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THE THEOREM OF FINE AND WILF FOR RELATIONAL PERIODS

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Abstract. We consider relational periods, where the relation is a compatibility relation on words induced by a relation on letters. We prove a variant of the theorem of Fine and Wilf for a (pure) period and a relational period.

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1. Introduction

In 1999 Berstel and Boasson introduced the notion of a *partial word*. In their paper [1] they studied periodicity properties of partial words and presented a variant of the theorem of Fine and Wilf for partial words with one hole. Further results with more holes and on periodicity properties of partial words in general can be found in [2,3,5–7,12,13]. The motivation for this research comes partly from the study of biological sequences such as DNA, RNA and proteins [4,11].

In the article [9] we introduced *word relations* as compatibility relations of words induced by a relation on letters. We showed that partial words can be seen as words with a special word relation. The study of relational codes and relationally free monoids continued in [10]. In this article we will consider relational periods of words. We shall prove a variant of the theorem of Fine and Wilf as an example of an interaction property between a (pure) period and a relational period.

2. Word relations

For a relation $R \subseteq X \times X$ we often write $x\,R\,y$ instead of $(x, y) \in R$. The restriction of R on $Y \subseteq X$ is $R_Y = R \cap (Y \times Y)$. A relation R is a compatibility

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relation on letters if it is both reflexive and symmetric, *i.e.*, (i) $\forall x \in X : \ x\,R\,x$, and (ii) $\forall x, y \in X : \ x\,R\,y \implies y\,R\,x$. The *identity relation* on a set X is defined by $\iota_X = \{(x,x) \mid x \in X\}$ and the *universal relation* on X is defined by $\Omega_X = \{(x,y) \mid x,y \in X\}$. Subscripts are often omitted when they are clear from the context. Clearly, both ι_X and Ω_X are compatibility relations on X.

A compatibility relation $R \subseteq A^+ \times A^+$ on the set of all nonempty words over an alphabet A will be called a *word relation* if it is induced by a relation on letters, *i.e.*,

$$a_1 \cdots a_m \, R \, b_1 \cdots b_n \quad \Longleftrightarrow \quad m = n \text{ and } a_i \, R \, b_i \text{ for all } i = 1, 2, \ldots, m$$

whenever $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$. The restriction of R on letters, denoted by R_A, is called the *generating relation* of R. Words u and v satisfying $u\,R\,v$ are said to be *compatible* or, more precisely, *R-compatible*. If the words are not compatible, they are said to be *incompatible*.

Since a word relation R is induced by its restriction on letters, it can be presented by listing all pairs $\{a,b\}$ $(a \neq b)$ such that $(a,b) \in R_A$. We use the notation

$$R = \langle r_1, \ldots, r_n \rangle,$$

where $r_i = (a_i, b_i) \in A \times A$ for $i = 1, 2, \ldots, n$, to denote that R is the word relation generated by the symmetric closure of $\iota_A \cup \{r_1, \ldots, r_n\}$.

Example 2.1. In the binary alphabet $A = \{a, b\}$ the compatibility relation

$$R = \langle (a,b) \rangle = \{(a,a), (b,b), (a,b), (b,a)\}$$

makes all words with equal length compatible with each other. In the ternary alphabet $\{a, b, c\}$, where

$$S = \langle (a,b) \rangle = \{(a,a), (b,b), (a,b), (b,a), (c,c)\},$$

we have $abba\,S\,baab$ but, for instance, words abc and cac are not S-compatible.

Partial words can be interpreted as words with a word relation. The next example will express this in more detail.

Example 2.2. A *partial word* of length n over an alphabet A is a partial function

$$w \colon \{1, 2, \ldots, n\} \to A.$$

The domain $D(w)$ of w is the set of positions $p \in \{1, 2, \ldots, n\}$ such that $w(p)$ is defined. The set $H(w) = \{1, 2, \ldots, n\} \setminus D(w)$ is the set of *holes* of w. To each partial word we may associate a total word $w_\Diamond$ over the extended alphabet $A_\Diamond = A \cup \{\Diamond\}$. This *companion* of w is defined by

$$w_\Diamond(p) \quad = \quad \begin{cases} w(p) & \text{if } p \in D(w), \\ \Diamond & \text{if } p \in H(w). \end{cases}$$

Thus, the holes are marked with the "do not know" symbol $\diamondsuit$. Clearly, partial words are in one-to-one correspondence with words over $A_\diamondsuit$.

The compatibility relation of partial words is defined as follows. Let x and y be two partial words of equal length. The word y is said to *contain* the word x if $D(x) \subseteq D(y)$ and $x(k) = y(k)$ for all k in $D(x)$. Two partial words x and y are said to be *compatible* if there exists a partial word z such that z contains both x and y. Then we write $x \uparrow y$.

From another viewpoint partial words with compatibility relation $\uparrow$ can be seen as words over the alphabet $A_\diamondsuit$ with the relation

$$R_\uparrow = \langle \{(\diamondsuit, a) \mid a \in A\} \rangle.$$

Two partial words $x_\diamondsuit$ and $y_\diamondsuit$ are compatible if and only if $x_\diamondsuit\, R_\uparrow y_\diamondsuit$. Namely, two partial words are compatible if for $1 \leq i \leq |x_\diamondsuit| = |y_\diamondsuit|$ we have $x_\diamondsuit(i) = y_\diamondsuit(i)$ or at least one of the letters $x_\diamondsuit(i)$ and $y_\diamondsuit(i)$ is $\diamondsuit$. For more details, see [10].

3. Relational period

Let $x = x_1 \cdots x_n$ be a word over the alphabet A. An integer $p \geq 1$ is a *(pure) period* of x if, for all $i, j \in \{1, 2, \ldots, n\}$, we have

$$i \equiv j \pmod p \implies x_i = x_j.$$

In this case, the word x is called *(purely) p-periodic*. The smallest integer which is a period of x is called *the (minimal) period* of x. Here we denote it by $\pi(x)$, or shortly, π if the word x is clear from the context.

For words with compatibility relation R on letters, we will now define *relational periods*.

Definition 3.1. Let R be a compatibility relation on an alphabet A. For a word $x = x_1 \cdots x_n \in A^+$, an integer $p \geq 1$ is a *(relational) R-period* of x if, for all $i, j \in \{1, 2, \ldots, n\}$, we have

$$i \equiv j \pmod p \implies x_i \, R \, x_j.$$

For a word x *the minimal (relational) R-period* is denoted by $\pi_R(x)$, or shortly, π_R if the word x is clear from the context. Note that a (pure) period is a relational R-period with $R = \iota$. Note also that for the universal similarity relation $\overline{\Omega} = \langle \Omega_A \rangle$, we clearly have $\pi_{\overline{\Omega}}(x) = 1$ for any word x.

Example 3.2. Define the following compatibility relations on the alphabet $A = \{a, b, c\}$: $R = \langle (b, c) \rangle, S = \langle (a, b) \rangle, T = \langle (a, c) \rangle$. Consider the word $x = abcba$. We clearly have

$$\pi = \pi_R = 4 > \pi_S = 3 > \pi_T = 2 > \pi_{\overline{\Omega}} = 1\cdot$$

Note that in this example the universal relation on A is the only relation such that the minimal relational period of x is one. Indeed, if one is an R'-period of x, then $x_1 \, R' \, x_2$, $x_1 \, R' \, x_3$ and $x_2 \, R' \, x_3$. In other words, $R' = \langle (a,b),(a,c),(b,c)\rangle = \langle \Omega_A \rangle = \overline{\Omega}$.

The following result is an easy consequence of the definition of a relational period.

Proposition 3.3. *Let R and S be compatibility relations on A such that $R \subseteq S$. Then every R-period of a word x is an S-period of x. Moreover, $\pi_R \geq \pi_S$.*

Since $\iota \subseteq R$ by the reflexivity of a compatibility relation R, we have the following corollary.

Corollary 3.4. *Every pure period of a word x is a relational period. Thus, for a word x and for a compatibility relation R, we always have $\pi \geq \pi_R$.*

As an example of the use of relational periods we will consider periods of partial words.

Example 3.5. In [1] a partial word w is said to have a *(partial) period* p if, for all $i,j \in D(w)$,

$$i \equiv j \pmod p \implies w(i) = w(j).$$

Consider now the companion of a partial word over the alphabet $A_\diamondsuit$. Recall that

$$R_\uparrow = \langle \{(\diamondsuit, a) \mid a \in A\}\rangle.$$

The number i belongs to $D(w)$ if and only if $w_\diamondsuit(i) \neq \diamondsuit$. Thus

$$i,j \in D(w) \iff w_\diamondsuit(i), w_\diamondsuit(j) \in A_\diamondsuit \setminus \{\diamondsuit\} = A.$$

If a partial word w has a period p, then for all $i,j \in D(w)$, we have

$$i \equiv j \pmod p \implies w_\diamondsuit(i) \, R_\uparrow \, w_\diamondsuit(j),$$

since $R_\uparrow \cap (A \times A) = \iota$. If i and j do not both belong to $D(w)$, then $w_\diamondsuit(i)$ or $w_\diamondsuit(j)$ is $\diamondsuit$ and $w_\diamondsuit(i) \, R_\uparrow \, w_\diamondsuit(j)$ is clear by the definition of the relation $R_\uparrow$. Thus, p is a relational $R_\uparrow$-period of $w_\diamondsuit$. On the other hand, if p is a relational $R_\uparrow$-period of $w_\diamondsuit$, then it is a partial period of w, since we have to consider only positions without holes and $R_\uparrow \cap (A \times A) = \iota$. In other words, we have showed that these two definitions of periods are equivalent.

Note that there exists also a weaker period of partial words. After Berstel and Boasson [1] a partial word w is said to have a *local period* p if

$$i, i+p \in D(w) \implies w(i) = w(i+p).$$

This can be expressed using compatibility relation $R_\uparrow$ similarly to the example above.

4. Fine and Wilf's theorem

The theorem of Fine and Wilf [8] is well-known in combinatorics on words:

Theorem 4.1. *If a word x has periods p and q, and the length of x is at least $p+q-\gcd(p,q)$, then also $\gcd(p,q)$ is a period of x.*

Berstel and Boasson gave the following variant of this theorem for partial words with one hole in [1]. Recall that $H(w)$ denotes the set of holes of w.

Theorem 4.2. *Let w be a partial word of length n with local periods p and q. If $H(w)$ is a singleton and if $n \geq p+q$, then $\gcd(p,q)$ is a (partial) period of w.*

Furthermore, they showed that the bound $p+q$ on the length is sharp. For partial words with more holes, the theorem of Fine and Wilf was considered, for example, in [7] and [3]. There it is shown that local periods p and q make a sufficiently long word to have also the period $\gcd(p,q)$ when certain unavoidable cases *(special words)* are excluded. The bound on the length depends on the number of holes in the word. Another result concerning interaction properties of partial periods was given in [12,13]. Shur and Gamzova found bounds for the length of a word with k holes such that partial periods p and q imply the partial period $\gcd(p,q)$.

These results concerning periods of partial words are special cases of interaction properties of relational periods. Since the interaction bounds of partial periods depend on the number of holes, this means that the interaction of periods in the case of arbitrary relational periods must depend on the number of occurrences of certain letters. Namely, any non-transitive compatibility relation R must have letter relations $(x_1,x_2),(x_2,x_3) \in R$, but $(x_1,x_3) \notin R$ for some letters x_1,x_2,x_3. Then the role of the letter x_2 in $R_{\{x_1,x_2,x_3\}}$ is exactly the same as the role of the hole $\diamondsuit$ in $R_{\uparrow}$. Hence, by the results of partial words, interaction bounds for R-periods will depend on the number of occurrences of x_2. This indicates that the situation can be very complicated when we consider general non-transitive relations. The following example shows that without any additional assumption we cannot find a general bound for the interaction of relational periods.

Example 4.3. Let $R = \langle (a,b),(b,c) \rangle$. There exists an infinite (not necessarily ultimately periodic) word

$$w = w_1 w_2 w_3 \cdots = acb^{6i_1-2}acb^{6i_2-2}\cdots,$$

where the numbers $i_j \geq 1$ are chosen freely. Now w has R-periods 2 and 3. Namely,

$$w_1w_3w_5\cdots \in \{a,b\}^*, \quad w_2w_4w_6\cdots \in \{b,c\}^*,$$

and

$$w_1w_4w_7\cdots \in \{a,b\}^*, \quad w_2w_5w_8\cdots \in \{b,c\}^*, \quad w_3w_6w_9\cdots \in \{b\}^*.$$

However, 1 is not a relational R-period of the word w. For example, $(w_1,w_2) = (a,c) \notin R$. Furthermore, all numbers $2,3,4,\ldots$ are R-periods of the ultimately periodic word $w' = acbbb\cdots$, but 1 is not an R-period of w'.

Nonetheless, some period interaction results can be obtained. If the relation R is an equivalence relation, the situation is reduces to Theorem 4.1.

Proposition 4.4. *Let R be an equivalence relation. If a word x has R-periods p and q and the length of the word is at least $p+q-\gcd(p,q)$, then $\gcd(p,q)$ is an R-period of x. The bound on the length is strict.*

Proof. Let R be an equivalence relation on the alphabet A and let x be a word over the alphabet with R-periods p and q and of length $n \geq p+q-\gcd(p,q)$. Suppose that A has m equivalence classes and let their set of representatives be $\{a_1,\dots,a_m\}$. Let $B = \{b_1,\dots,b_m\}$ be another alphabet. Consider now a letter-to-letter morphism $\varphi\colon A^* \to B^*$, where for every $i \in \{1,2,\dots,m\}$, each letter belonging to an equivalence class of a_i is mapped to b_i. This mapping is clearly well defined. Then $w = \varphi(x) = w_1 \cdots w_n$ is a word over B^*. Let $i,j \in \{1,2,\dots,n\}$ satisfy $i \equiv j \pmod{p}$. Since $x_i \, R \, x_j$ by the assumption, we have $w_i = \varphi(x_i) = \varphi(x_j) = w_j$ by the definition of the morphism φ. Thus, also w has the period p. Similarly, the word w is q-periodic. By the theorem of Fine and Wilf (Th. 4.1), we therefore conclude that w is also $\gcd(p,q)$-periodic. Let now $i,j \in \{1,2,\dots,n\}$ satisfy $i \equiv j \pmod{\gcd(p,q)}$. Then $w_i = w_j$. By the definition of φ this means that $x_i = \varphi^{-1}(w_i)$ and $x_j = \varphi^{-1}(w_j)$ belong to the same equivalence class. Hence, $x_i \, R \, x_j$. This means that $\gcd(p,q)$ is a relational R-period of the word x. Of course, the bound $p+q-\gcd(p,q)$ is the best possible, since there are counter examples of the original theorem of Fine and Wilf with length $p+q-\gcd(p,q)-1$ and our statement coincides with Theorem 4.1 by choosing $R = \iota$. $\square$

As was mentioned above, the theorem of Fine and Wilf cannot be generalized for relational periods (neither to local periods) of a non-transitive compatibility relation unless some restrictions on the number of relations (holes) and exclusions of some special cases are given. On the other hand, it might be possible to get new variations of the theorem by assuming some restrictions on compatibility relations. For example, by assuming that one of the periods is pure and only the other one is relational by the relation $R \neq \iota$ we get a theorem similar to that of Fine and Wilf. The sufficient and necessary lower bounds on the length of the word w considered in the theorem are given in Table 1.

Theorem 4.5. *Let P and Q be positive integers with $\gcd(P,Q) = d$. Denote $P = pd$ and $Q = qd$. Suppose that a word w has a (pure) period Q and a relational R-period P. Let $B = B(p,q)$ be defined by Table 1. If $|w| \geq Bd$, then also $\gcd(P,Q) = d$ is an R-period of the word w. This bound on the length is sharp.*

In order to make the proof of this theorem more readable, we first prove two propositions concerning the case $d = 1$. The first one says that our lower bounds $B(p,q)$ are sufficient.

Proposition 4.6. *Let p and q be positive integers and let $\gcd(p,q) = 1$. Suppose that a word w has a (pure) period q and a relational R-period p. Let $B = B(p,q)$ be defined as in Table 1. If $|w| = B$, then 1 is a relational R-period of w.*

TABLE 1. Table of lower bounds $B(p,q)$.

$B(p,q)$	$p<q$	$p>q$
p, q odd	$\frac{p+1}{2}q$	$q+\frac{q-1}{2}p$
p odd, q even	$\frac{p+1}{2}q$	$\frac{p+1}{2}q$
p even, q odd	$q+\frac{q-1}{2}p$	$q+\frac{q-1}{2}p$

Proof. The word w is a rational power of a word of length q. Thus in w there are at most q different letters. We show that a letter in an arbitrary position $s \in \{1,2,\ldots,q\}$ is R-compatible with all the other letters of the word w. If $q=2$, then there are at most two letters in w and we may use Proposition 4.4. Hence, let us assume that $q \geq 3$.

We make the following definitions. Let b be an integer in $\{1,2,\ldots,q\}$ such that $b \equiv B \pmod{q}$ and define

$$s' = \begin{cases} B-b+s & \text{if } s \in \{1,2,\ldots,b\}, \\ B-q-b+s & \text{if } s \in \{b+1,b+2,\ldots,q\}. \end{cases}$$

By the definition, s' is the last position in w such that $s' \equiv s \pmod{q}$. Note that since $B \geq q \geq b$, we have

$$0 < s \leq B-b+s \leq B,$$

if $s \in \{1,2,\ldots,b\}$ and

$$0 < s-b \leq B-q-b+s \leq B-b < B,$$

if $s \in \{b+1,b+2,\ldots,q\}$. Let us now define two sets

$$\begin{aligned} S_1 &= \left\{ s+ip \mid i=1,2,\ldots,\left\lfloor \frac{B-s}{p} \right\rfloor \right\}, \\ S_2 &= \left\{ s'-jp \mid j=0,1,\ldots,q-\left\lfloor \frac{B-s}{p} \right\rfloor -1 \right\}. \end{aligned}$$

Note that $0 < B-s < pq$, which implies that

$$1 \leq \left\lfloor \frac{B-s}{p} \right\rfloor \leq q-1 \quad \text{and} \quad q > q - \left\lfloor \frac{B-s}{p} \right\rfloor -1 \geq q-(q-1)-1 = 0.$$

Now we prove that all elements of S_1 and S_2 belong to the set $\{1, 2, \ldots, B\}$. For the set S_1 this is clear, since

$$\max(S_1) = s + \left(\left\lfloor \frac{B-s}{p} \right\rfloor\right) p \leq s + \frac{B-s}{p} p = s + B - s = B.$$

In order to prove that the minimal element of S_2 is always positive, we have to consider two different cases.

Case 1. Let us first assume that $B = \frac{p+1}{2}q$. Then $b = q$ and $s' = B - q + s$ for all $s \in \{1, 2, \ldots, q\}$. We have

$$\begin{aligned} \min(S_2) &= s' - \left(q - \left\lfloor \frac{B-s}{p} \right\rfloor - 1\right) p > s' - \left(q - 1 - \left(\frac{B-s}{p} - 1\right)\right) p \\ &= B - q + s - qp + B - s = 2B - (p+1)q \\ &= (p+1)q - (p+1)q = 0. \end{aligned}$$

Case 2. Let us then assume that $B = q + \frac{q-1}{2}p = \frac{p+2}{2}q - \frac{p}{2}$. Now

$$\left\lfloor \frac{B-s}{p} \right\rfloor = \frac{q-1}{2} + \left\lfloor \frac{q-s}{p} \right\rfloor \geq \frac{q-1}{2},$$

since q is odd and $q \geq s$. If $s \in \{1, 2, \ldots, b\}$, then

$$\begin{aligned} \min(S_2) &= s' - \left(q - \left\lfloor \frac{B-s}{p} \right\rfloor - 1\right) p \geq s' - \left(q - \frac{q-1}{2} - 1\right) p \\ &= q + \frac{q-1}{2}p - b + s - qp + \frac{q-1}{2}p + p \\ &= q - b + s \geq s > 0. \end{aligned}$$

If $s \in \{b+1, b+2, \ldots, q\}$, then

$$\begin{aligned} \min(S_2) &\geq s' - \left(q - \frac{q-1}{2} - 1\right) p \\ &= q + \frac{q-1}{2}p - q - b + s - qp + \frac{q-1}{2}p + p \\ &= s - b > 0. \end{aligned}$$

Next we show that the set $S_1 \cup S_2$ is a complete residue system modulo q for every chosen $s \in \{1, 2, \ldots, q\}$. The elements of S_1 are pairwise incongruent modulo q, since $\gcd(p, q) = 1$ and $\left\lfloor \frac{B-s}{p} \right\rfloor \leq q$. The same holds for S_2. Assume now that for some $i \in \{1, 2, \ldots, \left\lfloor \frac{B-s}{p} \right\rfloor\}$ and for some $j \in \{0, 1, \ldots, q - \left\lfloor \frac{B-s}{p} \right\rfloor - 1\}$ we have

$$s + ip \equiv s' - jp \pmod{q}. \tag{1}$$

TABLE 2. Table of critical positions $a(p, q)$.

$a(p,q)$	$p < q$	$p > q$
p, q odd	$\frac{q-p}{2}$	q
p odd, q even	$\frac{q}{2}$	$\frac{q}{2}$
p even, q odd	q	q

This is true if and only if $(i + j)p + s - s' \equiv 0 \pmod q$. Since $B \equiv b \pmod q$, we have $s' \equiv s \pmod q$ by the definition of s'. In other words, $(i+j)p+s-s' \equiv (i+j)p \pmod q$. Since $\gcd(p, q) = 1$, equation (1) is true if and only if $(i+j) \equiv 0 \pmod q$. But this is not possible, since

$$0 < (i + j) \leq \left\lfloor \frac{B-s}{p} \right\rfloor + q - \left\lfloor \frac{B-s}{p} \right\rfloor - 1 = q - 1 < q.$$

Hence, in $S_1 \cup S_2$ we have

$$|S_1 \cup S_2| = \left\lfloor \frac{B-s}{p} \right\rfloor + q - \left\lfloor \frac{B-s}{p} \right\rfloor = q$$

pairwise incongruent elements modulo q. Let b be a letter in position t and let a be the letter in position s (and s'). Now either an element in S_1 or in S_2 is congruent to t modulo q. Hence $a\,R\,b$. □

Next we will prove that our lower bounds are necessary.

Proposition 4.7. *Let p and q be positive integers such that $\gcd(p, q) = 1$. Let $B = B(p, q)$ be defined as in Table 1. There exists a word w and a compatibility relation R such that $|w| = B - 1$, w has a (pure) period q and an R-period p but 1 is not an R-period of w.*

Proof. Like in the proof of Proposition 4.6, let $b \in \{1, 2, \ldots, q\}$ satisfy $b \equiv B \pmod q$. In addition, we define so called *critical positions* $a(p, q)$ according to Table 2. We show that it is possible to construct a word w of length $|w| = B-1$ with a pure period q and an R-period p such that the letter in the critical position is not related to the letter in the position b. Note that all these critical positions are positive integers less than or equal to q. In the sequel we denote critical positions shortly by a.

Consider now the minimal solution (i, j) for the equation

$$a + iq \equiv b + jq \pmod p, \tag{2}$$

such that i and j are nonnegative integers. By the minimal solution we mean a solution where $\max(a + iq, b + jq)$ is as small as possible. Note that if $i > j$ for some solution, then $a + (i - j)q \equiv b \pmod p$ is a smaller solution. Similarly, if $j > i$, then $a \equiv b + (j - i)q \pmod p$ is a smaller solution. Thus the minimal solution is of the form where either $i = 0$ or $j = 0$.

Since $\gcd(p, q) = 1$, we know that $\{a + iq \mid i = 0, 1, \dots, p - 1\}$ and $\{b + iq \mid i = 0, 1, \dots, p - 1\}$ are complete residue systems modulo p. Hence there exists exactly one $j \in \{0, 1, \dots, p - 1\}$ satisfying $a \equiv b + jq \pmod p$ and exactly one $i \in \{0, 1, \dots, p - 1\}$ satisfying $a + iq \equiv b \pmod p$. Furthermore, for $j \in \{1, 2, \dots, p - 1\}$, we have

$$a \equiv b + jq \pmod p \implies a + (p - j)q = a + pq - jq \equiv b \pmod p,$$

and $p - j \in \{1, 2, \dots p - 1\}$. Hence, the minimal solution of (2) is either of the form $(0, j)$ or $(p - j, 0)$.

Now we prove that in all the cases of Tables 1 and 2 the minimal solution is

$$i = 0 \quad \text{and} \quad j = \frac{B - b}{q}.$$

Note that, since $B < pq$, we have $\frac{B-b}{q} \in \{1, 2, \dots, p - 1\}$.

Consider first those cases where $B = \frac{p+1}{2}q$ and consequently $b = q$. Let $j = \frac{B-b}{q}$.

Case 1. Let p and q be both odd and $p < q$. By Table 2, we have $a = \frac{q-p}{2}$. Now $b + jq = B$ and, since q is odd, it follows that

$$(b + jq) - a = \frac{p + 1}{2}q - \frac{q - p}{2} = \frac{q + 1}{2}p \equiv 0 \pmod p.$$

Hence, $(0, \frac{B-b}{q})$ is a solution. Furthermore,

$$jq = B - b = \frac{p + 1}{2}q - q = \frac{p - 1}{2}q$$

and

$$a + (p - j)q = a + pq - \frac{p - 1}{2}q = a + \frac{p + 1}{2}q = a + B > B.$$

Hence, in the solution $(p - \frac{B-b}{q}, 0)$, we have $\max(a + iq, b + jq) > B$ whereas in the solution $(0, \frac{B-b}{q})$, we have $\max(a + iq, b + jq) = B$. Thus, $(0, \frac{B-b}{q})$ is the minimal solution.

Case 2. Suppose that p is odd and q is even. By the parity of q, $a = \frac{q}{2}$ is an integer and

$$(b + jq) - a = \frac{p + 1}{2}q - \frac{q}{2} = \frac{q}{2}p \equiv 0 \pmod p.$$

Hence, $(0, \frac{B-b}{q})$ is a solution. Like in Case 1, we have

$$a + (p - j)q = a + B > B,$$

and therefore $(0, \frac{B-b}{q})$ is the minimal solution also in this case.

Consider next those cases where $B = q + \frac{q-1}{2}p$. According to Tables 1 and 2 we have $a = q$ and q is odd. Clearly, $(i, j) = (0, \frac{B-b}{q})$ is a solution, since

$$(b + jq) - a = q + \frac{q-1}{2}p - q = \frac{q-1}{2}p \equiv 0 \pmod{p}.$$

Like above $a + (p - j)q = a + pq - B + b$. By substituting a and B we get

$$a + (p - j)q = q + \left(\frac{q-1}{2}p + \frac{q-1}{2}p + p\right) - \left(q + \frac{q-1}{2}p\right) + b = B + (p - q) + b.$$

Case 3. Assume that $p > q$. Then $p - q$ is positive and $a + (p - j)q > B$. Thus the smallest solution is not $(p - j, 0)$.

Case 4. Assume that p is even, q is odd and $p < q$. Then $b = q - \frac{p}{2}$. Moreover,

$$a + (p - j)q = B + (p - q) + q - \frac{p}{2} = B + \frac{p}{2} > B,$$

and we conclude that $(0, \frac{B-b}{q})$ is the smallest solution also in this final case.

Define now a word w in the three letter alphabet $\{\alpha, \beta, \gamma\}$ by the rule

$$w = \begin{cases} (\gamma^{a-1}\alpha\gamma^{q-a-1}\beta)^{\frac{p+1}{2}} & \text{if } B = \frac{p+1}{2}q, \\ (\gamma^{b-1}\beta\gamma^{q-b-1}\alpha)^{\frac{B-b}{q}}\gamma^{b-1}\beta & \text{if } B = q + \frac{q-1}{2}p, \end{cases} \tag{3}$$

where $a = a(p, q)$ is given by Table 2. Define further that $w' = w\beta^{-1}$. If w has a relational R-period p, then by Proposition 4.6, it has also a relational R-period 1. However, by the considerations above, the word w' does not have an R-period 1 if α and β are unrelated by the compatibility relation R. Namely, the first occasion where the distance between the letters α and β in w is a multiple of p is the case where α is in the position a and β is in the position B. □

Now we are ready to prove our main theorem.

Proof of Theorem 4.5. Suppose that a word w has a pure period Q and a relational R-period P such that $\gcd(P, Q) = d$. Let $P = pd$, $Q = qd$ and $B = B(p, q)$. Let us consider a word

$$w^{(i)} = w_i w_{i+d} \cdots w_{i+k_i d}$$

where $1 \leq i \leq d$ and $k_i = \lfloor \frac{|w|-i}{d} \rfloor \geq B$. Now $\gcd(q, p) = 1$ and the word $w^{(i)}$ has a pure period q and a relational R-period p. The period q implies that the word $w^{(i)}$ is over the alphabet $A_i = \{w_i, w_{i+d}, \ldots, w_{i+(q-1)d}\}$. Since $B > q$, also the prefix of $w^{(i)}$ of length B is over the same alphabet. By Proposition 4.6, the prefix has a relational R-period 1. In other words, all letters A_i are compatible with each other. This means that also the whole word $w^{(i)}$ has 1 as an R-period. This is true for all $i = 1, 2, \ldots, d$. Consequently, d is a relational R-period of w.

In order to prove that the bound Bd on the length of w is necessary, we give an example of a word w' of length $Bd - 1$ such that it has a period Q and an

R-period P but no R-period d. Suppose that $w^{(d)}$ is given by equation (3) and $w^{(i)} = \gamma^B$ for $i = 1, 2, \ldots, d-1$. Let w be a perfect shuffle of these d sequences, *i.e.*,

$$w = w_1^{(1)} w_1^{(2)} \cdots w_1^{(d)} w_2^{(1)} w_2^{(2)} \cdots w_2^{(d)} \cdots w_B^{(1)} w_B^{(2)} \cdots w_B^{(d)}.$$

Define further that

$$w' = w\beta^{-1}.$$

Clearly w' has a period Q and an R-period P, but by the proof of Proposition 4.7, $\gcd(P, Q) = d$ is not an R-period of w', if α and β are unrelated by the compatibility relation R. $\square$

References

[1] J. Berstel and L. Boasson, Partial words and a theorem of Fine and Wilf. *Theoret. Comput. Sci.* **218** (1999) 135–141.

[2] F. Blanchet-Sadri, A periodicity result of partial words with one hole. *Comput. Math. Appl.* **46** (2003) 813–820.

[3] F. Blanchet-Sadri, Periodicity on partial words. *Comput. Math. Appl.* **47** (2004) 71–82.

[4] F. Blanchet-Sadri, Codes, orderings, and partial words. *Theoret. Comput. Sci.* **329** (2004) 177–202.

[5] F. Blanchet-Sadri and A. Chriscoe, Local periods and binary partial words: an algorithm. *Theoret. Comput. Sci.* **314** (2004) 189–216.

[6] F. Blanchet-Sadri and S. Duncan, Partial words and the critical factorization theorem. *J. Combin. Theory Ser. A* **109** (2005) 221–245.

[7] F. Blanchet-Sadri and R.A. Hegstrom, Partial words and a theorem of Fine and Wilf revisited. *Theoret. Comput. Sci.* **270** (2002) 401–419.

[8] N.J. Fine and H.S. Wilf, Uniqueness theorem for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965) 109–114.

[9] V. Halava, T. Harju and T. Kärki, Relational codes of words. *Theoret. Comput. Sci.* **389** (2007) 237–249.

[10] V. Halava, T. Harju and T. Kärki, Defect theorems with compatibility relation. *Semigroup Forum* **76** (2008) 1–24.

[11] P. Leupold, Partial words for DNA coding. *Lect. Notes Comput. Sci.* **3384** (2005) 224–234.

[12] A.M. Shur and Y.V. Gamzova, Periods' interaction property for partial words, in *Proceedings of Words'03*, edited by T. Harju and J. Karhumäki. *TUCS General Publication* **27** (2003) 75–82.

[13] A.M. Shur and Yu.V. Gamzova, Partial words and the interaction property of periods. *Izv. Math.* **68** (2004) 405–428.

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