu = 0$.

Theorem 1.1 (Chemin, Gamblin and X. Saint-Raymond, P. Zhang and Q. J. Qiu, C. Huang, P. Serfati). *There exists $T > 0$ (which can be taken arbitrarily large when $d = 2$) and a unique solution $v^0 \in L^\infty(0, T; Lip(\mathbb{R}^d))$ to the* **Euler** *equations:*

$$\partial_t v^0 + v^0 \cdot \nabla v^0 + \nabla p^0 = 0, \tag{3}$$

$$div\ v^0 = 0, \tag{4}$$

with v_0 as initial velocity. Moreover for each $t \in (0, T)$ the vorticity

$$\omega^0 := curl_x\ v^0(t) \tag{5}$$

is $C_c^{s,r}(\mathcal{O}_\pm(t))$ where $\mathcal{O}_\pm(t)$ are respectively the transported by the flow of $\mathcal{O}_{0,\pm}$ at time t that is $\mathcal{O}_\pm(t) := \mathcal{X}^0(t, \mathcal{O}_{0,\pm})$ where $\mathcal{X}^0$ is the flow of particle trajectories defined by $\partial_t \mathcal{X}^0(t, x) = v^0(t, \mathcal{X}^0(t, x))$ with initial data $\mathcal{X}^0(0, x) = x$. For each $t \in (0, T)$ the boundary $\partial\mathcal{O}_+(t)$ of the domain $\mathcal{O}_\pm(t)$ is $C^{s+1,r}$ and is given by the equation $\partial\mathcal{O}_+(t) = \{\varphi^0(t, .) = 0\}$, where

$$\varphi^0 \in L^\infty(0, T; C^{1,r}(\mathbb{R}^d)) \cap L^\infty(0, T; C^{s+1,r}(\mathcal{O}_\pm(t)))$$

verifies

$$D\varphi^0 = 0, \tag{6}$$

$$\varphi^0|_{t=0} = \varphi_0, \tag{7}$$

where D denote the vector field $D := \partial_t + v^0 \cdot \nabla$. Moreover $\mathcal{O}_\pm(t) = \{\pm\varphi^0(t, .) > 0\}$ and there exists $\eta > 0$ such that for $0 \leqslant t \leqslant T$, and x such that $|\varphi^0(t, x)| < \eta$ the vector $n(t, x) := \nabla_x \varphi^0(t, x)$ satisfies $n(t, x) \neq 0$. For each $t \in (0, T)$ the function $(\omega^0.n)(t, .)$ is $C^{0,r}$ on $\{|\varphi^0(t, .)| < \eta\}$. Finally the internal boundary $\partial\mathcal{O}_+(t)$ is analytic with respect to time and the restrictions on each side of the boundary of the flow $\mathcal{X}^0$ are also analytic with respect to time with values in $C^{s+1,r}$.

The two-dimensional case was proved first by Chemin [3] (see also his recent survey [4]). This result was very surprizing since it yielded a negative answer to a conjecture by Majda in [12] on the particular case of an initial vorticity ω_0 which is the characteristic function of a bounded domain of class $C^{s+1,r}$. The persistence of piecewise smoothness of the vorticity (Holder regularity up to the boundary) was proved later first by Depauw in the case $s = 0$ in [5] and by Huang [10] in the general case s in $\mathbb{N}$ with a Lagrangian approach. Chemin's approach was extended to the three dimensional case $d = 3$ by Gamblin and X. Saint-Raymond in [8]. The tangential smoothness in the general case $s \in \mathbb{N}$ was alluded in the comment (ii) of section $1.d$ of Gamblin and Saint-Raymond and rigorously proved by P. Zhang and Q. J. Qiu in [17]. The persistence of piecewise $C^{0,r}$ smoothness (the case $s = 0$) was proved by Huang in [11] by mean of a Lagrangian approach (see also [6] section 3.1). The proof of the persistence of higher order piecewise $C^{s,r}$ smoothness is given in [16]. The study of the smoothness of the boundary w.r.t. time began in Chemin's pioneering work [2] and was extended into analyticity w.r.t. time by P. Serfati [15] (see also its doctoral thesis).

Here we want to show that the solutions of the Navier-Stokes equations benefit from a *conormal* smoothing of the initial vorticity discontinuity into a layer of width $\sqrt{\nu t}$ around the hypersurface $\{\varphi^0(t, .) = 0\}$ where the discontinuity has been transported at the time t by the flow of the Euler equations. Hence the fluid vorticity

$$\omega^\nu := \operatorname{curl} v^\nu \tag{8}$$

depends -innerly- on the extra "fast" scale variable: $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ and will be described by an expansion of the form

$$\omega^\nu(t, x) \sim \Omega(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}}), \tag{9}$$

where the *viscous profile* $\Omega(t, x, X)$ admits some limits when $X \to \pm\infty$.

Remark 1.1. The idea to associate a viscous profile to an inviscid discontinuity seems to date back to Rankine [13] and is widely known when the discontinuity is a shock as for instance in compressible fluid mechanics (see the recent achievements by Guès, Métivier, Williams and Zumbrun, for instance in [9]). However since they are characteristic and conservative the vortex patches are very different from the shocks of the compressible fluid mechanic (which are noncharacteristic and dissipative). We therefore would like to precise that we borrow the words "viscous profile" to the setting of shocks profiles but that our setting is quite different. For instance extra scales involved are not the same in the two cases.

2. A highly simplified model

Let us first look at the 1d scalar heat equation:

$$\partial_t \omega^\nu = \nu \partial_x^2 \omega^\nu$$

which plays here the role of "baby model" for the Navier-Stokes equations. We prescribe as initial data a discontinuous vorticity: $\omega^\nu|_{t=0} = 1_{\mathbb{R}_+}$, where $1_{\mathbb{R}_+}$ denotes the characteristic function of $\mathbb{R}_+$. In the inviscid case $\nu = 0$ -which stands for (highly) simplified Euler equations- the solution is simply equal to the initial data $\omega^0(t, .) := 1_{\mathbb{R}_+}$ for any time, whereas for $\nu > 0$ and $t > 0$ one can explicitly compute the solutions ω^ν by convolution:

$$\omega^\nu(t, x) := \Omega(\frac{x}{\sqrt{\nu t}}) \text{ where } \Omega(X) := \frac{1}{\sqrt{\pi}} \int_{-\frac{X}{2}}^{\infty} e^{-y^2} dy.$$

Hence the initial discontinuity of the vorticity is smoothed out into a layer of size $\sqrt{\nu t}$ where occurs -smoothly- the transition between the values 0 and 1. It is useful to rewrite the ω^ν as:

$$\omega^\nu(t, x) := \omega^0(t, x) + \tilde{\Omega}_\pm(\frac{x}{\sqrt{\nu t}}) \quad \text{when} \quad \pm x > 0,$$

where

$$\tilde{\Omega}_\pm(X) := \frac{1}{\sqrt{\pi}} \int_{-\frac{X}{2}}^{\mp\infty} e^{-y^2} dy \quad \text{when} \quad \pm X > 0.$$

One then see the "viscous" solutions ω^ν as the sum of the "inviscid" solution ω^0 plus a "double initial-(internal) boundary layer" $\tilde{\Omega}_\pm$ which satisfies the double-ODE:

$$\partial_X^2 \tilde{\Omega}_\pm + \frac{X}{2}\partial_X \tilde{\Omega}_\pm = 0 \quad \text{when} \quad \pm X > 0,$$

and which matches the continuity conditions of ω^ν and $\partial_x \omega^\nu$ at the internal boundary $x = X = 0$ (for $t > 0$):

$$\omega^0|_{x=0^+} + \tilde{\Omega}_+|_{X=0^+} = 1 - 1/2 = 0 + 1/2 = \omega^0|_{x=0^-} + \tilde{\Omega}_-|_{X=0^-},$$

$$\partial_X \tilde{\Omega}_+|_{X=0^+} = \frac{1}{2\sqrt{\pi}} = \partial_X \tilde{\Omega}_-|_{X=0^-}$$

and vanishes $\tilde{\Omega}_\pm(X) \to 0$ in the limits $X \to \pm\infty$.

3. Inner scale in the Navier-Stokes equations

Of course the case of the Navier-Stokes equations is really much more complicated than the previous baby model. In particular the inviscid discontinuity moves: Theorem 1.1 states that the inviscid discontinuity occurs at the hypersurface $\{\varphi^0(t,.) = 0\}$ given by the eikonal equations (6)-(7) associated to the the particle derivative D. Therefore we expect that the solutions ω^ν of the Navier-Stokes equations with vortex patches as initial data can be described by an expansion of the form

$$\omega^\nu(t,x) \sim \omega^0(t,x) + \tilde{\omega}^\nu(t,x) \tag{10}$$

where $\tilde{\omega}^\nu$ denotes a perturbation mainly local and conormally self-similar, that is depending on the extra inner scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$, so that

$$\tilde{\omega}^\nu(t,x) := \tilde{\Omega}(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}), \tag{11}$$

with

$$\lim_{X \to \pm\infty} \tilde{\Omega}(t,x,X) = 0. \tag{12}$$

The consequences of the condition (12) on the profile $\tilde{\Omega}(t,x,X)$ at the level of the function $\tilde{\omega}^\nu$ are threefold:

1. For any $(t,\nu) \in (0,T) \times \mathbb{R}_+^*$, $\tilde{\omega}^\nu(t,x) \to 0$ when $\varphi^0 \to \pm\infty$. This was actually our motivation to impose the condition (12) on the profile $\tilde{\Omega}(t,x,X)$. It sounds natural that the viscous layer is confined to the neighbourhood of the hypersurface where the inviscid discontinuity occurs.

2. For any $t \in (0,T)$, for any $x \in \mathbb{R}^d \setminus \partial \mathcal{O}_+(t)$, $\tilde{\omega}^\nu(t,x) \to 0$ when $\nu \to 0^+$. This consequence is directly linked with another strong underlying motivation to this work that is the issue of the inviscid limit of the Navier-Stokes equation to the Euler ones. The "strength" of this inviscid limit (that is the space where it holds) does not depend only on the presence or not of material boundaries but also on the smoothness of the initial data.

3. For any $(x,\nu) \in \mathbb{R}^d \times \mathbb{R}_+^*$, $\tilde{\omega}^\nu(t,x) \to 0$ when $t \to 0^+$. This yields that the Navier-Stokes vorticities ω^ν have the same initial value than the Euler one ω^0.

4. Amplitudes

We now pay attention to the expected order of amplitudes of velocity and pressure profiles. In the full plane the Biot-Savart law has Fourier symbol $-\frac{\xi}{|\xi|^2} \wedge$. It is a pseudo-local operator of order -1 so that we expect that the velocity v^ν given by the Navier-Stokes equations can be described by an asymptotic expansion of the form:

$$v^\nu(t, x) \quad \sim \quad v^0(t, x) + \sqrt{\nu t} \, \tilde{V}(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}}), \tag{13}$$

where the profile $\tilde{V}(t, x, X)$ is also expected to satisfy

$$\lim_{X \to \pm\infty} \tilde{V}(t, x, X) \quad = \quad 0. \tag{14}$$

Plugging (10), (11) and (13) into the relations (8), taking into account (5) and equalling the leading order terms leads to

$$n \wedge \partial_X \tilde{V} = \tilde{\Omega}. \tag{15}$$

Hence the vorticity profile $\tilde{\Omega}$ has to satisfy the orthogonality condition:

$$\tilde{\Omega}.n = 0. \tag{16}$$

This condition is not a surprise: since w^0 is divergence free $w^0.n$ is continuous so that no (large amplitude) layer is expected on the normal component of the vorticity.

Now the pressure p^ν can be recovered from the velocity v^ν by applying the operator divergence to the equation (1) which yields the Laplace problem:

$$\triangle_x p^\nu = -\partial_i v_j^\nu . \partial_j v_i^\nu. \tag{17}$$

If the velocity v^ν satisfies the expansion (13), the r.h.s. of (17) should admit an expansion of the form:

$$\triangle_x p^\nu \quad \sim \quad \triangle_x p^0 + \tilde{F}(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}}), \tag{18}$$

where the function $\tilde{F}$ vanishes for $X \to \pm\infty$. Since the Laplacian is of order -2 we are lead to consider a perturbation of order νt on the pressure:

$$p^\nu(t, x) \quad \sim \quad p^0(t, x) + \nu t \tilde{P}(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}}), \tag{19}$$

where -once again- the fast scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ is expected to be a local inner scale (since the Laplacian operator is pseudo-local) so that

$$\lim_{X \to \pm\infty} \tilde{P}(t, x, X) \quad = \quad 0. \tag{20}$$

5. Looking for a profile equation

We plug the ansatz (13) and (19) into the equation (1), equalling the leading order terms, which are of order $\sqrt{\nu t}^0$:

$$Dv^0 + \nabla p^0 + D\varphi^0 . \partial_X \tilde{V} = 0. \tag{21}$$

which is satisfied since the velocity v^0 satisfies the Euler equations (3)-(4) and φ^0 satisfies the eikonal equation (6). At the following order $\sqrt{\nu t}$ we get the equality

$$D\tilde{V} + \tilde{V} \cdot n\, \partial_X \tilde{V} + \tilde{V} \cdot \nabla_x v^0 + \partial_X \tilde{P}\, n = \frac{1}{t}(|n|^2 \partial_X^2 \tilde{V} + \frac{X}{2} \partial_X \tilde{V} - \frac{1}{2}\tilde{V}). \qquad (22)$$

We now pay attention to the divergence free condition. Plugging the ansatz (13) into the equation (2), retaining the terms at order $\sqrt{\nu t}^0$ and taking into account that the velocity v^0 given by Euler is divergence free leads to the orthogonality equation: $n.\partial_X \tilde{V} = 0$, which by integration, with the condition (14) leads to the condition:

$$n.\tilde{V} = 0. \qquad (23)$$

An important consequence of the condition (23) is to kill the second term in (22) which is the only nonlinear one.

The equation (22) involves both $\tilde{V}$ and $\tilde{P}$. However the pressure in the NS equations is not truly an unknown but can be recovered from the velocity (as recalled in (17)) so that we expect that the same holds for the profiles. One way to proceed is to project normally the equation (22), to take into account that the (non-unit) normal vector $n(t, x)$ satisfies the equation:

$$Dn = -{}^t(\nabla v^0).n. \qquad (24)$$

and to use the condition (23) to get that

$$\partial_X \tilde{P} := -2 \frac{(\tilde{V} \cdot \nabla_x v^0).n}{|n|^2}. \qquad (25)$$

We now use the equation (25) to get rid of the pressure profile into the equation (22). Inverting the two sides and dividing by t, we have:

$$|n|^2 \partial_X^2 \tilde{V} + \frac{X}{2} \partial_X \tilde{V} - \frac{1}{2}\tilde{V} = t(D\tilde{V} + \tilde{V} \cdot \nabla_x v^0 - 2\frac{(\tilde{V} \cdot \nabla_x v^0).n}{|n|^2}n), \qquad (26)$$

The vector field n may vanish, away the patch boundary, hence so may do the coefficient in front of the leading order in the equation (26). To remedy to this we consider a function a in the space

$$\mathcal{B} := L^\infty\Big([0, T], C^{0,r}(\mathbb{R}^d)\Big) \cap L^\infty\Big(0, T; C^{s,r}\big(\mathcal{O}_\pm(t)\big)\Big) \qquad (27)$$

satisfying the condition

$$\inf_{[0,T] \times \mathbb{R}^d} a =: c > 0 \qquad (28)$$

and such that $a = |n|^2$ when $|\varphi^0| < \eta$, and we consider for the profile $V(t, x, X)$ the *linear* partial differential equation:

$$LV = 0 \qquad (29)$$

where the differential operator L is given by

$$L := \mathcal{E} - t(D + A)$$

where $\mathcal{E}$ and A are some operators of respective order 2 and 0 acting formally on functions $V(t, x, X)$ as follows:

$$\mathcal{E}V := a\partial_X^2 V + \frac{X}{2}\partial_X V - \frac{1}{2}V \text{ and } AV := V \cdot \nabla_x v^0 - 2\frac{(V \cdot \nabla_x v^0).n}{a}n.$$

The substitution of a instead of $|n|^2$ is almost harmless since their values are different only for $|\varphi^0| \geqslant \eta$, so that the corresponding values of the (expected, so far) solutions $V(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}})$ and $\tilde{V}(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}})$ respectively given by the equations (26) and (29) both tend to 0 as $\sqrt{\nu t}$ tends to 0, because of the condition (14).

6. Transmission conditions

Because of the parabolic nature of the Navier Stokes equations, we expect that v^ν and ω^ν are continuous including through $\varphi^0 = 0$ (these are the Rankine-Hugoniot conditions associated to the problem), which lead to the transmission conditions: $\tilde{V}$ and $\omega^0 + \tilde{\Omega}$ should be continuous, which (taking into account the equalities (15), (16) and (23)) is equivalent to the transmission conditions: $\tilde{V}$ and $n \wedge \omega^0 - |n|^2 \partial_X \tilde{V}$ should be continuous. More precisely this means a priori that

$$\tilde{V}|_{X=0^+, \varphi^0=0^+} - \tilde{V}|_{X=0^-, \varphi^0=0^-} = 0, \tag{30}$$

$$|n|^2 \partial_X \tilde{V}|_{X=0^+, \varphi^0=0^+} - |n|^2 \partial_X \tilde{V}|_{X=0^-, \varphi^0=0^-} = -(n \wedge \omega^0|_{\varphi^0=0^+} - n \wedge \omega^0|_{\varphi^0=0^-}) \tag{31}$$

Since X is the placeholder for $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ the function $\tilde{V}(t, x, X)$ needs to be defined only when X and $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ share the same sign. However it is useful to look for a profile $V(t, x, X)$ defined for (t, x, X) in the whole domain

$$\mathcal{D} := (0, T) \times \mathbb{R}^d \times \mathbb{R}.$$

As a consequence we will actually look at the following transmission conditions: for any $(t, x) \in (0, T) \times \mathbb{R}^d$,

$$[V] = 0 \quad \text{and} \quad [\partial_X V] = -\frac{n \wedge (\omega_+^0 - \omega_-^0)}{a}, \tag{32}$$

where the brackets denote the jump $[V] = V|_{X=0^+} - V|_{X=0^-}$ across $\{X = 0\}$ and where $\omega_\pm^0$ are two functions in $L^\infty\left((0, T), C^{s,r}(\mathbb{R}^d)\right)$ such that $\omega_\pm^0|_{\mathcal{O}_\pm(t)} = \omega^0$.

At the end the vorticity profile Ω in the expansion (9) would be constructed as

$$\Omega(t, x, X) := \omega_\pm^0(t, x) + n(t, x) \wedge \partial_X V(t, x, X), \quad \text{for} \quad \pm X > 0.$$

7. Well-posedness

In the previous sections we have derived the equations (29) and the transmission conditions (32). We want here to discuss their nature. First the condition (28) yields some ellipticity with respect to X for the operator $\mathcal{E}$. Roughly speaking the equation (29) is therefore hyperbolic in t, x and parabolic in t, X, but degenerates for $t = 0$ into an elliptic equation in X. In particular the hypersurface $\{t = 0\}$ is characteristic for the equation (29) so that none initial condition at $t = 0$ has to be prescribe. At the interface $\{X = 0\}$ the transmission conditions (32) are normal for the operator $\mathcal{E}$. As a matter of fact the equation (29), with the transmission conditions (32) on the interface $\{X = 0\}$ and the conditions (14) for X at infinities are well-posed in appropriate spaces.

Actually we will use here a L^2 setting, for two reasons: first in view to future extensions we want to give a claim hopefully robust. In particular it is well-known

since [1] that in (multi-dimensional) compressible fluid mechanics the inviscid system should be tackled in L^2-type spaces. This will to robustness is also the reason why we choose to put the emphasis on the velocity in this presentation, more than on the vorticity. The second reason for a L^2 setting is linked to the degeneracy at $t = 0$ of the equation (29), which leads to the existence of parasite solutions. For instance if we look for solutions V not depending on X and neglecting the term involving A the equation (29) simplifies into the Fuschian differential equation $t\partial_t V = -V/2$, which admits an infinity of solutions i.e. $V(t) = C/\sqrt{t}$, for $C \in \mathbb{R}$. However only one is in $L^2(0, T)$, corresponding to $C = 0$; and of course we expect that our scaling is relevant enough to have a solution with $L^2(0, T)$ smoothness (at least), even in the case of the full equation (29). Let us give a precise statement: denoting E_1 the space

$$E_1 := L^2\big((0, T) \times \mathbb{R}^d, H^1(\mathbb{R})\big)$$

we will prove:

Theorem 7.1. *For any $f \in E_1'$, for any $g \in L^2\big((0, T) \times \mathbb{R}^d\big)$ there exists exactly one solution $V(t, x, X) \in E_1$ of $LV = f$ on the both sides $\mathcal{D}_\pm := (0, T) \times \mathbb{R}^d \times \mathbb{R}_\pm^*$ with the transmission conditions $([V], [\partial_X V]) = (0, g)$ on the interface $\Gamma := (0, T) \times \mathbb{R}^d \times \{0\}$. In addition the function $\sqrt{t}\|V(t, ., .)\|_{L^2(\mathbb{R}^d \times \mathbb{R})}$ is continuous on $(0, T)$.*

Proof. The equation $LV = f$ is satisfied in the sense of distributions on both sides $\mathcal{D}_\pm$. Since V is in E_1 the jump $[V]$ is in $L^2(\Gamma)$. The sense given to the jump of the derivative $[\partial_X V]$ is actually a part of the problem. The idea is to give some sense by using the equation put in a weak form thanks to Green's formula.

We will explain this process a little bit more in details in a few lines but let us first point out here that because of its unbounded coefficient X the operator $\mathcal{E}$ does not strictly enter in the classical theory of elliptic operators (with, of course, t, x as parameter through the coefficient a). As a consequence we consider $\sigma > 0$ and a smooth function χ_σ such that $\chi_\sigma(X) = X$ for $|X| < \sigma$, $\chi_\sigma(X) = 3\sigma/2$ for $|X| > 2\sigma$ and $\|\chi_\sigma'\|_{L^\infty(\mathbb{R})} < 1$. We will work with the modified operators

$$L_\sigma := L + \frac{\chi_\sigma(X) - X}{2}\partial_X$$

whose coefficients are bounded. Even more the coefficients of the first order part are Lipschitz. We will also use the operators

$$\tilde{L} := L - a\partial_X^2 \text{ and } \tilde{L}_\sigma := \tilde{L} + \frac{\chi_\sigma(X) - X}{2}\partial_X.$$

Now we are ready to recast the problem into a weak form. For any V in the space $E_2 := \{V \in C_0(\mathcal{D}) / \ V|_{\mathcal{D}_\pm} \in C^\infty\}$ and W in $H^1(\mathcal{D})$ we have, integrating by parts, the following Green identity:

$$\sum_\pm \int_{\mathcal{D}_\pm} L_\sigma V.W = \int_{\mathcal{D}}(V.\tilde{L}_\sigma^* W - a\partial_X V.\partial_X W) - \int_\Gamma a[\partial_X V].W - T\int_{\tilde{\Gamma}} W.V \quad (33)$$

where $\tilde{\Gamma} := \{T\} \times \mathbb{R}^d \times \mathbb{R}$ and where $\tilde{L}_\sigma^*$ denotes the operator (the adjoint of $\tilde{L}_\sigma$)

$$\tilde{L}_\sigma^* V := -\frac{\chi_\sigma(X)}{2}\partial_X V - \frac{1}{2}(1 + \chi_\sigma'(X))V + t(DV + (1 + \operatorname{div} v^0 - A)V). \quad (34)$$

In fact less smoothness is needed. Let us introduce the Hilbert space $E_4 := \{V \in E_1 / \ L_\sigma V \in H^{-1}(\mathcal{D})\}$ endowed with the norm $\|V\|_{E_4} := \|V\|_{E_1} + \|L_\sigma V\|_{H^{-1}(\mathcal{D})}$.

Thanks to a classical lemma by Friedrichs [7] the space E_2 is dense in E_4. Moreover we have:

Lemma 7.1. *The map*

$$V \in E_2 \mapsto \tau := \begin{cases} a[\partial_X V] & \boldsymbol{on}\ \ \Gamma \\ TV & \boldsymbol{on}\ \ \tilde{\Gamma} \end{cases} \tag{35}$$

extends uniquely to a continuous linear map from E_4 to $H^{-\frac{1}{2}}(\Gamma \cup \tilde{\Gamma})$ and Green's identity (33) is still valid for any couple (V, W) in $E_4 \times H^1(\mathcal{D})$ in the generalized sense that

$$< L_\sigma V, W >_{H^{-1}(\mathcal{D}), H^1(\mathcal{D})} = \int_{\mathcal{D}} (V.\tilde{L}_\sigma^* W - a\partial_X V.\partial_X W) - < \tau, W|_{\Gamma \cup \tilde{\Gamma}} >_{H^{-\frac{1}{2}}(\Gamma \cup \tilde{\Gamma}), H^{\frac{1}{2}}(\Gamma \cup \tilde{\Gamma})}. \tag{36}$$

Proof. Let V be in E_2 and $\tilde{W}$ be in $H^{\frac{1}{2}}(\Gamma \cup \tilde{\Gamma})$. There exists a function W in $H^1(\mathcal{D})$ such that $W|_{\Gamma \cup \tilde{\Gamma}} = \tilde{W}$. From Green's identity (33) we infer that

$$|\int_{\Gamma \cup \tilde{\Gamma}} \tau \tilde{W}| \leqslant C\|V\|_{E_4}\|W\|_{H^1(\mathcal{D})} \leqslant C\|V\|_{E_4}\|\tilde{W}\|_{H^{\frac{1}{2}}(\Gamma \cup \tilde{\Gamma})}.$$

Hence by Hahn-Banach theorem we get the existence of a continuous extension, which is unique because of the density stated above. □

We therefore have given a meaning to the problem. This meaning can seem weak but the next result said that is actually quite strong. We denote here the space $H^{1,2}(\mathcal{D})$ of the functions $V \in H^1(\mathcal{D})$ such that $\partial_X^2 V|_{\mathcal{D}_\pm}$ are in L^2.

Lemma 7.2. *If $V \in E_4$ satisfies the jump conditions $[V] = 0$ and $[\partial_X V] = g$ on Γ in the sense given by Lemma 7.1 then there exists a sequence V^ε in $H^{1,2}(\mathcal{D})$ converging to V in E_4 and a sequence g^ε converging to g in $L^2((0,T) \times \mathbb{R}^d)$ such that $[V^\varepsilon] = 0$ and $[\partial_X V^\varepsilon] = g^\varepsilon$ on Γ.*

Proof. As this kind of process is very classical, see for instance Rauch [14], we only briefly sketch the proof. The idea is to construct the sequence V^ε by convoluting in the variables t, x only to preserve the jump conditions, to use Friedrichs lemma to prove the convergence in E_4 and then to gain the extra X derivative, that is to prove that the V^ε are in $H^{1,2}(\mathcal{D})$, thanks to the equation. □

We will now prove uniqueness as a consequence of the following estimate: for any function V in E_1 satisfying the transmission problem there holds

$$\|V\|_{E_1} \lesssim \|f\|_{E_1'} + \|g\|_{L^2(\mathbb{R}^d \times \mathbb{R})}. \tag{37}$$

The idea to prove this is to prove first that for smooth functions there holds

$$\|V\|_{E_1} + \sqrt{T}\|V(T,.,.)\|_{L^2(\mathbb{R}^d \times \mathbb{R})} \lesssim \|f\|_{E_1'} + \|g\|_{L^2(\mathbb{R}^d \times \mathbb{R})} \tag{38}$$

and then to proceed by density.

Now in order to prove the existence part of Theorem 7.1 we shall need another Green formula which involves the complete transposition of the operator L. At a smooth level that is for any V, W in the space E_2 this Green identity reads:

$$\sum_\pm \int_{\mathcal{D}_\pm} L_\sigma V.W = \int_{\mathcal{D}} L_\sigma^* W.V - \int_\Gamma a[\partial_X V].W + \int_\Gamma aV.[\partial_X W] - T\int_{\tilde{\Gamma}} W.V \tag{39}$$

where $L_\sigma^* = a\partial_X^2 + \tilde{L}_\sigma^*$ denotes the adjoint of the operator L_σ. Proceeding as previously we get the following weak extension of the previous Green identity:

Lemma 7.3. *The map*

$$V \in E_2 \mapsto \tau := \begin{cases} a[\partial_X V] & \boldsymbol{on} \ \ \Gamma \\ TV & \boldsymbol{on} \ \ \tilde{\Gamma} \end{cases} \tag{40}$$

extends uniquely to a continuous linear map from E_4 to $H^{-\frac{3}{4}}(\Gamma \cup \tilde{\Gamma})$ and Green's identity (39) is still valid for any couple V, W in $E_4 \times H^{1,2}(\mathcal{D})$ in the generalized sense that

$$< L_\sigma V, W >_{H^{-1}(\mathcal{D}), H^1(\mathcal{D})} = < L_\sigma^* W, V >_{E_1', E_1} - < \tau, W|_{\Gamma \cup \tilde{\Gamma}} >_{H^{-\frac{1}{2}}(\Gamma \cup \tilde{\Gamma}), H^{\frac{1}{2}}(\Gamma \cup \tilde{\Gamma})} \tag{41}$$

$$+ \int_\Gamma [\partial_X W].aV|_\Gamma.$$

It is therefore classical to infer the existence of a solution to the problem by using Riesz's theorem and the estimate (37) for the adjoint operator.

We then use again a sequence of approximation of the solution, and use the linearity of the problem together with the a-priori estimate (38) to show that this sequence is a Cauchy sequence in the space of the functions V such that $\sqrt{t}\|V(t,.,.)\|_{L^2(\mathbb{R}^d \times \mathbb{R})}$ is continuous on $(0, T)$. Completeness yields the conclusion.

We finally let σ goes to infinity. The estimate (38) is uniform with respect to σ. Using weak compactness we can pass to the weak limit so that we get a solution to the original equation. $\qquad \square$

In the case of the transmission conditions (32) the source terms are orthogonal to n. It is then possible to use the uniqueness part of the previous theorem to prove that the function $V(t, x, X) \cdot n(t, x)$ vanishes identically, what is self-consistent with the condition (23) found in section 5 when looking for a profile problem.

8. Smoothness

We now show that the solution inherits the smoothness with respect to the usual variables t, x from the coefficients; and which are piecewise smooth with respect to the fast variable X. To do that we define for any Frechet space E of functions depending on t, x and possibly on X the space

$$E_D := \{f \in E/ \ \exists C > 0/ \ (\frac{D^k f}{C^k k!})_{k \in \mathbb{N}} \text{ is bounded in } E\}.$$

The last claim of Theorem 1.1 says that ω^0 and n are in in the space $\mathcal{B}_D$ (see (27) for the definition of the space $\mathcal{B}$). It is also possible to construct the functions $\omega_\pm^0$ and a in the space $\mathcal{B}_D$.

We denote $p - \mathcal{S}(\mathbb{R})$ the space of the functions $f(X)$ whose restrictions to the half-lines $\mathbb{R}_\pm$ are in the Schwartz space of rapidly decreasing functions, and $\mathcal{A}$ the space (of the functions $f(t, x, X)$):

$$\mathcal{A} := L^\infty\Big((0, T), C^{0,r}\big(\mathbb{R}^d, p - \mathcal{S}(\mathbb{R})\big)\Big) \cap L^\infty\Big(0, T; C^{s,r}\big(\mathcal{O}_\pm(t), p - \mathcal{S}(\mathbb{R})\big)\Big).$$

Theorem 8.1. *There exists exactly one function $V(t, x, X) \in \mathcal{A}_D$ satisfying the equation (29) for $\pm X > 0$ and the transmission conditions (32).*

Proof. Lifting the r.h.s. of the transmission conditions (32) we reduce the problem to prove that the solution $\tilde{V}$ of

$$L\tilde{V} = \tilde{f} \text{ on } \mathcal{D}_{\pm}, \quad [\tilde{V}] = [\partial_X \tilde{V}] = 0 \text{ on } \Gamma, \tag{42}$$

with $\tilde{f} \in \mathcal{A}_D$, is also in $\mathcal{A}_D$.

Let us now first prove that $\tilde{V}$ is in $\mathcal{C}_D$ where we denote $\mathcal{C} := L^{\infty}\big((0,T) \times \mathbb{R}^d, H^1(\mathbb{R})\big)$. We establish an a-priori estimate, applying for any $k \in \mathbb{N}$ the field D^k to the problem (42) to get

$$(\mathcal{E} - k)\tilde{V}^{[k]} = \tilde{f}^{[k]} + t\tilde{V}^{[k+1]} + tA\tilde{V}^{[k]} \text{ on } \mathcal{D}_{\pm}, \quad [\tilde{V}^{[k]}] = [\partial_X \tilde{V}^{[k]}] = 0 \text{ on } \Gamma, \tag{43}$$

where we denote by $V^{[k]} := D^k\tilde{V}$ the kth iterated derivative of $\tilde{V}$ along D and where $\tilde{f}^{[k]} := \sum_{l=1}^3 \tilde{f}_l^{[k]}$, where $\tilde{f}_1^{[k]} := D^k\tilde{f}$ whereas $\tilde{f}_2^{[k]}$ and $\tilde{f}_3^{[k]}$ denote respectively the commutators:

$$\tilde{f}_2^{[k]} := [D^k, \mathcal{E}] = \sum_{l=0}^{k-1} \binom{k}{l} D^{k-l}a.\partial_X^2 \tilde{V}^{[l]}, \quad \tilde{f}_3^{[k]} := -[D^k, tA] = -\sum_{l=0}^{k-1} \binom{k}{l} D^{k-l}tA.\tilde{V}^{[l]}.$$

The last sums have to be omitted when $k = 0$. Here we have used that $[D^k, tD] = kD^k$. We now multiply the first equation of (43) by $\tilde{V}^{[k]}$ and we now integrate w.r.t. X only. This yields for any $t, x \in (0,T) \times \mathbb{R}^d$ the estimate:

$$\int_{\mathbb{R}} a|\partial_X \tilde{V}^{[k]}|^2 + (k + \tfrac{3}{4}) \int_R |\tilde{V}^{[k]}|^2 \leqslant |\int_{\mathbb{R}} \tilde{f}^{[k]}.\tilde{V}^{[k]}| + \int_{\mathbb{R}} t|\tilde{V}^{[k+1]}.\tilde{V}^{[k]}| + C_1 \int_{\mathbb{R}} t|\tilde{V}^{[k]}|^2. \tag{44}$$

Using the condition (28) the l.h.s. of (44) is larger than

$$c\int_{\mathbb{R}} |\partial_X \tilde{V}^{[k]}|^2 + \frac{3}{4}(k+1) \int_R |\tilde{V}^{[k]}|^2.$$

We now bound the r.h.s. of (44). Using Cauchy-Schwarz and Young inequalities we have

$$\int_{\mathbb{R}} |\tilde{f}_1^{[k]}.\tilde{V}^{[k]}| \leqslant \frac{4}{k+1} \int_{\mathbb{R}} |\tilde{f}^{[k]}|^2 + \frac{k+1}{4} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2. \tag{45}$$

Integrating by parts yields

$$|\int_{\mathbb{R}} \tilde{f}_2^{[k]}.\tilde{V}^{[k]}| \leqslant \sum_{l=0}^{k-1} \binom{k}{l} \int_{\mathbb{R}} |D^{k-l}a\partial_X \tilde{V}^{[l]}\partial_X \tilde{V}^{[k]}|. \tag{46}$$

Since a is analytic, there exists $C_a > 0$ s.t. for any $l \in \mathbb{N}$ $||D^l a||_{\mathcal{B}} \leqslant C_a^l l!$ so that using Cauchy-Schwarz and Young inequalities yield

$$|\int_{\mathbb{R}} \tilde{f}_2^{[k]}.\tilde{V}^{[k]}| \leqslant C_2(\sum_{l=0}^{k-1} \frac{k!}{l!}C_a^{k-l}||\partial_X \tilde{V}^{[l]}||)^2 + \frac{c}{2} \int_{\mathbb{R}} |\partial_X \tilde{V}^{[k]}|^2, \tag{47}$$

where we denote here $||f|| := (\int_{\mathbb{R}} |f(t,x,X)|^2 dX)^{\frac{1}{2}}$. In a similar way there exists $C_3, C_A > 0$ s.t.

$$|\int_{\mathbb{R}} \tilde{f}_3^{[k]}.\tilde{V}^{[k]}| \leqslant C_3(\sum_{l=0}^{k-1} \frac{k!}{l!}C_A^{k-l}||\tilde{V}^{[l]}||)^2 + \frac{1}{8} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2. \tag{48}$$

Finally for $0 < t < \underline{T}$ we have

$$|\int_{\mathbb{R}} t\tilde{V}^{[k+1]}.\tilde{V}^{[k]}| \leqslant \frac{k+1}{8} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2 + \frac{4\underline{T}}{k+1} \int_{\mathbb{R}} |\tilde{V}^{[k+1]}|^2. \tag{49}$$

Hence

$$\frac{c}{2}\int_{\mathbb{R}}|\partial_X \tilde{V}^{[k]}|^2 + \frac{k+1}{4}\int_{\mathbb{R}}|\tilde{V}^{[k]}|^2 \leqslant \frac{4}{k+1}\int_{\mathbb{R}}|\tilde{f}^{[k]}|^2 + C_2(\sum_{l=0}^{k-1}\frac{k!}{l!}C_a^{k-l}||\partial_X \tilde{V}^{[l]}||)^2$$

$$+C_3(\sum_{l=0}^{k-1}\frac{k!}{l!}C_A^{k-l}||\tilde{V}^{[l]}||)^2 + \frac{4\underline{T}}{k+1}\int_{\mathbb{R}}|\tilde{V}^{[k+1]}|^2 + C_1\underline{T}\int_{\mathbb{R}}|\tilde{V}^{[k]}|^2$$

thus we infer -keeping the notations C_1-C_3 for their squareroots- that

$$\frac{c}{4}||\partial_X \tilde{V}^{[k]}|| + \frac{\sqrt{k+1}}{8}||\tilde{V}^{[k]}|| \leqslant \frac{2}{\sqrt{k+1}}||\tilde{f}^{[k]}|| + C_2\sum_{l=0}^{k-1}\frac{k!}{l!}C_a^{k-l}||\partial_X \tilde{V}^{[l]}|| \qquad (50)$$

$$+C_3\sum_{l=0}^{k-1}\frac{k!}{l!}C_A^{k-l}||\tilde{V}^{[l]}|| + \sqrt{\frac{4\underline{T}}{k+1}}||\tilde{V}^{[k+1]}|| + C_1\sqrt{\underline{T}}||\tilde{V}^{[k]}||.$$

We introduce the functions

$$a_k(t,x) := \frac{||\tilde{V}^{[k]}||}{k!C^k}, \ b_k(t,x) := \frac{||\partial_X \tilde{V}^{[k]}||}{k!C^k\sqrt{k+1}} \text{ and } f_k(t,x) := \frac{||\tilde{f}^{[k]}||}{(k+1)!C^k},$$

where C is a positive real which will be chosen in a few lines. Dividing the estimate (50) by $k!C^k\sqrt{k+1}$ yields

$$\frac{c}{4}b_k + \frac{1}{8}a_k \leqslant 2f_k + C_2\sum_{l=0}^{k-1}(\frac{C_a}{C})^{k-l}b_l + C_3\sum_{l=0}^{k-1}(\frac{C_A}{C})^{k-l}a_l + \sqrt{4\underline{T}}Ca_{k+1} + C_1\sqrt{\underline{T}}a_k.(51)$$

We choose C large enough so that $\max(\frac{C_2}{\frac{C}{C_A}-1}, \frac{C_3}{\frac{C}{C_a}-1}) \leqslant \min(\frac{c}{8}, \frac{1}{16})$ and then T is chosen small enough so that $\sqrt{4\underline{T}}C \leqslant \frac{1}{64}$ and that $C_1\sqrt{\underline{T}} \leqslant \frac{1}{64}$. Hence summing over $k \in \mathbb{N}$ the estimates (51) yield the a-priori estimate: for any $t, x \in (0, \underline{T}) \times \mathbb{R}^d$

$$\sum_{k\in\mathbb{N}}(\frac{c}{8}b_k + \frac{1}{32}a_k) \leqslant 2\sum_{k\in\mathbb{N}}f_k. \qquad (52)$$

We now define the iterative scheme $(\tilde{V}^n)_{n\in\mathbb{N}}$ by setting $\tilde{V}^0$ as the solution of

$$\mathcal{E}\tilde{V}^0 = \tilde{f} \text{ on } \mathcal{D}_\pm, \ [\tilde{V}^0] = [\partial_X \tilde{V}^0] = 0 \text{ on } \Gamma,$$

and $\tilde{V}^{n+1}$ as the solution of

$$\mathcal{E}\tilde{V}^{n+1} = \tilde{f} + t(D+A)\tilde{V}^n \text{ on } \mathcal{D}_\pm, \ [\tilde{V}^{n+1}] = [\partial_X \tilde{V}^{n+1}] = 0 \text{ on } \Gamma.$$

Let us now only briefly explain our strategy to conclude and refer to [16] for the complete proof. We first prove that $\tilde{V}^0$ is in $\mathcal{C}_D$. Then proceeding as in the proof of the estimate (52) we infer the convergence of the iterative scheme to a solution $\tilde{V}$ of the problem (42). Using several time slices yields that V is in $\mathcal{C}_D$. Now to prove Theorem 8.1, we increase the smoothness with respect to x thanks to a Paley-Littlewood spectral localization. If we denote by $\tilde{\mathcal{B}}$ the space of the functions $f(t, x, X)$ with $\mathcal{B}$ smoothness in t, x with values in $H^1(\mathbb{R})$, this yields that $\tilde{V}$ is in $\tilde{\mathcal{B}}_D$. Then we prove by induction that $X^k\tilde{V}$ is in $\tilde{\mathcal{B}}_D$ for all k in $\mathbb{N}$. Finally we use the equation to increase by induction the number of derivatives with respect to X to get that $\tilde{V}$ is in $\tilde{\mathcal{A}}_D$.

$\square$

9. Complete expansion

If piecewise smoothness of the initial data is sufficient it is possible to continue the expansion with respect to νt of the solutions of the Navier-Stokes equations. At the extreme limit if the initial data is piecewise smooth on each side of the interface $\{\varphi^0 = 0\}$ -that is if $s = +\infty$- then it is possible to write a complete formal asymptotic expansion of the Navier-Stokes velocities of the form:

$$v^\nu(t,x) = v^0(t,x) + \sum_{j \geqslant 1} \sqrt{\nu t}^j \; V^j(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}) + O(\sqrt{\nu t}^\infty), \tag{53}$$

where the profile V^1 is the one constructed in the previous sections that is $V^1 := V$. This corresponds to a complete asymptotic expansion of the vorticity of the form:

$$\omega^\nu(t,x) = \sum_{j \geqslant 0} \sqrt{\nu t}^j \; \Omega^j(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}) + O(\sqrt{\nu t}^\infty), \tag{54}$$

As a teaser for our paper [16] where the complete construction is done, we give an insight of the method. We follow a Rankine-Hugoniot approach looking for some profiles V^j such that the expansion (53) solve -formally- the equations (1)-(2) on each side $\{\pm\varphi^0 > 0\}$ plus the continuity of the velocity, of the pressure and of the vorticity on the interface $\{\varphi^0 = 0\}$. Taking the divergence and the normal scalar product of the equation (1) yields the pressure problem:

$$\Delta p^\nu = -\text{div} \; (v^\nu \cdot \nabla v^\nu) \tag{55}$$

$$[p^\nu] = 0, \tag{56}$$

$$[\partial_n \; p^\nu] = [(-v^\nu \cdot \nabla v^\nu + \nu \Delta v^\nu).n], \tag{57}$$

where the notation $[\cdot]$ stands for the jump across the interface $\{\varphi^0 = 0\}$ (that is for a piecewise smooth function $f(t,x)$: we denote $[f] := f|_{\varphi^0=0^+} - f|_{\varphi^0=0^-}$). If we plug the expansion (53) into the r.h.s. of the equations (55)-(57) we are led to look for a pressure expansion of the form

$$p^\nu(t,x) = p^0(t,x) + \sum_{j \geqslant 2} \frac{\sqrt{\nu t}^j}{t} \; P^j(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}) + O(\sqrt{\nu t}^\infty), \tag{58}$$

The profiles above are of the following form: for $\pm X > 0$,

$$U(t,x,X) \quad := \quad \underline{U}(t,x) + \tilde{U}(t,x,X), \tag{59}$$

where the function $\tilde{U}(t,x,X)$ is rapidly decreasing when $\pm X \to \infty$, and the letter U is the placeholder for the Ω^j, the V^j and the P^j. We will refer to the term $\underline{U}$ as the regular part and to the term $\tilde{U}$ as the layer part. The layer part $\tilde{P}^2$ is equal to $\tilde{P}^2 = tP$ where P is the profile of the previous sections. This possibility to be smoothly factorized by t is very particular to the order 2. It comes from the fact that the (a-priori) main contribution given by the Laplacian in the r.h.s. of the equation (57) vanishes thanks to the orthogonality property (23). Furthermore integrating the equation (25), for $\pm X > 0$, between X and $\pm\infty$, and taking the condition at infinity (20) into account, yields

$$\tilde{P} := -2 \int_X^{\pm\infty} \frac{(\tilde{V} \cdot \nabla_x v^0).n}{|n|^2}, \tag{60}$$

so that the profile $\tilde{P}$ is discontinuous across $\{X = 0\}$. Hence to satisfy the pressure continuity we have to add to the layer part $\tilde{P}^2$ a regular part $\underline{P}^2$.

To determine the velocity and pressure profiles we proceed iteratively, determining at the step j the velocity profile V^j, the regular part of the pressure profile $\underline{P}^j$ and the layer part of the following pressure profile $\tilde{P}^{j+1}$, from the profiles already known by the previous steps.

10. Stability

To describe the stability of these expansions we introduce the set $\mathcal{F}$ of the families $(f^\nu(t,x))_\nu$ of the smooth functions such that for any $s' \in \mathbb{N}$, for any $r' \in (0,1)$, the sequence

$$(\sqrt{\nu t}^{s'+r'} \|f^\nu\|_{L^\infty((0,T),C^{s',r'}(\mathbb{R}^3))})_{0<\nu<\nu_0}$$

is bounded. Then we have

Theorem 10.1. *There exists $\nu_0 > 0$ such that for $0 < \nu < \nu_0$ for all $k \in \mathbb{N}$ for any $(t,x) \in (0,T) \times \mathbb{R}^d$*

$$\omega^\nu(t,x) = \sum_{j=0}^{k} \sqrt{\nu t}^j \; \Omega^j(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}) + \sqrt{\nu t}^{k+1} \omega_R^\nu \tag{61}$$

with $(\omega_R^\nu)_{0<\nu<\nu_0}$ in $\mathcal{F}$.

Let us point out that if it is well-known that for any $\nu \geq 0$ there exists $T^\nu > 0$ and a unique solution $v^\nu \in L^\infty(0,T^\nu; Lip(\mathbb{R}^d))$ solution of the equations (1)-(2) with v_0 as initial velocity, Theorem 10.1 proves that the lifetime $T^\nu > 0$ can be extended such that $T^\nu \geq T$. The lifetime of such expansions is the one of the solution of the Euler equation ("the ground state") which traps the main part of the nonlinearity of the problem.

We refer again to our paper [16] for the proof of Theorem 10.1 and more generally for a more detailed treatment of this topic.

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