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EQUATIONS ON PARTIAL WORDS*

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Abstract. It is well-known that some of the most basic properties of words, like the commutativity ($xy = yx$) and the conjugacy ($xz = zy$), can be expressed as solutions of word equations. An important problem is to decide whether or not a given equation on words has a solution. For instance, the equation $x^m y^n = z^p$ has only periodic solutions in a free monoid, that is, if $x^m y^n = z^p$ holds with integers $m, n, p \geq 2$, then there exists a word w such that x, y, z are powers of w. This result, which received a lot of attention, was first proved by Lyndon and Schützenberger for free groups. In this paper, we investigate equations on *partial words*. Partial words are sequences over a finite alphabet that may contain a number of "do not know" symbols. When we speak about equations on partial words, we replace the notion of equality ($=$) with compatibility ($\uparrow$). Among other equations, we solve $xy \uparrow yx$, $xz \uparrow zy$, and special cases of $x^m y^n \uparrow z^p$ for integers $m, n, p \geq 2$.

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1. Introduction

An important topic in algorithmic combinatorics on words is the *satisfiability problem for equations on words*, that is, the problem to decide whether or not a given equation on the free monoid has a solution. The problem was proposed in 1954 by Markov [29] and remained open until 1977 when Makanin answered it positively [28]. However, Makanin's algorithm is one of the most complicated algorithms ever presented and has at least exponential space complexity [25]. Rather recently, Plandowski showed, with a completely new algorithm, that the problem is actually in polynomial space [31,32]. However, the structure of the solutions cannot be found using Makanin's algorithm. Even for rather short instances of equations, for which the existence of solutions may be easily established, the structure of the solutions may be very difficult to describe. In particular, Hmelevskii proved that the set of solutions of $xyz = zvx$ cannot be described using only finitely many parameters, contrary to the case of equations in three unknowns [22] (see [17] for a short, elementary proof of Hmelevskii's result).

It is well known that some of the most basic properties of words, like the *commutativity* and the *conjugacy* properties, can be expressed as solutions of word equations. Two words x and y commute, namely $xy = yx$, if and only if x and y are powers of the same word, that is, there exists a word z such that $x = z^m$ and $y = z^n$ for some integers m and n. Two words x and y are *conjugate* if there exist words v and w such that $x = vw$ and $y = wv$. The latter is equivalent to the existence of a word z satisfying $xz = zy$ in which case there exist words v, w such that $x = vw$, $y = wv$, and $z = (vw)^n v$ for some nonnegative integer n. The equation $x^m y^n = z^p$ has only periodic solutions in a free semigroup, that is, if $x^m y^n = z^p$ holds with integers $m, n, p \geq 2$, then there exists a word w such that x, y, z are powers of w. This result, which received a lot of attention, was first proved by Lyndon and Schützenberger for free groups [27]. Proofs for free semigroups appear in [14,21,24].

In this paper, we investigate *equations on partial words*. When we speak about them, we replace the notion of *equality* with the notion of *compatibility*. A fundamental difference between equality and compatibility is that the latter is not transitive which makes this paper's results on partial words nontrivial adaptations of the corresponding results on words. Reference [23] presents some motivation from molecular biology for studying this type of equations on partial words. The contents of our paper are summarized as follows: Section 2 is devoted to reviewing basic concepts on words and partial words. There, we define in particular the containment relation ($\subset$) and the compatibility relation ($\uparrow$) on partial words. In Section 3, we give a result that expounds on the idea of the specialty of partial words satisfying the equation $x^m \uparrow y^n$. This result provides motivation on the conditions for when x and y are contained in powers of a common word. Section 4 reviews results on the equation $xy \uparrow yx$ on partial words that will be needed in later sections of our paper. In Section 5, we investigate the conjugacy equation $xz \uparrow zy$ on partial words. Our result is based on a decomposition of partial words x, y, z satisfying $xz \uparrow zy$ into $x = v_0 w_0$, $y = w_{m+1} v_{m+2}$ and $z = v_1 w_1 v_2 w_2 \dots v_m w_m v_{m+1}$

where $|v_i| = |z| \bmod |x|$ and $|w_i| = |x| - |v_i|$ for all i. We also study the system of equations $z \uparrow z'$ and $xz \uparrow z'y$. If $z = z'$, then this implies $xz \uparrow zy$. In Section 6, the equation $x^2 \uparrow y^m z$ on partial words is solved. This result is the first step for studying the equation $x^m y^n \uparrow z^p$ discussed in Section 7.

2. Preliminaries

Herein lies a brief description of terms and notations used for words and partial words.

2.1. Words

Let A be a nonempty finite set of symbols called an *alphabet*. Symbols in A are called *letters* and any finite sequence over A is called a *word* over A. The *empty word*, that is the word containing no letter, is denoted by ϵ. For any word u over A, $|u|$ denotes the number of letters occurring in u and is called the *length* of u. In particular, $|\epsilon| = 0$. The set of all words over A is denoted by A^*. If we define the operation of two words u and v of A^* by juxtaposition (or concatenation), then A^* is a monoid with identity ϵ. We call $A^+ = A^* \setminus \{\epsilon\}$ the *free semigroup generated by* A and A^* *the free monoid generated by* A. The set A^* can also be viewed as $\bigcup_{n\geq 0} A^n$ where $A^0 = \{\epsilon\}$ and A^n is the set of all words of length n over A.

A word of length n over A can be defined by a total function $u : \{0, \ldots, n-1\} \to A$ and is usually represented as $u = a_0 a_1 \ldots a_{n-1}$ with $a_i \in A$. A *period* of u is a positive integer p such that $a_i = a_{i+p}$ for $0 \leq i < n - p$. For a word u, the powers of u are defined inductively by $u^0 = \epsilon$ and, for any $i \geq 1$, $u^i = uu^{i-1}$. The *reversal* of u, denoted by $rev(u)$, is defined as follows: if $u = \epsilon$, then $rev(\epsilon) = \epsilon$, and if $u = a_0 a_1 \ldots a_{n-1}$, then $rev(u) = a_{n-1} \ldots a_1 a_0$. A word u is a *factor* of the word v if there exist words x, y such that $v = xuy$. The factor u is called *proper* if $u \neq \epsilon$ and $u \neq v$. The word u is a *prefix* (respectively, *suffix*) of v if $x = \epsilon$ (respectively, $y = \epsilon$).

A nonempty word u is *primitive* if there exists no word v such that $u = v^n$ with $n \geq 2$. Note the fact that the empty word is not primitive. If u is a nonempty word, then there exist a unique primitive word v and a unique positive integer n such that $u = v^n$.

2.2. Partial words

A *partial word* u of length n over A is a partial function $u : \{0, \ldots, n-1\} \to A$. For $0 \leq i < n$, if $u(i)$ is defined, then we say that i belongs to the *domain* of u, denoted by $i \in D(u)$, otherwise we say that i belongs to the *set of holes* of u, denoted by $i \in H(u)$. A word over A is a partial word over A with an empty set of holes (we sometimes refer to words as *full* words).

If u is a partial word of length n over A, then the *companion* of u denoted by $u_\diamond$, is the total function $u_\diamond : \{0, \ldots, n-1\} \to A \cup \{\diamond\}$ defined by

$$u_\diamond(i) = \begin{cases} u(i) & \text{if } i \in D(u), \\ \diamond & \text{otherwise.} \end{cases}$$

The bijectivity of the map $u \mapsto u_\diamond$ allows us to define for partial words concepts such as concatenation, powers, reversals, factors, prefixes, suffixes, etc in a trivial way. For instance, the reversal of u is defined by $(rev(u))_\diamond = rev(u_\diamond)$. The character $\diamond \notin A$ is viewed as a "do not know" character. The word $u_\diamond = abb\diamond bbcbb$ is the companion of the partial word u of length 9 where $D(u) = \{0, 1, 2, 4, 5, 6, 7, 8\}$ and $H(u) = \{3\}$. The length of the companion of a partial word u, also called the length of u, is denoted by $|u|$, and the set of distinct letters in A occurring in $u_\diamond$ is denoted by $\alpha(u)$. The set of all partial words over A with an arbitrary number of holes is denoted by $W(A)$. It is a monoid under the operation of concatenation with identity ϵ.

A *period* of a partial word u is a positive integer p such that $u(i) = u(j)$ whenever $i, j \in D(u)$ and $i \equiv j \bmod p$. In this case, we call u *p-periodic*. The smallest period of u is called the *minimal period* of u and is denoted by $p(u)$. A *weak period* of u is a positive integer p such that $u(i) = u(i+p)$ whenever $i, i+p \in D(u)$. In this case, we call u *weakly p-periodic*. The smallest weak period of u is called the *minimal weak period* of u and is denoted by $p'(u)$. Note that every weakly p-periodic full word is p-periodic but this is not necessarily true for partial words. Also even if the length of a partial word u is a multiple of a weak period of u, then u is not necessarily a power of a shorter partial word.

If u and v are partial words of equal length, then u is said to be contained in v denoted by $u \subset v$, if all symbols in $D(u)$ are in $D(v)$ and $u(i) = v(i)$ for all $i \in D(u)$. The order $u \subset v$ on partial words is obtained when we let $\diamond \prec a$ and $a \preceq a$ for all $a \in A$.

A partial word u is *primitive* if there exists no word v such that $u \subset v^n$ with $n \geq 2$. Note that if v is primitive and $v \subset u$, then u is primitive as well. It was shown in [4] that if u is a nonempty partial word, then there exist a primitive word v and a positive integer n such that $u \subset v^n$. However uniqueness does not hold as seen with the partial word u where $u_\diamond = \diamond a$ (here $u \subset a^2$ and $u \subset ba$ for distinct letters a, b). There, it was also shown that for partial words u and v, if there exists a primitive word x such that $uv \subset x^n$ for some positive integer n, then there exists a primitive word y such that $vu \subset y^n$. Moreover, if uv is primitive, then vu is primitive. These results extend similar results for words [35]. Also, it is immediate that if u is a primitive partial word, then $rev(u)$ is also primitive.

The partial words u and v are called *compatible*, denoted by $u \uparrow v$, if there exists a partial word w such that $u \subset w$ and $v \subset w$. We denote by $u \vee v$ the least upper bound of u and v. In other words, $u \subset u \vee v$ and $v \subset u \vee v$ and $D(u \vee v) = D(u) \cup D(v)$. As an example, $u_\diamond = aba\diamond\diamond a$ and $v_\diamond = a\diamond\diamond b\diamond a$ are the companions of two partial words u and v that are compatible and $(u \vee v)_\diamond = abab\diamond a$.

The following lemmas are useful for computing with partial words.

Lemma 1 (rules [1]). *Let u, v, w, x, y be partial words.*
Multiplication: *If $u \uparrow v$ and $x \uparrow y$, then $ux \uparrow vy$.*
Simplification: *If $ux \uparrow vy$ and $|u| = |v|$, then $u \uparrow v$ and $x \uparrow y$.*
Weakening: *If $u \uparrow v$ and $w \subset u$, then $w \uparrow v$.*

Lemma 2 (lemma of Lévi [1]). *Let u, v, x, y be partial words such that $ux \uparrow vy$.*
- *If $|u| \geq |v|$, then there exist partial words w, z such that $u = wz$, $v \uparrow w$, and $y \uparrow zx$.*
- *If $|u| \leq |v|$, then there exist partial words w, z such that $v = wz$, $u \uparrow w$, and $x \uparrow zy$.*

For convenience, we will refer to a partial word over A as a word over the enlarged alphabet $A \cup \{\diamond\}$, where the additional symbol $\diamond$ plays a special role. This allows us to say for example "the partial word $ab\diamond a\diamond b$" instead of "the partial word with companion $ab\diamond a\diamond b$".

3. The equation $x^m \uparrow y^n$ on partial words

In this section, we investigate the equation $x^m \uparrow y^n$ on partial words. The equation $x^m = y^n$ on words is well known. Indeed, if x and y are words, then $x^m = y^n$ for some positive integers m, n if and only if there exists a word z such that $x = z^k$ and $y = z^l$ for some integers k, l. When dealing with partial words x and y, if there exists a partial word z such that $x \subset z^k$ and $y \subset z^l$ for some integers k, l, then $x^m \uparrow y^n$ for some positive integers m, n. Indeed, by the multiplication rule, $x^l \subset z^{kl}$ and $y^k \subset z^{kl}$, showing that $x^l \uparrow y^k$. For the converse, it is beneficial to define the following manipulation of a partial word x. For a positive integer p and an integer $0 \leq i < p$, define $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ as

$$x(i)x(i+p)x(i+2p)\dots x(i+jp)$$

where j is the largest nonnegative integer such that $i + jp < |x|$. We shall call this the ith residual word of x modulo p.

Lemma 3 (equivalent condition for periodicity). *A partial word x is p-periodic if and only if $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is 1-periodic for all $0 \leq i < p$.*

Lemma 4 (equivalent condition for weak periodicity). *A partial word x is weakly p-periodic if and only if $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is weakly 1-periodic for all $0 \leq i < p$.*

Using the multiplication and the simplification rules, we can demonstrate the following lemma. Consequently, if $x^{m'} \uparrow y^{n'}$ and $\gcd(m', n') \neq 1$, then $x^m \uparrow y^n$ where $m = m'/\gcd(m', n')$ and $n = n'/\gcd(m', n')$. And therefore the assumption that $\gcd(m, n) = 1$ may be made without losing generality.

Lemma 5 (scaling). *Let x, y be partial words and let m, n and p be positive integers. Then $x^m \uparrow y^n$ if and only if $x^{mp} \uparrow y^{np}$.*

Lemma 6. *Let x, y be partial words and let m, n be positive integers such that $x^m \uparrow y^n$ with $\gcd(m,n) = 1$. Call $|x|/n = |y|/m = p$. If there exists an integer i such that $0 \leq i < p$ and $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is not 1-periodic, then $D(y\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right])$ is empty.*

Proof. Assume that there is an integer i such that $0 \leq i < p$ and $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is not 1-periodic. Then for some j and k such that $i + jp$ and $i + kp$ are in the domain of x,

$$x(i + jp) \neq x(i + kp).$$

Now assume that $D(y\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right])$ is not empty, that is, there is a constant l such that

$$y(i + (l + j)p) \neq \diamond.$$

Since one of the occurrences of $y(i + (l + j)p)$ in y^n is compatible with $x(i + jp)$, we have $y(i + (l + j)p) = x(i + jp)$ and hence

$$y(i + (l + j)p) \uparrow x((i + jp + l'|y|) \bmod |x|)$$

for all l'. Now we make the claim that there exists an l' such that

$$(i + jp + l'|y|) \equiv (i + kp) \bmod |x|. \tag{1}$$

Since $|y| = mp$ and $|x| = np$, (1) becomes

$$(j + l'm)p \equiv kp \bmod np$$

which may be reduced to

$$k - j \equiv l'm \bmod n.$$

Since $\gcd(m,n) = 1$, such an l' exists that satisfies our claim. Therefore

$$x(i + kp) = x((i + jp + l'|y|) \bmod |x|) \uparrow y(i + (l + j)p) = x(i + jp) \tag{2}$$

but we assumed earlier that $x(i + jp) \neq x(i + kp)$ and that $i + jp$ and $i + kp$ were both in the domain of x. Therefore the compatibility relation in (2) is a contradiction. □

Lemma 7. *Let x be a partial word, let m, p be positive integers, and let i be an integer such that $0 \leq i < p$. Then the relation*

$$x^m\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] = x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] x\left[\begin{smallmatrix} (i - |x|) \bmod p \\ p \end{smallmatrix}\right] x\left[\begin{smallmatrix} (i - 2|x|) \bmod p \\ p \end{smallmatrix}\right] \cdots x\left[\begin{smallmatrix} (i - (m-1)|x|) \bmod p \\ p \end{smallmatrix}\right]$$

holds.

Proof. The proof is by induction on m. Consider the case of $m = 2$. Note that

$$x^2\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] = x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] y$$

for some partial word y. Let k be the largest nonnegative integer such that $i+kp < |x|$. Then

$$y(0) = x(j)$$

where $j = (i+(k+1)p) \bmod |x|$. Therefore $i - j + (k+1)p = |x|$ by the definition of k and so

$$j = (i - |x|) \bmod p.$$

Hence $y = x\left[\begin{smallmatrix} j \\ p \end{smallmatrix}\right] = x\left[\begin{smallmatrix} (i-|x|) \bmod p \\ p \end{smallmatrix}\right]$ and the basis follows. Assume the relation holds for $m \leq n$. Then

$$x^{n+1}\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] = x^n\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] x\left[\begin{smallmatrix} (i-n|x|) \bmod p \\ p \end{smallmatrix}\right] =$$

$$x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] x\left[\begin{smallmatrix} (i-|x|) \bmod p \\ p \end{smallmatrix}\right] \cdots x\left[\begin{smallmatrix} (i-(n-1)|x|) \bmod p \\ p \end{smallmatrix}\right] x\left[\begin{smallmatrix} (i-n|x|) \bmod p \\ p \end{smallmatrix}\right]$$

which proves the lemma. □

Note that for any partial word x and positive integers m, p such that $|x|$ is divisible by p,

$$x^m\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] = (x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right])^m$$

where $0 \leq i < p$.

The following concept of a "1-pair" of partial words is basic in this paper.

Definition 1. Let x, y be partial words and let m, n be positive integers such that $x^m \uparrow y^n$ with $\gcd(m,n) = 1$. If for all $i \in H(x)$ the word

$$y^n\left[\begin{smallmatrix} i \\ |x| \end{smallmatrix}\right] = y^n(i)y^n(i+|x|)\dots y^n(i+(m-1)|x|)$$

is 1-periodic and for all $i \in H(y)$ the word

$$x^m\left[\begin{smallmatrix} i \\ |y| \end{smallmatrix}\right] = x^m(i)x^m(i+|y|)\dots x^m(i+(n-1)|y|)$$

is 1-periodic, then the pair (x, y) is called a **1-pair**.

Theorem 1. *Let x, y be partial words and let m, n be positive integers such that $x^m \uparrow y^n$ with $\gcd(m,n) = 1$. If (x, y) is a 1-pair, then there exists a partial word z such that $x \subset z^k$ and $y \subset z^l$ for some integers k, l.*

Proof. Since $\gcd(m,n) = 1$, there exists an integer p such that $\frac{|x|}{n} = \frac{|y|}{m} = p$. Now assume there exists an integer i such that $0 \leq i < p$ and $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is not 1-periodic. Then by Lemma 6, $i + jp \in H(y)$ for $0 \leq j < m$ which by the assumption that (x, y) is a 1-pair implies that $x^m\left[\begin{smallmatrix} i+jp \\ |y| \end{smallmatrix}\right]$ must be 1-periodic for any choice of j. Note that $|y| = mp$ and similarly $|x| = np$. Therefore by Lemma 7,

$$x^m\left[\begin{smallmatrix} i+jp \\ mp \end{smallmatrix}\right] = x\left[\begin{smallmatrix} i+jp \\ mp \end{smallmatrix}\right] x\left[\begin{smallmatrix} (i+jp-|x|) \bmod mp \\ mp \end{smallmatrix}\right] \cdots x\left[\begin{smallmatrix} (i+jp-(m-1)|x|) \bmod mp \\ mp \end{smallmatrix}\right].$$

Clearly

$$i + jp - l|x| = i + (j - ln)p$$

for all l. For $0 \leq j < m$, we claim that $\{(j-ln) \bmod m \mid 0 \leq l < m\} = \{0, 1, \dots, m-1\}$. Indeed, assuming there exist $0 \leq l_1 < l_2 < m$ such that

$$(j - l_1 n) \equiv (j - l_2 n) \bmod m$$

we get that m divides $(l_1 - l_2)n$, and since $\gcd(m,n) = 1$, that m divides $(l_1 - l_2)$, whence $l_1 = l_2$. So there exist $j_0, j_1, \ldots, j_{m-1}$ such that $j_0 = j$ and $\{j_0, j_1, \ldots, j_{m-1}\} = \{0, 1, \ldots, m-1\}$ and

$$x^m\left[\begin{smallmatrix} i+jp \\ mp \end{smallmatrix}\right] = x\left[\begin{smallmatrix} i+j_0p \\ mp \end{smallmatrix}\right] x\left[\begin{smallmatrix} i+j_1p \\ mp \end{smallmatrix}\right] \cdots x\left[\begin{smallmatrix} i+j_{m-1}p \\ mp \end{smallmatrix}\right].$$

Since $x^m\left[\begin{smallmatrix} i+jp \\ mp \end{smallmatrix}\right]$ is 1-periodic, there exists a letter a such that for all $0 \leq k < m$,

$$x\left[\begin{smallmatrix} i+j_kp \\ mp \end{smallmatrix}\right] \subset a^{m_{j_k}}$$

for some integer m_{j_k}. This contradicts our assumption that there is an i for which $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is not 1-periodic (here $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] = x(i)x(i+p)\ldots x(i+(n-1)p) \subset a^n$). Therefore $x\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is 1-periodic for all $0 \leq i < p$. By the equivalent condition for periodicity, this implies that x is p-periodic. The same argument holds for y, and since $x^m \uparrow y^n$, the result that there exists a word z of length p such that $x \subset z^n$ and $y \subset z^m$ is proven. □

The example $x^2 = (a \diamond b)^2 \uparrow (acbadb)^1 = y^1$ shows that the assumption of (x,y) being a 1-pair is necessary in Theorem 1. Here $y(1)y(4) = cd$ is not 1-periodic and there exists no partial word z as desired.

Corollary 1. *Let x and y be primitive partial words such that (x,y) is a 1-pair. If $x^m \uparrow y^n$ for some positive integers m and n, then $x \uparrow y$.*

Proof. Suppose to the contrary that $x \not\uparrow y$. Since (x,y) is a 1-pair, there exists a word z such that $x \subset z^k$ and $y \subset z^l$ for some integers k, l. Since $x \not\uparrow y$, we get $k \neq l$. But then x or y is not primitive, a contradiction. □

Note that if both x and y are full words, then (x,y) is a 1-pair. Corollary 1 hence implies that if x, y are primitive full words satisfying $x^m = y^n$ for some positive integers m and n, then $x = y$.

We conclude this section by further investigating the equation $x^2 \uparrow y^m$ on partial words where m is a positive integer.

Proposition 1. *Let x, y be partial words. Then $x^2 \uparrow y^m$ for some positive integer m if and only if there exist partial words $u, v, u_0, v_0, \ldots, u_{m-1}, v_{m-1}$ such that $y = uv$,*

$$x = (u_0v_0)\ldots(u_{n-1}v_{n-1})u_n = v_n(u_{n+1}v_{n+1})\ldots(u_{m-1}v_{m-1})$$

where $0 \leq n < m$, $u \uparrow u_i$ and $v \uparrow v_i$ for all $0 \leq i < m$, and where one of the following holds:

- *$m = 2n$ and $u = \epsilon$.*
- *$m = 2n+1$ and $|u| = |v|$.*

Proof. Note that if the conditions hold, then trivially $x^2 \uparrow y^m$ for some positive integer m. If $x^2 \uparrow y^m$ for some positive integer m, then we consider the cases where m is even or odd. If $m = 2n+1$ for some integer n, then there exist partial words u, v such that $y = uv, x \uparrow (uv)^n u$ and $x \uparrow v(uv)^n = (vu)^n v$. From this, we deduce that $|u| = |v|$. Now note that x may be factored as $x = (u_0v_0)\ldots(u_{n-1}v_{n-1})u_n =$

$v_n(u_{n+1}v_{n+1})\dots(u_{m-1}v_{m-1})$ where $u_i \uparrow u$ and $v_i \uparrow v$ for all $0 \le i < m$, If $m = 2n$ for some n, then $x \uparrow y^n$ and set $u = \epsilon$ in the above. □

4. The equation $xy \uparrow yx$ on partial words

It is well known that two nonempty words x and y commute if and only if there exists a word z such that $x = z^m$ and $y = z^n$ for some integers m, n. When dealing with two nonempty partial words x and y, the existence of a word z satisfying $x \subset z^m$ and $y \subset z^n$ for some integers m, n certainly implies $xy \uparrow yx$. To extend the converse to partial words, we first consider xy to have at most one hole.

Theorem 2 (commutativity one hole [1])**.** *Let x and y be nonempty partial words such that xy has at most one hole. If $xy \uparrow yx$, then there exists a word z such that $x \subset z^m$ and $y \subset z^n$ for some integers m, n.*

As stated in [1], Theorem 2 is false if xy has two holes. Take for example $x = \diamond bb$ and $y = abb\diamond$. To extend this theorem to the case when xy has at least two holes, we may assume $|x| \le |y|$. The extension is based on the concept of xy not being (k, l)-special where k, l denote the lengths of x, y respectively. For $0 \le i < k+l$, we define the sequence of i relative to k, l as $seq_{k,l}(i) = (i_0, i_1, i_2, \dots, i_n, i_{n+1})$ where $i_0 = i = i_{n+1}$ and where

For $1 \le j \le n, i_j \ne i$.
For $1 \le j \le n+1, i_j$ is defined as

$$i_j = \begin{cases} i_{j-1} + k & \text{if } i_{j-1} < l, \\ i_{j-1} - l & \text{otherwise.} \end{cases}$$

For example, if $k = 6$ and $l = 8$, then $seq_{(6,8)}(0) = (0, 6, 12, 4, 10, 2, 8, 0)$. Now, the concept of (k, l)-special is defined as follows.

Definition 2 ((k, l)-special [5])**.** Let k, l be positive integers satisfying $k \le l$ and let z be a partial word of length $k + l$. We say that z is (k, l)-special if there exists $0 \le i < k$ such that $seq_{k,l}(i) = (i_0, i_1, i_2, \dots, i_n, i_{n+1})$ contains (at least) two positions that are holes of z while $z_\diamond(i_0)z_\diamond(i_1)\dots z_\diamond(i_{n+1})$ is not 1-periodic.

If $k = 6$ and $l = 8$, then $z = acbca\diamond\diamond cbc\diamond cac$ is $(6, 8)$-special since $seq_{6,8}(0)$ contains the positions 6 and 10 which are in $H(z) = \{5, 6, 10\}$ while

$$z_\diamond(0)z_\diamond(6)z_\diamond(12)z_\diamond(4)z_\diamond(10)z_\diamond(2)z_\diamond(8)z_\diamond(0) = a\diamond aa\diamond bba$$

is not 1-periodic.

Theorem 3 (commutativity arbitrary number of holes [5])**.** *Let x, y be nonempty partial words such that $|x| \le |y|$. If $xy \uparrow yx$ and xy is not $(|x|, |y|)$-special, then there exists a word z such that $x \subset z^m$ and $y \subset z^n$ for some integers m, n.*

The concept of $\{k, l\}$-special and the following two lemmas will be useful in the sequel.

Definition 3 ($\{k,l\}$-special [10])**.** Let k, l be positive integers satisfying $k \leq l$ and let z be a partial word of length $k+l$. We say that z is $\{k,l\}$-special if there exists $0 \leq i < k$ such that $seq_{k,l}(i)$ satisfies the condition of Definition 2 or the condition of containing two consecutive positions that are holes of z.

If $k = 6$ and $l = 8$, then $z = \diamond babab\diamond\diamond ababab$ is $\{6,8\}$-special (but is not $(6,8)$-special). Indeed, $seq_{6,8}(0)$ contains the consecutive positions 0 and 6 that are holes of z.

Lemma 8 [1]**.** *Let x, y be nonempty words and let z be a partial word with at most one hole. If $z \subset xy$ and $z \subset yx$, then $xy = yx$.*

Lemma 9 [10]**.** *Let x, y be nonempty words and let z be a non $\{|x|,|y|\}$-special partial word. If $z \subset xy$ and $z \subset yx$, then $xy = yx$.*

Note that in Lemma 9, the assumption of z being non $\{|x|,|y|\}$-special cannot be replaced by the weaker assumption of z not being $(|x|,|y|)$-special. To see this, consider the partial words $x = ababab$, $y = cbabablab$, and $z = \diamond babab\diamond\diamond ababab$. Here, $z \subset xy$ and $z \subset yx$, but $xy \neq yx$.

The concept of (k,l)-special partial word, which relates to commutativity, turned out to be foundational in the design of our linear time algorithm for testing primitivity on partial words [5].

5. The equation $xz \uparrow zy$ on partial words

In this section, we consider the conjugacy property of partial words. Two partial words x and y are conjugate if there exist partial words v and w such that $x \subset vw$ and $y \subset wv$ [10]. It turns out that if the partial words x and y are conjugate, then there exists a partial word z satisfying the conjugacy equation $xz \uparrow zy$. The equation $xz = zy$ on words is well known. Indeed, if z is a word and x, y are nonempty words such that $xz = zy$, then there exist words v, w satisfying $x = vw$, $y = wv$, and $z = (vw)^n v$ for some integer $n \geq 0$. For partial words, the next similar result follows via the assumption of $xz \vee zy$ being $|x|$-periodic.

Theorem 4 [10]**.** *Let x, y, z be partial words with x, y nonempty. If $xz \uparrow zy$ and $xz \vee zy$ is $|x|$-periodic, then there exist words v, w such that $x \subset vw$, $y \subset wv$, and $z \subset (vw)^n v$ for some integer $n \geq 0$.*

As noted in [10], if z is a full word, then the assumption $xz \uparrow zy$ implies the one of $xz \vee zy$ being $|x|$-periodic and the following corollary holds. Note that Corollary 2 does not necessarily hold if z is not full even if x, y are full. The partial words $x = a, y = b$, and $z = \diamond bb$ provide a counterexample.

Corollary 2 [10]**.** *Let x, y be nonempty partial words, and let z be a full word. If $xz \uparrow zy$, then there exist words v, w such that $x \subset vw$, $y \subset wv$, and $z \subset (vw)^n v$ for some integer $n \geq 0$.*

First, we investigate the equation $xz \uparrow zy$ on partial words under the missing assumption of $xz \vee zy$ being $|x|$-periodic. The following two results give equivalences for conjugacy.

Theorem 5. *Let x, y and z be partial words such that $|x| = |y| > 0$. Then $xz \uparrow zy$ if and only if xzy is weakly $|x|$-periodic.*

Proof. Let m be defined as $\lfloor \frac{|z|}{|x|} \rfloor$ and n as $|z| \bmod |x|$. Then let $x = v_0w_0, y = w_{m+1}v_{m+2}$ and $z = v_1w_1v_2w_2 \dots v_mw_mv_{m+1}$ where each v_i has length n and each w_i has length $|x| - n$. We may now align xz and zy one above the other in the following way:

$$\begin{array}{cccccccccc} v_0 & w_0 & v_1 & w_1 & \dots & v_{m-1} & w_{m-1} & v_m & w_m & v_{m+1} \\ v_1 & w_1 & v_2 & w_2 & \dots & v_m & w_m & v_{m+1} & w_{m+1} & v_{m+2}. \end{array} \tag{3}$$

Assume $xz \uparrow zy$. Then the partial words in any column in (3) are compatible by simplification. Therefore for all i such that $0 \le i \le m+1$, $v_i \uparrow v_{i+1}$ and for all j such that $0 \le j \le m$, $w_j \uparrow w_{j+1}$. Thus $xz \uparrow zy$ implies that xzy is weakly $|x|$-periodic. Conversely, assume xzy is weakly $|x|$-periodic. This implies that $v_iw_i \uparrow v_{i+1}w_{i+1}$ for all i such that $0 \le i \le m$. Note that $v_{m+1}w_{m+1}v_{m+2}$ being weakly $|x|$-periodic, as a result $v_{m+1} \uparrow v_{m+2}$. This shows that $xz \uparrow zy$ which completes the proof. □

Theorem 6. *Let x, y and z be partial words such that $|x| = |y| > 0$. Then the following hold:*

1. *If $xz \uparrow zy$, then xz and zy are weakly $|x|$-periodic.*
2. *If xz and zy are weakly $|x|$-periodic and $\lfloor \frac{|z|}{|x|} \rfloor > 0$, then $xz \uparrow zy$.*

Proof. The proof is similar to that of Theorem 5. □

In Theorem 6(2), the assumption $\lfloor \frac{|z|}{|x|} \rfloor > 0$ is necessary. To see this, consider $x = aa$, $y = ba$ and $z = a$. Here, xz and zy are weakly $|x|$-periodic, but $xz \not\uparrow zy$.

Second, we consider solving the system of equations $z \uparrow z'$ and $xz \uparrow z'y$. Note that when $z = z'$, this system reduces to $xz \uparrow zy$. Let m be defined as $\lfloor \frac{|z|}{|x|} \rfloor$ and n as $|z| \bmod |x|$. Then let $x = v_0w_0, y = w_{m+1}v_{m+2}$, $z = v_1w_1v_2w_2 \dots v_mw_mv_{m+1}$, and $z' = v'_1w'_1v'_2w'_2 \dots v'_mw'_mv'_{m+1}$ where each v_i, v'_i has length n and each w_i, w'_i has length $|x| - n$. The $|x|$-*pshuffle* and $|x|$-*sshuffle* of xz and $z'y$ are defined as

$$\begin{aligned} &\mathrm{pshuffle}_{|x|}(xz, z'y) \\ &\qquad = v_0w_0v'_1w'_1v_1w_1v'_2w'_2 \dots v_{m-1}w_{m-1}v'_mw'_mv_mw_mv'_{m+1}w_{m+1}v_{m+1} \\ &\mathrm{sshuffle}_{|x|}(xz, z'y) = v_{m+1}v_{m+2}. \end{aligned}$$

Theorem 7. *Let x, y, z and z' be partial words such that $|x| = |y| > 0$ and $|z| = |z'| > 0$. Then $z \uparrow z'$ and $xz \uparrow z'y$ if and only if pshuffle$_{|x|}(xz, z'y)$ is weakly $|x|$-periodic and sshuffle$_{|x|}(xz, z'y)$ is $(|z| \bmod |x|)$-periodic.*

Proof. We may align z and z' (respectively, xz and $z'y$) one above the other in the following way:

$$\begin{array}{cccccccccc} v_1 & w_1 & v_2 & w_2 & \dots & v_{m-1} & w_{m-1} & v_m & w_m & v_{m+1} \\ v'_1 & w'_1 & v'_2 & w'_2 & \dots & v'_{m-1} & w'_{m-1} & v'_m & w'_m & v'_{m+1} \end{array} \tag{4}$$

$$\begin{array}{cccccccccc} v_0 & w_0 & v_1 & w_1 & \dots & v_{m-1} & w_{m-1} & v_m & w_m & v_{m+1} \\ v'_1 & w'_1 & v'_2 & w'_2 & \dots & v'_m & w'_m & v'_{m+1} & w_{m+1} & v_{m+2}. \end{array} \tag{5}$$

Assume $z \uparrow z'$ and $xz \uparrow z'y$. Then the partial words in any column in (4) (respectively, (5)) are compatible using the simplification rule. Therefore for all $0 \leq i < m$, $v_i w_i \uparrow v'_{i+1} w'_{i+1}$ and $v'_{i+1} w'_{i+1} \uparrow v_{i+1} w_{i+1}$. Also, we have $w_m \uparrow w_{m+1}$ and the following sequence of compatibility relations: $v_m \uparrow v'_{m+1}$, $v'_{m+1} \uparrow v_{m+1}$, and $v_{m+1} \uparrow v_{m+2}$. Thus, $\mathrm{pshuffle}_{|x|}(xz, z'y)$ is weakly $|x|$-periodic and $\mathrm{sshuffle}_{|x|}(xz, z'y)$ is $(|z| \bmod |x|)$-periodic. The converse follows symmetrically. □

The results in this section find some nice applications. In [11] for example, Blanchet-Sadri and Wetzler consider one of the most fundamental results on periodicity of words, namely the critical factorization theorem. Given a word w and nonempty words u, v satisfying $w = uv$, the *minimal local period* associated to the factorization (u, v) is the length of the shortest square at position $|u| - 1$. The critical factorization theorem shows that for any word, there is always a factorization whose minimal local period is equal to the minimal period of the word [12,13]. Blanchet-Sadri and Wetzler give a version of the critical factorization theorem for partial words (the one-hole case was considered earlier by Blanchet-Sadri and Duncan [8]). Their proof, which provides an efficient algorithm that computes a critical factorization when one exists, is based on the conjugacy equation on partial words.

6. The equation $x^2 \uparrow y^m z$ on partial words

In this section, we investigate the equation $x^2 \uparrow y^m z$ on partial words where it is assumed that m is a positive integer and z is a prefix of y. This equation has nontrivial solutions (a solution is *trivial* if x, y, z are contained in powers of a common word). Indeed, consider the compatibility relation $(a \diamond\diamond a)^2 \uparrow (aab)^2 aa$ where $x = a \diamond\diamond a$, $y = aab$ and $z = aa$. The equation $x^2 \uparrow y^m z$ will play a crucial role in the study of the equation $x^m y^n \uparrow z^p$ in the next section.

In order to characterize all solutions of the equation $x^2 \uparrow y^m z$, we need the concept of a "1-triple" of partial words.

Definition 4. Let x, y, z be partial words such that z is a proper prefix of y. Then (x, y, z) is a **1-triple** if for some positive integer m there exist partial words $u, v, u_0, v_0, \ldots, u_{m-1}, v_{m-1}, z_x$ such that $u \neq \epsilon$, $v \neq \epsilon$, $y = uv$,

$$x = (u_0 v_0) \ldots (u_{n-1} v_{n-1}) u_n \tag{6}$$

$$= v_n (u_{n+1} v_{n+1}) \ldots (u_{m-1} v_{m-1}) z_x \tag{7}$$

where $0 \leq n < m$, $u \uparrow u_i$ and $v \uparrow v_i$ for all $0 \leq i < m$, $z \uparrow z_x$, and where one of the following holds:

- $m = 2n$, $|u| < |v|$, and there exist partial words u', u'_n such that $z_x = u' u_n$, $z = u u'_n$, $u \uparrow u'$ and $u_n \uparrow u'_n$;
- $m = 2n + 1$, $|u| > |v|$, and there exist partial words v'_{2n} and z'_x such that $u_n = v_{2n} z_x$, $u = v'_{2n} z'_x$, $v_{2n} \uparrow v'_{2n}$ and $z_x \uparrow z'_x$.

Theorem 8. *Let x, y, z be partial words such that z is a proper prefix of y. Then $x^2 \uparrow y^m z$ for some positive integer m if and only if (x, y, z) is a 1-triple.*

Proof. Note that if the conditions hold, then trivially $x^2 \uparrow y^m z$ for some positive integer m. If $x^2 \uparrow y^m z$ for some positive integer m, then there exist partial words u, v and an integer n such that $y = uv, x \uparrow (uv)^n u$ and $x \uparrow v(uv)^{m-n-1}z$. Thus $|x| = n(|u| + |v|) + |u| = (m - n - 1)(|u| + |v|) + |v| + |z|$ which clearly shows

$$|z| = (2n - m + 2)|u| + (2n - m)|v|. \tag{8}$$

This determines a relationship between m and n. There are two cases to consider which correspond to assumptions on $|u|$ and $|v|$. Under the assumption $|u| = |v|$ we see that z must be either empty or equal to y which is a contradiction. If we assume $|u| < |v|$, then (8) shows $|z| = 2|u|$, and if we assume $|u| > |v|$, then $|z| = |u| - |v|$. Now note that x^2 may be factored in the following way:

$$x^2 = (u_0v_0)\dots(u_{n-1}v_{n-1})(u_nv_n)(u_{n+1}v_{n+1})\dots(u_{m-1}v_{m-1})z_x.$$

Here $u_i \uparrow u$ and $v_i \uparrow v$ and $z_x \uparrow z$. From this it is clear that (6) and (7) are satisfied.

Note that $u \neq \epsilon$ (otherwise $|u| < |v|$, in which case $|z| = 2|u| = 0$), and also $v \neq \epsilon$ (otherwise, $|u| > |v|$, in which case $|z| = |u| - |v| = |y|$). First assume $|u| < |v|$, equivalently $|z| = 2|u|$ and $m = 2n$. Note that the suffix of length $|u|$ of z_x must be u_n and therefore is compatible with u. The prefix of length $|u|$ of z must be u itself since z is a prefix of y. Thus $z_x = u'u_n$ and $z = uu'_n$ where $u \uparrow u'$ and $u_n \uparrow u'_n$ which is one of our assertions. Now assume $|u| > |v|$, that is $|z| = |u| - |v|$ and $m = 2n + 1$. Note by cancellation that $u_n = v_{2n}z_x$. Since $u_n \uparrow u$, we can rewrite u as $v'_{2n}z'_x$ where $v_{2n} \uparrow v'_{2n}$ and $z_x \uparrow z'_x$, which is our other assertion. □

Corollary 3. *Let x, y be partial words such that $|x| \geq |y| > 0$ and let z be a prefix of y. Assume that $x^2 \uparrow y^m z$ for some positive integer m. Referring to the notation of Theorem 8 (when $z \neq \epsilon$ and $z \neq y$) or referring to the notation of Proposition 1 (otherwise), both $w \uparrow uv$ and $w \uparrow vu$ hold where w denotes the prefix of length $|y|$ of x. Moreover, u and v are contained in powers of a common word if ($z = \epsilon$ and $m = 2n$) or ($z = y$ and $m + 1 = 2n$). This is also true if any of the following six conditions hold with $u \neq \epsilon$ and $v \neq \epsilon$:*

1. *y is full and w has at most one hole.*
2. *y is full and w is not $\{|u|, |v|\}$-special.*
3. *w is full and y has at most one hole.*
4. *w is full, and either ($|u| \leq |v|$ and uv is not $(|u|, |v|)$-special) or ($|v| \leq |u|$ and vu is not $(|v|, |u|)$-special).*
5. *$uv \uparrow vu$ and y has at most one hole.*
6. *$uv \uparrow vu$, and either ($|u| \leq |v|$ and uv is not $(|u|, |v|)$-special) or ($|v| \leq |u|$ and vu is not $(|v|, |u|)$-special).*

Proof. We first show the result when z is a proper prefix of y (or when $z \neq \epsilon$ and $z \neq y$). This part of the proof refers to the notation of Theorem 8. If $m > n+1$, then from the fact that $y = uv$ and $x \uparrow y^n u$ and $x \uparrow vy^{m-n-1}z$, we get $w \uparrow uv$ and $w \uparrow vu$. If on the other hand $m = n+1$, then $x \uparrow yu$ and $x \uparrow vz$. It follows that $|u| < |z|$ and we also get $w \uparrow uv$ and $w \uparrow vu$. For Statement 1, since u, v are full, we get $w \subset uv$ and $w \subset vu$ and by Lemma 8, $uv = vu$ and u, v are powers of a common word. For Statement 2, the result follows similarly since $uv = vu$ by Lemma 9. For Statement 3, we get $uv \uparrow vu$. By Theorem 2, u and v are contained in powers of a common word. Statement 4 follows similarly using Theorem 3. Statement 5 follows similarly as Statement 3, and Statement 6 as Statement 4.

We now show the result when $z = \epsilon$ (the case when $z = y$ is just a special case). This part of the proof refers to the notation of Proposition 1. The result trivially holds if $m = 2n$ since $u = \epsilon$ in this case. If $m = 2n+1$, then from the fact that $y = uv$ and $x \uparrow y^n u$ and $x \uparrow vy^n$, we get $w \uparrow uv$ and $w \uparrow vu$. The rest of the proof follows similarly as above. □

Corollary 4. *Let x, y, z be partial words such that z is a prefix of y. Assume that x, y are primitive and that $x^2 \uparrow y^m z$ for some integer $m \geq 2$. If x has at most one hole and y is full, then $x \uparrow y$.*

Proof. We show that $z = \epsilon$ and $m = 2$ (the result will then follow by simplification). Suppose to the contrary that $z \neq \epsilon$ or $m > 2$. In either case, we have $|x| > |y| > 0$. By Corollary 3, u and v are contained in powers of a common word, say $u \subset t^k$ and $v \subset t^l$ for some word t and nonnegative integers k, l. Indeed, this is trivially true when either $u = \epsilon$ or $v = \epsilon$. When both $u \neq \epsilon$ and $v \neq \epsilon$, Condition 1 of Corollary 3 is satisfied. Since $y = uv$ and y is primitive, we have ($k = 0$ and $l = 1$) or ($k = 1$ and $l = 0$). In the former case, $u = \epsilon$ and in the latter case, $v = \epsilon$. By Theorem 8, $z = \epsilon$ or $z = y$. If $z = \epsilon$, then $m > 2$. If m is even, then by Proposition 1, $m = 2n$ and $u = \epsilon$. Therefore, $x = v_0 \dots v_{n-1}$ with $n > 1$, and $x \subset v^n$ leading to a contradiction with the fact that x is primitive. If m is odd, then $m = 2n+1$ by Proposition 1 and $|u| = |v| = 0$ leading to a contradiction with the fact that $|y| = |uv| > 0$. Now, if $z = y$, then $x^2 \uparrow y^{m+1}$. If $m+1 = 2n$, then $u = \epsilon$ and $n > 1$, and if $m+1 = 2n+1$, then $|u| = |v| = 0$. In either case, we get a contradiction as above. □

Corollary 5 [14]**.** *Let x, y, z be words such that z is a prefix of y. If x, y are primitive and $x^2 = y^m z$ for some integer $m \geq 2$, then $x = y$.*

Note that Corollaries 4 and 5 do not hold when $m = 1$. Indeed, the words $x = aba$, $y = abaab$ and $z = a$ provide a counterexample. Also, Corollary 4 does not hold when x is full and y has one hole as is seen by setting $x = abaabb$, $y = ab\diamond$ and $z = \epsilon$.

7. The equation $x^m y^n \uparrow z^p$ on partial words

For integers $m \geq 2, n \geq 2$ and $p \geq 2$, Lyndon and Schützenberger showed that the equation $x^m y^n = z^p$ possesses a solution in a free group only when x, y, and z

are each a power of a common element [27]. The result is true in a free monoid as well [14]. The equation $x^m y^n \uparrow z^p$ in a free monoid $W(A)$ certainly has a solution when x, y, and z are contained in powers of a common word (we call such solutions the *trivial* solutions). However, there may be nontrivial solutions as is seen with the compatibility relation $(a \diamond b)^2 (b \diamond a)^2 \uparrow (abba)^3$. In this section, we characterize some of the solutions of the equation $x^m y^n \uparrow z^p$ for the case where $p \geq 4$. The characterization is stated as Theorem 9 which we show with a series of case proofs. We reduce the number of cases by using the following lemma.

Lemma 10. *Let x, y, z be partial words and let m, n, p be positive integers. If $x^m y^n \uparrow z^p$, then $(rev(y))^n (rev(x))^m \uparrow (rev(z))^p$.*

It will turn out that, in a free monoid $W(A)$, the equation $x^m y^n \uparrow z^p$, where $m \geq 2, n \geq 2$ and $p \geq 4$, may have solutions of the following types.

Definition 5 (type 1)**.** There exists a partial word w such that x, y, z are contained in powers of w. We call such solutions the *trivial* or *type 1* solutions.

Definition 6 (type 2)**.** The partial words x, y, z satisfy $x \uparrow z$ and $y \uparrow z$. We call such solutions the *type 2* solutions.

If z is full, then type 2 solutions are trivial solutions.

Theorem 9 ($p \geq 4$)**.** *Let x, y, z be primitive partial words such that (x, z) and (y, z) are 1-pairs. Let m, n, p be integers such that $m \geq 2, n \geq 2$ and $p \geq 4$. Then the equation $x^m y^n \uparrow z^p$ has only solutions of type 1 or type 2 unless $x^2 \uparrow z^k z_p$ for some integer $k \geq 2$ and nonempty prefix z_p of z, or $z^2 \uparrow x^l x_p$ for some integer $l \geq 2$ and nonempty prefix x_p of x.*

Proof. By Lemma 10, we need only examine the case when $|x^m| \geq |y^n|$. Now assume $x^m y^n \uparrow z^p$ has some solution that is not of type 1 or type 2. Our assumption on the lengths of x^m and y^n implies that $|x^m| \geq |z^2|$ and therefore either $|x^2| \geq |z^2|$ or $|x^2| < |z^2|$. Hence one of the following equations will be satisfied: $x^2 \uparrow z^k z_p$ for some integer $k \geq 2$ and prefix z_p of z, or $z^2 \uparrow x^l x_p$ for some integer $l \geq 2$ and prefix x_p of x.

Consider the case where z_p or x_p is the empty word. In either case, Corollary 1 implies that $x \uparrow z$. From $x^m y^n \uparrow z^p$ and $x \uparrow z$, using Lemma 1, we get $y^n \uparrow z^{p-m}$. Using Corollary 1 again, we have $y \uparrow z$. Hence these cases form type 2 solutions. □

Corollary 6 [14]**.** *Let x, y, z be primitive words and let m, n, p be integers such that $m \geq 2, n \geq 2$ and $p \geq 4$. Then the equation $x^m y^n = z^p$ has no nontrivial solutions.*

Proof. As in the proof of Theorem 9, we need only examine the case when $|x^m| \geq |y^n|$. This assumption leads to either $x^2 = z^k z_p$ for some integer $k \geq 2$ and prefix z_p of z, or $z^2 = x^l x_p$ for some integer $l \geq 2$ and prefix x_p of x. In either case, we have $x = z$ by Corollary 5, and from the equation $x^m y^n = z^p$, we get $y^n = z^{p-m}$ and $y = z$ since y, z are primitive full words. □

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