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Analyticity for certain solutions of nonhypoelliptic differential operators on the Heisenberg group

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We consider left invariant differential operators on the Heisenberg group G with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, $[\mathfrak{g}_2, \mathfrak{g}] = 0$, where $X_1, X_2, \ldots, X_{2n}$ is a basis of $\mathfrak{g}_1$ and T a basis of $\mathfrak{g}_2$. Let p be an elliptic, homogeneous non-commuting polynomial in $2n$ variables, i.e. $p(\xi_1, \xi_2, \ldots, \xi_{2n}) \geq C\,|\xi|^d$, $C > 0$. An operator of the form $L = p(X_1, X_2, \ldots, X_{2n})$ will be said to be homogeneous and elliptic in the generating directions. It is known that L is analytic-hypoelliptic and C^∞ hypoelliptic if and only if the L^2 nullspace of L is nontrivial (see [10], [9], [7], [8], and [5]). The results announced here show that even if L is not hypoelliptic, it has a left inverse, modulo the projection onto its kernel, which preserves real analyticity, locally. More precisely, our main result is the following.

Theorem 1. Let L be a homogeneous, left invariant differential operator on the Heisenberg group G elliptic in the generating directions. Then there are distributions k_1 and k_2 such that

(1) $$Lf * k_1 = f - \Pi_1 f$$

(2) $$L(f * k_2) = f - \Pi_2 f$$

for $f \in C_0^\infty(G)$, where $\prod_1$ and $\prod_2$ are the orthogonal projections onto the L^2 nullspaces of L and its adjoint L^*, respectively, and ($*$) denotes group convolution. Furthermore, the operators $f \to f * k_i$ and $f \to \prod_i f$, $i = 1,2$, all preserve analyticity, locally.

Corollary. If u and f are smooth functions of compact support on G and

$$(3) \qquad Lu = f \quad \text{in} \quad U,$$

where U is an open set, then $u_1 = (I - \prod) u$ is analytic in every subset of U where f is, and u_1 also satisfies (3).

In the special case where $L = \square_b^0$, the boundary Laplacian operator acting on 0-forms (see [2]) the analog of Theorem 1 was given by Greiner, Kohn, and Stein [4], who derived explicit formulas for k_i and $\prod_i$. The analyticity of the projections $\prod_i$ was proved by Geller [3], who also proved the existence of distributions k_i, satisfying (1) and (2) and preserving local smoothness. The general result was conjectured by Stein [3]. See also Melin [6] for related results.

To prove Theorem 1, we use a standard reduction to the case where L is self adjoint and of high degree, in addition to satisfying the conditions of Theorem 1. The following is partly based on an idea of Beals and Greiner [1].

Theorem 2. Let L be a self-adjoint differential operator of high homogeneous degree d satisfying the conditions of Theorem 1. Then there is a closed contour Γ around 0 in $\mathbb{C}$ such that $L_\alpha = L - \alpha(-iT)^{d/2}$ is hypoelliptic for all $\alpha \in \Gamma$. There exist distributions k_α, $\alpha \in \Gamma$, *such that* $L_\alpha k_\alpha = \delta$ *and for any* $f \in C_0^\infty(G)$ *and any multi-index* β *the function* $\alpha \to \|D^\beta(f * k_\alpha)\|_{L^\infty}$ *is bounded for* α *on* Γ. *Hence define* $K,S\colon C_0^\infty(G) \to C^\infty(G)$ *by*

$$Kf = \frac{1}{2\pi i}\int_\Gamma \alpha^{-1}\, f * k_\alpha d_\alpha$$

and

$$Sf = \frac{1}{2\pi i}\int_\Gamma T^{d/2}\, f * k_\alpha d_\alpha$$

Then

$$(4) \qquad LKf = K^* Lf = f - Sf, \qquad f \in C_0^\infty(G)\ ,$$

and $S = \Pi$, *the orthogonal projection onto the* L^2 *kernel of* L. *Furthermore,* K *and* S *preserve real analyticity, locally.*

The proof of Theorem 2 first requires constructing the k_α. For this we follow the method given by Métivier [7], checking that the k_α so obtained vary well with α. The first identity in (4) follows from the self adjointness of L, while the second is immediately obtained by writing $L = L_\alpha + \alpha(-iT)^{d/2}$. The proof that $S = \Pi$ is accomplished by applying the irreducible unitary representations to both operators. Then the equality reduces to a resolvant identity, and the original identity follows by the Plancherel formula for G.

Finally, to show that K and S preserve real analyticity, it suffices to show that the operators $f \to f * k_\alpha$, each of which preserves real analyticity, satisfy estimates uniform in α for α on Γ. For this, we use the methods of the second author [9] to estimate the L^2 norms of derivatives of $f * k_\alpha$, checking again that the constants obtained may be chosen independent of α.

References

[1] R. Beals and P. Greiner, (unpublished).

[2] G. Folland E. M. Stein, "Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group", Comm. Pure Appl. Math 27 (1974) 429-522.

[3] D. Geller, "Local solvability and homogeneous distributions on the Heisenberg group", Comm. P.D.E. 5 (5), (1980) 475-560.

[4] P. Greiner, J. J. Kohn, and E. M. Stein, "Necessary and sufficient conditions for solvability of the Lewy equation", Proc. Nat. Acad. Sci. 72, (9) (1975) 3287-3289.

[5] B. Helffer et J. Nourrigat, "Caractérisation des opérateurs hypoélliptiques homogènes invariants à gauche sur une groups de Lie graduée", Comm. P.D.E. 4 (8) (1979) 899-958.

[6] A. Melin, "Parametrix constructions for some classes of right-invariant differential operators on the Heisenberg group" (preprint).

[7] G. Métivier, "Hypoélliptícité analytique sur les groupes nilpotents de rang 2", Duke Math. J. 47, 1, (1980) 195-221.

[8] C. Rockland, "Hypoellipticity on the Heisenberg group: representation-theoretic criteria" Trans. Amer. Math. Soc. 240 (1978) 1-52.

[9] D. Tartakoff, "The analytic hypoellipticity of $\bar{\partial}_b$ and related operators on non-degenerate C-R manifolds", Acta Math. 145, (3-4), (1980) 177-203.

[10] F. Trèves, "Analytic hypoellipticity of a class of pseudodifferential operators", Comm. P.D.E. 3 (1978) 475-642.